IMS Collections
Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz
Vol. 4 (2008) 301–314

DOI: 10.1214/074921708000000444

Estimates of Dynamic *VaR* and Mean Loss Associated to Diffusion Processes

Laurent Denis,[1,*] Begoña Fernández,[2,*] and Ana Meda[2,*]

Université d'Evry-Val-d'Essonne, and Universidad Nacional Autónoma de México.

Abstract: Let X_t be a stochastic process driven by a differential equation of the form $dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,\ t > 0$, and let $X^{\star}_{s,t} = \sup_{s \le u \le t} X_u$, be the maximum of the diffusion. In this work we obtain bounds for the tail distribution of $X^{*}_{s,t}$, define several dynamic *VaR* type quantiles for this process and give upper and lower bounds for both, the *VaR* quantile and the conditioned mean loss associated to it. The results we obtain are based in the change of time property of the Brownian Motion, and can be applied to a a large class of examples used in Finance, in particular where $\sigma(t, X_t) = \sigma_t X_t^{\gamma}$, where $0 \le \gamma < 1$. The estimates we obtain are sharp. We discuss carefully the Geometric Brownian Motion, the Cox-Ingersoll-Ross and the Vasicek type models, and give an application to Russian options.

1. Introduction

For Risk Theory it is of interest to estimate high quantiles (Value at Risk: *VaR*) and mean loss given that an extreme event has occurred. One approach to deal with this is to use Extreme Value Theory by fitting Fréchet, Gumbel or Weibul distributions to approximate *VaR* and the Generalized Pareto distribution to fit the conditional loss distribution (see, for example, an excellent account in the book [EKM]; in references therein and subsequent work by the authors). An enormous amount of work in this direction has been done also for time series (see, for example [McNeil]). The behavior of extremes for diffusion processes has been studied by Davis (1982, [Dav]) who found a distribution F_t which is the asymptotic limit for the distribution of the maxima of the process as t tends to infinity. On the other hand, Borkovec and Klüppelberg (1998,[BoKl]) described the tail behavior of the limit F_t -using again Extreme Value Theory- in terms of the coefficients of the equation and proved that the number of $\epsilon-$upcrossings of certain level converges to a homogeneous Poisson Process as t tends to infinity and mentioned that the applications of these results to study risk measures of financial products was work in progress.

Another point of view is that of Talay and Zheng (2002, [TaZh]), who combined Monte-Carlo Methods with the Euler discretization to calculate *VaR* for diffusion processes that have densities (uniformly elliptic, or a more general setting as in Bally and Talay [BaTa1] and [BaTa2]). They applied their results to find *VaR* for portfolios.

*This work was partially supported by Grants CONACYT 37922E, and PAPIIT-DGAPA-UNAM, IN103606, México.

[1]Département de Mathématiques Equipe "Analyse et probabilités" Université d'Evry-Val-d'Essonne, Bâtiment Maupertuis rue du pére Jarlan, 91025 Evry Cedex France, e-mail: ldenis@univ-evry.fr

[2]Departamento de Matemáticas, Facultad de Ciencias, UNAM. Circuito Exterior s/n Ciudad Universitaria, Coyoacán, 04510, México DF, Mexico, e-mail: bff@hp.fciencias.unam.mx; amg@hp.fciencias.unam.mx

Keywords and phrases: dynamic *VaR*, mean loss, diffusion process

In this work we consider continuous processes driven by a differential stochastic equation $dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,\ t > 0$, define *VaR* quantiles for $\sup_{s\le u\le t} X_u$ and give bounds for both, the *VaR* quantile and the conditioned mean loss associated to it. The second quantity is generally considered better than *VaR* not only because it is subadditive, but also because it provides useful extra information. Our approach is completely different to that of the authors mentioned above and can be applied to a general class of diffusions, in particular to typical examples in Finance where $\sigma(t, X_t) = \sigma_t X_t^\gamma$, with $0 \le \gamma < 1$ and σ_t bounded. We use the change of time property of the Brownian Motion to give upper and lower bounds for the tail distribution of the process $\sup_{s\le u\le t} X_u$, and apply those results to obtain estimates for different measures of risk. We pay special attention to the Geometric Brownian Motion, the Vasicek and the Cox-Ingersoll-Ross *type models*. It is important to remark that our estimates are sharp, as can be seen in all the corollaries where we combine upper and lower bounds. This property –sharpness of the bounds, that is important by itself, seems relevant for practitioners, who in general obtain bounds for the coefficients of the equations they work with. We can also note that the coefficients that define the processes (σ and b in equation (1) below) can be random, as long as they satisfy certain hypotheses defined in Section 2.

The same type of analysis has already been done for diffusion processes with jumps in [DFM], using other techniques. Those bounds, if restricted to processes without jumps as in the present setting, are not as good as the estimates found here. Additionally, in this paper three important classes of examples are carefully discussed.

The structure of the paper is as follows: In Section 2 we state the notation, the basic assumptions we will use (Hypotheses **(UB)** and **(LB)**), and we describe the main examples we discuss all along the paper. Section 3 has the upper and lower estimates for the tail distribution of $\sup_{s\le u\le t} X_u$ based in the change of time property of the Brownian Motion. Also in that Section are the computations of the estimates for the main examples, including an application to Russian Options. We devote Section 4 to the definition of different kinds of dynamic *VaR* quantiles, and give general upper and lower estimates for them, based on the previous results. Finally, in Section 5, we consider the expected shortfall as well as a second order *VaR* conditioned on the past, and obtain the corresponding estimations.

2. Preliminaries: Hypotheses and Notation

We consider a one-dimensional Brownian Motion W and a diffusion process defined by the following differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt,\ t > 0, \tag{1}$$

which models financial assets such as interest rate. The functions σ and b are measurable, real valued, and so that equation (1) admits a unique solution which is a diffusion process with state space $\mathbb{R}$ (see for example [ReYor] Chapter IX, or [KarShr] Chapter 6).

All along this paper we shall denote by **(UB)** the hypotheses to have upper bounds, and by **(LB)** (correspondingly, to obtain lower bounds) to the following sets of conditions:

Hypotheses (UB):

1. for all $z \in \mathbb{R}$, and $t \geq 0$, $b(t,z)$ is unifomly bounded above $b(t,z) \leq b^*$ by the constant $b^* \geq 0$.
2. for all $z \in \mathbb{R}$, and uniformly on $t \geq 0$, $|\sigma(t,z)| \leq \sqrt{a^*}|z|^\gamma$, where $a^* > 0$, and $\gamma \in [0,1)$ are constants.

Observe that in the case $\gamma = 0$, Hypothesis 2. of **(UB)** means that σ is bounded. The case $\gamma = 1$ is not considered in **(UB)**: it shall be treated separately, for example in what we call the *Geometric Brownian type process* case.

Hypotheses (LB):

1. For all $z \in \mathbb{R}$, $t \geq 0$, $b_* \leq b(t,z)$ with the constant $b_* \leq 0$.
2. For all $z \in \mathbb{R}$, $t \geq 0$, $\sqrt{a_*} \leq \sigma(t,z)$, where $a_* > 0$ is constant.

Note that Hypotheses **(LB)** are just uniform lower bounds on the coefficients.

In a natural way, for all $x \in \mathbb{R}$, we shall denote by P_x the probability associated to X such that

$$P_x(X_0 = x) = 1.$$

Adopting conventional notations we define, for all $0 \leq s < t$:

$$X^\star_{s,t} = \sup_{s \leq u \leq t} X_u,$$

and we put $X^\star_t = X^\star_{0,t}$.

Set, $\forall z \in \mathbb{R}$, $a(t,z) = \sigma^2(t,z)$. Let us fix $q = 1-\alpha$ in $(0,1)$.

Let us denote by $\bar{\Phi}$ the tail of the standard normal distribution function defined on $\mathbb{R}$:

$$\bar{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{u^2}{2}} du.$$

For any real number r, we shall denote by r^+ (resp. r^-) its positive part (resp. negative part) so that $r = r^+ - r^-$.

We are interested in finding upper bounds for some dynamic measures of risk as the quantile type (*VaR*) and expected shortfall (analogous to those defined in [McNeil] for time series), but for the maximum of the process between times s and t.

We will define a quantile for the process $(X^\star_{s,t})$ given that the underlying process is observed at time s and found equal to m. In order to do that we set, for $x, m \in \mathbb{R}$ and $0 < s < t$,

$$\overline{VaR}^{s,t}_{m,\alpha}(X) = \inf\{z \in \mathbb{R},\ P_x(X^\star_{s,t} \leq z \mid X_s = m) \geq q\}.$$

Another definition for *VaR* (as in [TaZh]) is

$$VaR^{s,t}_{m,\alpha}(X) = \inf\{z, P_x(X_t \leq z \mid X_s = m) \geq q\}.$$

One can easily verify that $VaR^{s,t}_{m,\alpha}(X) \leq \overline{VaR}^{s,t}_{m,\alpha}(X)$, so all the upper estimates that follow are also valid for $VaR^{s,t}_{m,\alpha}(X)$.

We shall prove general results under **(UB)** or **(LB)**, and apply them either directly or after a change of variable. This will be done mainly to three classes of examples, which are:

Example 1: *The Geometric Brownian type process.*

In this example, X satisfies the SDE

$$\frac{dX_t}{X_t} = \sigma_t dW_t + b_t dt, \tag{2}$$

with $X_0 = m$, $m > 0$, where σ_t and b_t are real functions defined on $\mathbb{R}$. In the context of the original model (1), $b(t,z) = b_t z$, and $\sigma(t,z) = \sigma_t z$.

Example 2: *The Cox-Ingersoll-Ross (CIR) type model.*

The CIR process X is solution to the following SDE:

$$dX_t = \sqrt{a^* X_t} dW_t + (b^* - c^* X_t)dt, \tag{3}$$

with $X_0 = m$, $m > 0$ and $a^*, b^*, c^* \in \mathbb{R}^+$.

It can be shown (see [IkWa]), that for all $m \in \mathbb{R}^+$ this equation admits a unique $\mathbb{R}^+$–valued solution with $X_0 = m$. This process satisfies **(UB)** with $\gamma = 1/2$:

$$b(t,z) = b(z) = b^* - c^* z^+ \leq b^*, \text{ and } \sigma(t,z) = \sigma(z) = \sqrt{a^*|z|}.$$

Let us remak that in general, this model is used in theory of interest rates.

Example 3: *The Vasicek type model.*

In this model X satisfies the SDE

$$dX_t = \sigma dW_t + (\beta - \mu_t X_t)dt, \tag{4}$$

where $\sigma \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, and μ_t is a positive continuous function.

Let us remark that not all the examples above satisfy Hypotheses **(UB)** or **(LB)** directly.

3. Estimates for the distribution of $X^\star$ and applications

3.1. Estimates for the distribution of the sup

Lemma 3.1. *Let X be a solution of equation (1), and let us assume* **(UB)**. *Then, for $t > 0$, $m \in \mathbb{R}$ and $z \in \mathbb{R}$ we have*

$$P_m(X_t^\star \geq z) \leq 2\bar{\Phi}\left(\frac{(z - m - b^* t)^+}{\sqrt{a^* t}|z|^\gamma}\right).$$

Proof. If $z < m$, the result is clear.

If $z \geq m$, then (wlog) we can assume that

$$\forall t \geq 0,\ \forall y \geq z,\ a(t,y) = a(t,z).$$

We have

$$P_m(X_t^\star \geq z) \leq P_m\left(\sup_{0 \leq s \leq t} \int_0^s \sigma(u, X_u)dW_u \geq z - m - b^* t\right).$$

Let us denote

$$\forall s \geq 0,\ R_s = \int_0^s \sigma(u, X_u)dW_u.$$

By the change of time property, there exists a Brownian Motion $\tilde{B}$ such that

$$\forall s \geq 0,\ R_s = \tilde{B}_{\langle R,R\rangle_s}.$$

This yields

$$\begin{aligned} P_m(X_t^\star \geq z) &\leq P_m\left(\sup_{0 \leq s \leq t} \tilde{B}_{\langle R,R\rangle_s} \geq z - m - b^* t\right) \\ &\leq P_m\left(\sup_{0 \leq u \leq a^* t|z|^{2\gamma}} \tilde{B}_u \geq z - m - b^* t\right) \end{aligned}$$

since for all $s \in [0,t]$, $\langle R, R\rangle_s \leq a^* t|z|^{2\gamma}$.

As

$$P_m\left(\sup_{0\leq u\leq a^*t|z|^{2\gamma}} \tilde{B}_u \geq z-m-b^*t\right) = P_m(|\tilde{B}_{a^*t|z|^{2\gamma}}| \geq z-m-b^*t),$$

we get the result. □

To get a lower bound, we assume conditions **(LB)**, and we obtain the following:

Lemma 3.2. *Let X be a solution of equation (1), and let us assume conditions* (**LB**). *Then, for all $z, m \in \mathbb{R}$ and $t \geq 0$,*

$$2\bar{\Phi}\left(\frac{(z-m-b_*t)^+}{\sqrt{a_*t}}\right) \leq P_m(X_t^\star \geq z).$$

Proof. We have

$$P_m\left(\sup_{s\in[0,t]} \int_0^s \sigma(u, X_u)dW_u \geq z-m-b_*t\right) \leq P_m(X_t^\star \geq z).$$

By making the same change of variable as in the previous proof, with the same notation, and as

$$a_*t \ \leq \int_0^t \sigma^2(u,X_u)du = \langle R,R\rangle_t \text{ a.e.},$$

we deduce, using the Reflection Principle, that

$$\begin{aligned}
2\bar{\Phi}\left(\frac{(z-m-b_*t)^+}{\sqrt{a_*t}}\right) &= P_m(|\tilde{B}_{a_*t}| \geq z-m-b_*t) \\
&= P_m\left(\sup_{s\in[0,a_*t]} \tilde{B}_s \geq z-m-b_*t\right) \\
&\leq P_m\left(\sup_{s\in[0,\langle R,R\rangle_t]} \tilde{B}_s \geq z-m-b_*t\right) \\
&= P_m(X_t^\star \geq z),
\end{aligned}$$

and so the proof is complete. □

Combining both results above, we have

Corollary 3.3. *Let X be a solution of equation (1), and assume both* **(UB)** *and* **(LB)**. *Then, for all $z \geq m$,*

$$2\bar{\Phi}\left(\frac{(z-m-b_*t)^+}{\sqrt{a_*t}}\right) \leq P_m(X_t^\star \geq z) \leq 2\bar{\Phi}\left(\frac{(z-m-b^*t)^+}{\sqrt{a^*t}|z|^\gamma}\right).$$

3.2. Application to the Examples

We keep notations of section 2.

Example 1:*Brownian Geometric type process* In this example, the idea is to apply Itô's formula to the process

$$\forall t \geq 0,\ Y_t = \ln(X_t).$$

Indeed, under P_m,

$$Y_t = \ln m + \int_0^t \sigma_s \, dB_s + \int_0^t \tilde{b}_s \, ds,$$

where

$$\forall s \geq 0, \; \tilde{b}_s = b_s - \frac{\sigma_s^2}{2}.$$

As a consequence of Corollary 3.3, we have

Proposition 3.4. *In* **Example 1***, if one assumes that there exist constants* $0 < a_* \leq a^*$ *and* $b_* \leq b^*$ *such that for all* $t \geq 0$

$$b_* \leq b_t \leq b^* \text{ and } a_* \leq \sigma_t^2 \leq a^*,$$

then for all $z \geq m$ *and* $t \geq 0$

$$2\bar{\Phi}\left(\frac{(\ln z - \ln m + t(b_* - \frac{a^*}{2})^-)^+}{\sqrt{a_* t}}\right) \leq P_m(X_t^\star \geq z) \leq 2\bar{\Phi}\left(\frac{(\ln z - \ln m - t(b^* - \frac{a_*}{2})^+)^+}{\sqrt{a^* t}}\right).$$

American options are good candidates for an application of this result, in particular, the so called Russian option:

Application: Let us consider a Russian option, whose underlying asset X is a Brownian geometric type process satisfying all the Hypotheses of Proposition 3.4. Assume that its maturity is at time $t > 0$, and recall that the pay-off at each $s > 0$ is given by

$$f_s = M_0 \bigvee \sup_{u \leq s} X_u.$$

If we assume that M_0 is fixed, the quantity $P_m(X_t^\star \geq M_0)$ represents the risk for the seller that the option is exercised at a price greater than M_0, and Proposition 3.4 gives an estimate of such risk.

Example 3: *The Vasicek type model*

We introduce the process:

$$Y_t = e^{\int_0^t \mu_s ds} X_t, \quad t \geq 0.$$

One can easily verify that for all $t \geq 0$:

$$Y_t = m + \int_0^t \sigma e^{\int_0^s \mu_r dr} \, dW_s + \int_0^t \beta e^{\int_0^s \mu_r dr} \, ds,$$

this yields

Proposition 3.5. *If* X *denotes the Vasicek type process as introduced in Example 3, for all* $z \geq 0$ *and* m *in* $\mathbb{R}$*, we have:*

$$2\bar{\Phi}\left(\frac{(e^{\int_0^t \mu_s ds} z - m - \beta t)^+}{\sigma\sqrt{t}}\right) \leq P_m(X_t^\star \geq z) \leq 2\bar{\Phi}\left(\frac{(z - m - \beta t e^{\int_0^t \mu_s ds})^+}{\sigma e^{\int_0^t \mu_s ds}\sqrt{t}}\right).$$

Proof. It is an immediate consequence of:

$$P_m(Y_t^* \geq e^{\int_0^t \mu_s ds} z) \leq P_m(X_t^\star \geq z) \leq P_m(Y_t^* \geq z).$$

□

4. Estimates of different kinds of *VaR*

Along this section we assume that X satisfies equation (1) and the coefficients σ and b do not depend on t.

We make the last assumption in order to apply the Markov property, and consider dynamic *VaR*; but all the results in this Section remain valid if $s = 0$, without the extra condition on the coefficients.

4.1. Dynamic VaR

4.1.1. Estimates

Thanks to the estimates of the previous section, we are able to estimate $\overline{VaR}^{s,t}_{m,\alpha}(X)$:

Theorem 4.1. *Let us assume* **(UB)**. *Then, for all* $0 < s < t$,

$$\overline{VaR}^{s,t}_{m,\alpha}(X) \le r,$$

where r *is the unique root on* $[b^*(t-s)+m, +\infty[$ *of the following equation:*

$$z - |z|^\gamma \sqrt{a^*(t-s)}\bar{\Phi}^{-1}(\alpha/2) - m - b^*(t-s) = 0. \tag{5}$$

Proof. Thanks to the Markov property we have

$$P_x(X^\star_{s,t} < z \mid X_s = m) = P_m(X^\star_{t-s} < z),$$

so $P_x(X^\star_{s,t} < z \mid X_s = m) > q$ if and only if $P_m(X^\star_{t-s} \ge z) \le \alpha$, and because of Lemma 3.1 this is implied, if $z \ge m$, by the following inequality:

$$2\bar{\Phi}\left(\frac{(z-m-b^*(t-s))^+}{\sqrt{a^*(t-s)}|z|^\gamma}\right) \le \alpha.$$

Since $\gamma < 1$, the left member of the previous inequality is equal to one for $z \le m + b^*(t-s)$ and goes to zero when $z \to \infty$. Then there exists at least one root of the equation (5). The uniqueness is easy to verify. $\square$

Corollary 4.2. **(i)** *If* $\gamma = 0$, *that is* σ *bounded, there is the following estimate:*

$$\overline{VaR}^{s,t}_{m,\alpha}(X) \le m + b^*(t-s) + \sqrt{a^*(t-s)}\bar{\Phi}^{-1}(\alpha/2).$$

(ii) *If* $\gamma = 1/2$ *and* $m > 0$, *which corresponds to the CIR type model, we have:*

$$\begin{aligned}\overline{VaR}^{s,t}_{m,\alpha}(X) &\le m + b^*(t-s) + \frac{1}{2}a^*(t-s)(\bar{\Phi}^{-1})^2(\alpha/2)\\ &+ \frac{1}{2}\bar{\Phi}^{-1}(\alpha/2)\sqrt{a^*(t-s)\left(a^*(t-s)(\bar{\Phi}^{-1})^2(\alpha/2) + 4(m + b^*(t-s))\right)}.\end{aligned}$$

Proof. One just has to calculate the root r of equation (5) in both cases. $\square$

Theorem 4.3. *Let us assume* (**LB**). *Then for all* $0 \le s < t$,

$$m + b_*(t-s) + \sqrt{a_*(t-s)}\bar{\Phi}^{-1}(\alpha/2) \le \overline{VaR}^{s,t}_{m,\alpha}(X).$$

Proof. It is just a consequence of Lemma 3.2. □

Let us sum up the results we obtained in the case where σ and b are bounded and $s = 0$ (see the comment at the beginning of this section):

Proposition 4.4. *Assume* **(UB)** *with* $\gamma = 0$ *and* **(LB)***, so we consider the process*

$$X_t = m + \int_0^t \sigma(u, X_u)dW_u + \int_0^t b(u, X_u)du, \quad t \geq 0.$$

Then for all $t > 0$,

$$m + b_* t + \sqrt{a_* t}\bar{\Phi}^{-1}(\alpha/2) \leq \overline{VaR}^{0,t}_{m,\alpha}(X) \leq m + b^* t + \sqrt{a^* t}\bar{\Phi}^{-1}(\alpha/2).$$

Remark: This last inequality proves that in this case ($\gamma = 0$), the estimates we got are sharp.

4.1.2. Application to examples

Example 1:*Brownian Geometric type process*

In this case, as previously, we consider the process $Y = \ln X$ and keep the same notation.

We have, for all $0 \leq s < t$:

$$\overline{VaR}^{s,t}_{m,\alpha}(X) = e^{\overline{VaR}^{s,t}_{m,\alpha}(Y)}.$$

This leads to the following estimate, that we state for the case $s = 0$ so that coefficients σ an b may depend on t:

Proposition 4.5. *In Example 1, if one assumes that there exist constants* $0 < a_* \leq a^*$ *and* $b_* \leq b^*$ *such that for all* $t \geq 0$

$$b_* \leq b_t \leq b^* \text{ and } a_* \leq \sigma_t^2 \leq a^*,$$

then for all $t \geq 0$

$$me^{-t(b_* - \frac{a^*}{2})^- + \sqrt{a_* t}\bar{\Phi}(\frac{\alpha}{2})} \leq VaR^{0,t}_{m,\alpha}(X) \leq me^{t(b^* - \frac{a_*}{2})^+ + \sqrt{a^* t}\bar{\Phi}(\frac{\alpha}{2})}.$$

Example 3: *The Vasicek type model*

In this case we consider μ not depending on t.

$$Y_t = e^{\mu t} X_t, \quad t \geq 0.$$

Theorem 4.6. *If* $VaR^{0,t}_{m,\alpha}(e^{-\mu t}Y) \geq 0$ *then, for* $t \geq 0$,

$$e^{-\mu t}\left(m + \beta + \sigma\sqrt{t}\bar{\Phi}^{-1}\left(\frac{\alpha}{2}\right)\right) \leq \overline{VaR}^{0,t}_{x,\alpha}(X) \leq m + \beta e^{\mu t} + e^{\mu t}\sigma\sqrt{t}\bar{\Phi}^{-1}\left(\frac{\alpha}{2}\right).$$

Proof. Following the same argument of Proposition 4.4 we have:

$$VaR^{0,t}_{m,\alpha}(e^{-\mu t}Y) \leq \overline{VaR}^{0,t}_{x,\alpha}(X) \leq VaR^{0,t}_{m,\alpha}(Y),$$

and

$$m + \beta + \sigma\sqrt{t}\bar{\Phi}^{-1}\left(\frac{\alpha}{2}\right) \leq VaR^{0,t}_{m,\alpha}(Y) \leq m + \beta e^{\mu t} + e^{\mu t}\sigma\sqrt{t}\bar{\Phi}^{-1}\left(\frac{\alpha}{2}\right),$$

then we get the desired inequality. □

5. Expected shortfalls and other kinds of VaR

In this Section we assume again that X satisfies equation (1), the coefficients σ and b do not depend on t, and $m > 0$. The assumption on m is natural, and simplifies the calculations since applying the estimates of Lemma 3.1 we drop the absolute value.

5.1. Mean of the excess distribution over the threshold $\overline{VaR}^{s,t}_{m,\alpha}(X)$

We want to measure the expected shortfall of $X^\star_{s,t}$ given that the process X_t exceeds *VaR* between times s and t. In order to do that we find bounds for the excess distribution (see [EKM] and references therein) and find estimates for what we call a "second order" VaR. More precisely, we define for all $0 < s < t$:

$$\overline{VVaR}^{s,t}_{m,\alpha}(X) = \inf\{z \in \mathbb{R},\ P_x(X^\star_{s,t} < z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) \geq q\}.$$

Notice that for $z < \overline{VaR}^{s,t}_{m,\alpha}(X)$, $P_x(X^\star_{s,t} < z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) = 0$.

Lemma 5.1.

$$P_x(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) \leq \frac{P_m(X^\star_{t-s} \geq z)}{\alpha}$$

with equality if $P_m(X^\star_{t-s} = \overline{VaR}^{s,t}_{m,\alpha}(X)) = 0$.

Proof. Thanks to the Markov property, if $z \geq \overline{VaR}^{s,t}_{m,\alpha}(X)$ one has

$$\begin{aligned} P_x(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) &= \frac{P_x(X^\star_{s,t} \geq z \mid X_s = m)}{P_x(X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X) \mid X_s = m)} \\ &= \frac{P_m(X^\star_{t-s} \geq z)}{P_m(X^\star_{t-s} \geq \overline{VaR}^{s,t}_{m,\alpha}(X))} \\ &\leq \frac{P_m(X^\star_{t-s} \geq z)}{\alpha}. \end{aligned}$$

As $P_m(X^\star_{t-s} \geq \overline{VaR}^{s,t}_{m,\alpha}(X)) = \alpha$ if $P_m(X^\star_{t-s} = \overline{VaR}^{s,t}_{m,\alpha}(X)) = 0$, the last assertion of the Lemma is clear. □

So, we have the following

Corollary 5.2. *For all* $0 < s < t$, *we have*

$$\overline{VVaR}^{s,t}_{m,\alpha}(X) \leq \overline{VaR}^{s,t}_{m,\alpha^2}(X),$$

with equality if $P_m(X^\star_{t-s} = \overline{VaR}^{s,t}_{m,\alpha}(X)) = 0$.

Proof.

$$\begin{aligned} \overline{VVaR}^{s,t}_{m,\alpha}(X) &= \inf\{z, P_x(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) \leq \alpha\} \\ &\leq \inf\left\{z, \frac{P_m(X^\star_{t-s} \geq z)}{\alpha} \leq \alpha\right\} \\ &= \inf\{z, P_x(X^\star_{s,t} \geq z \mid X_s = m) \leq \alpha^2\} \\ &= \overline{VaR}^{s,t}_{m,\alpha^2}. \end{aligned}$$

□

We also are able to estimate the mean loss under this conditional probability:

Proposition 5.3. *Let us assume hypotheses* **(UB)**, *then if* $m \leq \overline{VaR}^{s,t}_{m,\alpha}(X)$

$$E_x\big(X^\star_{s,t} \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m\big) = \overline{VaR}^{s,t}_{m,\alpha}(X) + R,$$

where

$$R \leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{(z - m - b^*(t-s))^+}{\sqrt{a^*(t-s)}z^\gamma}\right) dz.$$

Proof. We have

$$\begin{aligned} E_x\big(X^\star_{s,t} \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m\big) &= \overline{VaR}^{s,t}_{m,\alpha}(X) \\ + \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} P_x\big(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m\big)\, dz. \end{aligned}$$

From Lemma 5.1, and Lemma 3.1, we have

$$\begin{aligned} &\int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} P_x\big(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m\big)\, dz \\ &\leq \frac{1}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} P_m\left(X^\star_{t-s} \geq z\right)\, dz \\ &\leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{(z - m - b^*(t-s))^+}{\sqrt{a^*(t-s)}z^\gamma}\right) dz \\ &\leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{z - m - b^*(t-s)}{\sqrt{a^*(t-s)}z^\gamma}\right) dz. \end{aligned}$$

□

Remark: This estimation seems complicated but, from a numerical point of view, it is easy to simulate since we assumed $m > 0$ and then, $\overline{VaR}^{s,t}_{m,\alpha}(X) \geq 0$, integrating by parts we have:

$$\begin{aligned} R \leq K \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} &\left((1-\gamma)z^{-\gamma+1} + \gamma(m + b^*(t-s))z^{-\gamma}\right) e^{-\frac{(z-m-b^*(t-s))^2}{2a^*(t-s)z^{2\gamma}}}\, dz \\ &- \frac{2\overline{VaR}^{s,t}_{m,\alpha}(X)}{\alpha} \bar{\Phi}\left(\frac{\overline{VaR}^{s,t}_{m,\alpha}(X) - m - b^*(t-s)}{\sqrt{a^*(t-s)}\left(\overline{VaR}^{s,t}_{m,\alpha}(X)\right)^\gamma}\right), \end{aligned}$$

where

$$K = \frac{2}{\alpha\sqrt{2\pi a^*(t-s)}}.$$

Moreover, if $\gamma = 0$, we have a more tractable formula:

Corollary 5.4. *Let us assume* **(UB)**, $\gamma = 0$, *then the constant* R *as in Proposition 5.3 satisfies*

$$\begin{aligned} R \leq &\frac{2\sqrt{a^*(t-s)}}{\alpha\sqrt{2\pi}} e^{-\frac{(\overline{VaR}^{s,t}_{m,\alpha}(X) - m - b^*(t-s))^2}{2a^*(t-s)}} \\ &-2\frac{\overline{VaR}^{s,t}_{m,\alpha}(X) - m - b^*(t-s)}{\alpha} \bar{\Phi}\left(\frac{\overline{VaR}^{s,t}_{m,\alpha}(X) - m - b^*(t-s)}{\sqrt{a^*(t-s)}}\right). \end{aligned}$$

Proof. If $\gamma = 0$, we have

$$R \leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{z-m-b^*(t-s)}{\sqrt{a^*(t-s)}}\right) dz$$
$$= \frac{2\sqrt{a^*(t-s)}}{\alpha} \int_{\frac{\overline{VaR}^{s,t}_{m,\alpha}(X)-m-b^*(t-s)}{\sqrt{a^*(t-s)}}}^{+\infty} \bar{\Phi}(u)\, du$$

which yields the result.

□

One can consider the risk at time t as defined by Talay and Zheng, and so define

$$VVaR^{s,t}_{m,\alpha}(X) = \inf\left\{z \in \mathbb{R},\ P_x(X_t < z \mid X_t \geq VaR^{s,t}_{m,\alpha}(X), X_s = m) \geq q\right\}.$$

The same arguments as those we used all along this section yield

Lemma 5.5. *For all* $0 < s < t$, *we have*

$$VVaR^{s,t}_{m,\alpha}(X) \leq VaR^{s,t}_{m,\alpha^2},$$

with equality if $P_m\left(X_{t-s} = VaR^{s,t}_{m,\alpha}(X)\right) = 0$.

Proposition 5.6. *If* $VaR^{s,t}_{m,\alpha}(X) \geq m$:

$$E_x\left(X_t \mid X_t \geq VaR^{s,t}_{m,\alpha}(X), X_s = m\right) \leq VaR^{s,t}_{m,\alpha}(X) + R,$$

where

$$R \leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{(z-m-b^*(t-s))^+}{\sqrt{a^*(t-s)}|z|^\gamma}\right) dz.$$

Example 1:

Consider that X is the Geometric Brownian Motion, i.e. where the coefficients σ and b are constant and do not depend on t, then we have the following estimate:

Proposition 5.7. *If* $\overline{VaR}^{s,t}_{m,\alpha}(X) \geq m$, *then for all* $0 \leq s < t$

$$E_x(X^\star_{s,t} \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) = \overline{VaR}^{s,t}_{m,\alpha}(X) + R$$

where

$$R \leq \frac{2}{\alpha} \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{(\ln z - \ln m - (b-\frac{\sigma^2}{2})^+(t-s))^+}{\sqrt{\sigma^2(t-s)}}\right) dz$$
$$\leq \frac{2}{\alpha}\left\{K\bar{\Phi}\left(\frac{\ln \overline{VaR}^{s,t}_{m,\alpha}(X) - \ln m - (b-\frac{\sigma^2}{2})^+(t-s) - \sigma^2(t-s)}{\sqrt{\sigma^2(t-s)}}\right)\right.$$
$$\left. - \overline{VaR}^{s,t}_{m,\alpha}(X)\bar{\Phi}\left(\frac{\ln \overline{VaR}^{s,t}_{m,\alpha}(X) - \ln m - (b-\frac{\sigma^2}{2})^+(t-s)}{\sqrt{\sigma^2(t-s)}}\right)\right\}$$

where

$$K = e^{-\frac{\sigma^2}{2}(t-s)-(\ln m+(b-\frac{\sigma^2}{2})^+(t-s))}.$$

Proof. As in the previous proofs, we start with

$$E_x(X^\star_{s,t} \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m) = \overline{VaR}^{s,t}_{m,\alpha}(X) + \int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} P_x(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m)\, dz.$$

For all $z \geq \overline{VaR}^{s,t}_{m,\alpha}(X)$ thanks to Proposition 3.4 we have

$$\begin{aligned} &P_x\left(X^\star_{s,t} \geq z \mid X^\star_{s,t} \geq \overline{VaR}^{s,t}_{m,\alpha}(X), X_s = m\right) \\ &\leq \frac{1}{\alpha} P_m(X^\star_{t-s} \geq z) = \frac{1}{\alpha} P_m(Y^\star_{t-s} \geq \ln z) \\ &\leq \frac{2}{\alpha}\bar{\Phi}\left(\frac{(\ln z - \ln m - (b - \frac{\sigma^2}{2})^+(t-s))^+}{\sqrt{\sigma^2(t-s)}}\right). \end{aligned}$$

This yields,

$$\begin{aligned} &\int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} P_x(X^\star_{s,t} \geq z)\, dz \\ &\leq \frac{2}{\alpha}\int_{\overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} \bar{\Phi}\left(\frac{(\ln z - \ln m - (b - \frac{\sigma^2}{2})^+(t-s))^+}{\sqrt{\sigma^2(t-s)}}\right) dz \\ &= \frac{2}{\alpha}\int_{\ln \overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} e^u \bar{\Phi}\left(\frac{u - \ln m - (b - \frac{\sigma^2}{2})^+(t-s)}{\sqrt{\sigma^2(t-s)}}\right) du \\ &= \frac{2}{\alpha}\Bigg\{ - \overline{VaR}^{s,t}_{m,\alpha}(X)\bar{\Phi}\left(\frac{\ln \overline{VaR}^{s,t}_{m,\alpha}(X) - \ln m - (b - \frac{\sigma^2}{2})^+(t-s)}{\sqrt{\sigma^2(t-s)}}\right) \\ &\qquad + \frac{1}{\sqrt{2\pi\sigma^2(t-s)}}\int_{\ln \overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} e^u e^{-\frac{(u - \ln m - (b - \frac{\sigma^2}{2})^+(t-s))^2}{2\sigma^2(t-s)}}\, du\Bigg\}. \end{aligned}$$

One can verify that

$$\begin{aligned} &\frac{1}{\sqrt{2\pi\sigma^2(t-s)}}\int_{\ln \overline{VaR}^{s,t}_{m,\alpha}(X)}^{+\infty} e^u e^{-\frac{(u - \ln m - (b - \frac{\sigma^2}{2})^+(t-s))^2}{2\sigma^2(t-s)}}\, du \\ &= K\bar{\Phi}\left(\frac{\ln \overline{VaR}^{s,t}_{m,\alpha}(X) - \ln m - (b - \frac{\sigma^2}{2})^+(t-s) - \sigma^2(t-s)}{\sqrt{\sigma^2(t-s)}}\right), \end{aligned}$$

where

$$K = e^{-\frac{\sigma^2}{2}(t-s) - (\ln m + (b - \frac{\sigma^2}{2})^+(t-s))};$$

this ends the proof. □

5.2. Second Order VaR conditioned on the past

In this Section we want to measure the VaR of $X^\star_{s,t}$ given that the process X exceeded VaR before time s. This would help to know how risky the future is (up to time $t > s$), given that you already exceeded VaR in the past.

In a natural way we define

$$\widetilde{VaR}^{s,t}_{x,\alpha}(X) = \inf\left\{z \in \mathbb{R},\ P_x\left(X^\star_{s,t} < z \mid X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)\right) \geq q\right\}.$$

Theorem 5.8. *Let us assume* **(UB)**,

$$\widetilde{VaR}^{s,t}_{x,\alpha}(X) \leq r,$$

where r *is the unique root on* $[b^*t + \overline{VaR}^{0,s}_{x,\alpha}(X), +\infty[$ *of the following equation:*

$$z - |z|^\gamma \sqrt{a^* t}\bar{\Phi}^{-1}(\alpha/2) - \overline{VaR}^{0,s}_{x,\alpha}(X) - b^* t = 0. \tag{6}$$

Proof. Let $z > \overline{VaR}^{0,s}_{x,\alpha}(X)$:

$$P_x\left(X^\star_{s,t} \geq z \mid X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)\right) = \frac{P_x\left(X^\star_{s,t} \geq z, X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)\right)}{P_x(X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X))}.$$

We now introduce

$$T = \inf\left\{u > 0,\ X_u = \overline{VaR}^{0,s}_{x,\alpha}(X)\right\},$$

denote by μ the law of T under P_x. Then

$$\begin{aligned}
P_x\left(X^\star_{s,t} \geq z, X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)\right) &= \int_0^s P_x\left(X^\star_{s,t} \geq z \mid T = r\right)\mu(dr) \\
&\leq \int_0^s P_x\left(X^\star_{r,t} \geq z \mid T = r\right)\mu(dr) \\
&= \int_0^s P_{\overline{VaR}^{0,s}_{x,\alpha}(X)}\left(X^\star_{t-r} \geq z\right)\mu(dr) \\
&\leq \int_0^s 2\bar{\Phi}\left(\frac{(z - \overline{VaR}^{0,s}_{x,\alpha}(X) - b^*(t-r))^+}{\sqrt{a^*(t-r)}|z|^\gamma}\right)\mu(dr) \\
&\leq 2\bar{\Phi}\left(\frac{(z - \overline{VaR}^{0,s}_{x,\alpha}(X) - b^* t)^+}{\sqrt{a^* t}|z|^\gamma}\right)P_x(T \leq s) \\
&= 2\bar{\Phi}\left(\frac{(z - \overline{VaR}^{0,s}_{x,\alpha}(X) - b^* t)^+}{\sqrt{a^* t}|z|^\gamma}\right)P_x(X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)).
\end{aligned}$$

So,

$$P_x\left(X^\star_{s,t} \geq z \mid X^\star_s \geq \overline{VaR}^{0,s}_{x,\alpha}(X)\right) \leq 2\bar{\Phi}\left(\frac{(z - \overline{VaR}^{0,s}_{x,\alpha}(X) - b^* t)^+}{\sqrt{a^* t}|z|^\gamma}\right),$$

and we conclude as in Theorem 4.1. ☐

Remark: In other words, the bound we obtain is the same as the one we got for $\overline{VaR}^{0,t}_{\overline{VaR}^{0,s}_{x,\alpha}(X),\alpha}(X)$.

As previously, for $\gamma = 0$ or $\gamma = 1/2$ we are able to calculate this bound and this yields:

Corollary 5.9. **(i)** *If* $\gamma = 0$, *that is* σ *bounded, there is the following estimate:*

$$\widetilde{VaR}^{s,t}_{x,\alpha}(X) \leq \overline{VaR}^{0,s}_{x,\alpha}(X) + b^* t + \sqrt{a^* t}\bar{\Phi}^{-1}(\alpha/2).$$

(ii) *If* $\gamma = 1/2$ *we have:*

$$\begin{aligned}
\widetilde{VaR}^{s,t}_{x,\alpha}(X) &\leq \overline{VaR}^{0,s}_{x,\alpha}(X) + b^* t + \frac{1}{2}a^* t(\bar{\Phi}^{-1})^2(\alpha/2) \\
&\quad + \frac{1}{2}\bar{\Phi}^{-1}(\alpha/2)\sqrt{a^* t\left(a^* t(\bar{\Phi}^{-1})^2(\alpha/2) + 4(\overline{VaR}^{0,s}_{x,\alpha}(X) + b^* t\right)}.
\end{aligned}$$

References

[BaTa1] Bally, V. and Talay, D. (1996). The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function. *Probab. Theory Related Fields* **104** (1) 43–60.

[BaTa2] Bally, V. and Talay, D. (1996). The law of the Euler scheme for stochastic differential equations. II. Convergence rate of the density. *Monte Carlo Methods Appl.* **2** (2) 93–128.

[BoKl] Borkovec, M. and Klüppelberg, C. (1998). Extremal behavior of diffusion models in finance. *Extremes* **1** (1) 47–80.

[Dav] Davis, R. (1982). Maximum and minimum of one-dimensional diffusions. *Stoch. Proc. Appl.* **13** 1–9.

[DFM] Denis, L., Fernández, B. and Meda, A. (2006). Estimation of Value at Risk for diffusion processes with jumps and their ruin probabilities. Preprint.

[EKM] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1999). *Modelling Extremal Events for Insurance and Finance.* Springer Verlag, Berlin-Heidelberg-New York.

[IkWa] Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Tokyo.

[KarShr] Karatzas, I. and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus.* Springer Verlag, Berlin, Heidelberg, New York.

[McNeil] McNeil, A. (2000). Extreme value theory for risk managers. In *Extremes and Integrated Risk Management* (P. Embrechts, ed.) 3–18. Risk Books, Risk Waters Group, London.

[ReYor] Revuz, D. and Yor, M. (1994). *Continuous Martingales and Brownian Motion.* Springer Verlag, Berlin, Heidelberg, New York.

[TaZh] Talay, D. and Zheng, Z. (2003). *Quantiles of the Euler scheme for diffusion processes and financial applications.* Conference on Applications of Malliavin Calculus in Finance (Rocquencourt, 2001). *Math. Finance* **13** (1) 187–199.