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On time changing continuous martingales to Brownian motion

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Abstract: A short variation of the original proof of Dubins and Schwarz of their result, that all continuous martingales can be time changed to Brownian motion, is given.

It would be hard to overstate my debt to Herman Rubin, whose office is just down the hall. My discussions with him range from twenty minutes on, say, infinitely divisible processes or the Fisk-Stratonovich integral (Fisk was Herman's student.) which completely changed the way I understood subjects I thought I knew, to countless small enlightenments, to providing an instant solution to a homework problem, due that day, that I realized, right before class, was a lot harder than I thought.

This paper provides a short variant of the original proof of the result of Dubins and Schwarz [2] that continuous martingales with unbounded paths can be time changed to standard Brownian motion. See [3] for a discussion of this theorem. We first consider the case that the paths of M are not constant on any open interval, and then discuss the general case. The embedding scheme used here was also used in [2]. The novelty is the use of the lemma below.

Theorem. *Let M_t, $t \geq 0$, be a continuous martingale satisfying $M_0 = 0$, $\sup_t |M_t| = \infty$, and $P(M_s = M_a, a < s < b) = 0$ for all $0 < a < b$. Then there are stopping times η_t, $t \geq 0$, which strictly and continuously increase from 0 to infinity, such that M_{η_t}, $t \geq 0$, is Brownian motion.*

Proof. Let $u_0^M = 0$, and $u_{k+1}^M = inf\{t > u_k : |M_t - M_{u_k^M}| = 1\}$, $k \geq 0$, and let $v_{n,j}^M = u_j^{2^n M}$, if $n, j \geq 0$. We drop the superscript M for the rest of this paragraph. Then M_{u_j}, $j \geq 0$, is a fair random walk, and $M_{v_{n,j}}$, $j \geq 0$, has the distribution of a fair random walk divided by 2^n. Of course the distribution of the $v_{n,j}$ is different for different martingales, but the distribution of the ordering of these times is not. To be precise, the probability of any event in the algebra of events generated by the events $\{v_{i,j} < v_{k,l}\}$ has the same probability for all martingales M. To see this, it helps to first check that $P(v_{1,3} < v_{0,1}) = 1/2$, since the random walk $M_{v_{0,j}}$, $j \geq 0$, is embedded in the random walk $M_{v_{1,j}}$, $j \geq 0$ by the discrete analog of the times $v_{0,k}$, and the probability of the analogous event for these walks is 1/2. Now since the walks $M_{v_{k,j}}$, $j \geq 0$, can for $0 \leq k < n$ all be embedded in the walk $M_{v_{n,j}}$, $j \geq 0$, which is of course the same walk for any M, the probability of an event in the algebra is the probability of an event for discrete random walk. $\square$

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Lemma. *For $0 \leq n < \infty$, let $t_{n,j}, 0 \leq j < \infty$, be a sequence, and suppose*

(i) $0 = t_{0,n}$, $n \geq 0$,

(ii) $t_{n,j} < t_{n,j+1}$, $j \geq 0$,

(iii) *for all j and n, $t_{n,j}$ is one of the numbers $t_{n+1,k}$, $k \geq 0$,*

(iv) *the set of all the $t_{n,j}$ is dense in $[0, \infty)$.*

Then a sequence a_n, $n \geq 0$, of nonnegative numbers converges if and only if given m there is a j such that $t_{m,j} < a_k < t_{m,j+2}$ for all large enough k. Furthermore if K is a positive integer, an increasing nonnegative function f on $[0, K]$ is continuous if and only if given $n > 0$ there is m such that for each i, $0 \leq i < Km$, there is $j = j(i)$ such that $t_{n,j} < f(i/m)$ and $f((i+1)/m) < t_{n,j+2}$.

This lemma is obvious. Now let $v^M_{n,2^{2n}}$ play the role of a_n and $v^M_{n,j}$ have the role of $t_{n,j}$ in this lemma. The conditions (i)–(iv) are easy to check, using the absence of flat spots for iv). The lemma implies that whether or not $v^M_{n,2^{2n}}$ converges (a.s.) depends only on the distribution of the order of the $v^M_{i,j}$. Since this latter distribution does not depend on M, we have either convergence for all M or no M. But if M is a Brownian motion B, we do have convergence, to 1. For, following Skorohod (see [1]), $v^B_{n,2^{2n}}$ has the distribution of the average of 2^{2n} iid random variables each having the distribution of $u^B_1 := u$. Since $Eu = 1$ and since the variance of u is finite, easily shown upon noting that $P(u > k+1 | u > k) < P(|Z| < 2)$, $k > 0$, where Z is standard normal, Chebyshev's inequality gives this convergence to 1. Similarly $\lim_{n\to\infty} v^M_{n,[t2^{2n}]} := \eta^M_t$ exists, where $[\,]$ denotes the greatest integer function. Now the distribution of $M_{\eta^M_1}$ is the limit of the distributions of $M_{v^M_{n,2^{2n}}}$, since M has continuous paths, and thus is the same for all martingales M, and this limit can be identified, by taking $M = B$, as standard normal. All the joint distributions can be similarly treated, and so $M_{\eta^M_t}$ is Brownian motion. This implies that η^M_t is strictly increasing. To see that it is continuous, use the last sentence of the lemma. An argument like that just given shows that continuity on $[0, K]$ for any K, and thus continuity on $[0, \infty)$, either holds for all or no M. And $\eta^B_t = t$. Finally, since η^M_t is continuous and strictly increasing, $\eta^M_t = sup_{k>0} \lim_{n\to\infty} v^M_{n,[(1-\frac{1}{k})2^{2n}t]}$, and so is a stopping time. $\square$

In case the paths of M have flat spots, remove them. Let A stand for the union of the open intervals on which M is constant. Let $h(t) = \inf\{y : |(0,y) \bigcap A^c| = t\}, 0 \leq t \leq \infty$, where $|\,|$ is Lebesgue measure and the c denotes complement, so that if we define $N_t = M_{h(t)}$, N is continuous with no flat spots. Whether or not N is a martingale, random walks can be embedded in it, since they can be embedded M. Thus just as above, $N_{\eta^N_t}$ is Brownian motion. Put $\mu_t = h(\eta^N_t)$. Then μ is left continuous and strictly increasing, and M_{μ_t} is Brownian motion. And $\mu_t = sup_{k>0} \limsup_{n\to\infty} v^M_{n,[(1-\frac{1}{k})2^{2n}t]}$, so μ_t is a stopping time.

References

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