CHAPTER 5

Decomposable Measures

The main purpose of this chapter is to discuss the extent to which Theorem 4.3 can be generalized to arbitrary Radon measures (rather than probability measures) and to cases where G is not compact. Such generalizations can be used in the derivation of densities of maximal invariants as well as in other areas. The approach here is modelled after that described in Andersson (1982). Other possible approaches to this problem are described in Wijsman (1986) (the global cross section approach using some Lie group theory) and Farrell (1985) (a measure-theoretic cross section approach developed by Schwartz (1966), unpublished).

The general method of averaging over a group to obtain a density of a maximal invariant is due to Stein (1956). However, there are mathematical problems to overcome. The approaches described in Farrell (1985) and Wijsman (1986) have their advantages and disadvantages as does the method to be described here. As far as I know, there are no "practical" problems where one of the methods can be applied but the others cannot. The ease with which the methods apply depends on the problem at hand and the method most familiar to you. It should be noted however that all of the methods require some regularity conditions which are essential.

In the first section of this chapter, we treat the compact group case. A version of Theorem 4.3 is established for Radon measures. However the methods and the language are quite different here because the methods used in Chapter 4 do not carry over easily to the case when the group is noncompact. The noncompact case is discussed in Section 5.2. In Section 5.3, a representation of the density of a maximal invariant due to Andersson (1982) is established. This representation is used to provide a proof of an important result due to Wijsman (1967) on ratios of densities of a maximal invariant statistic.

5.1. The compact group case. Throughout this section $\mathbf{X}$ is a locally compact Hausdorff space with a countable base for the topology (so the topology is a metric topology). Also, G is a compact topological group which acts topologically on $\mathbf{X}$. Thus, the map $(g, x) \to gx$ from $G \times \mathbf{X}$ to $\mathbf{X}$ is continuous.

Rather than introduce a particular measurable cross section, we will consider the quotient space $\mathbf{X}/G$ whose points are the equivalence classes $\{gx|g \in G\}$. That is, the points in $\mathbf{X}/G$ are just the orbits. The natural projection π on $\mathbf{X}$ to $\mathbf{X}/G$ given by

$$\pi(x) = G \cdot x = \{gx|g \in G\}$$

plays an important role in what follows. Observe that π is a maximal invariant function. Thus, a real valued function f defined on $\mathbf{X}$ is invariant iff there exists a real valued function f^* on $\mathbf{X}/G$ such that

$$f(x) = f^*(\pi(x)). \tag{5.1}$$

Let ν denote the unique invariant probability measure on G. Temporarily ignoring a host of technical considerations, look at the function

$$x \to \int_G f(gx)\nu(dg).$$

This function is clearly G-invariant, so it can be thought of as a map which sends f into a function f^* defined on $\mathbf{X}/G$. Thus, Theorem 4.3 can be written

$$\int_{\mathbf{X}} f(x)P(dx) = \int_{\mathbf{X}/G} f^*(y)Q(dy).$$

Equivalently, if J is the invariant integral defined by an invariant P and J_1 is the integral defined by Q, the above is

$$J(f) = J_1(T(f)), \tag{5.2}$$

where $T(f) = f^*$ is the mapping which sends f into f^*. Theorem 4.3 shows that if J is an invariant integral (corresponding to a probability measure), then there exists an integral J_1 (corresponding to a probability measure on $\mathbf{X}/G$) such that (5.2) holds. The validity of (5.2) for an arbitrary invariant integral J is the question to which we now turn.

Here is the general plan of attack. First, when $\mathbf{X}/G$ has the quotient topology (which makes $\mathbf{X}/G$ a locally compact Hausdorff space), it will be shown that the mapping T described above maps $K(\mathbf{X})$ *onto* $K(\mathbf{X}/G)$. Thus, given an invariant integral J defined on $K(\mathbf{X})$, we *define* J_1 on $K(\mathbf{X}/G)$ via the equation

$$J(f) = J_1(T(f)), \qquad f \in K(\mathbf{X}).$$

Of course, there is some work involved in showing that J_1 is well defined and J_1 is an integral. The uniqueness of J_1 follows because T is onto and hence (5.2) holds. We now turn to the technical details.

First, a few words about the quotient topology for $\mathbf{X}/G$. Recall that $\mathbf{X}$ is a locally compact Hausdorff space with a countable base for the topology and G is a compact topological group with a countable base for its topology. The action $(g, x) \to gx$ is assumed to be continuous on $G \times \mathbf{X}$ to $\mathbf{X}$. A subset $U \subset \mathbf{X}/G$ is open (in the quotient topology) iff $\pi^{-1}(U)$ is open in $\mathbf{X}$ where π is the natural projection on $\mathbf{X}$ to $\mathbf{X}/G$. Because G is compact and the topology for $\mathbf{X}$ has a countable base, the quotient topology (i) is a locally compact Hausdorff topology and (ii) has a countable base. Thus, the quotient space $\mathbf{X}/G$ is of the same type as the space $\mathbf{X}$.

Next, we consider the function T defined on $K(\mathbf{X})$ by

$T(f)$ is the unique function f^* defined on $\mathbf{X}/G$ which satisfies

$$\int_G f(gx)\nu(dg) = f^*(\pi(x)).$$

THEOREM 5.1. *The function T maps $K(\mathbf{X})$ onto $K(\mathbf{X}/G)$. Also, T satisfies*

$$\begin{aligned} T(\alpha f_1 + \beta f_2) &= \alpha T(f_1) + \beta T(f_2) \\ T(f) &\geq 0 \quad \text{when } f \geq 0, \end{aligned} \tag{5.3}$$

for $\alpha, \beta \in R^1$ and $f \in K(\mathbf{X})$.

PROOF. For $f \in K(\mathbf{X})$, the continuity of the function

$$x \to \int_G f(gx)\nu(dg)$$

is easily established using the bounded convergence theorem. Thus, by definition of the quotient topology, f^* is continuous. To show f^* has compact support when f has compact support $V \subset \mathbf{X}$, first note that the set

$$G \cdot V = \{y | y = gx \text{ for some } g \in G,\ x \in V\}$$

is a compact subset of $\mathbf{X}$ because $G \cdot V$ is the continuous image of the compact set $G \times V \subset G \times \mathbf{X}$. Thus $\pi(G \cdot V)$ is a compact set in $\mathbf{X}/G$ since π is continuous. If $y \notin \pi(G \cdot V)$, then $\pi^{-1}(y) = G \cdot x$ for some $x \notin V$. Thus for all $g \in G$, $f(gx) = 0$ because $gx \notin G \cdot V$ and f vanishes outside $G \cdot V$. Therefore $f^*(y) = 0$ for y outside the compact set $\pi(G \cdot V)$.

To show T is onto, let $f^* \in K(\mathbf{X}/G)$ and consider

$$f(x) = f^*(\pi(x)), \qquad x \in \mathbf{X}.$$

That f is continuous is obvious. If the compact set $V \subset \mathbf{X}/G$ contains the support of f^*, then $\pi^{-1}(V)$ is easily shown to be compact in $\mathbf{X}$. Obviously, if $x \notin \pi^{-1}(V)$, then $f(x) = 0$ so $\pi^{-1}(V)$ supports f. Because f defined above is G-invariant,

$$\int f(gx)\nu(dg) = f(x) = f^*(\pi(x)),$$

so $T(f) = f^*$. Hence T is onto.

That T satisfies the relations in (5.3) is easily proved. This completes the proof. □

The results of Theorem 5.1 provide the key technical step in the following generalization of Theorem 4.3.

THEOREM 5.2. *Suppose J is a G-invariant integral on $K(\mathbf{X})$. Then there exists a unique integral J_1 on $K(\mathbf{X}/G)$ such that*

$$J(f) = J_1(T(f)), \qquad f \in K(\mathbf{X}). \tag{5.4}$$

Conversely, for each integral J_1 on $K(\mathbf{X}/G)$, the integral J on $K(\mathbf{X})$ defined by

$$J(f) = J_1(T(f))$$

is G-invariant.

PROOF. Given $f^* \in K(\mathbf{X}/G)$, define J_1 by

$$J_1(f^*) = J(f)$$

for any f such that $T(f) = f^*$. To show that J_1 is well defined, suppose $T(f_1) = T(f_2) = f^*$. With $f_3 = f_1 - f_2$, obviously $T(f_3) = 0$. It must be shown that this implies $J(f_3) = 0$. First, represent J by its associated Radon measure μ:

$$J(f) = \int_{\mathbf{X}} f(x)\mu(dx).$$

The invariance of J yields

$$J(f) = \int_{\mathbf{X}} f(gx)\mu(dx), \qquad g \in G,$$

so that

$$J(f) = \int_{\mathbf{X}}\int_G f(gx)\nu(dg)\mu(dx).$$

Thus, if $T(f_3) = 0$,

$$T(f_3)(\pi(x)) = \int_G f_3(gx)\nu(dg) = 0,$$

which shows that $J(f_3) = 0$. Hence J_1 is well defined. The linearity of J_1 and the inequality $J_1(f^*) \geq 0$ for $f^* \geq 0$ are easily established. Hence J_1 is an integral and (5.4) holds. The uniqueness of J_1 follows from the fact that T is an onto map.

For the converse, just observe that $T(f) = T(L_g f)$ for $f \in K(\mathbf{X})$ and $g \in G$. Hence J defined by

$$J(f) = J_1(T(f))$$

is G-invariant. □

When the relationship (5.4) holds for $f \in K(\mathbf{X})$, then of course (5.4) holds for all integrable functions f because both sides of (5.4) are integrals. Thus, under the assumptions of Theorem 5.2, Equation (5.4) can be used for integrable functions as well as functions in $K(\mathbf{X})$.

Our first application of Theorem 5.2 involves finding the density function of a maximal invariant. Consider X taking values in a space $\mathbf{X}$ and assume that X has a density p with respect to a Radon measure μ.

THEOREM 5.3. *Assume that the dominating measure μ is invariant under the compact group G and let J denote the integral defined by μ. Write $J = J_1 \circ T$*

as in Equation (5.4) *and let* μ_1 *be the Radon measure associated with* J_1 *defined on the quotient space. Then the density function of* $Y = \pi(X)$ *with respect to* μ_1 *is* $p^* = T(p)$.

PROOF. Let f^* be any bounded measurable function defined on $\mathbf{X}/G$. It suffices to show that

$$\mathscr{E}f^*(Y) = \int_{\mathbf{X}/G} f^*(y)p^*(y)\mu_1(dy).$$

First observe that $f^*\pi$ is a bounded measurable function defined on $\mathbf{X}$ and $T((f^*\pi)p) = f^*T(p)$. Therefore,

$$\begin{aligned}\mathscr{E}f^*(Y) &= \mathscr{E}f^*(\pi(X)) = J((f^*\pi)p) = J_1(T((f^*\pi)p)) \\ &= J_1(f^*T(p)) = J_1(f^*p^*) = \int f^*(y)p^*(y)\mu_1(dy),\end{aligned}$$

where the third equality follows from Equation (5.4) in Theorem 5.2. This completes the proof. □

The application of Theorem 5.3 requires two separate steps—the calculation of the induced measure μ_1 and the evaluation of p^* which involves the calculation of

$$\int_G p(gx)\nu(dg).$$

In concrete problems, one often chooses a particular representation of the quotient space $\mathbf{X}/G$ [that is, some one-to-one function of $\pi(x)$] to facilitate the discussion of the density of a maximal invariant. In symbols, suppose $k: \mathbf{X}/G \to Z$ is a one-to-one onto bimeasurable function. Thus $Z = k(\pi(X))$ is a maximal invariant. Let δ be the image of μ_1 under k, that is, $\delta(B) = \mu_1(k^{-1}(B))$. A direct calculation shows that $\tilde{p}(z) = p^*(k^{-1}(z))$ is the density of Z with respect to δ.

The implication of this is that the calculation of the density of *any* maximal invariant function, say $k_0(X) = W$, involves (i) the calculation of the induced measure μ_0 given by $\mu_0(B) = \mu(k_0^{-1}(B))$ and (ii) writing the function

$$x \to \int p(gx)\nu(dg)$$

as a function of $w = k_0(x)$, say $p_0(w)$.

Here is the classical example of the noncentral Wishart density [Herz (1955) and James (1954)] to which the above argument applies.

EXAMPLE 5.1. Let $\mathbf{X}$ be the space of $n \times p$ real matrices of rank p and take μ to be Lebesgue measure on $\mathbf{X}$. The group $G = O_n$ acts on $\mathbf{X}$ via matrix multiplication:

$$x \to gx, \qquad g \in O_n.$$

Obviously μ is an invariant measure. A standard choice for a maximal invariant is

$$s = k_0(x) = x'x,$$

so s takes values in S_p^+—the set of $p \times p$ positive definite real matrices. To find the induced measure μ_0 on S_p^+ defined by

$$\mu_0(B) = \mu(k_0^{-1}(B)),$$

first observe that μ_0 is characterized by the equation

$$\int f^*(k_0(x))\,dx = \int f^*(s)\mu_0(ds), \tag{5.5}$$

which holds for all bounded integrable f^* on S_p^+. To solve (5.5) for μ_0, take f^* to be of the form

$$f^*(k_0(x)) = f_1^*(k_0(x))\phi(k_0(x)), \tag{5.6}$$

where

$$\phi(s) = (\sqrt{2\pi})^{-np}\exp\left[-\tfrac{1}{2}\operatorname{tr} s\right].$$

Then $\phi(k_0(x))$ is the density of a normal distribution on $\mathbf{X}$ (with mean zero and covariance equal to the identity). Thus, the left side of (5.5) is

$$\int f_1^*(k_0(X))\phi(k_0(x))\,dx$$

which is the expectation of

$$f_1^*(k_0(X)) = f_1^*(S),$$

where $S = X'X$. Now, standard multivariate arguments show that the density of S (with respect to Lebesgue measure ds on S_p^+) is

$$p_1^*(s) = \omega(n, p)|s|^{(n-p-1)/2}\exp\left[-\tfrac{1}{2}\operatorname{tr} s\right],$$

where $\omega(n, p)$ is the Wishart constant

$$[\omega(n, p)]^{-1} = \pi^{p(p-1)/4}2^{np/2}\prod_{j=1}^{p}\Gamma\left(\frac{n-j+1}{2}\right).$$

Thus, the right side of (5.5) is just the expectation of $f_1^*(S)$ relative to the density $p_1^*(s)$. This yields the equation

$$\int f_1^*(s)\phi(s)\mu_0(ds) = \int f_1^*(s)p_1^*(s)\,ds,$$

which holds for all bounded measurable f_1^*. Therefore

$$\mu_0(ds) = \frac{p_1^*(s)}{\phi(s)}\,ds = \omega(n, p)(\sqrt{2\pi})^{np}|s|^{(n-p-1)/2}\,ds. \tag{5.7}$$

Now, let $p(x)$ be a density of X with respect to Lebesgue measure on $\mathbf{X}$. Theorem 5.3 and the succeeding discussion show that the density of $X'X = S$ with respect to μ_0 is calculated by evaluating the integral

$$\int_{O_n} p(gx)\nu(dg)$$

and writing the answer as a function of $s \in S_p^+$. For this particular case, x can be written (uniquely) as

$$x = h\begin{pmatrix} s^{1/2} \\ 0 \end{pmatrix}$$

where $s = x'x$ is in S_p^+ and $h \in O_n$. Therefore the density of S is

$$p^*(s) = \int_{O_n} p\left(gh\begin{pmatrix} s^{1/2} \\ 0 \end{pmatrix}\right)\nu(dg) = \int_{O_n} p\left(g\begin{pmatrix} s^{1/2} \\ 0 \end{pmatrix}\right)\nu(dg). \tag{5.8}$$

The particular choice of p which leads to the noncentral Wishart distribution is

$$p_0(x) = (\sqrt{2\pi})^{-np}\exp\left[-\tfrac{1}{2}\operatorname{tr}(x-\theta)'(x-\theta)\right], \tag{5.9}$$

where θ is an $n \times p$ matrix. Of course this choice of p_0 corresponds to X having a $N(\theta, I_n \otimes I_p)$ distribution. Substitution of (5.9) into (5.8) yields

$$p^*(s) = (\sqrt{2\pi})^{-np}\exp\left[-\tfrac{1}{2}\operatorname{tr} s - \tfrac{1}{2}\operatorname{tr}\theta'\theta\right]\int_{O_n}\exp[\operatorname{tr} gz]\nu(dg), \tag{5.10}$$

where

$$z = \begin{pmatrix} s^{1/2} \\ 0 \end{pmatrix}\theta'.$$

Thus, the difficulty is the evaluation of

$$\psi(z) = \int_{O_n}\exp[\operatorname{tr} gz]\nu(dg). \tag{5.11}$$

It is the attempted evaluation of (5.11) which led to the development of zonal polynomials by Alan James and others. The reader can find an excellent discussion of zonal polynomials in Muirhead (1982), Farrell (1985) and Takemura (1984). This subject is not discussed further here. □

5.2. The noncompact case. In this section, we discuss the validity of Theorem 5.2 when the group G is not necessarily compact. Throughout this section, the group G is assumed to be a locally compact, σ-compact topological group whose topology has a countable base. The basic approach here is to make some modifications in the method of proof used in the compact case so that versions of both Theorems 5.1 and 5.2 become valid for noncompact groups.

The first problem to overcome concerns the appropriate definition of the function T discussed in Theorem 5.1. Thus, consider the group G acting topologically on the space $\mathbf{X}$. Let ν_l denote left Haar measure on G and let Δ be the modular function of G. Then

$$\nu_r(dg) = \frac{1}{\Delta(g)}\nu_l(dg)$$

is a right Haar measure on G. First observe that for $f \in K(\mathbf{X})$, the function

$$x \to \int f(gx)\nu_r(dg) \tag{5.12}$$

is an invariant function of x—assuming the integral makes sense. (We will discuss some conditions under which the integral is well defined a bit later, but for now assume everything is all right.) Thus, (5.12) provides the appropriate definition of the function T which supposedly maps $K(\mathbf{X})$ into $K(\mathbf{X}/G)$ (assuming the quotient space is a "reasonable" space and ignoring the continuity and compact support questions). Therefore $T(f)$ is defined to be the function f^* on $\mathbf{X}/G$ which satisfies

$$\int f(gx)\nu_r(dg) = f^*(\pi(x)), \tag{5.13}$$

where π is the natural projection from $\mathbf{X}$ to the quotient space $\mathbf{X}/G$ given by

$$\pi(x) = G \cdot x = \{gx | g \in G\}.$$

If J_1 is an integral defined on $K(\mathbf{X}/G)$, then the expression $J_1(T(f))$ should define an integral on $K(\mathbf{X})$. But, for $h \in G$,

$$T(L_h f) = \Delta^{-1}(h)T(f)$$

because

$$\begin{aligned}
\int (L_h f)(gx)\nu_r(dg) &= \int f(h^{-1}gx)\nu_r(dg) \\
&= \int f(h^{-1}gx)\Delta^{-1}(g)\nu_l(dg) \\
&= \Delta^{-1}(h)\int f(h^{-1}gx)\Delta^{-1}(h^{-1}g)\nu_l(dg) \\
&= \Delta^{-1}(h)\int f(gx)\Delta^{-1}(g)\nu_l(dg) \\
&= \Delta^{-1}(h)\int f(gx)\nu_r(dg).
\end{aligned}$$

Hence, if $J_1(T(f))$ does define an integral on $K(\mathbf{X})$, this integral is relatively invariant with multiplier Δ^{-1}. The whole point of this discussion is that the natural definition of T in the noncompact case leads to integrals $J_1(T(f))$ which are relatively invariant with multiplier Δ^{-1}. *Therefore, the only types of integrals J which can possibly have the representation $J(f) = J_1(T(f))$ must be relatively invariant with multiplier Δ^{-1}, given our definition of T.* This issue did not arise in the compact case because there are no nontrivial multipliers when the group is compact.

Now, consider G acting on $\mathbf{X}$ and suppose the integral J defined on $K(\mathbf{X})$ is relatively invariant with multiplier Δ^{-1}. We now want to discuss conditions under which a representation for J of the above type holds. Here is a simple example which shows that some additional assumptions need to be made.

EXAMPLE 5.2. Let $\mathbf{X} = R^1$ and take $G = R^2$ with addition as the group operation. For $g = (a_1, a_2) \in R^2$, the action of G on $\mathbf{X}$ is defined by

$$(a_1, a_2)x = x + a_1.$$

Obviously Lebesgue measure on $\mathbf{X}$ is invariant under this group action. Because G is a commutative group, the modulus of G is identically 1 and Lebesgue measure dg on G is right- and left-invariant. The difficulty here arises with the definition of T. Consider $f \geq 0$, $f \in K(\mathbf{X})$ and form

$$\int_G f(gx)\,dg = \int\int f(x + a_1)\,da_1\,da_2.$$

This integral is $+\infty$ as long as f is not 0, and there is just no way to patch things. The problem is that when $f \geq 0$ is not 0, for each x, the set

$$\{g | f(gx) \geq \varepsilon\}$$

has infinite measure for some $\varepsilon > 0$. This sort of situation must be ruled out in order to have a representation theorem. Because G acts transitively on $\mathbf{X}$ in this example, the trouble is not with the quotient space (or its topology), but the trouble is the size of the group compared to the size of the space. Indeed, for every interval $[c, d] \subset R^1$ with $c < d$ and for every $x \in R^1$,

$$\{g | gx \in [c, d]\} = \{(a_1, a_2) | a_1 \in [c - x, d - x]\} = [c - x, d - x] \times R^1$$

has infinite measure in G.

In order that the integral in (5.13)

$$\int_G f(gx)\nu_r(dg)$$

be well defined, it is sufficient that sets of the form

$$\{g | gx \in C\}$$

be compact when $C \subset \mathbf{X}$ is compact. This type of condition leads to a version of Theorem 5.2. □

Here is the condition which excludes the sort of situation encountered in Example 5.1. Consider the topological group G which acts topologically on the space $\mathbf{X}$.

Definition 5.1. The group G acts *properly* on $\mathbf{X}$ if the mapping ψ defined on $G \times \mathbf{X}$ to $\mathbf{X} \times \mathbf{X}$ by

$$\psi(g, x) = (gx, x)$$

is a *proper* mapping, that is, if the inverse image under ψ of each compact set in $\mathbf{X} \times \mathbf{X}$ is a compact set in $G \times \mathbf{X}$.

Here is the analog of Theorem 5.1:

Theorem 5.4. *Assume that G acts properly on $\mathbf{X}$. Then the quotient space $\mathbf{X}/G$ with its quotient topology is a locally compact Hausdorff space with a countable base for open sets. The mapping T on $K(\mathbf{X})$ given by $T(f) = f^*$, where f^* is the unique function satisfying*

$$\int f(gx)\nu_r(dg) = f^*(\pi(x)),$$

is well defined. Further T maps $K(\mathbf{X})$ onto $K(\mathbf{X}/G)$ and satisfies condition (5.3).

The above theorem is a conglomeration of results from Bourbaki. See Andersson (1982) for further discussion and references to the relevant portions of Bourbaki. Also, see the discussion in Wijsman (1985).

THEOREM 5.5. *Assume G acts properly on $\mathbf{X}$ and suppose that J is a relatively invariant integral on $K(\mathbf{X})$ with multiplier Δ^{-1}. Then there exists a unique integral J_1 on $K(\mathbf{X}/G)$ such that*

$$J(f) = J_1(T(f)), \qquad f \in K(\mathbf{X}). \tag{5.14}$$

Conversely, for each integral J_1 defined on $K(\mathbf{X}/G)$, J defined by (5.14) *is relatively invariant with multiplier Δ^{-1}.*

PROOF. The proof is essentially the same as the proof of Theorem 5.2 once it is noted that

$$T(L_h f) = \Delta^{-1}(h)T(f).$$

This relation was established earlier. The details are left to the reader. □

Before turning to some applications of Theorem 5.5, we first discuss a useful sufficient condition that G acts properly on $\mathbf{X}$. This condition also has the advantage of being a bit easier to understand than the condition of properness.

Given two subsets $A, B \subset \mathbf{X}$, set

$$(A, B) = \{g \in G | gA \cap B \neq \phi\}.$$

THEOREM 5.6. *Assume that for any compact subsets $A, B \subset \mathbf{X}$, the set (A, B) is a compact subset of G. Then G acts properly on $\mathbf{X}$.*

This theorem, which comes from Bourbaki, is given as Lemma 1.1 in Wijsman (1985) where a proof can be found. The assumption that (A, B) is compact whenever A and B are compact has some interesting consequences. For example, take $A = B = \{x\}$. Then

$$(\{x\}, \{x\}) = \{g | gx = x\},$$

which is the isotropy subgroup of x. Hence all of the isotropy subgroups are compact under the assumption of Theorem 5.6. When G is a "nice" subset of some Euclidean space, then the assumption that (A, B) is compact just means that (A, B) is a bounded set in the Euclidean space. (A, B) is always closed because the group action is continuous. Thus, in this case, the compactness of (A, B) is equivalent to the assertion that the collection of g's which "move" some point in A into B is bounded.

Finally, we turn to the question of a representation theorem for integrals (Radon measures) which are relatively invariant with a multiplier χ_0 on G. Thus suppose an integral J_0 on $K(\mathbf{X})$ is given by

$$J_0(f) = \int f(x)\mu_0(dx)$$

and

$$J_0(L_h f) = \int f(h^{-1}x)\mu_0(dx) = \chi_0(h)\int f(x)\mu_0(dx).$$

When G acts properly on $\mathbf{X}$ and when $\chi_0 = \Delta^{-1}$, then and only then does Theorem 5.5 apply directly. However, when $\chi_0 \neq \Delta^{-1}$, it is possible to change the measure μ_0 into a new measure to which Theorem 5.5 applies. Here are the details of that modification. Because the group G is assumed to be σ-compact, it follows that given any multiplier χ on G there exists a positive continuous function ϕ defined on $\mathbf{X}$ which satisfies

$$\phi(gx) = \chi(g)\phi(x) \tag{5.15}$$

for $x \in \mathbf{X}$ and $g \in G$. [See Bourbaki (1963), Proposition 7, Section 2, 4°.] Now, define a new measure μ by

$$\mu(dx) = \frac{1}{\phi(x)}\mu_0(dx), \tag{5.16}$$

where ϕ satisfies (5.15) with $\chi = \chi_0\Delta$. The claim is that the integral J defined by

$$J(f) = \int f(x)\mu(dx)$$

is relatively invariant with multiplier Δ^{-1}. This claim is verified as follows. For $h \in G$,

$$\begin{aligned} J(L_h f) &= \int f(h^{-1}x)\mu(dx) = \int f(h^{-1}x)\frac{1}{\phi(x)}\mu_0(dx) \\ &= \int f(h^{-1}x)\frac{1}{\phi(hh^{-1}x)}\mu_0(dx) \\ &= \frac{1}{\chi_0(h)\Delta(h)}\int f(h^{-1}x)\frac{1}{\phi(h^{-1}x)}\mu_0(dx) \\ &= \frac{\chi_0(h)}{\chi_0(h)\Delta(h)}\int f(x)\frac{1}{\phi(x)}\mu_0(dx) = \Delta^{-1}(h)J(f). \end{aligned}$$

Thus, under the assumption that G acts properly on $\mathbf{X}$, Theorem 5.5 can be applied to J. Summarizing this discussion gives:

THEOREM 5.7. *Assume G acts properly on $\mathbf{X}$ and let*

$$J_0(f) = \int f(x)\mu_0(dx)$$

be relatively invariant with multiplier χ_0. Pick ϕ to be a positive continuous function on $\mathbf{X}$ satisfying (5.15) with $\chi = \chi_0\Delta$. Then the integral

$$J(f) = \int f(x)\frac{1}{\phi(x)}\mu_0(dx)$$

is relatively invariant with multiplier Δ^{-1}. *Further* J_0 *has the representation*

$$J_0(f) = J(f\phi) = J_1(T(f\phi)) \tag{5.17}$$

for some integral J_1 *defined on* $K(\mathbf{X}/G)$.

PROOF. This follows immediately from the discussion above and Theorem 5.5. □

It should be mentioned that $T(f\phi)$ in (5.17) takes a special form because ϕ satisfies (5.15). Recall that for $f_1 \in K(\mathbf{X})$, $T(f_1) = f_1^*$ is the function in $K(\mathbf{X}/G)$ which satisfies the equation

$$\int f_1(gx)\nu_r(dg) = f_1^*(\pi(x)).$$

Substituting $f_1 = f\phi$ and using (5.15) yields

$$\begin{aligned}(f\phi)^*(\pi(x)) &= \int f(gx)\phi(gx)\nu_r(dg) \\ &= \phi(x)\int f(gx)\chi_0(g)\Delta(g)\nu_r(dg) \\ &= \phi(x)\int f(gx)\chi_0(g)\nu_l(dg),\end{aligned} \tag{5.18}$$

where $\nu_l = \Delta\nu_r$ is a left Haar measure on G.

5.3. The Wijsman representation. The main result in this section is a version of the Wijsman representation for the ratio of densities of a maximal invariant. The version presented here, under the assumption that G acts properly on $\mathbf{X}$, is due to Andersson (1982) and is an easy consequence of Equation (5.18) following Theorem 5.7.

Our first goal is to establish a version of Theorem 5.3 for the noncompact case. Throughout this section, G is a locally compact, σ-compact topological group which acts topologically on $\mathbf{X}$. *It is also assumed that the action of G on $\mathbf{X}$ is proper, so Theorems* 5.5 *and* 5.6 *are valid.* As usual π denotes the natural projection of $\mathbf{X}$ onto $\mathbf{X}/G$.

THEOREM 5.8. *Consider a random variable X with values in $\mathbf{X}$ and assume that X has a density p with respect to a measure μ_0 which is relatively invariant with multiplier χ_0. As in Theorem* 5.7, *let ϕ be a positive continuous function satisfying* (5.15) *with $\chi = \chi_0\Delta$ so that the equation*

$$J_0(f) = J_1(T(f\phi)) \tag{5.19}$$

holds for some integral

$$J_1(f^*) = \int_{\mathbf{X}/G} f^*(y)\mu_1(dy)$$

defined on $K(\mathbf{X}/G)$. Then the density function of the maximal invariant $Y =$

$\pi(X)$ with respect to μ_1 is

$$(p\phi)^* = T(p\phi).$$

PROOF. The argument is a minor variation of that given in the proof of Theorem 5.3. It suffices to show that for $f^* \in K(\mathbf{X}/G)$,

$$\mathscr{E}f^*(Y) = \int f^*(y)(p\phi)^*(y)\mu_1(dy).$$

This equality follows from

$$\begin{aligned}\mathscr{E}f^*(Y) &= \mathscr{E}f^*(\pi(X))\\ &= J_0((f^*\pi)p) = J_1(T((f^*\pi)p\phi))\\ &= J_1(f^*T(p\phi)) = J_1(f^*(p\phi)^*)\\ &= \int f^*(y)(p\phi)^*(y)\mu_1(dy).\end{aligned}$$

This completes the proof. □

Here is Andersson's (1982) version of Wijsman's theorem:

THEOREM 5.9. *Under the assumptions of Theorem 5.8, let p_1 and p_2 be two possible densities of X. Then for each $x \in \mathbf{X}$ such that the denominator is positive, the ratio of the densities $(p_2\phi)^*/(p_1\phi)^*$ of the maximal invariant $\pi(X)$ is*

$$r(\pi(x)) = \frac{\int p_2(gx)\chi_0(g)\nu_l(dg)}{\int p_1(gx)\chi_0(g)\nu_l(dg)}. \tag{5.20}$$

PROOF. According to Theorem 5.8, when X has density p_i, the density of $\pi(X)$ with respect to μ_1 is $(p_i\phi)^*$. However, Equation (5.18) shows that

$$(p_i\phi)^*(\pi(x)) = \phi(x)\int p_i(gx)\chi_0(g)\nu_l(dg).$$

With division of $(p_2\phi)^*(\pi(x))$ by $(p_1\phi)^*(\pi(x))$, the result follows. □

Applications of this result to robustness and decision theory results are given in later chapters. The paper of Wijsman (1985) contains some very useful methods of verifying that certain group actions are proper. This paper also contains a number of important examples to which we refer later.