LOCALIZATION AND DECAY OF CORRELATIONS FOR A PINNED LATTICE FREE FIELD IN DIMENSION TWO

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We prove that the two-dimensional harmonic crystal with a weak local pinning to a wall has finite variance and exponentially decaying correlations, regardless how weak the pinning is. The proof is based on an improved pressure estimate and an application of reflection positivity.

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1 A survey on models and questions

There is a natural class of generalizations of the standard random walks to higher dimensional "time", which are mainly considered in mathematical physics. To motivate them, we consider first a standard real valued random walk $X_0 = 0, X_1, \dots, X_n$. For simplicity, we assume that the distribution of the increments has a symmetric density f which is bounded and positive everywhere. Therefore, we can write $f(x) = \exp(-\phi(x))$, where ϕ is bounded from below, and symmetric. The joint density of $(X_1, \dots, X_n)$ is then $(x_1, \dots, x_n) \to \exp(-\sum_{i=1}^n \phi(x_i - x_{i-1}))$ where we put $x_0 = 0$. For the higher dimensional versions introduced below, it is usually more natural to look at random walks which are tied down at the endpoint, i.e. conditioned on $X_{n+1} = 0$. As we have assumed the increments to have a density, there is no problem to define that properly: The conditioned random walk has just the n-dimensional density

$$\frac{1}{Z_n} \exp\left[-\sum_{i=1}^{n+1} \phi(x_i - x_{i-1})\right] \tag{1}$$

where we now set $x_0 = x_{n+1} = 0$, and where Z_n is the appropriate norming

$$Z_n = \int \cdots \int \exp\left[-\sum_{i=1}^{n+1} \phi(x_i - x_{i-1})\right] dx_1 \dots dx_n$$

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in order that (1) becomes a probability density. As we have to do this norming anyway, we can as well shift ϕ around by some constant, and therefore assume that $\phi(0) = 0$.

Using this formulation, one easily sees how a generalization to higher dimensional "time" should look like. We replace the discrete time interval $\{1, \dots, n\}$ by a discrete box

$$V_n = \{-n, -n+1, \dots, n-1, n\}^d,$$

define the outer boundary by $\partial V_n = \{x \in \mathbb{Z}^d \backslash V_n : \exists y \in V_n \text{ with } |x-y| = 1\}$, and then our measure P_n on $\mathbb{R}^{V_n}$ by its density with respect to Lebesgue measure:

$$P_n(dx) = \frac{1}{Z_n} \exp\left[-H_\phi^{(n)}(x)\right] \prod_{i \in V_n} dx_i,$$

where the so called "Hamiltonian" is $H_\phi^{(n)}(x) = \sum_{i,j \in V_n \cup \partial V_n,\ |i-j|=1} \phi(x_i - x_j)$, and where again, we put zero boundary conditions $x \equiv 0$ on ∂V_n. Z_n is the appropriate norming constant, in order that P_n is a probability measure. They are usually called *gradient models* for obvious reasons. The corresponding family of one dimensional projections $X_i : \mathbb{R}^{V_n} \to \mathbb{R}$, $i \in V_n$, is called a "random field" or a "random surface". We could of course as well consider $\mathbb{Z}$-valued random fields. In the mathematical physics literature, such models are often called SOS-interfaces, and there is a huge literature about them. (SOS means "solid on solid", but the background of this name is somewhat obscure.) Evidently these random fields are natural generalizations of the standard simple random walks, but it should be emphasized, that they may have quite different properties in higher dimensions than they do in dimension 1.

The special case which is easiest is the so called harmonic case, where $\phi(x)$ is proportional to x^2. By scaling ϕ, the proportionality factor is at our disposal. For later convenience, we take $\phi(x) = \frac{1}{8d}x^2$. In that case, of course, the measure is just Gaussian. It is sometimes called the "harmonic crystal" or the lattice massless free field. The covariances are easy to calculate. One of the nice features of the model is that one has a *random walk representation* for them:

$$E_n(X_i X_j) = \mathbb{E}_i^{\mathrm{RW}} \left(\sum_{i=0}^{\tau_{V_n}} 1_{\eta_i = j} \right), \tag{2}$$

where $\eta_0, \eta_1, \dots$ is a standard symmetric random walk on $\mathbb{Z}^d$, starting at i under $\mathbb{P}_i^{\mathrm{RW}}$, and τ_{V_n} is the first exit time from V_n. This representation follows by observing that the covariance matrix of the field is the inverse of the discrete Laplacian on V_n with Dirichlet boundary condition, and this inverse is just the right hand side of (2). The random walk representation is very useful for the investigation of properties of this random field. For instance, by standard properties of random walks, one derives that

$$\mathrm{Var}_n(X_0) \sim \begin{cases} n & \text{for } d = 1 \\ \log n & \text{for } d = 2 \\ O(1) & \text{for } d \geq 3 \end{cases} .$$

The one dimensional case is of course just the standard fluctuation of a random walk. One sees that the behavior of the random field in higher dimensions is quite different from the standard one dimensional random walk. Properties of the above type have been proved for more general cases, and not just for the harmonic one. See for instance [3]. It should be emphasized that although the random surface is localized for $d \geq 3$ in the sense that the fluctuations are of order 1 (which is in striking contrast to the one dimensional situation), the surface continues to have long range correlations: From the random walk representation it is evident that for points i, j which are not close to the boundary ∂V_n ($i, j \in \frac{1}{2}V_n$, say), the covariances $E_n(X_i X_j)$ are of order $|i-j|^{-d+2}$.

For dimensions $d \geq 3$ it is easy to see and well known, that there exists a limit P_∞, a measure on $\mathbb{R}^{\mathbb{Z}^d}$, which is just the centered Gaussian measure whose covariances are given by the Green's function of the discrete Laplacian $G(i,j) = \mathbb{E}_i^{\mathrm{RW}}\left(\sum_{k=0}^{\infty} 1_{\eta_k = j}\right)$. An important development in recent years led to the discovery that more general gradient fields with ϕ convex possess also random walk representations, see [18], [23], [9]. However, the random walks which have to be used in these cases are random walks in random environments, where the random environment is generated by an auxiliary diffusion process on $\mathbb{R}^{V_n}$. Although many important properties have been extended in this way from the Gaussian case to more general ones, the fine properties of this more complicated random walk are difficult to discuss and many questions remain open. We will not pursue that line here.

In mathematical physics, one has investigated questions about such surfaces which are quite natural in this context, but which have not attracted much attention in the standard literature of random walks. Some of these problems center around the interaction of the surface with $\{x \in \mathbb{R}^{V_n} : x_i = 0\ \forall i \in V_n\}$, which is sometimes called a "wall". For instance, one asks how the properties of the surface change if there is a small attraction to this wall, or if one considers only random surfaces being on one side of the wall. Often, there appear qualitative transitions of the "macroscopic" behavior if some of the parameters are changed smoothly. Examples are so called "wetting transitions", where the surface at specific values of the parameters ceases to cling to the wall. We briefly discuss the wetting transition in the last section.

There is a survey paper by Michael Fisher [14], his lectures on the occasion of the Boltzmann prize, where he introduced many of these problems and discussed them for the random walk case.

During the rest of this paper, we entirely stick to the harmonic case $\phi(x) = \frac{1}{8d}x^2$.

A relatively simple question is how the random surface is influenced by the presence of a local attraction to the above wall. There are several ways to build in such an attraction. The standard one is to modify the Hamiltonian H_ϕ by adding a potential $\sum_{i\in V_n} V(x_i)$ where V is a function which is symmetric and has its minimum at 0. If the potential V itself is quadratic, $V(x) = \frac{\mu}{2}x^2$, $\mu > 0$, say, we arrive at what in physical jargon is called the massive free field. We define the so called "massive Hamiltonian":

$$H_\mu^{(n)}(x) = \frac{1}{8d} \sum_{i,j\in V_n\cup\partial V_n,\ |i-j|=1} (x_i - x_j)^2 + \frac{\mu}{2}\sum_{i\in V_n} x_i^2 \tag{3}$$

and then the probability measure

$$P_{n,\mu}(dx) = \frac{1}{Z_{n,\mu}} \exp\left[-H_\mu^{(n)}(x)\right] \prod_{i\in V_n} dx_i \tag{4}$$

where

$$Z_{n,\mu} = \int \exp\left[-H_\mu^{(n)}(x)\right] dx. \tag{5}$$

$P_{n,\mu}$ is still Gaussian. The random walk representation needs only a simple modification: We have to replace the standard random walk $(\eta_k)_{k\geq 0}$ with one having a constant death rate. More precisely, the random walk has probability $\frac{\mu}{\mu+1}$ of disappearing into a "graveyard" at every step it makes. We can also formalize that by introducing a geometrically distributed random variable ζ, and replace τ_{V_n} in the formula (2) by $\tau_{V_n} \wedge \zeta$. ¿From this, one easily checks that this massive field has exponentially decaying correlations, uniformly in n.

Considerably more delicate is the case where V is flat at infinity, and does not grow. For instance, one can consider

$$V(x) = -\varepsilon 1_{[-a,a]}(x). \tag{6}$$

in which case the "pinned" probability measure is no longer Gaussian:

(7)

$$\hat{P}_n(dx) = \frac{1}{\hat{Z}_n} \exp\left[-\frac{1}{8d} \sum_{i,j\in V_n\cup\partial V_n,\ |i-j|=1} (x_i - x_j)^2 - \sum_{i\in V_n} V(x_i)\right] \prod_{i\in V_n} dx$$

(We will always write $\hat{P}$ for such a locally pinned measure.) It is much less clear, but true, that also in the case of a local pinning, the field is localized in a strong sense, meaning that

$$\sup_n \hat{E}_n(|X_0|^2) < \infty, \tag{8}$$

and there exist constants $c, C > 0$, depending on ε, a, such that

$$\sup_n \hat{E}_n(X_i X_j) \leq C \exp\left[-c\,|i-j|\right]. \tag{9}$$

We will discuss such a property here for the most delicate two-dimensional case. However, we will consider a slightly different form of local pinning, which is technically somewhat easier to handle. We describe that in the next section.

2 Statement of the result

The question of localization for the two dimensional harmonic crystal as introduced above was first studied in the papers [10, 11, 12, 13] and Lemberger [22]. They proved that $\hat{E}_n|X_0|$ is bounded no matter how small the potential is. They also proved in a mean field regime that the correlation $\hat{E}_n(X_k X_l)$ decays exponentially in $|k-l|$, uniformly in n. The exponential decay for the pinned measure given by (7) had however remained open for $d = 2$. This problem is addressed here (for a slightly modified model).

It is remarkable that also in the case of a non quadratic potential V, there is a random walk representation of covariances, at least if V is symmetric. This has been described in [5] (and in a less general way in [4]) where it is shown that the covariances $\hat{E}(X_k X_l)$ can be written as a sum over paths connecting k and l. By using Osterwalder-Schrader positivity this idea was embodied in a bound (see [5, lemma 4.1 and below]) which uses that the typical walk tries to minimize the number of points it occupies and thus resembles a geodesic. We will explain this below precisely for a slightly modified case, where the random walk representation is technically a bit simpler. The outcome is that $\hat{E}(X_k X_l)$ exhibits exponential decay and is bounded for $l = k$. Thus results similar to Dunlop et al. can be and were obtained in [5] but only for the easier case where $V(x)$ is even, monotone and increasing to infinity. The last condition was imposed because otherwise the required pressure estimates were lacking. In this note we use ideas developed in [1] to provide these pressure estimates which in turn then provide these decay properties. In comparison with the work of Dunlop et al. and Lemberger our arguments rest on special properties (even potential, Osterwalder-Schrader positivity and monotonicity), but it applies in a wide regime of coupling constants to potentials which are difficult to analyze by cluster expansions used by these authors. It also appears that the method presented here provides correct dependencies of the decay on the relevant parameters.

The method of [5] works directly only for periodic boundary conditions. This means that we identify n and $-n$ in $V_n = \{-n, -n+1, \dots, n\}^d$, getting a (discrete) torus T_n, but we loose any outer boundary, and therefore loose also any boundary condition. It is evident that in that case, a Gaussian measure with Hamiltonian $\frac{1}{8d}\sum_{k,l\in T_n, |k-l|=1}(x_k - x_l)^2$ no longer exists ($|k-l| = 1$ has to be interpreted on the torus). This is easiest remedied by introducing a (small) mass. Thus we start with the measure $P_{n,\mu}$ defined in (4), but with periodic boundary conditions (i.e. where V_n is replaced by T_n). As remarked in the previous section, this automatically has exponentially decaying correlations. The decay however disappears in the $\mu \to 0$ limit. For us, the "mass" parameter μ serves only to replace the 0-boundary condition which no longer can be implemented in the periodic case, and we want to have results which are *uniform* in μ, after taking the thermodynamic limit $n \to \infty$. The random walk for which (2) is correct is then simply a random walk on the torus with killing rate $\frac{\mu}{1+\mu}$. We can then perform the thermodynamic limit $n \to \infty$, and obtain a measure $P_{\infty,\mu}$ on $\mathbb{R}^{\mathbb{Z}^d}$. This measure evidently is a translation invariant centered Gaussian measure with exponentially decaying correlations, where the decay depends on the parameter μ, and disappears when $\mu \to 0$ and where $\lim_{\mu\to 0} \mathrm{Var}_{\infty,\mu}(X_0) = \infty$. What we will prove here is that if we introduce an additional pinning which acts only locally, as described in section 1, we get localization, i.e. estimates (8) and (9), which are *uniform* in μ.

We stick to the two-dimensional case which is the most delicate. (In three and higher dimensions, there is actually a simple domination argument, as has been remarked by Dima Ioffe, which does not work in the two dimensional case). We however change the pinning slightly, to make it purely local. Our measures will be

$$\hat{P}_{n,\mu,J}(dx) = \frac{1}{\hat{Z}_{n,\mu,J}} \exp\left[-H_\mu(x)\right] \prod_{i\in T_n} \left(dx_i + e^J \delta_0(dx_i)\right), \tag{10}$$

where

$$\hat{Z}_{n,\mu,J} = \int e^{-H_\mu(x)} \prod_{i\in T_n} \left(dx_i + e^J \delta_0(dx_i)\right). \tag{11}$$

The parameter $J \in \mathbb{R}$ regulates the strength of the pinning. The interesting case is when $\exp(J)$ is small, which means that the pinning is weak.

Theorem 2.1 *Assume $d = 2$. For all $J \in \mathbb{R}$ the field is localized in the sense that*

$$\sup_{\mu>0} \limsup_{n\to\infty} \hat{E}_{n,\mu,J} \left|X_0\right|^2 < \infty, \tag{12}$$

and

$$\sup_{\mu>0} \limsup_{n\to\infty} \hat{E}_{n,\mu,J}(X_i X_j) \leq C_J \exp\left[-c_J\left|i-j\right|\right] \tag{13}$$

for some positive constants c_J, C_J. Furthermore, for sufficiently large negative J

$$\sup_{\mu>0} \limsup_{n\to\infty} \hat{E}_{n,\mu,J}|X_0|^2 \leq K|J| \tag{14}$$

for some positive constant K.

3 Random Walk Expansion

The main advantage of using the "delta-pinned" measure (10) is that it has a more elementary random walk representation of the correlations than in the case of a general (symmetric) V in (7). It would however be only technically a bit more complicated to consider for instance the case (6) in combination with the random walk representation of [5]. On the other hand, it should be emphasized that the periodic boundary conditions are crucial for applying the chessboard estimates we are using here. For $A \subset T_n$ and $A^c = T_n \backslash A$, let

$$P_{A,\mu}(dx) = \frac{1}{Z_{A,\mu}} e^{-H_\mu(x)} \prod_{k\in A} dx_k \prod_{k\in A^c} \delta_0(dx_k).$$

This is a probability measure on $\mathbb{R}^{T_n}$, but restricted to $\mathbb{R}^A$, it is just the free field on A. In particular, we will have the random walk expansions for the covariances under $P_{A,\mu}$ exactly as (2), where only we have to replace τ_{V_n} by $\tau_A \wedge \zeta = \min(\tau_A, \zeta)$, ζ being the geometrically distributed killing from the positivity of μ. The measure, we are interested in, namely $\hat{P}_{n,\mu,J}$ can now easily be expanded in terms of these Gaussian measures: Expanding the product we get

$$\hat{P}_{n,\mu,J} = \sum_{A\subset T_n} e^{J|A^c|} \frac{Z_{A,\mu}}{\hat{Z}_{n,\mu,J}} P_{A,\mu} \tag{15}$$

The covariance is therefore

$$\hat{E}_{n,\mu,J}(X_i X_j) = \sum_{A\subset T_n} e^{J|A^c|} \frac{Z_{A,\mu}}{\hat{Z}_{n,\mu,J}} E_{A,\mu}(X_i X_j)$$

We insert the random walk expansion for the Gaussian expectation

$$E_{A,\mu}(X_i X_j) = \sum_{k=0}^{\infty} \mathbb{E}_i^{RW}(\mathbf{1}_{\eta_k=j} \mathbf{1}_{\tau_A\wedge\zeta>k}).$$

Resumming over A under the random walk expectation noting that the constraint $\mathbf{1}_{\tau_A \wedge \zeta > k}$ is the same as requiring A^c to be disjoint from the range of the walk $\eta_{[0,k]}$ and the walk not being killed by the clock, we get

$$\hat{E}_{n,\mu,J}(X_i X_j) = \sum_{k=0}^{\infty} \mathbb{E}_i^{\mathrm{RW}} \left(\mathbf{1}_{\eta_k = j} \mathbf{1}_{\zeta > k} \frac{\hat{Z}_{n,\mu,J}(\eta_{[0,k]})}{\hat{Z}_{n,\mu,J}} \right) \tag{16}$$

where, for any set $B \subset T_n$,

$$\hat{Z}_{n,\mu,J}(B) = \int e^{-H_\mu(x)} \prod_{k \in B} dx_k \prod_{k \in B^c} \left(dx_k + e^J \delta_0(dx_k) \right)$$

This is the random walk representation. It is essentially a special case of [5, Theorem 2.2] which applies to any even potential V. From this expression one also sees that the variables are always positively correlated.

4 Reduction to a pressure estimate

We define

$$\delta_J = \inf_{\mu > 0} \liminf_{n \to \infty} \frac{1}{|T_n|} \log \left[\frac{\hat{Z}_{n,\mu,J}}{Z_{n,\mu}} \right] \tag{17}$$

where $Z_{n,\mu}$ has been defined in (5). We prove in the next section the following

Proposition 4.1 *For all $J \in \mathbf{R}$, we have $\delta_J > 0$ and*

$$\liminf_{J \to -\infty} \frac{1}{J} \log \delta_J \geq 1$$

As a special case of the estimates in section 4 of [5], it follows that $\delta_J > 0$ implies that the variance of X_0 is bounded in the thermodynamic limit. Exponential decay of the covariance is also an easy consequence. For the convenience of the reader, we prove here how the theorem now follows from this proposition, in particular as the argument is simpler in our "delta pinning"-case than for a potential V of the type (6).

Following [5], we use Osterwalder-Schrader positivity in the form of the chessboard estimate [17], [15], [24], [16]

$$\hat{Z}_{n,\mu,J}(B) \leq \left[Z_{n,\mu}^{|B|} \hat{Z}_{n,\mu,J}^{|B^c|} \right]^{|T_n|^{-1}}$$

so that from (16), we get

$$\begin{aligned}\hat{E}_{n,\mu,J}(X_iX_j) &\le \sum_{k=0}^{\infty} \mathbb{E}_i^{\mathrm{RW}}\left[\mathbf{1}_{\eta_k=j}\mathbf{1}_{\zeta>k}\left[\frac{Z_{n,\mu}}{\hat{Z}_{n,\mu,J}}\right]^{|\eta[0,k]|/|T_n|}\right]\\ &\le \sum_{k=0}^{\infty} \mathbb{E}_i^{\mathrm{RW}}\left[\mathbf{1}_{\eta_k=j}\left[\frac{Z_{n,\mu}}{\hat{Z}_{n,\mu,J}}\right]^{|\eta[0,k]|/|T_n|}\right].\end{aligned}$$

We therefore have

$$\alpha(i,j) \stackrel{\mathrm{def}}{=} \sup_{\mu}\limsup_{n\to\infty}\hat{E}_{n,\mu,J}(X_iX_j) \le \sum_{k=0}^{\infty}\mathbb{E}_i^{\mathrm{RW}}\left\{\mathbf{1}_{\eta_k=j}e^{-\delta_J|\eta[0,k]|}\right\}, \tag{18}$$

where on the right hand side the random walk is on $\mathbb{Z}^2$. Remark that we are done now with the positive μ and therefore with the geometric clock ζ, and also with any finite size of the torus, and so we can now use just estimates for the standard random walk. Using this expression, we can easily estimate the covariances and the variance. Let τ_{V_N} be the first exit time from the box V_N. Then

$$\begin{aligned}\alpha(0,0) &\le \sum_{N=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{E}_0^{\mathrm{RW}}\left\{\mathbf{1}_{\eta_k=0}\mathbf{1}_{\tau_{V_{N+1}}>k\ge\tau_{V_N}}e^{-\delta_J|\eta[0,k]|}\right\}\\ &\le \sum_{N=0}^{\infty}\sum_{k=0}^{\infty}\mathbb{E}_0^{\mathrm{RW}}\left\{\mathbf{1}_{\eta_k=0}\mathbf{1}_{\tau_{V_{N+1}}>k\ge\tau_{V_N}}e^{-\delta_J N}\right\}.\end{aligned} \tag{19}$$

For $i,j\in V_N$ define

$$G_{V_N}(i,j) = \sum_{k=0}^{\infty}\mathbb{E}_i^{\mathrm{RW}}\left\{\mathbf{1}_{\eta_k=j}\mathbf{1}_{\tau_{V_N}>k}\right\}$$

This is a Green's function, the kernel of $(-\Delta_{V_N})^{-1}$, where Δ_{V_N} is the lattice Laplacian with Dirichlet boundary conditions at the boundary of V_N. From (19) we get:

$$\begin{aligned}\alpha(0,0) &\le \sum_{N=0}^{\infty}\left(G_{V_{N+1}}(0,0) - G_{V_N}(0,0)\right)e^{-\delta_J N}\\ &= \sum_{N=0}^{\infty}G_{V_N}(0,0)\left(e^{-\delta_J(N-1)} - e^{-\delta_J N}\right) \le \delta_J\sum_{N=0}^{\infty}G_{V_N}(0,0)e^{-\delta_J(N-1)}\end{aligned}$$

By [20, Theorem 1.6.6] we have the estimate

$$G_{V_N}(0,0) \le c\log N = c(\log N\delta_J - \log\delta_J),$$

and therefore

$$\alpha(0,0) \le c\left\{\delta_J \sum_{N=0}^{\infty} \log(\delta_J N) e^{-\delta_J (N-1)} - \delta_J \log \delta_J \sum_{N=0}^{\infty} e^{-\delta_J(N-1)}\right\} < \infty$$

which proves (12) of the theorem. Evidently, we have $\delta_J \to 0$ for $J \to -\infty$. As $\delta_J \to 0$, the two sums on the right hand side of the estimate are Riemann sum approximations to

$$\int_0^\infty \log t \; e^{-t} dt - \log \delta_J \int_0^\infty e^{-t} dt$$

and therefore

$$\alpha(0,0) \le \log \delta_J^{-1}(1 + o(-J))$$

as $J \to -\infty$. This together with (18) proves (14) of the theorem.

The decay of the correlations (13) follows by a simple modification. By translation invariance, we only have to consider $a(j,0)$. Then in the above estimates (19), we can restrict the summation over N to $N \ge |j_1| + |j_2|$ where $j = (j_1, j_2)$. Therefore we get

$$\alpha(j,0) \le c\left\{\delta_J \sum_{N=|j_1|+|j_2|}^{\infty} \log(\delta_J N) e^{-\delta_J (N-1)} - \delta_J \log \delta_J \sum_{N=|j_1|+|j_2|}^{\infty} e^{-\delta_J(N-1)}\right\}$$

from which (13) is immediate.

5 Estimates on δ_J, proof of proposition

We finish the proof of our main theorem by proving the proposition of the last section. In the sequel, $c > 0$ is a generic constant, not necessarily the same at different occurrences.

We subdivide T_n into boxes B of side length $2L$, where for convenience we assume that L divides n. The partition function $\hat{Z}_{n,\mu,J}$ is expanded as in (15). A lower bound for $\hat{Z}_{n,\mu,J}$ is then obtained by restricting the sum over A in the expansion to a special class $\mathfrak{A}_{\mathfrak{L}}$ of sets defined by: $A \in \mathfrak{A}_{\mathfrak{L}}$ if, for every box B, $B \cap A^c$ contains exactly one lattice point and this point lies within $L/2$ of the center of B. Thus

$$\frac{\hat{Z}_{n,\mu,J}}{Z_{n,\mu}} \ge \sum_{A \in \mathfrak{A}_{\mathfrak{L}}} e^{J|A^c|} \frac{Z_{A,\mu}}{Z_{n,\mu}} \ge L^{2(n/L)^2} e^{J(n/L)^2} \inf_{A \in \mathfrak{A}_{\mathfrak{L}}} \frac{Z_{A,\mu}}{Z_{n,\mu}}.$$

The proof is now easily finished using the following result:

Lemma 5.1 *There exists $L_o \in \mathbb{N}$ and depending on $\mu > 0$, there exists $n_o(\mu) \in \mathbb{N}$, such that for $n \geq n_o(\mu)$ and $L \geq L_o$*

$$\inf_{A \in \mathfrak{A}_L} \frac{Z_{A,\mu}}{Z_{n,\mu}} \geq \exp\left(- \left[\frac{n}{L}\right]^2 (c + \log\log L)\right)$$

We postpone the proof of this lemma for a moment, and proceed with the proof of the proposition. Evidently, the right hand side in the above lemma is μ-independent, but we have to remember that we have the restriction $n \geq n_o(\mu)$ which does not bother us, as the claimed μ-uniformity is after taking the thermodynamical limit only. From the lemma we get

$$\delta_J \geq L^{-2} \left(J - \log\log L - c + \log L^2\right)$$

The key point is that the entropy in the sets $\mathfrak{A}_{\mathfrak{L}}$ has given rise to the last term which dominates at large L, so by optimizing over L,

$$L^2 = e^{-J+o(J)} \text{ as } J \to -\infty$$

we achieve a strictly positive

$$\delta_J \geq e^{J+o(-J)}$$

and this implies the statement in the proposition.

Proof of Lemma 5.1 By definition of $\mathfrak{A}_{\mathfrak{L}}$, every box B contains exactly one point, call it k, which is not in $B \cap A$. In the ratio $Z_{B\cap A,\mu}/Z_{B,\mu}$ of partition functions below we integrate first out all the variables except $x = x_k$ which leads to a Gaussian law with variance $a^{-1} = G_B(k,k)$, and therefore

$$\frac{Z_{B\cap A,\mu}}{Z_{B,\mu}} = \frac{\int e^{-\frac{1}{2}ax^2}\delta_0(dx)}{\int e^{-\frac{1}{2}ax^2}dx} \geq c(\log L)^{-1},$$

where in the last inequality, we estimate the Green's function $G_B(k,k)$ using [20, Theorem 1.6.6]. This we do with every box B, and therefore, we get

$$\prod_B \frac{Z_{B\cap A,\mu}}{Z_{B,\mu}} \geq e^{-(n/L)^2(\log\log L + c)}$$

Noting that this is the right hand side of the expression in Lemma 5.1, we see that it is sufficient to prove that Q defined by

$$\exp(-\frac{1}{2}Q) \stackrel{\text{def}}{=} \frac{Z_{A,\mu}}{Z_{n,\mu}} \Big/ \prod_B \frac{Z_{B\cap A,\mu}}{Z_{B,\mu}} \tag{20}$$

satisfies

$$Q \leq c(n/L)^2.$$

These Gaussian partition functions can be integrated in terms of determinants of lattice Laplacians, which then can be expressed with a random walk representation. This is reviewed in details in [1], Section 4.1. The outcome is that

$$\log Z_{A,\mu} = \frac{|A|}{2} \log \frac{\pi}{2} + \frac{1}{2} \sum_{k \in A} \sum_{m=1}^{\infty} \frac{1}{2m} \mathbb{P}_k^{\mathrm{RW}} \left(\eta_{2m} = k, \tau_A > 2m, \zeta > 2m\right).$$

Formally, this is just coming from expanding the logarithm in the equality $\det[1 + A] = \exp(\mathrm{Tr} \log [I + A])$ and using a random walk representation for the resulting terms. Implementing the above expression into (20), we arrive at

$$Q = \sum_B \sum_{k \in B} \sum_{m=1}^{\infty} \frac{1}{2m} \mathbb{P}_k^{\mathrm{RW}} (\eta_{2m} = k, \tau_B \leq 2m, \tau_A \leq 2m, \zeta > 2m)$$

because the partition function $Z_{A,\mu}/Z_{n,\mu}$ involves the sum over all paths in T_n that leave A. Amongst these are paths that stay inside some box B but leave A at the single point in $B \cap A^c$. These are divided out by the denominator in Q so we are left with paths that exit A and whichever box B they started in. We can replace the random walk on the torus by the free random walk, making an error in the above expression of order $n^2 \exp[-c\mu n] \sum_{m=1}^{\infty} \mathbb{P}(\zeta > 2m)/2m$, which for any $\mu > 0$ is at most 1, if n is large enough, $n \geq n_o(\mu)$, say, and can therefore be neglected. After having made this replacement, we write

$$Q = Q^{\leq} + Q^{\geq} \text{ corresponding to } \sum_m = \sum_{m \leq L^2} + \sum_{m \geq L^2}$$

Thus

$$\begin{aligned} Q^{\geq} &\leq \sum_{k \in T_n} \sum_{m \geq L^2} \frac{1}{2m} \mathbb{P}_k^{\mathrm{RW}} (\eta_{2m} = k) \\ &\leq cn^2 \sum_{m \geq L^2} \frac{1}{2m} \frac{1}{m} \leq c \left[\frac{n}{L}\right]^2 \end{aligned}$$

where the second inequality rests on the local central limit theorem. Also

$$\begin{aligned} Q^{\leq} &\leq \sum_B \sum_{k \in T_n} \sum_{m \leq L^2} \frac{1}{2m} \mathbb{P}_k^{\mathrm{RW}} (\eta_{2m} = k, \tau_B \leq 2m, \tau_A \leq 2m) \\ &\leq \left[\frac{n}{L}\right]^2 \sum_{m \leq L^2} \mathbb{P}_0^{\mathrm{RW}} (\eta_{2m} = 0, \tau_B \leq 2m) \end{aligned}$$

Stop the walk on first exit from B and write it in terms of the hitting distribution on the boundary of the box. The sum over walks that start with this hitting distribution and return to 0 is bounded by $(1/m)\exp(-cL^2/m)$ by the local central limit theorem. Therefore

$$Q^{\leq} \leq \left[\frac{n}{L}\right]^2 \sum_{m\leq L^2} \frac{1}{m} e^{-cL^2/m} \leq c\left[\frac{n}{L}\right]^2$$

because as $L \to \infty$

$$\sum_{m\leq L^2} \frac{1}{m} e^{-cL^2/m} = \sum_{m\leq L^2} \frac{1}{m/L^2} e^{-cL^2/m} \frac{1}{L^2} \to \int_0^1 \frac{1}{t} e^{-1/t} dt$$

We have now proved $Q \leq c(n/L)^2$ as required. ∎

6 Concluding remarks

There have been some very recent developments concerning the topics of this paper. In particular, in Deuschel and Velenik [8] and Ioffe and Velenik [19], these authors have been able to prove (among other things) the exponential decay of correlations in two dimensional gradient models with convex interactions in quite some generality. They are using a refined version of the pressure estimate as developed here, and a renormalization procedure.

A particularly interesting topic with similar questions is the so called "wetting transition". Here one considers the random field which stays on one side of the "wall" $\{x : x_i \equiv 0\}$, say on the positive side. One therefore considers the probability measures $P_n^+(\cdot) \stackrel{\text{def}}{=} P_n(\cdot|\Omega^+)$, where $\Omega^+ \stackrel{\text{def}}{=} \{X_i \geq 0, \forall i\}$, and similarly for the pinned situation $\hat{P}_{n,J}^+(\cdot) \stackrel{\text{def}}{=} \hat{P}_{n,J}(\cdot|\Omega^+)$. (We consider again measures here with zero boundary conditions outside a box V_n). In the one dimensional case, the measure P_n^+ just describes a random walk tied down at the endpoints of the interval, and conditioned to stay positive. It is well known that after Brownian rescaling, this measure converges weakly to the law of the Brownian excursion with base line $[-1, 1]$. If we now consider the pinned measure $\hat{P}_{n,J}^+$ (still in one dimension), it is not clear from the outset if the paths under this (for n large) should still look like a Brownian excursion or if the walk sticks close to 0 due to the pinning. In his survey paper [14], Michael Fisher proved the somewhat surprising fact that there is a transition in the behaviour depending on the pinning parameter. (Fisher actually looked at a discrete random walk, but similar results can easily be proved for the situation described here, see [2]). It turns out that for small J, the pinning is not sufficient, and the path indeed just looks like if the pinning would not be present, i.e. like a Brownian excursion (after rescaling). On the other hand, if J is large, then the pinning takes over and the path becomes localized in the strong sense described by (8) and (9). This transition is what is called a "wetting transition". The phenomenon had actually been known before, see [6], [21].

The question arises if there is a similar transition occurring in higher dimensions. Quite recently, we were able to show that there is no such transition for $d \geq 3$ in the harmonic case [2]. On the other hand, Caputo and Velenik [7] showed very recently the existence of a wetting transition for $d = 2$, and actually in all dimensions for the continuous SOS case, where the interaction function ϕ of Section 1 is not quadratic but the absolute value. The results proved in these papers are however somewhat weak in the sense that only the pressure estimates have been provided. To be precise, it is proved in [2] that

$$\delta_J^+ = \liminf_{n\to\infty} \frac{1}{n^d} \log \left[\frac{\hat{Z}_{n,J}^+}{Z_n^+} \right] > 0 \tag{21}$$

for all $J \in \mathbb{R}$, and $d \geq 3$, where

$$Z_n^+ = \int_{\Omega^+} e^{-H(x)} \prod_{i\in V_n} dx_i$$

and

$$\hat{Z}_{n,T_n}^+ = \int_{\Omega^+} e^{-H(x)} \prod_{i\in V_n} \left(dx_i + e^J \delta_0(dx_i)\right),$$

and in [7] it is proved that $\delta_J^+ = 0$ for small enough J. The proof that $\delta_J^+ > 0$ for large enough J is actually very easy in all dimensions (see [1]). From $\delta_J^+ > 0$, it is easy to see that under $\hat{P}_{n,J}^+$, there is a positive density of zeros in the sense that there exists an $\varepsilon(J) > 0$ with

$$\lim_{n\to\infty} \hat{P}_{n,J}^+(\sharp\{i \in V_n : X_i = 0\} \geq \varepsilon(J)n^d\}) = 1.$$

It seems however to be quite delicate to conclude from that that one has a behavior of the type (8) and (9). In [22] such a result was proved by cluster expansion techniques for a slightly different model in a situation which would correspond to J being large. These methods however appear to be powerless to prove the result for arbitrary J for which $\delta_J^+ > 0$, which certainly should be true, but cannot be proved by the methods discussed in this paper (or any other methods presently available).

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