Recent Developments in Multiple Comparison Procedures
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Approximations to multivariate t integrals with application to multiple comparison procedures

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Abstract: Various multiple comparison procedures involve the evaluation of multivariate normal and t integrals with non-decomposable correlation matrices. While exact methods exist for their computations, it is sometimes necessary to consider simpler and faster approximations. We consider approximations based on approximations to the correlation matrix (methods which provide no error control) as well as inequality based methods (where, by definition, the sign of the error is known). Comparisons of different methods, to assess accuracy, are given for particular multiple comparison problems which require high-dimensional integrations.

1. Introduction

Several multiple comparison procedures involve the numerical computation of multidimensional integrals, which are not easily handled. A prominent example is to find simultaneous confidence intervals for all-pairwise comparisons in general linear models. Current integration methods typically behave well for low to moderate numbers of treatment groups (say less than 10, see Bretz, Hayter and Genz, 2001). Independently of such general integration routines, research also focused on approximating the arising integrals by lower order expressions. The Tukey–Kramer procedure is a well known approximation for the all-pairwise comparisons, where the true covariance matrix of the estimates is substituted by the identity matrix. Two questions arise from this approach. Firstly, because the Tukey–Kramer procedure has not been proved to be conservative for all designs, there is still room for other procedures (where an approximation is called conservative, if its critical values are larger than the true ones). Secondly, Somerville (1993) showed that the conservatism of the Tukey–Kramer procedure can be quite substantial, so that sharper methods may exist, which are still conservative. These facts motivate us to look at and compare different approximations to multivariate normal and t probabilities.

The paper is organized as follows. Section 2 introduces some basic notation. Section 3 presents the methods used for our numerical comparisons. The methods are divided into inequality based methods and methods making use of approximative correlation matrices. The results of the numerical study are given in Section 4. The final Section 5 gives some conclusions.

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2. Motivation

Consider the $k-$variate t integral

$$T_k(\mathbf{a},\mathbf{b};\mathbf{R},\nu)=\frac{\Gamma(\frac{\nu+k}{2})}{\Gamma(\frac{\nu}{2})\sqrt{|\mathbf{R}|(\nu\pi)^k}}\int_{\mathbf{A}}\left(1+\frac{\mathbf{x}^t\mathbf{R}^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+k}{2}}d\mathbf{x}, \tag{1}$$

with ν degrees of freedom and correlation matrix $\mathbf{R}=\{\rho_{ij}\}$; $\mathbf{x}^t$ is used to denote the transpose of $\mathbf{x}$. The integration region $\mathbf{A}=\{\mathbf{x}\in\mathbb{R}^k : \mathbf{a}\le\mathbf{C}\mathbf{x}\le\mathbf{b}\}$ is a convex polyhedron for a given $q\times k$ constraint matrix $\mathbf{C}$ and $\mathbf{a},\mathbf{b}\in\bar{\mathbb{R}}^q$. The main interest lies in determining a critical value $\mathbf{t}_{1-\alpha}=(t_{1-\alpha},\ldots,t_{1-\alpha})^t$, such that $T_k(-\mathbf{t}_{1-\alpha},\mathbf{t}_{1-\alpha};\mathbf{R},\nu)=1-\alpha$ for a given probability $\alpha\in(0,1)$. Because this problem requires repeated evaluations of $T_k(\mathbf{a},\mathbf{b};\mathbf{R},\nu)$ (Genz and Bretz, 2000), we focus on the numerical evaluation of integral (1).

It is well known that particular structures of $\mathbf{R}$ allow a substantial dimension reduction of the initial integration problem (Curnow and Dunnett, 1962). Consider as an example the two-way no-interaction model $Y_{ijl}=\mu+\alpha_i+\beta_j+\epsilon_{ijl}$, $i=1,\ldots,I, j=1,\ldots,J$ and $l=1,\ldots,n_{ij}$. Suppose we are interested in the many-to-one comparisons $\alpha_2-\alpha_1,\ldots,\alpha_I-\alpha_1$. Then, $\mathbf{R}=\mathbf{D}+\boldsymbol{\lambda}\boldsymbol{\lambda}^t$, where $D=\mathrm{diag}(1-\lambda_i^2)$ and $\lambda_i=(1+w_1/w_{i+1})^{-1/2}, i=1,\ldots,I-1$ as long as the cell sizes satisfy $n_{ij}=w_i n_{.j}$ for all i and j, $n_{.j}=\sum_i n_{ij}$ (Hsu, 1992). Thus, $\mathbf{R}$ is of one-factorial structure and the integral (1) is reduced to a two-dimensional integral, regardless of I,

$$\int_0^\infty\int_{-\infty}^\infty\prod_{i=1}^{I-1}\left[\Phi\left(\frac{b_i s-\lambda_i y}{\sqrt{1-\lambda_i^2}}\right)-\Phi\left(\frac{a_i s-\lambda_i y}{\sqrt{1-\lambda_i^2}}\right)\right]\varphi(y)\,dy\chi_\nu(s)\,ds, \tag{2}$$

where φ is the standard normal pdf and χ_ν is the $\sqrt{\chi^2_\nu}$ pdf and $\Phi'=\varphi$. But if above proportionality rule is violated for $I>4$, the integral (1) is not reducible in general (Hsu, 1992). A similar problem arises for all-pairwise comparisons in one-way layouts. As long as the group sample sizes n_i are equal to a common n, the integral (1) can be reduced to the double integral

$$I\int_0^\infty\int_{-\infty}^\infty\big[\Phi(y)-\Phi(y-|b|s)\big]^{I-1}\varphi(y)\,dy\chi_\nu(s)\,ds, \tag{3}$$

which can be efficiently computed (for the purpose of computing critical values, $-\mathbf{a}=\mathbf{b}=(b,\ldots,b)^t$). But this reduction fails if $n_i\neq n$ for some i.

3. Approximation methods

In the following we describe the two classes of approximations used in the numerical study in Section 4.

3.1. Inequality methods

Inequality methods provide lower or upper bounds for the true integral value.

Let $P(t)=T_k(-\mathbf{t},\mathbf{t};\mathbf{R},\nu)$, and define $A_j=\{\mathbf{x}:|(C\mathbf{x})_j|\le t\}$. In order to be consistent with other methods we assume the constraint matrix C has been scaled so that $C\mathbf{R}C^t$ is a correlation matrix. If we let

$$S_1(t)=\sum_{j=1}^q Prob\big(A_j^c(t)\big),$$

where $A_j^c(t)$ is the complement of the set $A_j(t)$, then the Bonferroni lower bound for $P(t)$ (see Hsu, 1996) is

$$L^{(1)}(t) = 1 - S_1(t) \le P(t).$$

A simple upper bound for $P(t)$ is

$$P(t) \le 1 - \min_j Prob\big(A_j^c(t)\big) = U^{(1)}(t).$$

Both of these bounds require only 1-dimensional distribution values. If $t_l^{(1)}$ and $t_u^{(1)}$ are determined by solving $U^{(1)}(t) = 1-\alpha$ and $L^{(1)}(t) = 1-\alpha$, respectively, then $t_l^{(1)} \le t_{1-\alpha} \le t_u^{(1)}$. The bounding interval for $t_{1-\alpha}$ can be found directly using the appropriate 1-dimensional inverse distribution function:

$$\big[t_l^{(1)}, t_u^{(1)}\big] = \left[t_\nu^{-1}\left(1-\frac{\alpha}{2}\right), t_\nu^{-1}\left(1-\frac{\alpha}{2q}\right)\right],$$

where

$$t_\nu(u) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}} \int_{-\infty}^{u} \left(1+\frac{s^2}{\nu}\right)^{-\frac{\nu+1}{2}} ds.$$

Shorter bounding intervals can be found using bivariate distribution values (Dunnett and Sobel, 1954) if a modified Bonferroni bound (Dawson and Sankoff, 1967) is combined with the Hunter-Worsley bound. These bounds are described in the book by Hsu (1996, Appendix A). If we define $S_2(t)$ by

$$S_2(t) = \sum_{j<i} Prob\big(A_j^c(t) \cap A_i^c(t)\big),$$

then the modified Bonferroni bounds and Hunter-Worsley guarantee that

$$\begin{aligned} L^{(2)}(t) &= 1 - S_1(t) + \sum_{(i,j)\in T^*} Prob\big(A_j^c(t) \cap A_i^c(t)\big) \\ &\le P(t) \le 1 - 2\frac{S_1(t) - S_2(t)/k}{k+1} = U^{(2)}(t). \end{aligned}$$

where $k = 1 + \lfloor 2S_2(t)/S_1(t) \rfloor$ and T^* is maximal spanning tree for the complete graph of order q with edge weights $Prob(A_j^c(t) \cap A_i^c(t))$. If we determine $t_l^{(2)}$, $t_u^{(2)}$ that satisfy $U^{(2)}(t_l^{(2)}) = 1-\alpha$ and $L^{(2)}(t_u^{(2)}) = 1-\alpha$, then $t_l^{(1)} \le t_l^{(2)} \le t_{1-\alpha} \le t_u^{(2)} \le t_u^{(1)}$. Starting with $[t_l^{(1)}, t_u^{(1)}]$, we use numerical optimization, applied to $L^{(2)}(t)$, to determine $t_u^{(2)}$. Then we use numerical optimization, applied to $U^{(2)}(t)$, starting with $[t_l^{(1)}, t_u^{(2)}]$, to determine $t_l^{(2)}$.

Even shorter bounding intervals for $t_{1-\alpha}$ can be determined for many problems if trivariate distribution values are used. If we define $S_3(t)$ by

$$S_3(t) = \sum_{k<j<i} Prob\big(A_k^c(t) \cap A_j^c(t) \cap A_i^c(t)\big),$$

then sharp bounds that use $S_1(t)$, $S_2(t)$ and $S_3(t)$ (see Boros and Prekopa, 1989) are given by

$$L^{(3)}(t) = 1 - S_1(t) + 2\frac{(2j-1)S_2(t) - 3S_3(t)}{j(j+1)},$$

and

$$U^{(3)}(t) = 1 - \frac{(i+2q-1)S_1(t) - 2((2i+q-2)S_2(t) - 3S_3(t))/i}{q(i+1)},$$

where $i = 1 + \lfloor 2((q-2)S_2(t) - 3S_3(2))/((q-1)S_1(t) - 2S_2(t))\rfloor$ and $j = 2 + \lfloor 3S_3(t)/S_2(t)\rfloor$. We can use numerical optimization applied to $L^{(3)}(t)$ and $U^{(3)}(t)$, to determine $t_u^{(3)}$ and $t_l^{(3)}$. Unfortunately, it is not always true that $L^{(2)}(t) \le L^{(3)}(t)$, so we cannot assume that $t_u^{(3)}$ will be closer to $t_{1-\alpha}$ than $t_u^{(2)}$. However, Boros and Prekopa (1989) show that $U^{(3)}(t) \le U^{(2)}(t)$, so $t_l^{(3)}$ will be at least as close to $t_{1-\alpha}$ as $t_l^{(2)}$. The cost of computing $S_3(t)$ can be quite high for problems where q is large because of the $q(q-1)(q-2)/6$ trivariate values needed for $S_3(t)$.

There are also bounds (*hybrid bounds*) that use only selected trivariate distribution values. The *cherry tree* bounds (see Bukszar and Prekopa, 2001), which we denote by $L^{(2,3)}(t)$, are lower bounds for $P(t)$ that require only $q-2$ trivariate values. The optimal cherry tree bound can be expensive to compute, but we use a bound defined by

$$L^{(2,3)}(t) = L^{(2)}(t) + \sum_{(i,j)\in T^{**}} \big(Prob\big(A_j^c(t) \cap A_i^c(t)\big) - Prob\big(A_{k_{(i,j)}}^c(t) \cap A_j^c(t) \cap A_i^c(t)\big)\big),$$

The edge set T^{**} contains $q-2$ of the $q-1$ edges in the $L^{(2)}(t)$ bound edge set T^*. For each edge in T^{**} the vertex $k_{(i,j)}$ is selected using the method described by Bukszar and Prekopa (2001). We always have $L^{(2,3)}(t) \ge L^{(2)}(t)$ because $Prob(A_j^c(t) \cap A_i^c(t)) \ge Prob(A_{k_{(i,j)}}^c(t) \cap A_j^c(t) \cap A_i^c(t))$. If we let $t_u^{(2,3)}$ be the point where $L^{(2,3)}(t) = 1-\alpha$, then $t_{1-\alpha} \le t_u^{(2,3)} \le t_u^{(2)} \le t_u^{(1)}$.

A class of hybrid upper bounds has been described by Tomescu (1986). Define

$$U^{(2,3)}(t) = 1 - S_1(t) + S_2(t) - \sum_{E(T_q^3)} Prob\big(A_k^c(t) \cap A_j^c(t) \cap A_i^c(t)\big),$$

where $E(T_q^3)$ is the set of $(q-1)(q-2)/2$ hyperedges (i,j,k) for a 3-hypertree. Tomescu shows that $P(t) \le U^{(2,3)}(t)$. The optimal hypertree bound can also be expensive to compute because all possible trivariate distribution values are needed. We use a bound determined using T^*. We successively delete $q-2$ terminal vertices from T^*. Each time a terminal vertex k (along with its edge) is deleted, that vertex is adjoined to each of the remaining T^* edges to form a set of hyperedges. The union of the sets of $q-2, q-3, \ldots, 1$ hyperedges found in this way form the hypertree T_q^3 that we use for $U^{(2,3)}(t)$. This bound is always less than or equal to the second order Bonferroni $1 - S_1(t) + S_2(t)$ bound but is not necessarily less than or equal to $U^{(2)}(t)$ or $U^{(3)}(t)$. We define $t_l^{(2,3)}$ to be the point where $U^{(2,3)}(t) = 1-\alpha$.

3.2. Approximate correlation matrix methods

A second class of approximations consists of methods that use an approximation of the form $\mathbf{R} = \mathbf{D} + \boldsymbol{\lambda}\boldsymbol{\lambda}^t$ for the correlation matrix, so the t-integrals take the form (2).

An important approximation for all-pairwise comparisons is the Tukey–Kramer (TK) method (Tukey, 1953; Kramer, 1957). Tukey and Kramer conjectured that the use of balanced critical values instead of the true ones is always conservative (i.e., the covariance matrix $\mathbf{V}$ of the estimates is replaced by the identity matrix). This was proved by Hayter (1984) in the unbalanced independent one-factorial design for

general I, where $\mathbf{V} = \text{diag}(n_i^{-1})$. A proof of this conjecture for general designs (and hence possibly non-diagonal $\mathbf{V}$) for $I = 3$ was given by Brown (1984). The GT2 procedure proposed by Hochberg (1974) and based on Šidák's inequality replaces $\mathbf{R}$ by the identity matrix. This method is known to be inferior to TK, but it is conservative for all $\mathbf{V}$. More recently, Iyengar (1988) and Iyengar and Tong (1989) investigated replacing ρ_{ij} by their common average $\bar{\rho} = \frac{2}{I(I-1)} \sum_{i<j} \rho_{ij}$. It has been shown by Iyengar (1988), that this approach is not necessarily conservative for all choices of $\mathbf{R}$, but no counter example has been found yet for the types of $\mathbf{R}$ that arise with multiple comparison problems. Royen (1987) and Hsu (1992) independently provided different techniques to find the 'closest' $\hat{\mathbf{R}}$, which still possesses the product correlation structure in equation (2). Both factor-analytic and linear programming methods were investigated by Hsu (1992) and Hsu and Nelson (1998). These methods are not applicable on all-pairwise comparison problems. We refer to the original articles for more details. For comparison reasons we also included the classical yet very conservative F-test of Scheffé. Finally, Solow (1990) described a simple way for approximating multivariate normal probabilities from univariate and bivariate marginal probabilities. Section 4 provides results from a generalization for multivariate t integrals of Solow's method. This generalized method uses a decomposition of (1) into a product of conditional probabilities and approximates each term in the product using conditional expectations.

Hochberg and Tamhane (1987, p. 145) considered several other approximations tailored to the many-to-one comparisons case. But these methods are usually inferior to the methods that we have described already, so we did not include them in our study. Another approach is to consider the spectral decomposition $\mathbf{R} = \sum_i e_i \mathbf{p}_i \mathbf{p}_i^t$, where e_i is the ith eigenvalue in decreasing order and $\mathbf{p}_i$ is the corresponding normalized eigenvector. A possible approximation is to replace the ρ_{ij} by $p_i p_j$ and apply (2), where p_i and p_j are the elements of $\mathbf{p}_1$. But this method did not perform well in our comparison study, so we omitted the results from the following section.

4. Numerical comparisons

We compared the different methods for the two situations discussed in Section 2. For the all-pairwise comparisons, we let $I = 4$ ($q = 6$) and $\mathbf{V} = \text{diag}(n_i^{-1})$. The specific values for n_i are given in Table 1. The table further presents the estimates $\hat{t}_{1-\alpha}$ for each method and the associated probability $T_k(-\hat{\mathbf{t}}_{1-\alpha}, \hat{\mathbf{t}}_{1-\alpha}; \mathbf{R}, \nu)$ are given in italics below. Critical values and probabilities were calculated with three significant digits of accuracy using the methods of Genz and Bretz (2002). For the $I = 4$ cases $t_l^{(2,3)}$ and $t_u^{(2,3)}$ seem to be good low-cost improvements over $t_l^{(2)}$ (Dawson–Sankoff) and $t_u^{(2)}$ (Hunter–Worsley). Further on, $t_l^{(2,3)}$ appears to always be better than $t_l^{(3)}$. The TK should not be used if the sample sizes differ by a large amount. Similar results apply to the other approximate correlation methods. We have not included $t_l^{(1)}$ values because these values never had an accuracy greater than one digit.

For the many-to-one comparisons we looked at $J = 2$ and $I = 5$ (if $I \leq 4$, then $q = I - 1 \leq 3$ and (2) holds for any $\mathbf{V}$ if the λ_i's are defined appropriately). Table 2 specifies the values for $n_{i\cdot} = \sum_j n_{ij}$. In all cases we set $n_{i2} = 2$, $n_{i1} = n_{i\cdot} - n_{i2}$, $i = 1, 2$ and $n_{i1} = 2$, $n_{i2} = n_{i\cdot} - n_{i1}$, $i = 3, 4, 5$ and. This ensures that the proportionality rule of Section 2 is violated. Similar results as for the all pairwise comparisons hold here. The hybrid bounds are found again to be good approximations to $t_{1-\alpha}$. The Hsu method is usually accurate to three significant digits and is a good competitor to $t_l^{(2,3)}$ and $t_u^{(2,3)}$. The Solow method is easily

Table 1: Numerical results for all-pairwise comparisons with one-way layout ($I = 4$)

$t_l^{(2)}$	$t_l^{(3)}$	$t_l^{(2,3)}$	$t_{1-\alpha}$	$t_u^{(2,3)}$	$t_u^{(3)}$	$t_u^{(2)}$	$t_u^{(1)}$	Šidák	TK	GT2	$\bar{\rho}$	F-test	n_1	n_2	n_3	n_4
2.672	2.683	2.686	2.693	2.725	2.695	2.751	2.792	2.784	2.693	2.775	2.699	2.932	10	10	10	10
0.948	*0.949*	*0.950*	*0.950*	*0.954*	*0.951*	*0.956*	*0.960*	*0.959*	*0.950*	*0.959*	*0.951*	*0.972*				
2.641	2.652	2.655	2.660	2.679	2.660	2.705	2.752	2.744	2.661	2.738	2.664	2.897	10	12	14	16
0.948	*0.950*	*0.950*	*0.950*	*0.953*	*0.950*	*0.955*	*0.960*	*0.959*	*0.950*	*0.959*	*0.951*	*0.972*				
2.619	2.628	2.634	2.638	2.654	2.638	2.677	2.729	2.721	2.643	2.716	2.644	2.876	10	14	18	22
0.948	*0.949*	*0.950*	*0.950*	*0.952*	*0.951*	*0.955*	*0.960*	*0.959*	*0.951*	*0.959*	*0.951*	*0.973*				
2.602	2.612	2.618	2.622	2.636	2.622	2.658	2.713	2.705	2.630	2.702	2.630	2.863	10	16	22	28
0.948	*0.949*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.960*	*0.959*	*0.951*	*0.959*	*0.951*	*0.973*				
2.589	2.601	2.607	2.610	2.622	2.611	2.643	2.702	2.694	2.621	2.691	2.620	2.853	10	18	26	34
0.947	*0.949*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.960*	*0.960*	*0.951*	*0.959*	*0.951*	*0.973*				
2.578	2.590	2.597	2.600	2.611	2.601	2.631	2.694	2.686	2.615	2.684	2.613	2.846	10	20	30	40
0.947	*0.949*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.961*	*0.960*	*0.952*	*0.960*	*0.952*	*0.974*				
2.519	2.538	2.552	2.554	2.560	2.555	2.578	2.663	2.655	2.589	2.654	2.584	2.818	10	40	70	100
0.945	*0.948*	*0.950*	*0.950*	*0.951*	*0.950*	*0.953*	*0.962*	*0.962*	*0.954*	*0.962*	*0.954*	*0.976*				
2.490	2.515	2.532	2.535	2.536	2.540	2.557	2.654	2.647	2.582	2.646	2.575	2.810	10	60	110	160
0.944	*0.948*	*0.950*	*0.950*	*0.951*	*0.951*	*0.953*	*0.964*	*0.963*	*0.956*	*0.963*	*0.955*	*0.976*				
2.472	2.501	2.521	2.524	2.529	2.526	2.545	2.650	2.643	2.579	2.642	2.571	2.806	10	80	150	220
0.943	*0.947*	*0.949*	*0.950*	*0.951*	*0.951*	*0.953*	*0.964*	*0.964*	*0.957*	*0.964*	*0.956*	*0.977*				
2.458	2.490	2.514	2.517	2.521	2.519	2.537	2.647	2.640	2.577	2.640	2.568	2.804	10	100	190	280
0.942	*0.946*	*0.949*	*0.950*	*0.950*	*0.951*	*0.953*	*0.965*	*0.964*	*0.957*	*0.964*	*0.956*	*0.977*				
2.546	2.560	2.570	2.571	2.581	2.573	2.596	2.675	2.667	2.599	2.666	2.597	2.828	80	40	20	10
0.947	*0.948*	*0.950*	*0.950*	*0.952*	*0.950*	*0.953*	*0.962*	*0.961*	*0.953*	*0.961*	*0.953*	*0.975*				
2.488	2.511	2.527	2.528	2.534	2.530	2.546	2.652	2.644	2.580	2.644	2.576	2.808	270	90	30	10
0.945	*0.948*	*0.950*	*0.950*	*0.951*	*0.950*	*0.952*	*0.964*	*0.963*	*0.956*	*0.963*	*0.956*	*0.977*				
2.454	2.484	2.505	2.506	2.511	2.509	2.521	2.644	2.637	2.574	2.637	2.569	2.801	640	160	40	10
0.943	*0.947*	*0.950*	*0.950*	*0.951*	*0.950*	*0.952*	*0.965*	*0.965*	*0.958*	*0.965*	*0.958*	*0.978*				
2.431	2.466	2.491	2.494	2.497	2.495	2.506	2.642	2.634	2.572	2.634	2.566	2.799	1250	250	50	10
0.941	*0.946*	*0.950*	*0.950*	*0.951*	*0.950*	*0.953*	*0.967*	*0.966*	*0.959*	*0.966*	*0.958*	*0.978*				
2.414	2.454	2.482	2.484	2.487	2.487	2.496	2.640	2.633	2.571	2.633	2.564	2.797	2160	360	60	10
0.940	*0.947*	*0.950*	*0.950*	*0.951*	*0.950*	*0.952*	*0.968*	*0.967*	*0.961*	*0.967*	*0.959*	*0.979*				
2.400	2.445	2.474	2.477	2.480	2.480	2.488	2.640	2.632	2.570	2.632	2.563	2.797	3430	490	70	10
0.939	*0.945*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.969*	*0.967*	*0.961*	*0.967*	*0.960*	*0.980*				
2.393	2.437	2.468	2.471	2.474	2.475	2.481	2.639	2.632	2.570	2.632	2.562	2.796	5120	640	80	10
0.940	*0.945*	*0.950*	*0.950*	*0.950*	*0.951*	*0.952*	*0.968*	*0.968*	*0.962*	*0.968*	*0.961*	*0.980*				
2.387	2.423	2.459	2.463	2.466	2.466	2.472	2.639	2.632	2.569	2.631	2.561	2.796	10000	1000	100	10
0.939	*0.945*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.970*	*0.969*	*0.963*	*0.968*	*0.962*	*0.980*				

Table 2: Numerical results for many-to-one comparisons with two-way layout ($I = 5$)

$t_l^{(2)}$	$t_l^{(3)}$	$t_l^{(2,3)}$	$t_{1-\alpha}$	$t_u^{(2,3)}$	$t_u^{(3)}$	$t_u^{(2)}$	$t_u^{(1)}$	Šidák	Hsu	Solow	$\bar{\rho}$	$n_{1.}$	$n_{2.}$	$n_{3.}$	$n_{4.}$	$n_{5.}$
2.176	2.209	2.211	2.215	2.224	2.219	2.254	2.321	2.313	2.214	2.212	2.217	10	10	10	10	10
0.946	*0.949*	*0.950*	*0.950*	*0.951*	*0.950*	*0.954*	*0.960*	*0.960*	*0.950*	*0.950*	*0.950*					
2.094	2.152	2.158	2.161	2.170	2.170	2.202	2.295	2.287	2.161	2.159	2.169	10	12	14	16	18
0.942	*0.949*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.963*	*0.962*	*0.950*	*0.950*	*0.951*					
2.034	2.117	2.125	2.128	2.137	2.141	2.170	2.282	2.275	2.129	2.126	2.141	10	14	18	22	26
0.939	*0.949*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.964*	*0.964*	*0.950*	*0.950*	*0.951*					
1.985	2.094	2.102	2.105	2.113	2.119	2.147	2.274	2.267	2.106	2.104	2.123	10	16	22	28	34
0.935	*0.948*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.966*	*0.965*	*0.950*	*0.950*	*0.952*					
1.969	2.074	2.084	2.088	2.096	2.104	2.129	2.269	2.261	2.088	2.087	2.109	10	18	26	34	42
0.936	*0.948*	*0.950*	*0.950*	*0.951*	*0.952*	*0.954*	*0.967*	*0.966*	*0.950*	*0.950*	*0.952*					
1.958	2.059	2.070	2.074	2.082	2.090	2.115	2.265	2.257	2.075	2.074	2.099	10	20	30	40	50
0.936	*0.948*	*0.949*	*0.950*	*0.951*	*0.951*	*0.954*	*0.967*	*0.967*	*0.950*	*0.950*	*0.953*					
1.903	1.984	2.006	2.010	2.016	2.023	2.045	2.251	2.244	2.010	2.012	2.054	10	40	70	100	130
0.937	*0.947*	*0.950*	*0.950*	*0.951*	*0.951*	*0.954*	*0.971*	*0.971*	*0.950*	*0.950*	*0.955*					
1.880	1.957	1.982	1.985	1.990	1.997	2.017	2.248	2.240	1.985	1.987	2.040	10	60	110	160	210
0.938	*0.947*	*0.951*	*0.950*	*0.951*	*0.952*	*0.953*	*0.973*	*0.972*	*0.950*	*0.950*	*0.956*					
1.867	1.939	1.969	1.972	1.977	1.982	2.000	2.246	2.239	1.972	1.973	2.032	10	80	150	220	290
0.938	*0.946*	*0.950*	*0.950*	*0.951*	*0.951*	*0.953*	*0.973*	*0.973*	*0.950*	*0.950*	*0.956*					
1.859	1.929	1.959	1.963	1.967	1.973	1.989	2.245	2.238	1.962	1.963	2.028	10	100	190	280	370
0.938	*0.946*	*0.950*	*0.950*	*0.950*	*0.951*	*0.953*	*0.974*	*0.973*	*0.950*	*0.950*	*0.957*					
2.164	2.176	2.180	2.179	2.182	2.180	2.194	2.253	2.245	2.180	2.178	2.200	160	80	40	20	10
0.948	*0.950*	*0.950*	*0.950*	*0.950*	*0.950*	*0.952*	*0.958*	*0.957*	*0.950*	*0.950*	*0.952*					
2.143	2.157	2.161	2.162	2.164	2.162	2.173	2.244	2.237	2.161	2.158	2.196	810	270	90	30	10
0.947	*0.950*	*0.950*	*0.951*	*0.950*	*0.950*	*0.951*	*0.959*	*0.958*	*0.950*	*0.950*	*0.954*					
2.129	2.146	2.150	2.150	2.152	2.152	2.161	2.242	2.235	2.151	2.148	2.194	2560	640	160	40	10
0.948	*0.950*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.960*	*0.959*	*0.950*	*0.949*	*0.954*					
2.120	2.139	2.145	2.145	2.146	2.145	2.154	2.242	2.234	2.144	2.141	2.194	6250	1250	250	50	10
0.947	*0.949*	*0.950*	*0.950*	*0.951*	*0.950*	*0.952*	*0.961*	*0.960*	*0.950*	*0.950*	*0.955*					
2.115	2.135	2.141	2.140	2.143	2.141	2.148	2.242	2.234	2.139	2.136	2.193	12960	2160	360	60	10
0.946	*0.949*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.961*	*0.960*	*0.950*	*0.950*	*0.956*					
2.108	2.129	2.135	2.135	2.137	2.136	2.144	2.242	2.234	2.135	2.134	2.193	24010	3430	490	70	10
0.947	*0.949*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.961*	*0.960*	*0.950*	*0.950*	*0.956*					
2.104	2.125	2.133	2.132	2.135	2.133	2.141	2.241	2.234	2.132	2.130	2.193	40960	5120	640	80	10
0.946	*0.949*	*0.950*	*0.950*	*0.950*	*0.949*	*0.951*	*0.962*	*0.961*	*0.950*	*0.950*	*0.956*					
2.097	2.121	2.127	2.128	2.129	2.128	2.135	2.241	2.234	2.128	2.125	2.192	100000	10000	1000	100	10
0.946	*0.949*	*0.950*	*0.950*	*0.950*	*0.950*	*0.951*	*0.962*	*0.961*	*0.950*	*0.949*	*0.958*					

implemented and performs well for the low I value that we have used for Table 2, but other tests have shown that its performance deteriorates rapidly with increasing I.

5. Conclusions

The present paper compared several methods for approximating multivariate t probabilities. The methods can be classified into inequality based methods which provide either an upper or a lower bound of the true integral value and other approximations, which mostly rely on replacing the true correlation matrix through a more convenient matrix. The numerical comparisons show that the true critical values are approximated with satisfactory accuracy for practical purposes. The modification described in this paper of the hybrid bounds by Tomescu (1986) are quickly implemented and run at low computation time. These hybrid bounds usually yielded three digit accurate values for the typical multiple comparison problems which we looked at. All results are also valid for multivariate normal problems.

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