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On the characteristic function of Pearson type IV distributions

Wei-Liem Loh[1]

National University of Singapore

Abstract: Using an identity of Stein (1986), this article gives an exact expression for the characteristic function of Pearson type IV distributions in terms of confluent hypergeometric functions.

1. Introduction

Pearson (1895) introduced a family of probability density functions where each member p of the family satisfies a differential equation

$$p^{(1)}(w) = -\frac{a+w}{a_2 w^2 + a_1 w + a_0} p(w), \tag{1}$$

for some constants a, a_0, a_1 and a_2. The Pearson family is very general and it includes many of the probability distributions in common use today. For example, the beta distribution belongs to the class of Pearson type I distributions, the gamma distribution to Pearson type III distributions and the t distribution to Pearson type VII distributions.

This article focuses on the Pearson type IV distributions. These distributions have unlimited range in both directions and are unimodal. In particular, Pearson type IV distributions are characterized by members satisfying (1) with $0 < a_2 < 1$ and the equation

$$a_2 w^2 + a_1 w + a_0 = 0$$

having no real roots. Writing $A_0 = a_0 - a_1^2 (4a_2)^{-1}$ and $A_1 = a_1 (2a_2)^{-1}$, it follows from (1) that a Pearson type IV distribution has a probability density function of the form

$$p(w) = \frac{A}{[A_0 + a_2(w+A_1)^2]^{1/(2a_2)}} \exp\left[-\frac{a-A_1}{\sqrt{a_2 A_0}} \arctan(\frac{w+A_1}{\sqrt{A_0/a_2}})\right], \qquad \forall w \in R,$$

where A is the normalizing constant. It is well known that Pearson type IV distributions are technically difficult to handle in practice [Stuart and Ord (1994), page 222]. Johnson, Kotz and Balakrishnan (1994), page 19, noted that working with $p(w)$ often leads to intractable mathematics, for example if one attempts to calculate its cumulative distribution function.

The main result of this article is an exact expression (see Theorem 2) for the characteristic function of a Pearson type IV distribution in terms of confluent hypergeometric functions. We note that we have been unable to find any non-asymptotic

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[1]Department of Statistics and Applied Probability, National University of Singapore, Singapore 117546, Republic of Singapore. e-mail: stalohwl@nus.edu.sg

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closed-form expression for the characteristic function of a Pearson type IV distribution in the literature.

The approach that we shall take is inspired by the results of Stein (1986) on the Pearson family of distributions. Since confluent hypergeometric functions have an extensive literature going back over two hundred years to Euler and Gauss, it is plausible that Theorem 2 may provide us with a way of understanding the behavior of Pearson type IV distributions better in a more rigorous manner.

For example, one possible use of Theorem 2 is that we can now apply Fourier analytic techniques in combination with Stein's method [see Stein (1986)] to obtain Pearson type IV approximations to the distribution of a sum of weakly dependent random variables. This work is currently in progress and hence will not be addressed here. The hope is that such a Pearson type IV approximation would have the same order of accuracy as that of an one-term Edgeworth expansion [see, for example, Feller (1971), page 539] with the (often desirable) property that the Pearson type IV approximation is a probability distribution whereas the one-term Edgeworth expansion is not.

We should also mention that besides one-term Edgeworth approximations, gamma and chi-square approximations exist in the literature [see, for example, Shorack (2000), page 383]. The latter approximations typically have the same order of accuracy as the former. However, gamma and chi-square approximations are supported on the half real line and may be qualitatively inappropriate for some applications.

Finally throughout this article, $\mathcal{I}\{.\}$ denotes the indicator function and for any function $h : R \to R$, we write $h^{(r)}$ as the rth derivative of h (if it exists) whenever $r = 1, 2, \cdots$.

2. Pearson type IV characteristic function

We shall first state an identity of Stein (1986) for Pearson type IV distributions.

Theorem 1 (Stein). *Let p be the probability density function of a Pearson type IV distribution satisfying*

$$p^{(1)}(w) = -\frac{(2\alpha_2 + 1)w + \alpha_1}{\alpha_2 w^2 + \alpha_1 w + \alpha_0} p(w), \qquad \forall w \in R, \tag{2}$$

for some constants α_0, α_1 and α_2. Then for a given bounded piecewise continuous function $h : R \to R$, the differential equation

$$(\alpha_2 w^2 + \alpha_1 w + \alpha_0) f^{(1)}(w) - w f(w) = h(w), \qquad \forall w \in R, \tag{3}$$

has a bounded continuous and piecewise continuously differentiable solution $f : R \to R$ if and only if

$$\int_{-\infty}^{\infty} h(w)p(w)dw = 0. \tag{4}$$

When (4) is satisfied, the unique bounded solution f of (3) is given by

$$\begin{aligned} f(w) &= \int_{-\infty}^{w} \frac{h(x)}{\alpha_2 x^2 + \alpha_1 x + \alpha_0} \exp\left(\int_x^w \frac{y dy}{\alpha_2 y^2 + \alpha_1 y + \alpha_0} \right) dx \\ &= -\int_{w}^{\infty} \frac{h(x)}{\alpha_2 x^2 + \alpha_1 x + \alpha_0} \exp\left(-\int_w^x \frac{y dy}{\alpha_2 y^2 + \alpha_1 y + \alpha_0} \right) dx, \qquad \forall w \in R. \end{aligned}$$

We refer the reader to Stein (1986), Chapter 6, for the proof of Theorem 1.

Let Z be a random variable having probability density function p where p satisfies (2).

Proposition 1. *Let Z be as above and ψ_Z be its characteristic function. Then ψ_Z satisfies the following homogeneous second order linear differential equation:*

$$\psi_Z^{(1)}(t) + t\alpha_0\psi_Z(t) - t\alpha_2\psi_Z^{(2)}(t) - it\alpha_1\psi_Z^{(1)}(t) = 0, \qquad \forall t \in R. \tag{5}$$

Proof. Since $\psi_Z(t) = Ee^{itZ}$, $t \in R$, we observe from Theorem 1 that

$$\begin{aligned}
& \int_{-\infty}^{\infty} [(\alpha_2 w^2 + \alpha_1 w + \alpha_0)\frac{d}{dw}(e^{itw}) - we^{itw}]p(w)dw \\
= \ & \int_{-\infty}^{\infty} [it(\alpha_2 w^2 + \alpha_1 w + \alpha_0)e^{itw} - we^{itw}]p(w)dw \\
= \ & (\alpha_2 w^2 + \alpha_1 w + \alpha_0)e^{itw}p(w)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}[(2\alpha_2 + 1)w + \alpha_1]e^{itw}p(w)dw \\
& - \int_{-\infty}^{\infty}(\alpha_2 w^2 + \alpha_1 w + \alpha_0)e^{itw}p^{(1)}(w)dw \\
= \ & 0.
\end{aligned}$$

Hence we conclude that

$$-it\alpha_2\psi_Z^{(2)}(t) + t\alpha_1\psi_Z^{(1)}(t) + it\alpha_0\psi_Z(t) + i\psi_Z^{(1)}(t) = 0, \qquad \forall t \in R.$$

This proves Proposition 1. □

Definition. Following Slater (1960), pages 2 to 5, we define the confluent hypergeometric function (with complex-valued parameters a and b) to be a power series in x of the form

$${}_1F_1(a; b; x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{j!(b)_j},$$

where $(a)_j = a(a+1)\cdots(a+j-1)$, etc. and b is not a negative integer or 0. We further define

$$U(a; b; x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}{}_1F_1(a; b; x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}{}_1F_1(1+a-b; 2-b; x).$$

Remark. It is well known [see for example Theorem 2.1.1 of Andrews, Askey and Roy (1999)] that the series ${}_1F_1(a; b; x)$ [and hence $U(a; b; x)$] converges absolutely for all x.

The theorem below establishes an explicit expression for $\psi_Z(t)$.

Theorem 2. *Let ψ_Z be as in Proposition 1,*

$$\begin{aligned}
\Delta &= \sqrt{4\alpha_0\alpha_2 - \alpha_1^2}, \\
r &= \frac{\sqrt{4\alpha_0\alpha_2 - \alpha_1^2}}{2\alpha_2} + \frac{i\alpha_1}{2\alpha_2}, \\
\bar{r} &= \frac{\sqrt{4\alpha_0\alpha_2 - \alpha_1^2}}{2\alpha_2} - \frac{i\alpha_1}{2\alpha_2}, \\
\nu &= 1 + \frac{1}{\alpha_2},
\end{aligned} \tag{6}$$

and $k\alpha_2 \neq 1$ *for all* $k = 1, 2, \cdots$. *Then for* $t \in R$, *we have*

$$\begin{aligned}\psi_Z(t) &= \frac{e^{-r|t|}\Gamma(\nu - r\Delta^{-1})}{\Gamma(\nu)} U(-\frac{r}{\Delta}; 1-\nu; \frac{\Delta|t|}{\alpha_2}) \mathcal{I}\{t \geq 0\} \\ &\quad + \frac{e^{-\bar{r}|t|}\Gamma(\nu - \bar{r}\Delta^{-1})}{\Gamma(\nu)} U(-\frac{\bar{r}}{\Delta}; 1-\nu; \frac{\Delta|t|}{\alpha_2}) \mathcal{I}\{t < 0\}.\end{aligned}$$

Remark. We would like to add that the confluent hypergeometric function $U(.;,;.)$ is available in a number of mathematical software packages. For example in *Mathematica* [Wolfram (1996)],

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HypergeometricU[a, b, x]
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is the command to evaluate $U(a; b; x)$.

Proof of Theorem 2. We observe from (5) that for all $t \in R$,

$$t\psi_Z^{(2)}(t) + (-\frac{1}{\alpha_2} + \frac{it\alpha_1}{\alpha_2})\psi_Z^{(1)}(t) - \frac{t\alpha_0}{\alpha_2}\psi_Z(t) = 0. \tag{7}$$

STEP 1. Suppose that $t > 0$. We seek a solution of the above differential equation that has the form

$$\psi(t) = e^{-rt}\sum_{j=0}^{\infty} c_j t^j, \qquad \forall 0 < t < \infty,$$

for complex constants $c_0, c_1, \cdots$. Observing that

$$\begin{aligned}\psi^{(1)}(t) &= -re^{-rt}\sum_{j=0}^{\infty} c_j t^j + e^{-rt}\sum_{j=1}^{\infty} jc_j t^{j-1}, \\ \psi^{(2)}(t) &= r^2 e^{-rt}\sum_{j=0}^{\infty} c_j t^j - 2re^{-rt}\sum_{j=1}^{\infty} jc_j t^{j-1} + e^{-rt}\sum_{j=2}^{\infty} j(j-1)c_j t^{j-2},\end{aligned}$$

and substituting these expressions into the left hand side of (7), we have

$$\begin{aligned}& r^2 e^{-rt}\sum_{j=0}^{\infty} c_j t^{j+1} - 2re^{-rt}\sum_{j=1}^{\infty} jc_j t^j + e^{-rt}\sum_{j=2}^{\infty} j(j-1)c_j t^{j-1} \\ & +(-\frac{1}{\alpha_2} + \frac{it\alpha_1}{\alpha_2})(-re^{-rt}\sum_{j=0}^{\infty} c_j t^j + e^{-rt}\sum_{j=1}^{\infty} jc_j t^{j-1}) - \frac{t\alpha_0 e^{-rt}}{\alpha_2}\sum_{j=0}^{\infty} c_j t^j \qquad (8) \\ = {} & 0, \qquad \forall 0 < t < \infty.\end{aligned}$$

Equating the coefficient of t^0 in (8) to zero, we have

$$\frac{rc_0}{\alpha_2} - \frac{c_1}{\alpha_2} = 0,$$

and equating the coefficient of $t^j, j = 1, 2, \cdots$, in (8) to zero, we have

$$\begin{aligned}& r^2 c_{j-1} - 2rjc_j + j(j+1)c_{j+1} + \frac{rc_j}{\alpha_2} - \frac{(j+1)c_{j+1}}{\alpha_2} - \frac{i\alpha_1 rc_{j-1}}{\alpha_2} + \frac{i\alpha_1 jc_j}{\alpha_2} - \frac{\alpha_0 c_{j-1}}{\alpha_2} \\ & = 0.\end{aligned}$$

This implies that $c_1 = rc_0$, and in general for $j = 2, 3, \cdots$,

$$\begin{aligned} c_j &= \frac{1}{j[1-(j-1)\alpha_2]}\left\{c_{j-1}[r - 2r(j-1)\alpha_2 + i(j-1)\alpha_1]\right. \\ &\quad \left. + c_{j-2}(-\alpha_0 + r^2\alpha_2 - ir\alpha_1)\right\}, \end{aligned}$$

whenever $k\alpha_{n,2} \neq 1, k = 1, 2, \cdots$. We observe from (6) that r satisfies

$$r^2\alpha_2 - i\alpha_1 r - \alpha_0 = 0.$$

Since $4\alpha_0\alpha_2 > \alpha_1^2$ (from the definition of Pearson type IV distributions), we conclude that

$$\begin{aligned} c_j &= \frac{c_{j-1}}{j[1-(j-1)\alpha_2]}[r - 2r(j-1)\alpha_2 + i(j-1)\alpha_1] \\ &= c_{j-2}\prod_{k=j-1}^{j}\left\{\frac{r - 2r(k-1)\alpha_2 + i(k-1)\alpha_1}{k[1-(k-1)\alpha_2]}\right\} \\ &= c_0\prod_{k=1}^{j}\left\{\frac{r - 2r(k-1)\alpha_2 + i(k-1)\alpha_1}{k[1-(k-1)\alpha_2]}\right\}, \qquad \forall j = 1, 2, \cdots, \end{aligned}$$

and hence for $t \geq 0$,

$$\begin{aligned} \psi(t) &= c_0 e^{-r|t|}\sum_{j=0}^{\infty}\frac{|t|^j}{j!}\prod_{k=1}^{j}\left\{\frac{r - 2r(k-1)\alpha_2 + i(k-1)\alpha_1}{1-(k-1)\alpha_2}\right\} \\ &= c_0 e^{-r|t|}\sum_{j=0}^{\infty}\frac{|t|^j}{j!}\prod_{k=1}^{j}\left\{\frac{(k-1)\Delta - r}{\alpha_2(k-1)-1}\right\} \\ &= c_0 e^{-r|t|}{}_1F_1(-\frac{r}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta|t|}{\alpha_2}). \end{aligned} \tag{9}$$

STEP 2. Suppose that $t < 0$. Writing $\xi = -t$ and $u_Z(\xi) = \psi_Z(t)$, we have

$$\psi_Z^{(1)}(t) = \frac{du_Z(\xi)}{d\xi}\frac{d\xi}{dt} = -u_Z^{(1)}(\xi),$$

and

$$\psi_Z^{(2)}(t) = \frac{d}{d\xi}(-\frac{du_Z(\xi)}{d\xi})\frac{d\xi}{dt} = u_Z^{(2)}(\xi).$$

Consequently, (5) now takes the form

$$\xi u_Z^{(2)}(\xi) + (-\frac{1}{\alpha_2} - \frac{i\xi\alpha_1}{\alpha_2})u_Z^{(1)}(\xi) - \frac{\xi\alpha_0}{\alpha_2}u_Z(\xi) = 0, \qquad \forall\xi > 0. \tag{10}$$

We seek a solution of the above differential equation that has the form

$$u(\xi) = e^{-\bar{r}\xi}\sum_{j=0}^{\infty}d_j\xi^j, \qquad \forall 0 < \xi < \infty,$$

for complex constants $d_0, d_1, \cdots$. Arguing as in Step 1, we observe that for $t = -\xi < 0$,

$$\begin{aligned} u(\xi) &= d_0 e^{-\bar{r}|t|}\sum_{j=0}^{\infty}\frac{|t|^j}{j!}\prod_{k=1}^{j}\left\{\frac{(k-1)\Delta - \bar{r}}{\alpha_2(k-1)-1}\right\} \\ &= d_0 e^{-\bar{r}|t|}{}_1F_1(-\frac{\bar{r}}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta|t|}{\alpha_2}). \end{aligned} \tag{11}$$

Since a solution of (7) is continuous at $t = 0$, we have $c_0 = d_0$. Thus we conclude from (9) and (11) that a solution of (7) is

$$\psi(t) = e^{-r|t|}{}_1F_1(-\frac{r}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta|t|}{\alpha_2})\mathcal{I}\{t \geq 0\} + e^{-\bar{r}|t|}{}_1F_1(-\frac{\bar{r}}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta|t|}{\alpha_2})\mathcal{I}\{t < 0\}. \tag{12}$$

Step 3. Suppose that $t > 0$. We seek a solution of (7) that has the form

$$\tilde{\psi}(t) = e^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j}, \qquad \forall 0 < t < \infty,$$

for complex constants $c_0, c_1, \cdots$. Observing that

$$\begin{aligned}
\tilde{\psi}^{(1)}(t) &= -re^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j} + e^{-rt}\sum_{j=0}^{\infty}(\nu+j)c_j t^{\nu+j-1}, \\
\tilde{\psi}^{(2)}(t) &= r^2 e^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j} - 2re^{-rt}\sum_{j=0}^{\infty}(\nu+j)c_j t^{\nu+j-1} \\
&\quad + e^{-rt}\sum_{j=0}^{\infty}(\nu+j)(\nu+j-1)c_j t^{\nu+j-2},
\end{aligned}$$

and substituting these expressions into the left hand side of (7), we have

$$\begin{aligned}
& r^2 e^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j+1} - 2re^{-rt}\sum_{j=0}^{\infty}(\nu+j)c_j t^{\nu+j} + e^{-rt}\sum_{j=0}^{\infty}(\nu+j)(\nu+j-1)c_j t^{\nu+j-1} \\
&\quad - \frac{1}{\alpha_2}[-re^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j} + e^{-rt}\sum_{j=0}^{\infty}(\nu+j)c_j t^{\nu+j-1}] \\
&\quad + \frac{i\alpha_1}{\alpha_2}[-re^{-rt}\sum_{j=0}^{\infty} c_j t^{\nu+j+1} + e^{-rt}\sum_{j=0}^{\infty}(\nu+j)c_j t^{\nu+j}] - \frac{\alpha_0 e^{-rt}}{\alpha_1}\sum_{j=0}^{\infty} c_j t^{\nu+j+1} \\
&\quad = 0, \quad \forall 0 < t < \infty.
\end{aligned} \tag{13}$$

Equating the coefficient of t^{ν} in (13) to zero, we have

$$-2r\nu c_0 + (\nu+1)\nu c_1 + \frac{rc_0}{\alpha_2} - \frac{(\nu+1)c_1}{\alpha_2} + \frac{i\alpha_1\nu c_0}{\alpha_2} = 0,$$

and equating the coefficient of $t^{\nu+j-1}$, $j = 2, 3, \cdots$, in (13) to zero, we have

$$\begin{aligned}
& r^2 c_{j-2} - 2r(\nu+j-1)c_{j-1} + (\nu+j)(\nu+j-1)c_j + \frac{rc_{j-1}}{\alpha_2} - \frac{(\nu+j)c_j}{\alpha_2} \\
& + \frac{i\alpha_1}{\alpha_2}[-rc_{j-2} + (\nu+j-1)c_{j-1}] - \frac{\alpha_0 c_{j-2}}{\alpha_2} = 0.
\end{aligned}$$

This gives

$$c_1 = \frac{2r\alpha_2 + r - i\alpha_1\nu}{\alpha_2(1+\nu)}c_0,$$

and in general for $j = 2, 3, \cdots$,

$$\begin{aligned} c_j &= \frac{1}{j(\nu+j)\alpha_2}\{[2(\nu+j-1)r\alpha_2 - r - i\alpha_1(\nu+j-1)]c_{j-1} \\ &\quad -(\alpha_2 r^2 - i\alpha_1 r - \alpha_0)c_{j-2}\} \\ &= c_0 \prod_{k=1}^{j} \frac{2(\nu+k-1)r\alpha_2 - r - i\alpha_1(\nu+k-1)}{k(\nu+k)\alpha_2}. \end{aligned}$$

Hence for $t \geq 0$, we have

$$\begin{aligned} \tilde{\psi}(t) &= c_0 e^{-rt} \sum_{j=0}^{\infty} t^{\nu+j} \prod_{k=1}^{j} \frac{2(\nu+k-1)r\alpha_2 - r - i\alpha_1(\nu+k-1)}{k(\nu+k)\alpha_2} \\ &= c_0 t^{\nu} e^{-rt} {}_1F_1(\nu - \frac{r}{\Delta}; \nu+1; \frac{\Delta t}{\alpha_2}). \end{aligned} \tag{14}$$

STEP 4. Suppose that $t < 0$. Writing $\xi = -t$ and $u_Z(\xi) = \psi_Z(t)$, we seek a solution of (10) that has the form

$$\tilde{u}(\xi) = e^{-\bar{r}\xi} \sum_{j=0}^{\infty} d_j \xi^{\nu+j}, \qquad \forall 0 < \xi < \infty,$$

for complex constants $d_0, d_1, \cdots$. Arguing as in Step 3, we observe that for $t = -\xi < 0$,

$$\begin{aligned} \tilde{u}(\xi) &= d_0 |t|^{\nu} e^{-\bar{r}|t|} \sum_{j=0}^{\infty} \frac{|t|^j}{j!} \prod_{k=1}^{j} \frac{(\nu+k-1)\Delta - \bar{r}}{(\nu+k)\alpha_2} \\ &= d_0 |t|^{\nu} e^{-\bar{r}|t|} {}_1F_1(\nu - \frac{\bar{r}}{\Delta}; \nu+1; \frac{\Delta |t|}{\alpha_2}). \end{aligned} \tag{15}$$

Since a solution of (7) is continuous at $t = 0$, we have $c_0 = d_0$. Thus we conclude from (14) and (15) that a solution of (7) is

$$\begin{aligned} \psi(t) &= |t|^{\nu} e^{-r|t|} {}_1F_1(\nu - \frac{r}{\Delta}; \nu+1; \frac{\Delta |t|}{\alpha_2}) \mathcal{I}\{t \geq 0\} \\ &\quad + |t|^{\nu} e^{-\bar{r}|t|} {}_1F_1(\nu - \frac{\bar{r}}{\Delta}; \nu+1; \frac{\Delta |t|}{\alpha_2}) \mathcal{I}\{t < 0\}. \end{aligned} \tag{16}$$

As the solutions in (12) and (16) are independent, the general solution of (7) is given by

$$\begin{aligned} \psi(t) &= \left\{ A e^{-r|t|} {}_1F_1(-\frac{r}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta |t|}{\alpha_2}) \right. \\ &\quad \left. + B |t|^{\nu} e^{-r|t|} {}_1F_1(\nu - \frac{r}{\Delta}; \nu+1; \frac{\Delta |t|}{\alpha_2}) \right\} \mathcal{I}\{t \geq 0\} \\ &\quad + \left\{ \tilde{A} e^{-\bar{r}|t|} {}_1F_1(-\frac{\bar{r}}{\Delta}; -\frac{1}{\alpha_2}; \frac{\Delta |t|}{\alpha_2}) \right. \\ &\quad \left. + \tilde{B} |t|^{\nu} e^{-\bar{r}|t|} {}_1F_1(\nu - \frac{\bar{r}}{\Delta}; \nu+1; \frac{\Delta |t|}{\alpha_2}) \right\} \mathcal{I}\{t < 0\}, \end{aligned}$$

where $A, \tilde{A}, B$ and $\tilde{B}$ are arbitrary constants. Consequently since $\psi_Z(0) = 1$, we have $A = \tilde{A} = 1$ and

$$\begin{aligned}\psi_Z(t) &= \Big\{e^{-r|t|}{}_1F_1(-\frac{r}{\Delta}; 1-\nu; \frac{\Delta|t|}{\alpha_2}) \\ &\qquad + B|t|^\nu e^{-r|t|}{}_1F_1(\nu - \frac{r}{\Delta}; \nu+1; \frac{\Delta|t|}{\alpha_2})\Big\}\mathcal{I}\{t \geq 0\} \\ &\quad + \Big\{e^{-\bar{r}|t|}{}_1F_1(-\frac{\bar{r}}{\Delta}; 1-\nu; \frac{\Delta|t|}{\alpha_2}) \\ &\qquad + \tilde{B}|t|^\nu e^{-\bar{r}|t|}{}_1F_1(\nu - \frac{\bar{r}}{\Delta}; \nu+1; \frac{\Delta|t|}{\alpha_2})\Big\}\mathcal{I}\{t < 0\}, \qquad (17)\end{aligned}$$

for some constants B and $\tilde{B}$.

STEP 5. To complete the proof of Theorem 2, it suffices to determine the constants B and $\tilde{B}$ in (17). We observe from Slater (1960), page 60, that for $x \to \infty$,

$${}_1F_1(a; b; x) = x^{a-b}e^x\frac{\Gamma(b)}{\Gamma(a)}(1 + O(|x|^{-1}).$$

Hence it follows from (17) that as $t \to \infty$,

$$\begin{aligned}\psi_Z(t) &= e^{-rt}e^{\Delta t/\alpha_2}\Big\{(\frac{\Delta t}{\alpha_2})^{\nu-1-r\Delta^{-1}}\frac{\Gamma(-\alpha_2^{-1})}{\Gamma(-r\Delta^{-1})} \\ &\qquad + Bt^\nu(\frac{\Delta t}{\alpha_2})^{-1-r\Delta^{-1}}\frac{\Gamma(\nu+1)}{\Gamma(\nu - r\Delta^{-1})}\Big\}(1+o(1)) \\ &= e^{-rt}e^{\Delta t/\alpha_2}t^\nu(\frac{\Delta t}{\alpha_2})^{-1-r\Delta^{-1}}\Big\{(\frac{\Delta}{\alpha_2})^\nu\frac{\Gamma(-\alpha_2^{-1})}{\Gamma(-r\Delta^{-1})} \\ &\qquad + B\frac{\Gamma(\nu+1)}{\Gamma(\nu - r\Delta^{-1})}\Big\}(1+o(1)).\end{aligned}$$

Since $\lim_{t\to\infty}\psi_{Z_n}(t) = 0$, we have

$$B = -(\frac{\Delta}{\alpha_2})^\nu\frac{\Gamma(-\alpha_2^{-1})\Gamma(\nu - r\Delta^{-1})}{\Gamma(\nu+1)\Gamma(-r\Delta^{-1})}. \qquad (18)$$

Similarly as $t \to -\infty$,

$$\begin{aligned}\psi_Z(t) &= e^{-\bar{r}|t|}e^{\Delta|t|/\alpha_2}\Big\{(\frac{\Delta|t|}{\alpha_2})^{\nu-1-\bar{r}\Delta^{-1}}\frac{\Gamma(-\alpha_2^{-1})}{\Gamma(-\bar{r}\Delta^{-1})} \\ &\qquad + \tilde{B}|t|^\nu(\frac{\Delta|t|}{\alpha_2})^{-1-\bar{r}\Delta^{-1}}\frac{\Gamma(\nu+1)}{\Gamma(\nu - \bar{r}\Delta^{-1})}\Big\}(1+o(1)) \\ &= e^{-\bar{r}t}e^{\Delta|t|/\alpha_2}|t|^\nu(\frac{\Delta|t|}{\alpha_2})^{-1-\bar{r}\Delta^{-1}}\Big\{(\frac{\Delta}{\alpha_2})^\nu\frac{\Gamma(-\alpha_2^{-1})}{\Gamma(-\bar{r}\Delta^{-1})} \\ &\qquad + \tilde{B}\frac{\Gamma(\nu+1)}{\Gamma(\nu - \bar{r}\Delta^{-1})}\Big\}(1+o(1)).\end{aligned}$$

Since $\lim_{t\to-\infty}\psi_Z(t) = 0$, we have

$$\tilde{B} = -(\frac{\Delta}{\alpha_2})^\nu\frac{\Gamma(-\alpha_2^{-1})\Gamma(\nu - \bar{r}\Delta^{-1})}{\Gamma(\nu+1)\Gamma(-\bar{r}\Delta^{-1})}. \qquad (19)$$

Theorem 2 now follows from (17), (18), (19), the definition of $U(.;.;.)$ and Euler's reflection formula, namely

$$\Gamma(x)\Gamma(1-x) = \frac{x}{\sin(\pi x)},$$

[see, for example, Theorem 1.2.1 of Andrews, Askey and Roy (1999)]. □

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