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Some properties of the arc-sine law related to its invariance under a family of rational maps*

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Abstract: This paper shows how the invariance of the arc-sine distribution on $(0, 1)$ under a family of rational maps is related on the one hand to various integral identities with probabilistic interpretations involving random variables derived from Brownian motion with arc-sine, Gaussian, Cauchy and other distributions, and on the other hand to results in the analytic theory of iterated rational maps.

1. Introduction

Lévy [20, 21] showed that a random variable A with the arc-sine law

$$P(A \in da) = \frac{da}{\pi\sqrt{a(1-a)}} \qquad (0 < a < 1) \tag{1}$$

can be constructed in numerous ways as a function of the path of a one-dimensional Brownian motion, or more simply as

$$A = \tfrac{1}{2}(1 - \cos\Theta) \stackrel{d}{=} \tfrac{1}{2}(1 - \cos 2\Theta) = \cos^2\Theta \tag{2}$$

where Θ has uniform distribution on $[0, 2\pi]$ and $\stackrel{d}{=}$ denotes equality in distribution. See [31, 7] and papers cited there for various extensions of Lévy's results. In connection with the distribution of local times of a Brownian bridge [29], an integral identity arises which can be expressed simply in terms of an arc-sine variable A. Section 5 of this note shows that this identity amounts to the following property of A, discovered in a very different context by Cambanis, Keener and Simons [6, Proposition 2.1]: for all real a and c

$$\frac{a^2}{A} + \frac{c^2}{1-A} \stackrel{d}{=} \frac{(|a| + |c|)^2}{A}. \tag{3}$$

As shown in [6], where (3) is applied to the study of an interesting class of multivariate distributions, the identity (3) can be checked by a computation with densities, using (2) and trigonometric identities. Here we offer some derivations of (3) related

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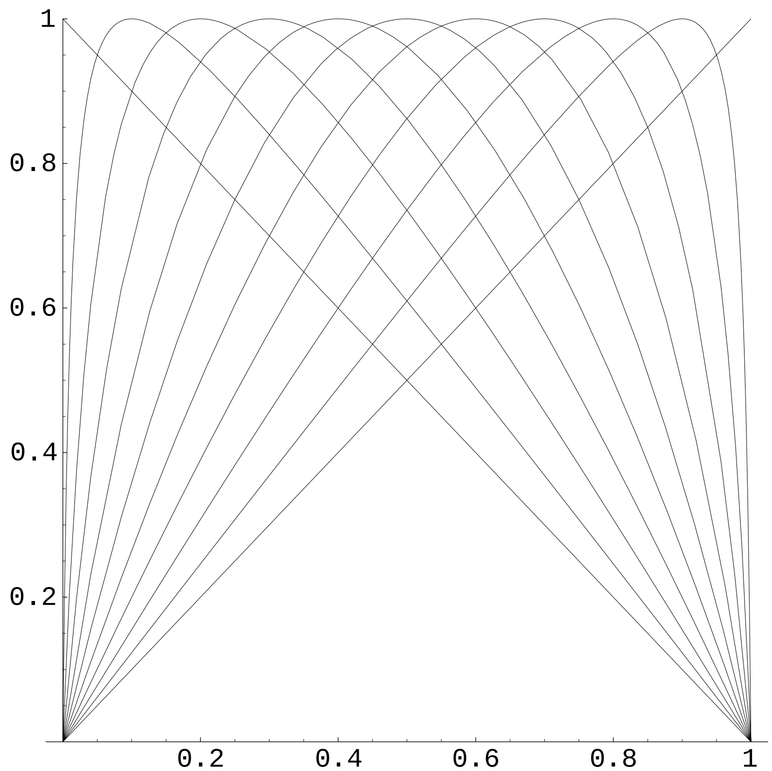

Figure 1: Graphs of $Q_u(a)$ for $0 \le a \le 1$ and $u = k/10$ with $k = 0, 1, \ldots, 10$.

to various other characterizations and properties of the arc-sine law. For $u \in [0,1]$ define a rational function

$$Q_u(a) := \left(\frac{u^2}{a} + \frac{(1-u)^2}{1-a} \right)^{-1} = \frac{a(1-a)}{u^2 + (1-2u)a} \tag{4}$$

So (3) amounts to $Q_u(A) \stackrel{d}{=} A$, as restated in the following theorem. It is easily checked that Q_u increases from 0 to 1 over $(0,u)$ and decreases from 1 to 0 over $(u,1)$, as shown in the above graphs.

Theorem 1. *For each $u \in (0,1)$ the arc-sine law is the unique absolutely continuous probability measure on the line that is invariant under the rational map $a \to Q_u(a)$.*

The conclusion of Theorem 1 for $Q_{1/2}(a) = 4a(1-a)$ is a well known result in the theory of iterated maps, dating back to Ulam and von Neumann [38]. As indicated in [3] and [22, Example 1.3], this case follows immediately from (2) and the ergodicity of the Bernoulli shift $\theta \mapsto 2\theta \pmod{2\pi}$. This argument shows also, as conjectured in [15, p. 464 (A3)] and contrary to a footnote of [37, p. 233], that the arc-sine law is not the only non-atomic law of A such that $4A(1-A) \stackrel{d}{=} A$. For the argument gives $4A(1-A) \stackrel{d}{=} A$ if $A = (1 - \cos 2\pi U)/2$ for any distribution of U on $[0,1]$ with $(2U \bmod 1) \stackrel{d}{=} U$, and it is well known that such U exist with singular continuous distributions, for instance $U = \sum_{m=1}^{\infty} X_m 2^{-m}$ for X_m independent Bernoulli(p) for any $p \in (0,1)$ except $p = 1/2$. See also [15] and papers cited there for some related characterizations of the arc-sine law, and [13] where this property of the arc-sine law is related to duplication formulae for various special functions defined by Euler integrals.

Section 2 gives a proof of Theorem 1 based on a known characterization of the standard Cauchy distribution. In terms of a complex Brownian motion Z, the connection between the two results is that the Cauchy distribution is the hitting distribution on $\mathbb{R}$ for $Z_0 = \pm i$, while the arc-sine law is the hitting distribution on $[0, 1]$ for $Z_0 = \infty$. The transfer between the two results may be regarded as a consequence of Lévy's theorem on the conformal invariance of the Brownian track. In Section 4 we use a closely related approach to generalize Theorem 1 to a large class of functions Q instead of Q_u. The result of this section for rational Q can also be deduced from the general result of Lalley [18] regarding Q-invariance of the equilibrium distribution on the Julia set of Q, which Lalley obtained by a similar application of Lévy's theorem.

2. Proof of Theorem 1

Let A have the arc-sine law (1), and let C be a standard Cauchy variable, that is

$$P(C \in dy) = \frac{dy}{\pi(1+y^2)} \qquad (y \in \mathbb{R}). \tag{5}$$

We will exploit the following elementary fact [33, p. 13]:

$$A \overset{d}{=} 1/(1+C^2). \tag{6}$$

Using (6) and $C \overset{d}{=} -C$, the identity (3) is easily seen to be equivalent to

$$uC - (1-u)/C \overset{d}{=} C. \tag{7}$$

This is an instance of the result of E. J. G. Pitman and E. J. Williams [28] that for a large class of meromorphic functions G mapping the half plane $\mathbb{H}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ to itself, with boundary values mapping $\mathbb{R}$ (except for some poles) to $\mathbb{R}$, there is the identity in distribution

$$G(C) \overset{d}{=} \operatorname{Re} G(i) + (\operatorname{Im} G(i))C \tag{8}$$

where $i = \sqrt{-1}$ and $z = \operatorname{Re} z + i \operatorname{Im} z$. Kemperman [14] attributes to Kesten the remark that (8) follows from Lévy's theorem on the conformal invariance of complex Brownian motion Z, and the well known fact that for τ the hitting time of the real axis by Z, the distribution of Z_τ given $Z_0 = z$ is that of $\operatorname{Re} z + (\operatorname{Im} z)C$. As shown by Letac [19], this argument yields (8) for all *inner functions on* $\mathbb{H}^+$, that is all holomorphic functions G from $\mathbb{H}^+$ to $\mathbb{H}^+$ such that the boundary limit $G(x) := \lim_{y \downarrow 0} G(x + iy)$ exists and is real for Lebesgue almost every real x. In particular, (8) shows that

$$\text{if } G \text{ is inner on } \mathbb{H}^+ \text{ with } G(i) = i, \text{ then } G(C) \overset{d}{=} C. \tag{9}$$

As indicated by E. J. Williams [39] and Kemperman [14], for some inner G on $\mathbb{H}^+$ with $G(i) = i$, the property $G(C) \overset{d}{=} C$ characterizes the distribution of C among all absolutely continuous distributions on the line. These are the G whose action on $\mathbb{R}$ is ergodic relative to Lebesgue measure. Neuwirth [26] showed that an inner function G with $G(i) = i$ is ergodic if it is not one to one. In particular,

$$G_u(z) := uz - (1-u)/z \tag{10}$$

as in (7) is ergodic. The above transformation from (3) to (7) amounts to the semi-conjugacy relation

$$Q_u \circ \gamma = \gamma \circ G_u \text{ where } \gamma(w) := 1/(1+w^2). \tag{11}$$

So Q_u acts ergodically as a measure preserving transformation of $(0,1)$ equipped with the arc-sine law. It is easily seen that for $u \in (0,1)$ a Q_u-invariant probability measure must be concentrated on $[0,1]$, and Theorem 1 follows.

See also [35, p. 58] for an elementary proof of (7), [1, 23, 24, 2] for further study of the ergodic theory of inner functions, [16, 19] for related characterizations of the Cauchy law on $\mathbb{R}$ and [17, 9] for extensions to $\mathbb{R}^n$.

3. Further interpretations

Since $w \mapsto 1/(1+w^2)$ maps i to ∞, another application of Lévy's theorem shows that the arc-sine law of $1/(1+C^2)$ is the hitting distribution on $[0,1]$ of a complex Brownian motion plane started at ∞ (or uniformly on any circle surrounding $[0,1]$). In terms of classical planar potential theory [32, Theorem 4.12], the arc-sine law is thus identified as the *normalized equilibrium distribution* on $[0,1]$. The corresponding characterization of the distribution of $1-2A$ on $[-1,1]$ appears in Brolin [5], in connection with the invariance of this distribution under the action of Chebychev polynomials, as discussed further in the next section. Equivalently by inversion, the distribution of $1/(1-2A)$ is the hitting distribution on $(-\infty,1] \cup [1,\infty)$ for complex Brownian motion started at 0. Spitzer [36] found this hitting distribution, which he interpreted further as the hitting distribution of $(-\infty,1] \cup [1,\infty)$ for a Cauchy process starting at 0. This Cauchy process is obtained from the planar Brownian motion watched only when it touches the real axis, via a time change by the inverse local time at 0 of the imaginary part of the Brownian motion. The arc-sine law can be interpreted similarly as the limit in distribution as $|x| \to \infty$ of the hitting distribution of $[0,1]$ for the Cauchy process started at $x \in \mathbb{R}$. See also [30] for further results in this vein.

4. Some generalizations

We start with some elementary remarks from the perspective of ergodic theory. Let $\lambda(a) := 1 - 2a$, which maps $[0,1]$ onto $[-1,1]$. Obviously, a Borel measurable function $f^\dagger$ has the property

$$f^\dagger(A) \stackrel{d}{=} A \tag{12}$$

for A with arc-sine law if and only if

$$\tilde{f}(1-2A) \stackrel{d}{=} 1-2A \text{ where } \tilde{f} = \lambda \circ f^\dagger \circ \lambda^{-1}. \tag{13}$$

Let $\rho(z) := \frac{1}{2}(z + z^{-1})$, which projects the unit circle onto $[-1,1]$. It is easily seen from (2) that (13) holds if and only if there is a measurable map f from the circle to itself such that

$$f(U) \stackrel{d}{=} U \text{ and } \tilde{f} \circ \rho(u) = \rho \circ f(u) \text{ for } |u| = 1 \tag{14}$$

where U has uniform distribution on the unit circle. In the terminogy of ergodic theory [27], every transformation $f^\dagger$ of $[0,1]$ which preserves the arc-sine law is thus a *factor* of some non-unique transformation f of the circle which preserves Lebesgue measure. Moreover, this f can be taken to be *symmetric*, meaning

$$f(\overline{z}) = \overline{f(z)}.$$

If f acts ergodically with respect to Lebesgue measure on the circle, then $f^\dagger$ acts ergodically with respect to Lebesgue measure on $[0,1]$, hence the arc-sine law is the unique absolutely continuous $f^\dagger$-invariant measure on $[0,1]$. This argument is well known in case $f(z) = z^d$ for $d = 2,3,\ldots$, when it is obvious that (14) holds and well known that f is ergodic. Then $\tilde{f}(x) = T_d(x)$, the dth *Chebychev polynomial* [34] and we recover from (14) the well known result ([3],[34, Theorem 4.5]) that

$$T_d(1-2A) \stackrel{d}{=} 1-2A \qquad (d = 1,2,\ldots). \tag{15}$$

Let $\mathbb{D} := \{z : |z| < 1\}$ denote the unit disc in the complex plane. An *inner function on* $\mathbb{D}$ is a function defined and holomorphic on $\mathbb{D}$, with radial limits of modulus 1 at Lebesgue almost every point on the unit circle. Let $\phi(z) := i(1+z)/(1-z)$ denote the Cayley bijection from $\mathbb{D}$ to the upper half-plane $\mathbb{H}^+$. It is well known that the inner functions G on $\mathbb{H}^+$, as considered in Section 2, are the conjugations $G = \phi \circ f \circ \phi^{-1}$ of inner functions f on $\mathbb{D}$. So either by conjugation of (9), or by application of Lévy's theorem to Brownian motion in $\mathbb{D}$ started at 0,

$$\text{if } f \text{ is inner on } \mathbb{D} \text{ with } f(0) = 0, \text{ then } f(U) \stackrel{d}{=} U \tag{16}$$

where U is uniform on the unit circle. If f is an inner function on $\mathbb{D}$ with a fixed point in $\mathbb{D}$, and f is not one-to-one, then f acts ergodically on the circle [26]. The only one-to-one inner functions with $f(0) = 0$ are $f(z) = cz$ for some c with $|c| = 1$. By combining the above remarks, we obtain the following generalization of (15), which is the particular case $f(z) = z^d$:

Theorem 2. *Let f be a symmetric inner function on $\mathbb{D}$ with $f(0) = 0$. Define the transformation $\tilde{f}$ on $[-1,1]$ via the semi-conjugation*

$$\tilde{f} \circ \rho(z) = \rho \circ f(z) \text{ for } |z| = 1, \text{ where } \rho(z) := \tfrac{1}{2}\big(z + z^{-1}\big). \tag{17}$$

If A has arc-sine law then

$$\tilde{f}(1-2A) \stackrel{d}{=} 1-2A. \tag{18}$$

Except if $f(z) = z$ or $f(z) = -z$, the arc-sine law is the only absolutely continuous law of A on $[0,1]$ with this property.

It is well known that every inner function f which is continuous on the closed disc is a *finite Blaschke product*, that is a rational function of the form

$$f(z) = c \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a}_i z} \tag{19}$$

for some complex c and a_i with $|c| = 1$ and $|a_i| < 1$. Note that $f(0) = 0$ iff some $a_i = 0$, and that f is symmetric iff $c = \pm 1$ with some a_i real and the rest of the a_i forming conjugate pairs. In particular, if we take $c = 1, a_1 = 0, a_2 = a \in (-1,1)$, we find that the degree two Blaschke product

$$f_a(z) := z\frac{(z-a)}{(1-az)} = \frac{z-a}{z^{-1}-a}$$

for $a = 1 - 2u$ is the conjugate via the Cayley map $\phi(z) := i(1+z)/(1-z)$ of the function $G_u(w) = uw - (1-u)/w$ on $\mathbb{H}^+$, which appeared in Section 2. For $f = f_{1-2u}$ the semi-conjugation (17) is the equivalent via conjugation by ϕ of the semi-conjugation (11). So for instance

$$Q_u \circ \gamma \circ \phi = \gamma \circ \phi \circ f_{1-2u} \quad \text{where} \quad \gamma \circ \phi(z) = \frac{-(1-z)^2}{4z} \tag{20}$$

so that

$$\gamma \circ \phi(z) = \tfrac{1}{2}(1 - \operatorname{Re} z) \text{ if } |z| = 1,$$

and Theorem 1 can be read from Theorem 2.

Consider now a rational function R as a mapping from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ where $\overline{\mathbb{C}}$ is the Riemann sphere. A subset A of $\overline{\mathbb{C}}$ is *completely R-invariant* if A is both forward and backward invariant under R: for $z \in \overline{\mathbb{C}}$, $z \in A \Leftrightarrow R(z) \in A$. Beardon [4, Theorem 1.4.1] showed that for R a polynomial of degree $d \geq 2$, the interval $[-1,1]$ is completely R-invariant iff R is T_d or $-T_d$. A similar argument yields

Proposition 3. *Let f be a symmetric finite Blaschke product of degree d. Then there exists a unique rational function $\tilde{f}$ which solves the functional equation*

$$\tilde{f} \circ \rho(z) = \rho \circ f(z) \textit{ for } z \in \overline{\mathbb{C}}, \textit{ where } \rho(z) := \tfrac{1}{2}\big(z + z^{-1}\big). \tag{21}$$

This $\tilde{f}$ has degree d, and $[-1,1]$ is completely $\tilde{f}$-invariant. Conversely, if $[-1,1]$ is completely R-invariant for a rational function R, then $R = \tilde{f}$ for some such f.

Proof. Note that ρ maps the circle with ± 1 removed in a two to one fashion to $(-1,1)$, while ρ fixes ± 1, and maps each of $\mathbb{D}$ and $\mathbb{D}^* := \{z : |z| > 1\}$ bijectively onto $[-1,1]^c := \overline{\mathbb{C}} \backslash [-1,1]$. Let us choose to regard

$$\rho^{-1}(w) = w + i\sqrt{1 - w^2}$$

as mapping $[-1,1]^c$ to $\mathbb{D}$. Then $\tilde{f} := \rho \circ f \circ \rho^{-1}$ is a well defined mapping of $[-1,1]^c$ to itself. Because f is continuous and symmetric on the unit circle, this $\tilde{f}$ has a continuous extension to $\overline{\mathbb{C}}$, which maps $[-1,1]$ to itself. So $\tilde{f}$ is continuous from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, and holomorphic on $[-1,1]^c$. It follows that $\tilde{f}$ is holomorphic from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$, hence $\tilde{f}$ is rational. Clearly, $\tilde{f}$ leaves $[-1,1]$ completely invariant.

Conversely, if $[-1,1]$ is completely R-invariant for a rational function R, then we can define $f := \rho^{-1} \circ R \circ \rho$ as a holomorphic map $\mathbb{D}$ to $\mathbb{D}$. Because R preserves $[-1,1]$ we find that f is continuous and symmetric on the boundary of $\mathbb{D}$. Hence f is a Blaschke product, which must be symmetric also on $\mathbb{D}$ by the Cauchy integral representation of f. □

As a check, Proposition 3 combines with Theorem 2 to yield the special case $K = [-1,1]$ of the following result:

Theorem 4. Lalley [18] *Let K be a compact non-polar subset of $\mathbb{C}$, and suppose that K is completely R-invariant for a rational mapping R with $R(\infty) = \infty$. Then the equilibrium distribution on K is R-invariant.*

Proof. Lalley gave this result for $K = J(R)$, the Julia set of a rational mapping R, as defined in any of [5, 22, 4, 18], assuming that $R(\infty) = \infty \notin J(R)$. Then K is necessarily compact, non-polar, and completely R-invariant. His argument, which we now recall briefly, shows that these properties of K are all that is required for the conclusion. The argument is based on the fact [32, Theorem 4.12] that the normalized equilibrium distribution on K is the hitting distribution of K for a Brownian motion Z on $\overline{\mathbb{C}}$ started at ∞. Stop Z at the first time τ that it hits K. By Lévy's theorem, and the complete R-invariance of K, the path $(R(Z_t), 0 \leq t \leq \tau)$ has (up to a time change) the same law as does $(Z_t, 0 \leq t \leq \tau)$. So the distribution of the endpoint Z_τ is R-invariant. □

According to a well known result of Fatou [22, p. 57], the Julia set of a Blaschke product f is either the unit circle or a Cantor subset of the circle. According to Hamilton [11, p. 281], the former case obtains iff the action of f on the circle is ergodic relative to Lebesgue measure. Hamilton [12, p. 88] states that a rational map R has $[-1,1]$ as its Julia set iff R is of the form described in Proposition 3 for some symmetric and ergodic Blaschke product f. In particular, for the Chebychev polynomial T_d it is known [4] that $J(T_d) = [-1,1]$ for all $d \geq 2$, and [25, Theorem 4.3 (ii)] that $J(Q_u) = [0,1]$ for all $0 < u < 1$. Typically of course, the Julia set of a rational function is very much more complicated than an interval or smooth curve [22, 4, 8].

Returning to consideration of the arc-sine law, it can be shown by elementary arguments that if Q preserves the arc-sine law on $[0,1]$ and $Q(a) = P_2(a)/P_1(a)$ with P_i a polynomial of degree i, then $Q = Q_u$ or $1 - Q_u$ for some $u \in [0,1]$. This and all preceding results are consistent with the following:

Conjecture 5. *Every rational function R which preserves the arc-sine law on* $[0,1]$ *is of the form $R(a) = \frac{1}{2}(1-\tilde{f}(1-2a))$ where $\tilde{f}$ is derived from a symmetric Blaschke product f with $f(0) = 0$, as in Theorem* 2.

5. Some integral identities

Let $(B_t, t \geq 0)$ denote a standard one-dimensional Brownian motion. Let

$$\varphi(z) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}; \qquad \overline{\Phi}(x) := \int_x^\infty \varphi(z)\,dz = P(B_1 > x).$$

According to formula (13) of [29], the following identity gives two different expressions for the conditional probability density $P(B_U \in dx \,|\, B_1 = b)/dx$ for U with uniform distribution on $[0,1]$, assumed independent of $(B_t, t \geq 0)$:

$$\int_0^1 \frac{1}{\sqrt{u(1-u)}}\, \varphi\left(\frac{x - bu}{\sqrt{u(1-u)}}\right) du = \frac{\overline{\Phi}(|x| + |b-x|)}{\varphi(b)}. \tag{22}$$

The first expression reflects the fact that B_u given $B_1 = b$ has normal distribution with mean bu and variance $u(1-u)$, while the second was derived in [29] by consideration of Brownian local times. Multiply both sides of (22) by $\sqrt{2/\pi}$ to obtain the following identity for A with the arc-sine law (1): for all real x and b

$$E\left[\exp\left(-\frac{1}{2}\frac{(x-bA)^2}{A(1-A)}\right)\right] = 2\, e^{b^2/2}\, \overline{\Phi}(|x| + |b-x|). \tag{23}$$

Now

$$\frac{(x-bA)^2}{A(1-A)} = \frac{x^2}{A} + \frac{(x-b)^2}{1-A} - b^2 \stackrel{d}{=} \frac{(|x| + |b-x|)^2}{A} - b^2 \tag{24}$$

where the equality in distribution is a restatement of (3). So (23) amounts to the identity

$$E\left[\exp\left(-\frac{1}{2}\left(\frac{x^2}{A} + \frac{y^2}{1-A}\right)\right)\right] = 2\, \overline{\Phi}(|x| + |y|) \tag{25}$$

for arbitrary real x, y. Moreover, the identity in distribution (3) allows (25) to be deduced from its special case $y = 0$, that is

$$E\left[\exp\left(-\frac{x^2}{2A}\right)\right] = 2\overline{\Phi}(|x|), \tag{26}$$

which can be checked in many ways. For instance, $P(1/A \in dt) = dt/(\pi t\sqrt{t-1})$ for $t > 1$ so (26) reduces to the known Laplace transform [10, 3.363]

$$\frac{1}{2\pi}\int_1^\infty \frac{1}{t\sqrt{t-1}}\, e^{-\lambda t}\, dt = \overline{\Phi}(\sqrt{2\lambda}) \qquad (\lambda \geq 0). \tag{27}$$

This is verified by observing that both sides vanish at $\lambda = \infty$ and have the same derivative with respect to λ at each $\lambda > 0$. Alternatively, (26) can be checked as follows, using the Cauchy representation (6). Assuming that C is independent of B_1, we can compute for $x \geq 0$

$$E\left[\exp\left(-\frac{1}{2}\frac{x^2}{A}\right)\right] = e^{-\frac{1}{2}x^2} E\left[\exp(ixCB_1)\right] = e^{-\frac{1}{2}x^2} E\left[\exp(-x|B_1|)\right] = 2\overline{\Phi}(x). \tag{28}$$

We note also that the above argument allows (24) and hence (3) to be deduced from (23) and (26), by uniqueness of Laplace transforms.

By differentiation with respect to x, we see that (25) is equivalent to

$$E\left[\frac{x}{A}\exp\left(-\frac{1}{2}\left(\frac{x^2}{A} + \frac{y^2}{1-A}\right)\right)\right] = \sqrt{\frac{2}{\pi}}\, e^{-\frac{1}{2}(x+y)^2} \qquad (x > 0, y \geq 0). \tag{29}$$

That is to say, for each $x > 0$ and $y \geq 0$ the following function of $u \in (0,1)$ defines a probability density on $(0,1)$:

$$f_{x,y}(u) := \frac{x}{\sqrt{2\pi u^3(1-u)}}\, \exp\left[\frac{1}{2}\left((x+y)^2 - \frac{x^2}{u} - \frac{y^2}{1-u}\right)\right]. \tag{30}$$

This was shown by Seshadri [35, §p. 123], who observed that $f_{x,y}$ is the density of $T_{x,y}/(1+T_{x,y})$ for $T_{x,y}$ with the inverse Gaussian density of the hitting time of x by a Brownian motion with drift y. In particular, $f_{x,0}$ is the density of $x^2/(x^2 + B_1^2)$. See also [29, (17)] regarding other appearances of the density $f_{x,0}$.

6. Complements

The basic identity (3) can be transformed and checked in another way as follows. By uniqueness of Mellin transforms, (3) is equivalent to

$$\frac{u^2}{A\varepsilon_2} + \frac{(1-u)^2}{(1-A)\varepsilon_2} \stackrel{d}{=} \frac{1}{A\varepsilon_2} \tag{31}$$

where ε_2 is an exponential variable with mean 2, assumed independent of A. But it is elementary and well known that $A\varepsilon_2$ and $(1-A)\varepsilon_2$ are independent with the same distribution as B_1^2. So (31) amounts to

$$\frac{u^2}{X^2} + \frac{(1-u)^2}{Y^2} \stackrel{d}{=} \frac{1}{X^2} \tag{32}$$

where X and Y are independent standard Gaussian. But this is the well known result of Lévy[20] that the distribution of $1/X^2$ is stable with index $\frac{1}{2}$. The same argument yields the following multivariate form of (3): if $(W_1, \ldots, W_n)$ is uniformly distributed on the surface of the unit sphere in $\mathbb{R}^n$, then for $a_i \geq 0$

$$\sum_{i=1}^{n} \frac{a_i^2}{W_i^2} \stackrel{d}{=} \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{W_1^2}. \tag{33}$$

This was established by induction in [6, Proposition 3.1]. The identity (32) can be recast as

$$\frac{X^2Y^2}{a^2X^2 + c^2Y^2} \stackrel{d}{=} \frac{X^2}{(a+c)^2} \qquad (a, c > 0). \tag{34}$$

This is the identity of first components in the following bivariate identity in distribution, which was derived by M. Mora using the property (7) of the Cauchy distribution: for $p > 0$

$$\left(\frac{(XY(1+p))^2}{X^2 + p^2Y^2}, \frac{(X^2 - p^2Y^2)^2}{X^2 + p^2Y^2}\right) \stackrel{d}{=} \left(X^2, Y^2\right). \tag{35}$$

See Seshadri [35, §2.4, Theorem 2.3] regarding this identity and related properties of the inverse Gaussian distribution of the hitting time of $a > 0$ by a Brownian motion with positive drift. Given (X^2, Y^2), the signs of X and Y are chosen as if by two independent fair coin tosses, so (34) is further equivalent to

$$\frac{XY}{\sqrt{a^2X^2 + c^2Y^2}} \stackrel{d}{=} \frac{X}{a+c} \qquad (a, c > 0). \tag{36}$$

As a variation of (26), set $x = \sqrt{2\lambda}$ and make the change of variable $z = \sqrt{2\lambda u}$ in the integral to deduce the following curious identity: if X is a standard Gaussian then for all $x > 0$

$$E\left(\frac{x}{X\sqrt{X^2 - x^2}} \,\middle|\, X > x\right) \equiv \sqrt{\frac{\pi}{2}} \qquad (x > 0) \tag{37}$$

As a check, (37) for large x is consistent with the elementary fact that the distribution of $(x(X - x) \,|\, X > x)$ approaches that of a standard exponential variable ε_1 as $x \to \infty$. The distribution of $(x/(X\sqrt{X^2 - x^2}) \,|\, X > x)$ therefore approaches that of $1/\sqrt{2\varepsilon_1}$ as $x \to \infty$, and $E(1/\sqrt{2\varepsilon_1}) = \sqrt{\pi/2}$.

By integration with respect to $h(x)dx$, formula (37) is equivalent to the following identity: for all non-negative measurable functions h

$$\sqrt{\frac{2}{\pi}} E\left[\int_0^X \frac{xh(x)\,dx}{X\sqrt{X^2 - x^2}} 1(X \geq 0)\right] = E\left[\int_0^X h(x)\,dx\,1(X \geq 0)\right].$$

That is to say, for U with uniform $(0, 1)$ distribution, assumed independent of X,

$$\sqrt{\frac{1}{2\pi}} E\left[h\left(\sqrt{1 - U^2}\,|X|\right)\right] = E\left[\,|X|\,h(\,|X|\,U)\right].$$

Equivalently, for arbitrary non-negative measurable g

$$E\left[g\left((1 - U^2)X^2\right)\right] = \sqrt{2\pi} E\left[\,|X|\,h(X^2U^2)\right]. \tag{38}$$

Now $X^2 \stackrel{d}{=} A\varepsilon_2$ where ε_2 is exponential with mean 2, independent of A; and when the density of X^2 is changed by a factor of $\sqrt{2\pi}|X|$ we get back the density of ε_2. So the identity (38) reduces to

$$(1 - U^2)A\varepsilon_2 \stackrel{d}{=} U^2\varepsilon_2$$

and hence to

$$(1-U^2)A \stackrel{d}{=} U^2.$$

This is the particular case $a = b = c = 1/2$ of the well known identity

$$\beta_{a+b,c}\,\beta_{a,b} \stackrel{d}{=} \beta_{a,b+c}$$

for $a, b, c > 0$, where $\beta_{p,q}$ denotes a random variable with the beta(p, q) distribution on $(0, 1)$ with density at u proportional to $u^{p-1}(1-u)^{q-1}$, and it is assumed that $\beta_{a+b,c}$ and $\beta_{a,b}$ are independent.

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