The Square-root Game

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Abstract

In this work I give an elementary proof of the following : "The absolute 1/2 moment of the beta (1/4, 1/4) distribution about t is independent of t for $0 < t < 1$.

Keywords : beta distribution, two person game, Richardson extrapolation.

Theorem 1 *The absolute* $1/2$ *moment of the beta* $(1/4, 1/4)$ *distribution about t is independent of t for* $0 < t < 1$ *:*

$$\int_0^1 p(x)(|x-t|^{(1/2)})\,dx,$$

where

$$p(x) = [x(1-x)]^{(-3/4)},$$

is independent of t for $0 < t < 1$.
More generally, for any a with $0 < a < 1$, *the absolute* $a - th$ *moment of the beta* $((1-a)/2, (1-a)/2)$ *distribution about t is independent of t for* $0 < t < 1$.

This note, which I am pleased to write in honor of my old friend Tom Ferguson, is about the process that led to the Theorem.

Quite a few years ago, shortly after Tom Ferguson got his first PC, I asked him about the Square-root Game :

Square-root Game

Players I and II simultaneously choose numbers x and y in the unit inverval. Then II pays I the amount $|x-y|^{(1/2)}$.

Later that day, Tom told me that the value of the game is .59907, to 5 places. I asked him how he got such accuracy, since he could solve games only up to 30×30 on his machine. He said that he'd used Richardson extrapolation (which I'd never heard of). He told me a bit about Richardson extrapolation, and we turned to other rhings.

Then, in my Fall 1994 game theory class I assigned, as a homework problem, to solve the Square-root Game to 3 places. Several students succeeded, but one group of four students, working together, claimed to have solved the game to 15 places. According to them, if either player used the beta $(1/4, 1/4)$ strategy, then Player I's expected income, as calculated by Mathematica, was constant to 15 places, no matter what the other Player did. They could not prove that a beta $(1/4, 1/4)$ strategy gave a constant income, and neither could I.

Later I asked Jim Pitman about the more general case as stated in the Theorem, and he gave a not-quite-elementary proof. Later he found a generalization to higher dimensions in Landkof [1972]. Finally I found an elementary proof, given below.

The method the four students used to get their solution is simple and instructive.

1. They solved a discrete version, restricting each Player to the 21 choices $0, .05, \ldots, .95, 1$. The good strategy for each player was a U-shaped distribution, symmetric about $1/2$.

2. They calculated the variance of this distribution, and found the beta distribution symmetric about $1/2$ with the same variance. It was beta $(.2613, .2613)$.

3. They guessed that .2613 was trying to be .25, so tried beta$(1/4, 1/4)$ as a strategy.

Here is the proof of the Theorem. Fix $a, 0 < a < 1$, and put

$$f(t) = \int_0^1 p(x)(|x-t|^a)\,dx$$

where $p(x) = [x(1-x)]^{(-(a+1)/2)}$.

We must show that f is constant on $0 < t < 1$. Its derivative is

$$f'(t) = a\int_0^t [(t-x)^{(a-1)}]p(x)\,dx - a\int_t^1 [(x-t)^{(a-1)}]p(x)\,dx$$

With the change of variable $u = 1 - x$ in the second integral, we get

$$\begin{aligned} f'(t) &= a[\int_0^t [(t-x)^{(a-1)}]p(x)\,dx - \int_o^{1-t} [(1-t-u)^{(a-1)}]p(u)\,du] \\ &= a(F(t) - F(1-t)), \text{ where} \\ F(t) &= \int_0^t [(t-x)^{(a-1)}]p(x)\,dx. \end{aligned}$$

So we must show that $F(1-t) = F(t)$. To evaluate F, make the linear fractional change of variable $z = (t-x)/(t-tx) : x = t(1-z)/(1-tz)$ (see Carr, [1970], Formula 2342). We get

$$F(t) = [t(1-t)]^{((a-1)/2)} \int_0^1 [(1-z)^{(-(a+1)/2)}][z^{(a-1)}]\,dz$$

So $F(1-t) = F(t)$, proving the Theorem.
So the value of the Square-root game is I's expected income when he chooses x according to beta $(1/4, 1/4)$ and II chooses $y = 0$, namely

$$\begin{aligned} \Gamma(1/2)/\Gamma(1/4)\Gamma(1/4)) \int_0^1 (x^{(1/2)})([x(1-x)]^{(-3/4)})\,dx &= \Gamma(1/2)\Gamma(3/4)\Gamma(1/4) \\ &= .599070117367796.... \end{aligned}$$

So Tom's first five places were correct.

Aknowledgement

I thank the referee for his kind remarks.

References

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