CONFORMAL INVARIANCE, DROPLETS, AND ENTANGLEMENT

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Very brief surveys are presented of three topics of importance for interacting random systems, namely conformal invariance, droplets, and entanglement. For ease of description, the emphasis throughout is upon progress and open problems for the percolation model, rather than for the more general random-cluster model. Substantial recent progress has been made on each of these topics, as summarised here. Detailed bibliographies of recent work are included.

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1 Introduction

Rather than attempt to summarise the 'state of the art' in percolation and disordered systems, a task for many volumes, we concentrate in this short article on three areas of recent progress, namely conformal invariance, droplets, and entanglement. In each case, the target is to stimulate via a brief survey, rather than to present the details.

Much of the contents of this article may be expressed in terms of the random-cluster model, but for simplicity we consider here only the special case of percolation, defined as follows. Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d$; that is, $\mathcal{L}$ is an infinite, connected, locally finite graph embedded in $\mathbb{R}^d$ which is invariant under translation by any basic unit vector. We write $\mathcal{L} = (\mathcal{V}, \mathcal{E})$, and we choose a vertex of $\mathcal{L}$ which we call the *origin*, denoted 0. The *cubic lattice*, denoted $\mathbb{Z}^d$, is the lattice in $\mathbb{R}^d$ with integer vertices and with edges joining pairs of vertices which are Euclidean distance 1 apart.

Let $0 \leq p \leq 1$. In bond percolation on $\mathcal{L}$, each edge is designated *open* with probability p, and *closed* otherwise, different edges receiving independent designations. In site percolation, it is the vertices of $\mathcal{L}$ rather than its edges which are designated open or closed. In either case, for $A, B \subseteq V$, we write $A \leftrightarrow B$ if there exists an open path joining some $a \in A$ to some $b \in B$,

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and we write $A \leftrightarrow \infty$ if there exists an infinite open self-avoiding path from some vertex in A.

Let P_p denote the appropriate product measure, and let

$$\theta(p) = P_p(0 \leftrightarrow \infty)$$

be the probability that the origin lies in some infinite open self-avoiding path. The critical probability of the process is defined by

$$p_c = \sup\{p : \theta(p) = 0\}.$$

Note that the values of $\theta(p)$ and p_c depend on the choice of $\mathcal{L}$ and on the type (bond or site) of the process; we shall suppress this information whenever it is clear from the context.

For more information about the mathematics of percolation, see Grimmett (1997, 1999). Periodic reference will be made to a more general model called the *random-cluster model*. There is a sense in which the latter model includes percolation, Ising, and Potts models as special cases; the reader is referred to Grimmett (1995) for a recent account of random-cluster processes.

The next three sections contain summaries of recent progress on conformal invariance, the droplet problem, and the question of entanglement, respectively.

2 Conformal invariance

Consider a random spatial process in $\mathbb{R}^2$, perhaps a percolation process on some two-dimensional lattice. Let us assume the existence, in an appropriate sense, of the limit process obtained by a spatial re-scaling of the original process by an increasing sequence of factors. Under certain assumptions, the law of the limit process is expected to be invariant under conformal maps of the underlying space $\mathbb{R}^2$.

This remarkable speculation has emerged from conformal field theory, and is relevant to a variety of random processes including the percolation and Ising models. For simplicity, we consider here the case of percolation only.

Let $\mathcal{L}$ be a lattice in two dimensions, and let p_c be its critical probability (we shall not at this stage be specific whether it is *bond* or *site* percolation under consideration). Consider a percolation process on $\mathcal{L}$, with $p = p_c$. The hypothesis of universality suggests that the chances of long-range connections should, to some degree, be independent of certain 'local fluctuations' in $\mathcal{L}$. In particular, local deformations of space, within limits, are not expected to affect such probabilities. One family of local changes arises by local rotations and dilations, and such mappings of $\mathbb{R}^2$ constitute the set of conformal maps.

We make this specific in the manner surveyed and pursued by Langlands et al. (1992, 1994). Take a simple closed curve γ in the plane, and disjoint arcs α, β of γ. For a dilation factor $r \geq 1$, define

$$\pi_r(\gamma; \alpha, \beta) = P(r\alpha \leftrightarrow r\beta \text{ in } r\gamma)$$

where $P = P_{p_c}$ and the event in question is the event that there exists an open path of $\mathcal{L}$ whose intersection with the inside of γ contains an open connection between $r\alpha$ and $r\beta$. (See Figure 2.1.)

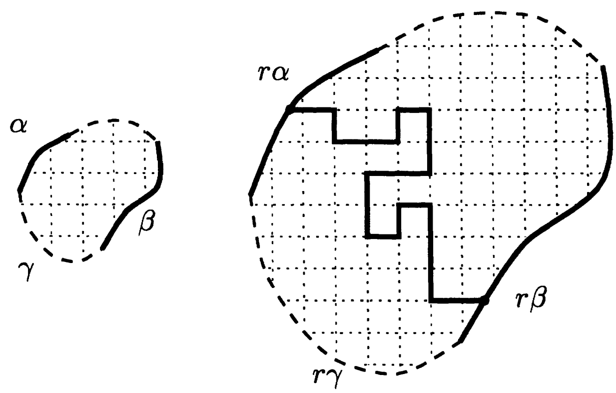

Figure 2.1. An illustration of the event that $r\alpha$ is joined to $r\beta$ within $r\gamma$, in the case of bond percolation on $\mathbb{Z}^2$.

The first conjecture is the existence of the limit

$$\pi(\gamma; \alpha, \beta) = \lim_{r \to \infty} \pi_r(\gamma; \alpha, \beta) \tag{2.1}$$

for all triples $(\gamma; \alpha, \beta)$. Some convention is needed in order to make sense of (2.1), arising from the fact that $r\gamma$ lives in the plane $\mathbb{R}^2$ rather than on the lattice $\mathcal{L}$; this poses no major problem. Only in very special cases is (2.1) known to hold. For example, in the case of bond percolation on $\mathbb{Z}^2$, self-duality enables a proof of the existence of the limit when γ is a square and α, β are opposite sides thereof; for this special case, the limit equals $1/2$.

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be conformal on the inside of γ and bijective on the curve γ itself. The hypothesis of conformal invariance states that

$$\pi(\gamma; \alpha, \beta) = \pi(\phi\gamma; \phi\alpha, \phi\beta) \tag{2.2}$$

for all such ϕ. This conjecture concerns only percolation of a specific type (bond or site) on a specific lattice. More general forms of the conjecture come readily to hand; see Langlands, Pouliot, and Saint-Aubin (1994). First, it is natural to extend the conjecture to include the existence (or not) of crossings between more than one pair of arcs of the curve γ. Secondly, the conjecture may be extended to include a hypothesis of universality.

Lengthy computer simulations, reported by Langlands, Pouliot, and Saint-Aubin (1994), support these conjectures. Particularly stimulating evidence is provided by a formula known as Cardy's formula. By following a sequence of transformations of models, and applying ideas of conformal field theory, Cardy (1992) was led to the following explicit formula for crossing probabilities between two arcs of a simple closed curve γ.

Let γ be a simple closed curve, and let z_1, z_2, z_3, z_4 be four points on γ in clockwise order. There is a conformal map ϕ on the inside of γ which maps to the unit disc, taking γ to its circumference, and the points z_i to the points w_i. There are many such maps, but the cross-ratio of such maps,

$$u = \frac{(w_4 - w_3)(w_2 - w_1)}{(w_3 - w_1)(w_4 - w_2)}, \tag{2.3}$$

is a constant satisfying $0 \le u \le 1$ (we think of z_i and w_i as points in the complex plane). We may parametrise the w_i as follows: we may assume that

$$w_1 = e^{i\delta}, \quad w_2 = e^{-i\delta}, \quad w_3 = -e^{i\delta}, \quad w_4 = -e^{-i\delta}$$

for some δ satisfying $0 \le \delta \le \frac{1}{2}\pi$. Note that $u = \sin^2 \theta$. We take α to be the segment of γ from z_1 to z_2, and β the segment from z_3 to z_4. Using the hypothesis (2.2) of conformal invariance, we deduce that $\pi(\gamma; \alpha, \beta)$ may be expressed as some function $f(u)$, where u is given in (2.3). Cardy (1992) has derived (non-rigorously) a differential equation for f, namely

$$u(1-u)f''(u) + \tfrac{2}{3}(1-2u)f'(u) = 0,$$

subject to the boundary conditions $f(0) = 0$, $f(1) = 1$. The solution is

$$f(u) = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}\, u^{1/3}\, {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; u), \tag{2.4}$$

where ${}_2F_1$ is a hypergeometric function. The derivation is somewhat speculative, but the predictions of the formula may be verified by Monte Carlo simulation (see Figure 3.2 of Langlands, Pouliot, and Saint-Aubin (1994)).

The function in (2.4) appears complicated, and calls for an intuitive motivation. It turns out that there is a special choice for the triple $(\gamma; \alpha, \beta)$ for which the formula in (2.4) takes an extremely simple form, as follows. Consider site percolation on the triangular lattice, illustrated in Figure 2.2; the critical probability of the process is $p_c = \frac{1}{2}$ (see Grimmett (1999), Section 11.9). Let x satisfy $0 \le x \le 1$. Take γ to be an equilateral triangle of unit side-length; let α be one side of γ, and β be a sub-interval of another side, with length x and having the vertex opposite to α as an endpoint. (See Figure 2.3.) It may be conjectured that

$$\pi(\gamma; \alpha, \beta) = x. \tag{2.5}$$

Subject to a suitable generalised form of the hypothesis of conformal invariance, (2.5) is equivalent to Cardy's formula. Conjecture (2.5) appears to be due to L. Carleson. It is supported by numerical simulations of Bain (1999) and probably others. Formula (2.5) may be justified by the self-matching property when $x = \frac{1}{2}$.

Note that (2.5) should be valid for other processes also, such as *bond percolation* on the triangular lattice when p equals its critical value.

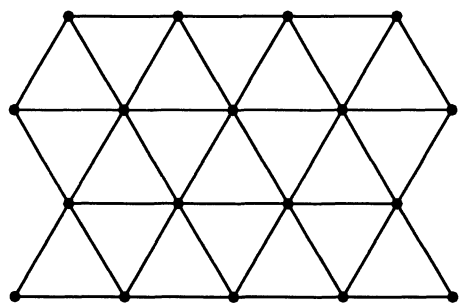

Figure 2.2. Part of the triangular lattice.

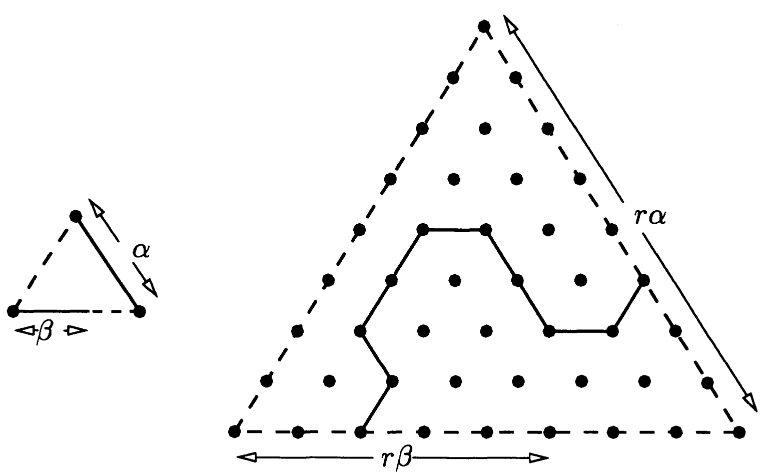

Figure 2.3. An illustration of the triple $(\gamma; \alpha, \beta)$ and of the event relevant to the probability $\pi_r(\gamma; \alpha, \beta)$.

The above 'calculations' are striking. As suggested by Aizenman (1995), similar calculations may well be possible for more complicated crossing probabilities than the cases discussed above. For example, Watts (1996) has performed numerical simulations which give support to a conjecture for the limiting probability that a large rectangle is crossed from left to right and simultaneously from top to bottom.

In the above formulation, the principle of conformal invariance is expressed in terms of a collection $\{\pi(\gamma; \alpha, \beta)\}$ of limiting 'crossing probabilities'. It would be useful to have a representation of these $\pi(\gamma; \alpha, \beta)$ as probabilities associated with a specific random variable on a specific probability space. Aizenman (1995) has made certain proposals about how this might be possible. In his formulation, we observe a bounded region $D_R = [0, R]^2$, and we shrink the lattice spacing a of bond percolation restricted to this domain. Let $p = p_c$, and let G_a be the graph of open connections of bond percolation with lattice spacing a on D_R. By describing G_a through the set of Jordan curves describing the realised paths, he has apparently obtained sufficient compactness to imply the existence of weak limits as $a \to 0$. Possibly there is a *unique* weak limit, and Aizenman has termed an object sampled according to this limit as the 'web'. The fundamental conjectures are therefore that there is a unique weak limit, and that this limit is conformally invariant. Further work in this direction may be found in Aizenman and Burchard (1999).

The quantities $\pi(\gamma; \alpha, \beta)$ should then arise as crossing probabilities in 'web-measure'. This geometrical vision may be useful to physicists and mathematicians in understanding conformal invariance.

Mathematicians have long been interested in the existence of long open connections in critical percolation models in $\mathbb{R}^d$ (see, for example, Kesten (1982), Kesten and Zhang (1993)). An overall description of such connections will depend greatly on whether d is small or large. When $d = 2$, a complex picture is expected, involving long but finite paths on all scales whose geometry may be described as 'fractal'. See Aizenman (1997, 1998) for accounts of the current state of knowledge. A particular question of interest is to ascertain the fractal dimension of the exterior boundary of a large droplet (see Section 3 of the current paper). Such questions are linked to similar problems for Brownian Motion in two dimensions. The (rigorous) conformal invariance of Brownian Motion has been used to derive certain exact calculations, some of which are rigorous, of various associated critical exponents (see Lawler and Werner (1998) and Duplantier (1999), for example). Such results support the belief that similar calculations are valid for percolation.

The picture for large d is expected to be quite different. Indeed, Hara and Slade (1999a, 1999b) have recently proved that, for large d, the two- and three-point connectivity functions of critical percolation converge to appropriate correlation functions of the process known as Integrated Super-Brownian Excursion.

In one interesting 'continuum' percolation model, conformal invariance may actually be proved rigorously. We drop points $\{X_1, X_2, \ldots\}$ in the plane $\mathbb{R}^2$ in the manner of a Poisson process with intensity λ. Now divide $\mathbb{R}^2$ into tiles $\{T(X_1), T(X_2), \dots\}$, where $T(X)$ is defined as the set of points in $\mathbb{R}^2$ which are no further from X than they are from any other point of the Poisson process (this is the 'Dirichlet' or 'Voronoi' tesselation). We designate each tile to be open with probability $\frac{1}{2}$ and closed otherwise. This continuum percolation model has a property of self-duality, and, using the conformal invariance and other properties of the Poisson point process, one may show in addition that it satisfies conformal invariance. See Aizenman (1998) and Benjamini and Schramm (1998).

We note that Langlands et al. (1999) have reported a largely numerical study of conformal invariance for the two-dimensional Ising model.

3 Droplets and large deviations

Consider the Ising model on a finite box B of the square lattice $\mathbb{Z}^2$ with $+$ boundary conditions, and suppose that the temperature T is low. (We omit a formal definition of the Ising model, which is known to many, and which is not central to this short review.) The origin may lie within some region whose interior spins behave as in the $-$ phase, but it is unlikely that such a region, or 'droplet', is large. What is the probability that this droplet is indeed large? Conditional on its being large, what is its approximate shape? For low T, such questions were answered by Dobrushin, Kotecký, and Shlosman (1992), who proved amongst other things that droplets have approximately the shape of what is termed a Wulff crystal (after Wulff (1901)). In later work, such results were placed in the context of the associated random-cluster model, and were proved for all subcritical T; see Ioffe (1994, 1995), Ioffe and Schonmann (1998), and the references therein.

In a parallel development for percolation on $\mathbb{Z}^2$, Alexander, Chayes, and Chayes (1990) explored the likely shape of a large finite open cluster when $p > p_c$. They established a Wulff shape, and proved in addition the existence of $\eta(p) \in (0, \infty)$ such that

$$-\frac{1}{\sqrt{n}} \log P_p(|C| = n) \to \eta(p) \quad \text{as } n \to \infty \tag{3.1}$$

where C denotes the set of vertices which are connected to the origin by open paths.

The geometrical framework for such results begins with a definition of 'surface tension'. The details of this are beyond this article, but the very rough idea is as follows. Let $\mathbf{k}$ be a unit vector, and let $\sigma(\mathbf{k}, p)$ denote 'surface tension in direction $\mathbf{k}$'. For the Ising model, $\sigma(\mathbf{k}, p)$ is defined in terms of the probability of the existence of a certain type of interface orthogonal to $\mathbf{k}$; for percolation, one considers the probability of a certain type of dual path of closed edges which is, in a sense to be defined, orthogonal to $\mathbf{k}$. When suitably defined, these probabilities decay exponentially, and the relevant exponents allow a definition of 'surface tension' in each case. Given a closed curve γ, one may define its energy $\eta(\gamma)$ as the integral along γ of $\sigma(\mathbf{k}, p)$, where $\mathbf{k}$ denotes the normal vector to γ. We say that γ encloses a 'Wulff crystal' if $\eta(\gamma) \leq \eta(\gamma')$ for all closed curves γ' enclosing the same area as γ.

We make this discussion of surface tension more concrete in the case of two dimensions, following Alexander, Chayes, and Chayes (1990). For a unit vector $\mathbf{k}$ and an integer n, let $[n\mathbf{k}]$ be a vertex of $\mathbb{Z}^2$ lying closest to $n\mathbf{k}$. The existence of the limit

$$\sigma(\mathbf{k}, p) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P_{1-p}(0 \leftrightarrow [n\mathbf{k}]) \right\}$$

follows by subadditivity, and this may be used as a definition of surface tension.

Consider now the percolation model on $\mathbb{Z}^2$ with $p > p_c$. If $|C| < \infty$, the origin lies within some closed dual circuit γ. For a wide variety of possible γ, the circuit γ contains with large probability a large open cluster of size approximately $\theta(p)|\operatorname{ins}(\gamma)|$, where $\operatorname{ins}(\gamma)$ denotes the inside of γ. It turns out that, amongst all γ with $\theta(p)|\operatorname{ins}(\gamma)| = n$, say, the γ having largest probability may be approximated by the Wulff crystal enclosing area $n/\theta(p)$. The length of such γ has order $\sqrt{n}$, and one is led towards (3.1). A substantial amount of work is required to make this argument rigorous.

It is a great advantage to work in two dimensions, and until recently there has been only little progress towards understanding how to prove such results in three dimensions. Topological and probabilistic problems intervened. However, a recent paper of Cerf (1998) has answered such problems, and has shown the way to a Wulff construction in three dimensions. Cerf has proved a large deviation principle from which the Wulff construction emerges. A key probabilistic tool is the 'coarse graining' of Pisztora (1996), which is itself based on the results of Grimmett and Marstrand (1990); see also Grimmett (1999, Section 7.4).

Cerf's paper has provoked a further look at the Ising model, this time in three dimensions. Bodineau (1999) has achieved a Wulff construction for low temperatures, and Cerf and Pisztora (1999) have proved such a result for all T smaller than a certain value T_{slab} believed equal to the critical temperature T_c. The latter paper used methods of Pisztora (1996) concerning 'coarse graining' for random-cluster models.

4 Entanglement

The theory of long-chain polymers has led to the study of entanglements in systems of random arcs of $\mathbb{R}^3$. Suppose that a set of arcs is chosen within $\mathbb{R}^3$ according to some given probability measure μ. Under what conditions on μ does there exist with strictly positive probability one or more infinite entanglements? Such a question was posed implicitly by Kantor and Hassold (1988), and has been studied further for bond percolation on the cubic lattice $\mathbb{Z}^3$ by Aizenman and Grimmett (1991), Holroyd (1998), and Grimmett and Holroyd (1999).

It is first necessary to decide on a definition of an 'entanglement'. Let $\mathbb{E}$ be the edge set of $\mathbb{Z}^3$. We think of an edge e as being the closed line segment of $\mathbb{R}^3$ joining its endpoints. For $E \subseteq \mathbb{E}$, we let $[E]$ be the union of the edges in E. The term 'sphere' is used to mean a subset of $\mathbb{R}^3$ which is homeomorphic to the unit sphere. The complement of a sphere S has two connected components, an unbounded *outside* denoted $\mathrm{out}(S)$, and a bounded *inside* denoted $\mathrm{ins}(S)$. For $E \subseteq \mathbb{E}$ and a sphere S, we say that S *separates* E if $S \cap [E] = \emptyset$ but $[E]$ has non-empty intersection with both inside and outside of S.

Let E be a non-empty finite subset of $\mathbb{E}$. We call E *entangled* if it is separated by no sphere. See Figure 4.1.

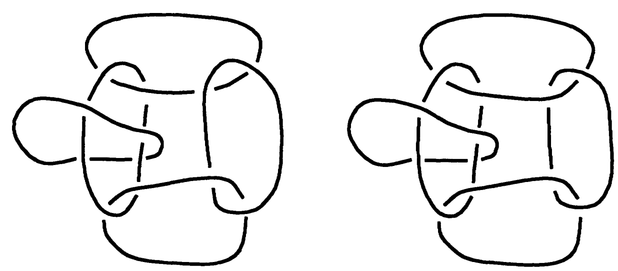

Figure 4.1. The left graph is not entangled; the right graph is entangled.

There appears to be no unique way of defining an infinite entanglement, and the 'correct' way is likely to depend on the application in question. Two specific ways propose themselves, and it turns out that the corresponding definitions are 'extreme' in a manner to be explained soon.

Let E be a (non-empty) finite or infinite subset of $\mathbb{E}$.

(a) We call E *strongly entangled* if, for every finite subset F of E, there exists a finite entangled subset F' of E satisfying $F \subseteq F'$.

(b) We call E *weakly entangled* if it is separated by no sphere.

Note that all connected graphs are entangled in both manners, and that a finite subset of $\mathbb{E}$ is strongly entangled if and only if it is weakly entangled.

Let $\mathcal{E}_f$ (respectively $\mathcal{E}_\infty$) be the collection of all strongly entangled sets of edges (respectively weakly entangled sets). It is proved in Grimmett and Holroyd (1999) that $\mathcal{E}_f \subseteq \mathcal{E}_\infty$, and that these sets are extreme in the sense that $\mathcal{E}_f \subseteq \mathcal{E} \subseteq \mathcal{E}_\infty$ for any collection $\mathcal{E}$ of non-empty subsets of $\mathbb{E}$ having the following three properties:

(i) the intersection of $\mathcal{E}$ with the set of finite graphs is exactly the set of finite entangled graphs;
(ii) if $E \in \mathcal{E}$, then E is separated by no sphere;
(iii) let $E_1, E_2, \ldots \in \mathcal{E}$ be a sequence such that, for every pair i and j, E_i and E_j have a common vertex; then $\bigcup_i E_i \in \mathcal{E}$.

Furthermore, $\mathcal{E}_f$ and $\mathcal{E}_\infty$ satisfy conditions (i)–(iii).

The reason for the notation $\mathcal{E}_f$ and $\mathcal{E}_\infty$ is that the notions of weak and strong entanglement arise naturally through a consideration of finite entanglements within the box $[-n,n]^3$ in the limit of large n, with 'free' and 'wired' boundary conditions respectively.

Let J_0 (respectively J_1) be the event that the origin of $\mathbb{Z}^3$ lies in an infinite, open, strongly (respectively weakly) entangled set E; note that J_0 and J_1 are increasing events. We define the *strong* and *weak entanglement probabilities* by

$$\theta_0(p) = P_p(J_0), \quad \theta_1(p) = P_p(J_1),$$

and the associated *entanglement critical points*

$$\begin{aligned} p_c^0 &= \sup\{p : \theta_0(p) = 0\}, \\ p_c^1 &= \sup\{p : \theta_1(p) = 0\}. \end{aligned}$$

Since $\mathcal{E}_f \subseteq \mathcal{E}_\infty$, it is immediate that $\theta_0(p) \le \theta_1(p)$, whence $p_c^0 \ge p_c^1$.

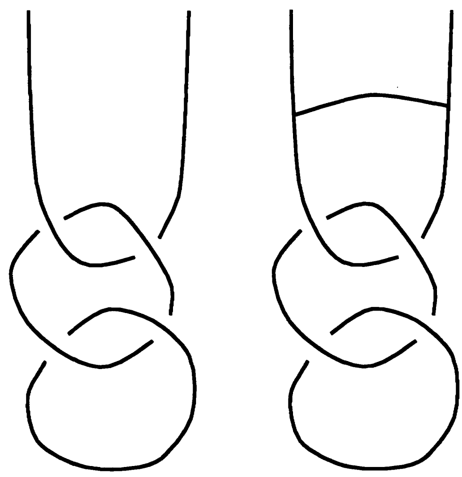

Figure 4.2. The left graph is weakly but not strongly entangled; the right graph is both strongly and weakly entangled.

It is proved in Grimmett and Holroyd (1999) that $\theta_0(p) = \theta_1(p)$ for all values of p sufficiently close to 1, and it may be conjectured that

$$\begin{aligned} p_c^0 = p_c^1 = p_c^{\text{ent}}, &\quad \text{for some } p_c^{\text{ent}} \in (0,1), \\ \theta_1(p) = \theta_0(p) &\quad \text{if } p > p_c^{\text{ent}}. \end{aligned}$$

Such mathematical questions were not treated in the initial paper of Kantor and Hassold (1988). Numerical work reported there suggested the existence of an 'entanglement critical point' p_c^{ent} satisfying $p_c^{\mathrm{ent}} \approx p_c - 1.8 \times 10^{-7}$. No formal definition of this critical point was presented, and indeed the discussion of this initial paper concerned only *finite* entanglements. The strict inequality $p_c^0 < p_c$ follows by the argument presented in Aizenman and Grimmett (1991). The complementary inequality $p_c^1 > 0$ has been proved by Holroyd (1998).

The list of open problems concerning entanglement in percolation includes: proving the almost sure equivalence of the notions of strong and weak entanglement, establishing an exponential tail for the size of the maximal finite entanglement containing the origin when $p < p_c^{\mathrm{ent}}$, and deciding whether or not there exists an infinite entanglement of a given type when p equals the appropriate critical value. Uniqueness of the infinite entanglement when p exceeds the corresponding critical value has been proved recently by Häggström (1999), but the critical process (when p equals the critical value) is still not understood.

The following combinatorial question may prove interesting. Let η_n be the number of finite entangled subsets of $\mathbb{E}$ which contain the origin and have exactly n edges. Does there exist a constant A such that $\eta_n \leq A^n$ for all n?

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