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Distributions of failure times associated with non-homogeneous compound Poisson damage processes

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Abstract: Failure time distributions are derived for non-homogeneous compound Poisson cumulative damage processes. We focus attention on Weibull type processes with exponential damage size. The hazard functions are illustrated and their asymptotic behavior investigated. Moment equations and maximum likelihood estimates are studied for the homogeneous case.

1. Introduction

Bogdanoff and Kozin, in their book (1985) define cumulative damage (CD) as the "irreversible accumulation of damage throughout life, that ultimately leads to failure". Such damage can be manifested by corrosion, cracks, physical wear in bearing, piston rings, locks, etc. We focus attention on damage processes that occur at random times, according to some non-homogeneous Poisson process. The amount of damage that accumulates follows a specified distribution. Thus, the amount of damage at time t, is a realization of a random process $\{Y(t), t \geq 0\}$, where $Y(t) \geq 0$ is a non-decreasing process with $Y(t) \to \infty$ a.s. as $t \to \infty$.

A system subjected to such a damage process fails at the first instant at which $Y(t) \geq \beta$, where $0 < \beta < \infty$ is a threshold specific to the system. Thus, the distribution of the failure times is a stopping time distribution. We present in the present paper the methodology of deriving these distributions. We are interested in particular in a family of non-homogeneous Poisson processes having an intensity function of the Weibull type, namely $\lambda(t) = (\lambda t)^\nu$, $0 < \lambda, \nu < \infty$. In Section 2 we specify compound non-homogeneous Poisson damage processes, and the distribution of the cumulative damage $Y(t)$, at time t. In Section 3 we derive the density and the reliability function of failure times driven by such processes. In particular we focus attention on cumulative Weibull processes with exponentially distributed damage amount in each occurrence. We investigate and illustrate the behavior of the distribution of failure times and the hazard function. In Section 4 we develop estimators of the parameters of the failure distribution in the homogeneous case ($\nu = 1$).

An extensive list of publications on damage processes is given in Bogdanoff and Kozin (1985). They provide empirical examples, and mention (p. 28) the non-homogeneous Poisson process with a Weibull intensity function. The theory for a discrete Markov chain model, having b states of damage is developed in this book. A recent paper on the subject is that of W. Kahle and H. Wendt (2000). They have modeled damage by a marked point process, and focus attention on

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doubly stochastic compound Poisson processes. Their formulation is close to ours, but they do not provide an explicit formula for the distribution of failure times. Other related papers are those concerned with shock models, like Esary, Marshall and Proshan (1973), Feng, Adachi and Kowada (1994), Shaked (1983), Soczyk (1987).

2. Compound cumulative damage processes

We consider cumulative damage processes (CDP) modeled by *non-homogeneous compound Poisson processes*. In this model, the system is subjected to shocks at random times, $0 < \tau_1 < \tau_2 < \cdots$, following a non-homogeneous Poisson process, with an *intensity function* $\lambda(t)$ (see Kao, 1997, pp. 56). The amount of damage to the system at the n-th shock is a random variable X_n, $n \geq 1$. We assume that $X_0 \equiv 0$, $X_1, X_2, \ldots$ are i.i.d., and that the sequence $\{X_n, n \geq 1\}$ is *independent* of $\{\tau_n, n \geq 1\}$.

Let $\{N(t), t \geq 0\}$ be a non-homogeneous Poisson counting process, with $N(0) = 0$ where

$$N(t) = \max\{n : \tau_n \leq t\}. \tag{1}$$

$\{N(t), t \geq 0\}$ is a process of independent increments such that, for any $0 \leq s < t < \infty$,

$$P\{N(t) - N(s) = n\} = e^{-(m(t)-m(s))}\frac{(m(t) - m(s))^n}{n!}, \tag{2}$$

$n = 0, 1, \ldots$, where $m(t) = \int_0^t \lambda(s)ds$, $0 \leq t < \infty$. The *compound damage process* (CDP) $\{Y(t), t \geq 0\}$ is defined as

$$Y(t) = \sum_{n=0}^{N(t)} X_n. \tag{3}$$

It is a compound non-homogeneous Poisson process. The compound Poisson Process (CPP) is the special case of a constant intensity function, $\lambda(t) = \lambda$, for all $0 < t < \infty$, $0 < \lambda < \infty$. We restrict attention in the present paper to the family of compound Weibull processes (CWP), in which $\lambda(t) = \lambda\nu(\lambda t)^{\nu-1}$, $0 < t < \infty$ for $0 < \lambda$, $\nu < \infty$. Furthermore, we assume that X_n, $n \geq 1$, are absolutely continuous random variables, having a common distribution function, F, and density f.

The cdf of $Y(t)$, at $t > 0$, has a discontinuity at $y = 0$, and is absolutely continuous on $0 < y < \infty$. It is given by

$$D(y; t) = \sum_{n=0}^{\infty} e^{-m(t)}\frac{(m(t))^n}{n!}F^{(n)}(y). \tag{4}$$

with $D(0; t) = \exp(-m(t))$, and $F^{(n)}$ is the n-fold convolution of F, i.e.,

$$F^{(n)}(y) = \begin{cases} F(y), & \text{if } n = 1 \\ \int_0^y f(x)F^{(n-1)}(y - x)dx, & \text{if } n \geq 2. \end{cases} \tag{5}$$

The defective density of $Y(t)$ on $(0, \infty)$ is

$$d(y; t) = \sum_{n=1}^{\infty} e^{-m(t)}\frac{(m(t))^n}{n!}f^{(n)}(y). \tag{6}$$

where $f^{(n)}$ is the n-fold convolution of the density f. We will use the notation $p(n;\mu)$ and $P(n;\mu)$ for the probability function and cdf, respectively, of the Poisson distribution with mean μ. Accordingly, the density of the CWP, at $0 < y < \infty$ and $0 < t < \infty$ is

$$d(y;t,\lambda,\nu) = \sum_{n=1}^{\infty} p(n;(\lambda t)^{\nu}) f^{(n)}(y), \tag{7}$$

and its cdf is

$$D(y;t,\lambda,\nu) = \sum_{n=0}^{\infty} p(n;(\lambda t)^{\nu}) F^{(n)}(y). \tag{8}$$

We consider a special case of these functions, when the amount of damage X_n is exponentially distributed, with parameter μ, i.e., $E\{X_n\} = \frac{1}{\mu}$. In this special case $f^{(n)}(y) = \mu p(n-1;\mu y)$ and $F^{(n)}(y) = 1 - P(n-1;\mu y)$. The results of this paper can be generalized to damage processes driven by compound renewal processes with any distribution F.

3. Cumulative damage failure distributions

A cumulative damage failure time is the stopping time

$$T(\beta) = \inf\{t > 0 : Y(t) \geq \beta\}, \tag{9}$$

where $0 < \beta < \infty$. Since $Y(t)$ is non-decreasing a.s., we immediately obtain that, in the continuous case,

$$P\{T(\beta) > t\} = D(\beta;t), \quad 0 < t < \infty. \tag{10}$$

This is the reliability (survival) function of the system. Thus, for the CWP, with general damage distribution,

$$P\{T(\beta) > t\} = \sum_{n=0}^{\infty} p(n;(\lambda t)^{\nu}) F^{(n)}(\beta). \tag{11}$$

In the special case of exponential damage distribution,

$$P\{T(\beta) > t\} = 1 - \sum_{n=1}^{\infty} p(n;(\lambda t)^{\nu}) P(n-1;\mu\beta). \tag{12}$$

We see in (3.4) that, in the exponential case, the distribution of $T(\beta)$ depends on μ and β only through $\zeta = \mu\beta = \beta/E\{X_1\}$. Accordingly, let $R(t;\lambda,\nu,\zeta)$ denote the reliability function of a system under CWP with exponential damage distribution (CWP/E).

Theorem 1. *Under* CWP/E *the reliability function is*

$$R(t;\lambda,\nu,\zeta) = \sum_{j=0}^{\infty} p(j;\zeta) P(j;(\lambda t)^{\nu}). \tag{13}$$

Proof. According to (3.4),

$$R(t;\lambda,\nu,\zeta) \quad = \quad 1 - \sum_{n=1}^{\infty} p(n;(\lambda t)^{\nu}) \sum_{j=0}^{n-1} p(j;\zeta)$$

$$= 1 - \sum_{j=0}^{\infty} p(j;\zeta) \sum_{n=j+1}^{\infty} p(n;(\lambda t)^{\nu})$$
$$= 1 - \sum_{j=0}^{\infty} p(j;\zeta)(1 - P(j;(\lambda t)^{\nu})).$$

This implies (3.5). □

It is obvious from (3.1) that $P\{T(\beta) < \infty\} = 1$ for any $0 < \beta < \infty$. This follows also from the following theorem.

Theorem 2. *Under* CWP/E*,* $R(0;\lambda,\nu,\zeta) = 1$*,* $R(t;\lambda,\nu,\zeta)$ *is strictly decreasing in* t*, for* (λ,ν,ζ) *fixed, and* $\lim_{t\to\infty} R(t;\lambda,\nu,\zeta) = 0$*, for any* (λ,ν,ζ) *in* $\mathbb{R}^{3+}$*.*

Proof. According to (3.5), since $\lim_{t\to 0} P(j;(\lambda t)^{\nu}) = 1$ for all $j = 0, 1, \ldots$ and any $0 < \lambda,\ \nu < \infty$, the bounded convergence theorem implies that

$$\lim_{t\to 0} R(t;\lambda,\nu,\zeta) = \sum_{j=0}^{\infty} p(j;\zeta) \lim_{t\to 0} P(j;(\lambda t)^{\nu}) = 1.$$

Furthermore, the Poisson family is an MLR family and $P(j;(\lambda t)^{\nu}) \downarrow t$. Hence, $R(t;\lambda,\nu,\zeta) \ \downarrow\ t$, i.e., $\frac{\partial}{\partial t} R(t;\lambda,\nu,\zeta) < 0$, for any fixed (λ,ν,ζ), $0 < \lambda,\nu,\zeta < \infty$. Finally, since $\lim_{t\to\infty} P(j;(\lambda t)^{\nu}) = 0$ for any fixed $j \geq 0$, $0 < \lambda,\nu < \infty$, the dominated convergence theorem implies that $\lim_{t\to\infty} R(t;\lambda,\nu,\zeta) = 0$, for any $0 < \lambda,\nu,\zeta < \infty$. □

Theorem 3. *Under* CWP/E*, the density of* $T(\zeta)$*,* $0 < \zeta < \infty$*, is*

$$f(t;\lambda,\nu,\zeta) = \lambda\nu(\lambda t)^{\nu-1} \sum_{j=0}^{\infty} p(j;\zeta) p(j;(\lambda t)^{\nu}), \tag{14}$$

and its m-th moment, $m \geq 1$*, is*

$$E\{(T(\zeta))^m\} = \frac{1}{\lambda^m} \sum_{j=0}^{\infty} p(j;\zeta) \frac{\Gamma(j+1+\frac{m}{\nu})}{\Gamma(j+1)}. \tag{15}$$

Proof. It is easy to verify that

$$\frac{\partial}{\partial\omega} P(j;\omega) = -p(j;\omega), \quad 0 < \omega < \infty.$$

Moreover,

$$f(t;\lambda,\nu,\zeta) = -\frac{\partial}{\partial t} P\{T(\beta) > t\}$$
$$= -\frac{\partial}{\partial t} \sum_{j=0}^{\infty} p(j;\zeta) P(j;(\lambda t)^{\nu}).$$

This implies (3.6), since $R(t;\lambda,\nu,\zeta)$ is an analytic function of t, or by bounded convergence. To prove (3.7) we write

$$E\{(T(\zeta))^m\} = \int_0^{\infty} t^m f(t;\lambda,\nu,\zeta) dt$$

$$
\begin{aligned}
&= \nu\lambda^{\nu}\sum_{j=0}^{\infty}p(j;\zeta)\frac{\lambda^{\nu j}}{j!}\int_0^{\infty}t^{m+\nu(j+1)-1}e^{-(\lambda t)^{\nu}}dt \\
&= \sum_{j=0}^{\infty}p(j;\zeta)\frac{\lambda^{\nu(j+1)}}{j!}\int_0^{\infty}u^{\frac{m}{\nu}+j}e^{-\lambda^{\nu}u}du \\
&= \frac{1}{\lambda^m}\sum_{j=0}^{\infty}p(j;\zeta)\frac{\Gamma(j+1+\frac{m}{\nu})}{\Gamma(j+1)}.
\end{aligned}
$$

□

Corollary. *In the homogeneous case ($\nu = 1$) with exponential damage, the expected value, variance and coefficient of skewness of $T(\zeta)$ are, correspondingly,*

$$E\{T(\beta) \mid \lambda, \nu = 1, \zeta\} = \frac{1+\zeta}{\lambda}, \tag{16}$$

$$V\{T(\zeta) \mid \lambda, \nu = 1, \zeta\} = \frac{1+2\zeta}{\lambda^2} \tag{17}$$

and

$$\gamma_1(T(\zeta)) = \frac{2(1+3\zeta)}{(1+2\zeta)^{3/2}}. \tag{18}$$

Notice also that equation (3.7) shows that moments of $T(\zeta)$ of all orders exist, since moments of all orders of the Poisson distribution exist. In Figure 1 we present several densities of $T(\zeta)$, for $\lambda = 1$, $\zeta = 5$ and $\nu = 1.1, 1, .9$. According to eq. (3.6),

$$\lim_{t\to 0} f(t;\lambda,\nu,\zeta) = \begin{cases} \infty, & \text{if } \nu < 1 \\ \lambda e^{-\zeta}, & \text{if } \nu = 1 \\ 0, & \text{if } \nu > 1. \end{cases} \tag{19}$$

Indeed, $\lim_{t\to 0} p(j;(\lambda t)^{\nu}) = I\{j = 0\}$, i.e., 1 if $j = 0$ and 0 otherwise. Thus, $\lim_{t\to 0}\sum_{j=0}^{\infty}p(j;\zeta)p(j;(\lambda t)^{\nu}) = p(0;\zeta) = e^{-\zeta}$. The densities $f(t;\lambda,\nu,\zeta)$ are uni-modal whenever $\nu \geq 1$, and bi-modal when $\nu < 1$. Figure 1 does not show the behavior of these densities in the interval $(0, 1)$. We see that the density becomes more symmetric as ζ grows. Indeed, $\frac{\partial}{\partial\zeta}\gamma_1(T(\zeta)) = -\frac{6\zeta}{(1+2\zeta)^{5/2}} < 0$ for all $0 < \zeta < \infty$.

From eq. (3.5) we obtain immediately that the reliability function $R(t;\lambda,\nu,\zeta)$, is a strictly increasing function of ζ, for each fixed (t,λ,ν). This result is obvious from (3.1) if $\mu = 1$. Generally, for fixed t,λ,ν $P(j;(\lambda t)^{\nu})$ is an increasing function of j. Hence, since the Poisson family $\{p(\cdot;\zeta), 0 < \zeta < \infty\}$ is a monotone likelihood ratio family (MLR), $E_{\zeta}\{P(J;(\lambda t)^{\nu})\}$ is an increasing function of ζ.

The hazard function under CWP/E damage processes is

$$h(t;\lambda,\nu,\zeta) = \frac{\lambda\nu(\lambda t)^{\nu-1}\sum_{j=0}^{\infty}p(j;\zeta)p(j;(\lambda t)^{\nu})}{\sum_{j=0}^{\infty}p(j;\zeta)P(j;(\lambda t)^{\nu})}. \tag{20}$$

We obtain from (3.11) since $\lim_{t\to 0} P(j;(\lambda t)^{\nu}) = 1$ for all $j \geq 0$, that,

$$\lim_{t\to 0} h(t;\lambda,\nu,\zeta\}) = \begin{cases} \infty, & \text{if } 0 < \nu < 1 \\ \lambda e^{-\zeta}, & \text{if } \nu = 1 \\ 0, & \text{if } \nu > 1. \end{cases} \tag{21}$$

In Figure 2 we illustrate the hazard function (3.12) for $\lambda = 1$, $\zeta = 5$ and $\nu = .53, .55, .57$.

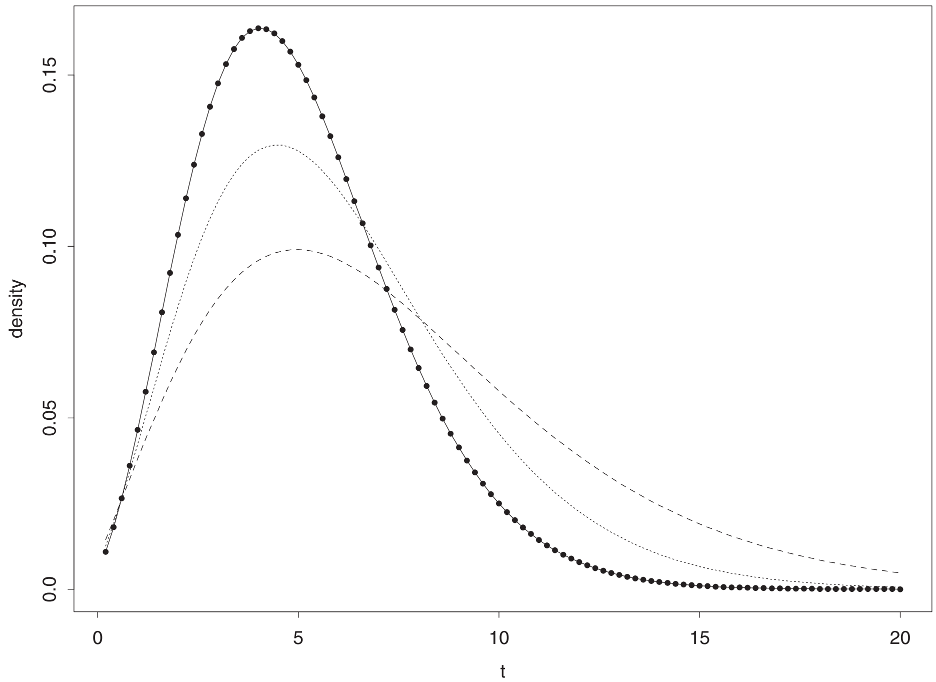

Figure 1: Densities of $T(\zeta)$, $\lambda = 1$, $\zeta = 5$, $\nu = 1.1$ —●—, $\nu = 1.0$ $\cdots$, $\nu = 0.9$ - - -

Similar types of hazard functions were discussed by Aalen and Gjesing (2003).

We examine now the asymptotic behavior of the hazard function (3.12), as $t \to \infty$. Make first the transformation $u = (\lambda t)^{\nu}$. In terms of u, the hazard function is

$$h^*(u; \lambda, \nu, \zeta) = \lambda \nu u^{1-1/\nu} \cdot \frac{E_\zeta\{p(J; u)\}}{E_\zeta\{P(J; u)\}}, \tag{22}$$

where $J \sim \text{Pois}(\zeta)$.

Theorem 4. *For a fixed λ, ν, ζ, the asymptotic behavior of the hazard function is*

$$\lim_{u\to\infty} h^*(u; \lambda, \nu, \zeta) = \begin{cases} \infty, & \textit{if } \nu > 1 \\ \lambda, & \textit{if } \nu = 1 \\ 0, & \textit{if } \nu < 1. \end{cases} \tag{23}$$

Proof. Since $p(j; u) \leq P(j; u)$ for $j = 0, 1, \ldots$ and each u, $0 < u < \infty$,

$$\overline{\lim_{u\to\infty}} \frac{E_\zeta\{p(J; u)\}}{E_\zeta\{P(J; u)\}} \leq 1. \tag{24}$$

We now prove that

$$\lim_{u\to\infty} \frac{E_\zeta\{p(J; u)\}}{E_\zeta\{P(J; u)\}} = 1. \tag{25}$$

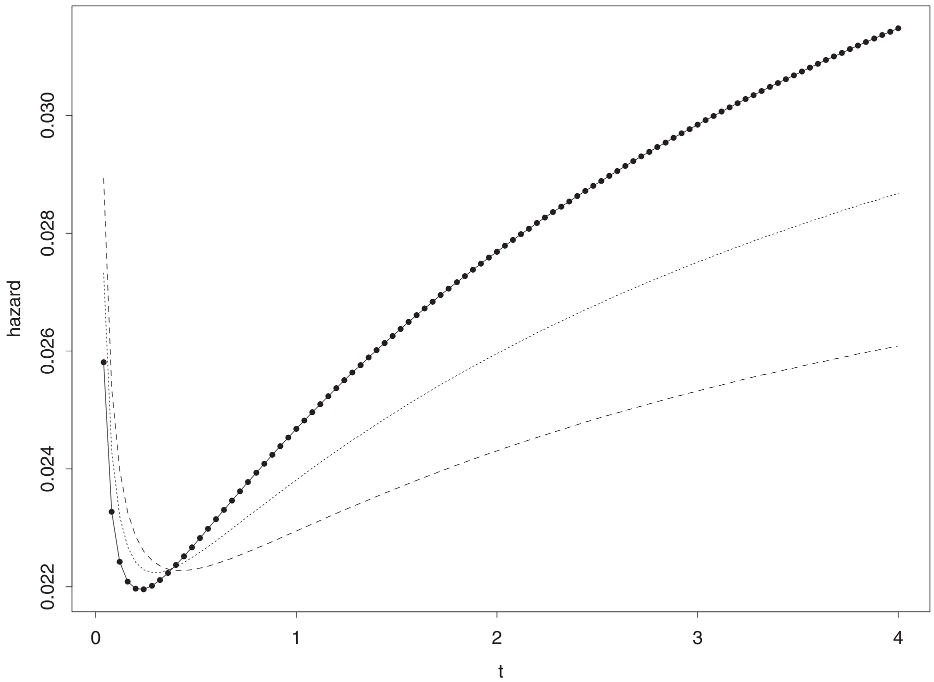

Figure 2. Hazard Functions, $\lambda = 1$, $\zeta = 5$, $\nu = .57$ —●—, $\nu = .55$ $\cdots$, $\nu = .53$ - - -

First, by dominated convergence, $\lim_{u\to\infty} E_\zeta(p(J;u)\} = E_\zeta\{\lim_{u\to\infty} p(J;u)\} = 0$. Similarly, $\lim_{u\to\infty} E_\zeta\{P(J;u)\} = 0$. By L'Hospital rule,

$$\begin{aligned}\lim_{u\to\infty} \frac{E_\zeta\{p(J;u)\}}{E_\zeta\{P(J;u)\}} &= \lim_{u\to\infty} \frac{\frac{d}{du}E_\zeta\{p(J;u)\}}{\frac{d}{du}E_\zeta\{P(J;u)\}} \\ &= \lim_{u\to\infty} \frac{E_\zeta\{p(J;u) - p(J-1;u)\}}{E_\zeta\{p(J;u)\}} \\ &= 1 - \lim_{u\to\infty} \frac{\sum_{j=0}^{\infty} p(n+1;\zeta)p(n;u)}{\sum_{n=0}^{\infty} p(n;\zeta)p(n;u)}.\end{aligned}$$

Furthermore,

$$\frac{\sum_{n=0}^{\infty} p(n+1;\zeta)p(n;u)}{\sum_{n=0}^{\infty} p(n;\zeta)p(n;u)} = \zeta \frac{\sum_{n=0}^{\infty} \frac{1}{n+1} p(n;\zeta)p(n;u)}{\sum_{n=0}^{\infty} p(n;\zeta)p(n;u)}$$

Fix a positive integer K (arbitrary). Then,

$$
\begin{aligned}
R(\zeta, u) &= \frac{\sum_{n=0}^{\infty} \frac{1}{n+1} p(n;\zeta) p(n;u)}{\sum_{n=0}^{\infty} p(n;\zeta) p(n;u)} \\
&\le \frac{\sum_{n=0}^{K} \frac{1}{n+1} p(n;\zeta) p(n;u) + \frac{1}{K+2} \sum_{n=K+1}^{\infty} p(n;\zeta) p(n;u)}{\sum_{n=0}^{K} p(n;\zeta) p(n;u) + \sum_{n=K+1}^{\infty} p(n;\zeta) p(n;u)}
\end{aligned} \tag{26}
$$

Finally, since $p(n;u) \to 0$ as $u \to \infty$ for each $n = 0, 1, \ldots$,

$$
\lim_{u\to\infty} \sum_{j=0}^{K} \frac{1}{j+1} p(j;\zeta) p(j;u) = \lim_{u\to\infty} \sum_{j=0}^{K} p(j;\zeta) p(j;u) = 0.
$$

Thus,

$$
\begin{aligned}
\overline{\lim_{u\to\infty}} R(\zeta; u) &\le \frac{1}{K+2} \lim_{u\to\infty} \frac{\sum_{j=K+1}^{\infty} p(j;\zeta) p(j;u)}{\sum_{j=K+1}^{\infty} p(j;\zeta) p(j;u)} \\
&= \frac{1}{K+2}, \quad \text{for all fixed } \zeta.
\end{aligned}
$$

□

In Figure 3 we illustrate a hazard function for $\lambda = 1$, $\zeta = 5$, $\nu = 0.5$.

4. Estimation of parameters

Let $T_1, T_2, \ldots, T_n$ be i.i.d. random failure times following CWP/E. The likelihood function of the parameters (λ, ν, ζ) is

$$
L(\lambda, \nu, \zeta; T_1, \ldots, T_n) = (\lambda)^{n\nu} \nu^n \left(\prod_{i=1}^{n} T_i^{\nu-1} \right) \cdot \prod_{i=1}^{n} \sum_{j=0}^{\infty} p(j;\zeta) p(j; (\lambda T_i)^{\nu}). \tag{27}
$$

Accordingly, the minimal sufficient statistic is the trivial one $(T_{(1)}, \ldots, T_{(n)})$, where $0 < T_{(1)} \le T_{(2)} \le \cdots \le T_{(n)}$.

4.1. Moment equations estimators of λ, ζ in the homogeneous case, $\nu = 1$.

Let $M_1 = \frac{1}{n} \sum_{i=1}^{n} T_i$ and $M_2 = \frac{1}{n} \sum_{i=1}^{n} T_i^2$ be the first two sample moments. The moment equations estimators (MEE) of λ and ζ are obtained by solving the equations,

$$
\frac{1 + \hat{\zeta}}{\hat{\lambda}} = M_1 \tag{28}
$$

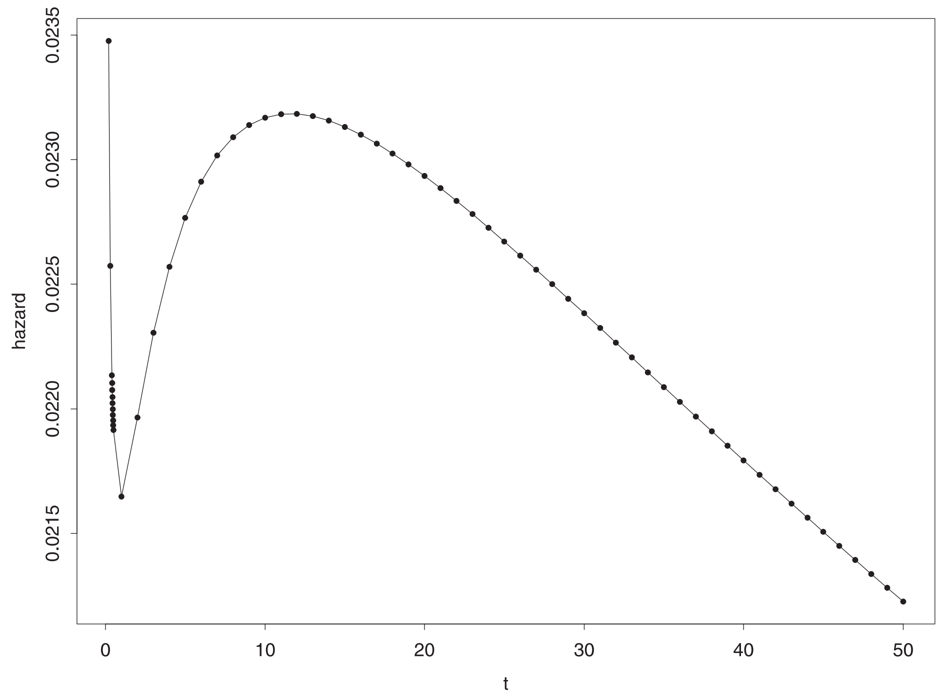

Figure 3. Hazard Function for $\lambda = 1$, $\nu = .5$, $\zeta = 5$

and

$$\frac{2 + 4\hat{\zeta} + \hat{\zeta}^2}{\hat{\lambda}^2} = M_2. \tag{29}$$

Or, equivalently,

$$\hat{\lambda} = \frac{1 + \hat{\zeta}}{M_1}, \tag{30}$$

and $\hat{\zeta}$ is the positive root of the quadratic equation

$$\hat{\zeta}^2 \left(1 - \frac{M_1^2}{M_2}\right) - 2\hat{\zeta}\left(\frac{2M_1^2}{M_2} - 1\right) - \left(\frac{2M_1^2}{M_2} - 1\right) = 0. \tag{31}$$

A real root exists provided $M_2 < 2M_1^2$. Since $2M_1^2 - M_2 \xrightarrow[n\to\infty]{a.s.} (\frac{\zeta}{\lambda})^2 > 0$, an MEE exists for n sufficiently large. It is given by

$$\hat{\zeta} = \frac{(2M_1^2 - M_2)^{1/2}(M_1 + (2M_1^2 - M_2)^{1/2})}{M_2 - M_1^2}. \tag{32}$$

Both $\hat{\lambda}$ and $\hat{\zeta}$ are strongly consistent estimators of λ and ζ, respectively. The mean squared errors of these estimators can be approximated by the delta method. We obtain

$$\text{MSE}\{\hat{\lambda}\} = \frac{\lambda^2}{n} \cdot \frac{1 + 12\zeta + 58\zeta^2 + 144\zeta^3 + 192\zeta^4 + 128\zeta^5 + 32\zeta^6}{\zeta^2(1 + 2\zeta)^4} + O\left(\frac{1}{n^2}\right), \tag{33}$$

and

$$\text{MSE}\{\hat{\zeta}\} = \frac{1}{n\zeta^2}(2(1+\zeta)^4 - (1+\zeta)^2 - \zeta^2) + O\left(\frac{1}{n^2}\right). \tag{34}$$

In the following table we compare the values of the MSE, as approximated by eq.'s (4.7) and (4.8), to those obtained by simulations. When $\nu = 1$ the distribution of T is like that of $\chi^2[2;\zeta]/(2\lambda)$, where $\chi^2[2;\zeta]$ is a non-central chi-square with 2 degrees of freedom, and parameter of non-centrality ζ. Thus

$$T \sim (N_1^2(\sqrt{\zeta}, 1) + N_2^2(\sqrt{\zeta}, 1))/(2\lambda),$$

where $N_i(\sqrt{\zeta}, 1)$ $(i = 1, 2)$ are i.i.d. normal random variables with mean $\sqrt{\zeta}$ and variance 1. 10,000 simulation runs yield the following results

Table 1. *MSE Values of the MEE By Delta Method and By Simulations*

λ	ζ	n	Delta Method		Simulation	
			$\hat{\lambda}$	$\hat{\zeta}$	$\hat{\lambda}$	$\hat{\zeta}$
1	5	50	0.0568	2.0248	0.0744	2.6286
		100	0.0284	1.0124	0.0322	1.1475
2	5	50	0.2272	2.0248	0.3058	2.6746
		100	0.1136	1.0124	0.1323	1.1828

We notice that the delta method for samples of size 50 or 100 is not sufficiently accurate. It yields values which are significantly smaller than those of the simulation. Also, since the MEE $\hat{\lambda}$ and $\hat{\zeta}$ are continuously differentiable functions of the sample moments M_1 and M_2, the asymptotic distributions of $\hat{\lambda}$ and $\hat{\zeta}$ are normal, with means λ and ζ and variances given by (4.7) and (4.8).

4.2. Maximum likelihood estimators, $\nu = 1$

The log-likelihood function of (λ, ζ), given $\mathbf{T}^{(n)}$ is

$$l(\lambda, \zeta; \mathbf{T}^{(n)}) = n \log \lambda + \sum_{i=1}^{n} \log E_\zeta\{p(J; \lambda T_i)\}, \tag{35}$$

where $J \sim \text{Pois}(\zeta)$. Accordingly, the score functions are

$$\frac{\partial}{\partial \lambda} l(\lambda, \zeta; \mathbf{T}^{(n)}) = \frac{n}{\lambda} - \sum_{i=1}^{n} T_i + \zeta \sum_{i=1}^{n} T_i W(\lambda, \zeta, T_i), \tag{36}$$

and

$$\frac{\partial}{\partial \zeta} l(\lambda, \zeta, \mathbf{T}^{(n)}) = -n + \lambda \sum_{i=1}^{n} T_i W(\lambda, \zeta, T_i), \tag{37}$$

where

$$W(\lambda, \zeta, T) = \frac{E_\zeta\{\frac{1}{1+J} p(J; \lambda T)\}}{E_\zeta\{p(J; \lambda T)\}}. \tag{38}$$

Let $\hat{\lambda}$ and $\hat{\zeta}$ be the maximum likelihood estimators (MLE) of λ and ζ, respectively.

From (4.10) and (4.11) we obtain that, as in (4.4),

$$\hat{\lambda} = \frac{1 + \hat{\zeta}}{M_1}. \tag{39}$$

Substituting $\hat{\lambda}$ in (4.11) we obtain the function

$$\overset{\bullet}{l}(\zeta) = (1+\zeta)\sum_{i=1}^{n} U_i W\left(\frac{1+\zeta}{M_1}, \zeta, M_1 U_i\right) - n, \tag{40}$$

where $U_i = T_i/M_1$. More specifically,

$$\overset{\bullet}{l}(\zeta) = (1+\zeta)\sum_{i=1}^{n} U_i \frac{E_\zeta\{\frac{1}{1+J}p(J;(1+\zeta)U_i)\}}{E_\zeta\{p(J;(1+\zeta)U_i)\}} - n. \tag{41}$$

Notice that $\overset{\bullet}{l}(0) = 0$. The MLE of ζ, $\hat{\zeta}$, is the positive root of $\overset{\bullet}{l}(\zeta) \equiv 0$. $N = 1,000$ simulation runs gave the following estimates of the MSE of $\hat{\lambda}$ and $\hat{\zeta}$, when $\lambda = 1$, $\zeta = 5$ and $n = 50$, namely:

$$\widehat{\text{MSE}}(\hat{\lambda}) = 0.06015 \quad \text{and} \quad \widehat{\text{MSE}}(\hat{\zeta}) = 2.13027.$$

As expected, these estimators of the MSE of $\hat{\lambda}$ and $\hat{\zeta}$ are smaller than those of the MEE estimates, given in Table 1. The asymptotic distribution of the MLE vector $(\hat{\lambda}, \hat{\zeta})$ is bivariate normal with mean (λ, ζ) and covariance matrix AV, which is the inverse of the Fisher information matrix. The asymptotic variance-covariance matrix of the MLE can be estimated by simulation. $N = 10,000$ simulation runs gave, for the case of $\lambda = 1$, $\zeta = 5$ the asymptotic variance-covariance matrix

$$AV = \frac{1}{n}\begin{bmatrix} 2.33917 & 13.04000 \\ 13.04000 & 83.30706 \end{bmatrix}.$$

Thus, the asymptotic variance of $\hat{\zeta}$ for $n = 50$ is $AV(\hat{\zeta}) = \frac{83.30706}{50} = 1.66614$. We see that the estimated variance of $\hat{\zeta}$ is, as in the case of the MEE, considerably larger than its asymptotic variance. The convergence is apparently very slow.

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