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Maximum Queue Length of a Fluid Model with an Aggregated Fractional Brownian Input

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Abstract: A fractional Brownian queueing model, that is, a fluid queue with an input of a fractional Brownian motion, has been applied in network modeling since the self-similarity and long-range dependence were observed in Internet traffic. In this paper, a fluid queue with an aggregated fractional Brownian input, which is a generalization of a fractional Brownian queueing model, is considered and the maximum queue length over a time interval $[0, t]$ is studied. The impact of an aggregated fractional Brownian input on the queue length process is analyzed and the main results on the maximum queue length are compared with some related known results in the literature.

1. Introduction

In the 1990s, researchers observed the properties of self-similarity and long-range dependence in Internet traffic. Since then, various models have been proposed to model these complex features. In [17], Norros proposed a fluid queueing model with an input of a fractional Brownian motion. Different from the traditional queueing models, a fluid model has an input process with a continuous sample path. Since a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ has the properties of self-similarity and long-range dependence, e.g., [7], [15], it is used to capture the complex features of Internet traffic.

A fractional Brownian queueing model is a useful model for analyzing the impact of self-similarity and long-range dependence on the queueing performance, however there are some generic shortcomings in this model. Firstly, since the input process is Gaussian, negative increments, which are not meaningful for a queueing model, can be observed at small time scales. Secondly, the actual Internet traffic is regulated by TCP/IP protocol, which is a closed-loop congestion control mechanism. A fractional Brownian queueing model, which is open-loop as are many queueing models, cannot capture the dynamics of Internet traffic over small time scales, i.e., less than the typical round trip packet time. Although the model has some shortcomings, it can be used to approximate other aspects of Internet traffic under certain circumstances. It has been empirically demonstrated in [8] that a fractional Brownian queueing model is appropriate for the backbone traffic, in which millions of independent flows are highly aggregated, traffic control on a single flow would not dominate the whole traffic and the time scale is larger than the typical round trip time. In recent network measurements [12], it was observed that for small time scales, less

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than a millisecond, the traffic in the Internet backbone is memoryless or of short memory; while for larger time scales, in milliseconds, the long-range dependence characterizes the backbone traffic. From a practical point of view, see [18], [21], a fractional Brownian queueing model is an approximation of Internet traffic and can produce meaningful results for queueing performance, such as inter-congestion event times and congestion durations, which are in a time scale larger than the typical round trip time.

In practice, it has been observed that the Hurst parameter estimated in network does not remain constant. For this reason, besides a fractional Brownian motion, other Gaussian processes have been proposed to model network traffic, such as an aggregation of independent fractional Brownian motions, [5], [19], [22, p 335] and an integrated Ornstein-Uhlenbeck process [3], [4], [5]. Here a queue with an aggregated fractional Brownian input is studied and the maximum queue length over a time interval $[0, t]$ is analyzed. For a queue with a single fractional Brownian input, i.e., a fractional Brownian model, the maximum queue length was considered in [23]. In this paper, the results of [23] are extended, and the impact of an aggregated fractional Brownian input on the queueing behavior is analyzed.

The structure of the paper is as follows: In Section 2, some preliminaries on a queueing model with an aggregated fractional Brownian input are given, the maximum queue length is defined and some results in the literature are reviewed. In Section 3, the main results are presented and compared with some known related results. Section 4 is devoted to the proofs of the main results, Theorems 3.1 and 3.2.

2. Preliminary

The definition of a fractional Brownian motion is as follows.

Definition 2.1. *A standard fractional Brownian motion with Hurst parameter* $H \in (0,1)$, $\{B^H(t), t \in [0,\infty)\}$ *on the complete probability space* $(\Omega, \mathscr{F}, P)$ *is a real-valued Gaussian process with continuous sample paths such that for* $s, t \in [0,\infty)$, $E[B^H(t)] = 0$ *and* $E[B^H(s)B^H(t)] = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}]$.

More properties of a fractional Brownian motion can be found in [7], [15] and the references therein. The queueing model is a single fluid queue with an infinite buffer size and a fixed service rate. Let $A(t) = mt + Y(t)$ be the cumulated arrivals to the queue up to time t, where m is a mean input rate and $Y = \{Y(t), t \geq 0\}$ is a continuous Gaussian process with stationary increments. Let μ denote a service rate and $c = \mu - m$ be the surplus rate. For the stability of the queue, it is assumed that $c > 0$.

In the case that Y is a fractional Brownian motion, this model is called a fractional Brownian queueing model, which is proposed by Norros [17] to capture the self-similarity of Internet traffic. Here a more general Gaussian process is considered initially. It is assumed that the input process Y is an aggregation of independent standard fractional Brownian motions, that is,

$$Y(t) = \sum_{i=1}^{N} \sigma_i B^{H_i}(t) \tag{1}$$

where for $i = 1, ..., N$, σ_i's are real-valued coefficients and $\{B^{H_i}(t), t \geq 0\}$ are independent fractional Brownian motions with Hurst parameters $H_i \in (0,1)$. Let

$\mathcal{J} \subset \{1, ..., N\}$ be the set of all indices j such that $H_j = \max_{1 \le i \le N} \{H_i\}$ and

$$\sigma = \sqrt{\sum_{i \in \mathcal{J}} \sigma_i^2}. \tag{2}$$

Note that for $N = 1$, the model is a fractional Brownian queueing model. Let $G = \sum_{i=1}^{N} \sigma_i^2$ and $\gamma = 2 \min_{1 \le i \le N} \{H_i\}$, then for $t \in [0, 1]$, $E\left[Y^2(t)\right] \le G t^{\gamma}$. Based on [14, Lemma 12.2.1], there exists a constant $C_{G,\gamma} > 0$, which only depends on G and γ, such that for all x,

$$P\left(\sup_{0 \le s \le 1} Y(s) > x\right) \le 4 \exp\left(-C_{G,\gamma} x^2\right). \tag{3}$$

Let $Q = \{Q(t), t \ge 0\}$ denote the queue length process. In the literature, the process Q is also called a workload process or a storage process. Suppose $Q(0) = 0$, then for each time $t \ge 0$, the queue length $Q(t)$ can be written as

$$Q(t) = Y(t) - ct + \sup_{0 \le s \le t} \left(-Y(s) + cs\right). \tag{4}$$

In general, for $0 \le s \le t$, $Q(t)$ can be written in terms of $Q(s)$ as

$$Q(t) = Y(t) - ct + \max\left\{\sup_{s \le r \le t} \left(-Y(r) + cr\right), Q(s) - \left(Y(s) - cs\right)\right\}. \tag{5}$$

Let $Q(\infty) \stackrel{d}{=} \lim_{t \to \infty} Q(t)$ be the steady state queue length where the limit denotes convergence in law. Let $M(t)$ denote the maximum of the queue length in $[0, t]$, that is,

$$M(t) = \max_{0 \le s \le t} Q(s). \tag{6}$$

The properties of $M(t)$ have been analyzed for different queueing models, see e.g. [1], [2], [11], and were applied in network systems to estimate certain traffic parameters. In the context of renewal processes, that is, the queue length process is renewal, some asymptotic properties of the maximum queue length were analyzed in [9].

To discuss asymptotic properties of $M(t)$, it is convenient to introduce a stationary version of the queue length process. It follows from [13], also see [9], [23], that one can construct a probability space supporting both the process $\{Y(t), t \ge 0\}$ and a stationary process $Q^* = \{Q^*(t), t \ge 0\}$ such that

(i) $Q^*(t) \stackrel{d}{=} Q(\infty)$ for all $t \ge 0$,

(ii) For $t \ge 0$,

$$Q^*(t) = Y(t) - ct + \max\left\{Q^*(0), \sup_{0 \le s \le t} \left(-Y(s) + cs\right)\right\}. \tag{7}$$

Remark 2.1. *Recall that $Q(t) = Y(t) - ct + \sup_{0 \le s \le t} (-Y(s) + cs)$, so it follows from (7) that for all $t \ge 0$, $Q^*(t) \ge Q(t)$.*

Let $M^*(t)$ be the maximum of the queue length process Q^* over an interval $[0, t]$, that is,

$$M^*(t) = \max_{0 \le s \le t} Q^*(s). \tag{8}$$

The following proposition is used in the proofs of the main results. It shows that the logarithmic overflow probability, i.e. $\log P\left(Q(\infty) > b\right)$, is asymptotically determined by the largest Hurst parameter H.

Proposition 2.1. *Let $Y(t) = \sum_{i=1}^{N} \sigma_i B^{H_i}(t)$ be defined as in (1), $H = \max_{1 \le i \le N} H_i$ and σ be as in (2). Let $Q(\infty)$ be the steady state queue length, then*

$$\lim_{b \to \infty} \frac{\log P\left(Q(\infty) > b\right)}{b^{2-2H}} = -\theta, \tag{9}$$

where

$$\theta = \frac{c^{2H}}{2\sigma^2 H^{2H}(1-H)^{2-2H}}. \tag{10}$$

The proof of this proposition is given in Section 4.1. Recently, the asymptotic overflow probability, i.e., $\lim_{b \to \infty} P(Q(\infty) > b)$, was obtained using a double sum method in [5].

The main results on the maximum queue length of a queue with an input of an aggregation of fractional Brownian motions are given in Theorem 3.1 and 3.2. Before the main results are presented, some results in the literature are reviewed. The maximum queue length of a fractional Brownian queueing model was discussed in [9], [23]. For $H = 1/2$, that is, the input traffic is a Brownian motion, the property of $M(t)$ was discussed in [9] using renewal theory. With some different approaches, the maximum queue length of a fractional Brownian queueing model with $H \in (1/2, 1)$ is analyzed in [23]. The results from [9] and [23] are summarized as follows. For brevity, let

$$\beta = \frac{1}{2-2H}. \tag{11}$$

Note that $\beta > 1/2$ since $H \in (0,1)$.

Theorem 2.1. *Let $Y(t) = \sigma B^H(t)$ where σ is a real-valued coefficient and $\{B^H(t), t \ge 0\}$ is a fractional Brownian motion with Hurst parameter $H \in [1/2, 1)$. Let $M(t)$ be defined in (6). Then,*
(i)

$$\lim_{t \to \infty} \frac{M(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta} \tag{12}$$

in L^p for each $p \in [1, \infty)$ where θ and β are given in (10) and (11), respectively.
(ii) For $H = 1/2$, the convergence of (12) holds also almost surely.

The above theorem shows that for a fractional Brownian queueing model, the maximum queue length $M(t)$ grows like $\left(\theta^{-1} \log t\right)^{\beta}$ for a large t.

3. Main Results

The following two theorems are the main results of this paper.

Theorem 3.1. *Let $Y(t) = \sum_{i=1}^{N} \sigma_i B^{H_i}(t)$ be defined as in (1), $H = \max_{1 \le i \le N} H_i$ and σ be defined in (2). Let $M(t)$ be defined in (6), then*

$$\lim_{t \to \infty} \frac{M(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta} \tag{13}$$

in L^p for each $p \in [1, \infty)$ where θ is given in (10) and β is given in (11).

The proof of this theorem is given in Section 4.1. According to Theorem 3.1, for a queue with an aggregated fractional Brownian input, the asymptotic behavior of $M(t)$ only depends on the largest Hurst parameter. For example, suppose that $Y(t) = 0.99B^{H_1}(t) + 0.01B^{H_2}(t)$ where $\{B^{H_1}(t), t \geq 0\}$ and $\{B^{H_2}(t), t \geq 0\}$ are independent fractional Brownian motions with $H_1 = 0.55$ and $H_2 = 0.95$, respectively. Since the coefficient of B^{H_1} is relatively large, when the transient behavior is considered, the component of B^{H_1} dominates the queueing performance, that is, the component of B^{H_2} can be ignored. However, when the asymptotic behavior is discussed, by Theorem 3.1, the maximum queue length will be dominated by the component of B^{H_2}. Therefore, even though the coefficient of B^{H_2} is relatively small, when large time periods are considered, the component of B^{H_2} is not negligible. In this example, since the coefficient of B^{H_2} is small, the convergence of the maximum queue length is slow and may be difficult to observe from simulations.

It can be observed that the first part of Theorem 2.1 is a special case of Theorem 3.1. The result of Theorem 3.1 can be further extended to a general Gaussian queueing model, that is, where Y is a general Gaussian process. Under some mild assumptions, it can be shown that asymptotically a suitably normalized maximum queue length is determined by a suitable function of the asymptotic variance of the input Y. The assumptions on Y are satisfied for most Gaussian processes that are applied to model network traffic in the literature, such as a fractional Brownian motion and an integrated Ornstein-Uhlenbeck process. The general result and the assumptions will be presented in a future paper. In the following, the maximum queue length of a fractional Brownian queueing model is revisited and a stronger result is obtained.

Theorem 3.2. *Let $Y(t) = \sigma B^H(t)$ where σ is a real-valued coefficient and $\{B^H(t), t \geq 0\}$ is a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$. Let $M(t)$ be defined in (6), then*

$$\lim_{t\to\infty} \frac{M(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta \quad \text{a.s.} \tag{14}$$

and in L^p for each $p \in [1,\infty)$ where θ is given in (10) and β is given in (11).

The proof of this theorem is given in Section 4.2. Theorem 3.2 extends the result of Theorem 2.1 in two directions: (i) the convergence result of (12) holds almost surely for any $H \in (0,1)$; (ii) the L^p convergence is true for $H \in (0,1/2)$. As discussed in [9], [23], [24], the maximum queue length $M(t)$ can be applied to estimate the overflow probability $P(Q(\infty) > b)$, which is important for the admission control in network systems. From Proposition 2.1, it is known that asymptotically the logarithmic overflow probability is essentially determined by H and θ. Assume that the Hurst parameter H of the input traffic is known, following Theorem 3.2, the value θ can be strongly consistently estimated by using the maximum queue length $M(t)$, i.e., $\lim_{t\to\infty} \frac{M(t)^{2-2H}}{\log t} = \frac{1}{\theta}$ a.s.

4. Proofs of the main results

The proofs of Theorems 3.1 and 3.2 are given in Section 4.1 and 4.2, respectively.

4.1. A queue with an aggregated fractional Brownian input

Proof of Proposition 2.1. It is sufficient to show that Y satisfies Hypotheses 2.1 and 2.3 in [6]. Apply the same notation as [6], let $v(t) = t^{2-2H}$ and $a(t) = t$. Note that $\sum_{i=1}^N \sigma_i^2 t^{2H_i} \sim \sigma^2 t^{2H}$, then

$$\begin{aligned}\lambda(\xi) &= \lim_{t\to\infty} \frac{\log E\left[\exp\left(\xi \frac{v(t)(Y(t)-ct)}{a(t)}\right)\right]}{v(t)} \\ &= \lim_{t\to\infty} \frac{-\xi c t^{2-2H} + \frac{1}{2}\xi^2 t^{2-4H}\sum_{i=1}^N \sigma_i^2 t^{2H_i}}{t^{2-2H}} \\ &= \frac{1}{2}\xi^2\sigma^2 - c\xi.\end{aligned}$$

So (i) and (ii) of Hypothesis 2.1 are satisfied. Let $h(t) = t^{2-2H}$ and $a^{-1}(t) = \sup\{s \in [0,\infty); a(s) \le t\}$. It can be verified that for $\xi > 0$, $g(\xi) = \lim_{t\to\infty} \frac{v\left(a^{-1}(t)/\xi\right)}{h(t)} = \xi^{2H-2}$, which satisfies Hypothesis 2.1(iii). Let $W_t = Y(t) - ct$. For $n \in \mathbb{Z}$, let $W_n^* = \sup_{0\le r<1} W_{n+r}$. Recall that Y has stationary increments and it is assumed that the components of Y, i.e., the fractional Brownian motions, are independent, then for $\xi > 0$,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H}\left(W_n^* - W_n\right)\right)\right]}{n^{2-2H}} \\ &= \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H}\sup_{0\le r\le 1}\left(Y(n+r) - c(n+r)\right) - Y(n) + cn\right)\right]}{n^{2-2H}} \\ &\le \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H}\sup_{0\le r\le 1} Y(r)\right)\right]}{n^{2-2H}} \\ &\le \sum_{i=1}^N \limsup_{n\to\infty} \frac{\log E\left[\exp\left(\xi n^{1-2H}\sup_{0\le r\le 1} B^{H_i}(r)\right)\right]}{n^{2-2H}}.\end{aligned}$$

From [16], it is obtained that for $i = 1, ..., N$,

$$\begin{aligned}E\left[\exp\left(\xi n^{1-2H}\sup_{0\le r\le 1} B^{H_i}(r)\right)\right] &= \int_0^\infty P\left(\sup_{0\le r\le 1} B^{H_i}(r) \ge \frac{\log x}{\xi n^{1-2H}}\right)dx \\ &\le 2\int_0^\infty P\left(B^{H_i}(1) \ge \frac{\log x}{\xi n^{1-2H}}\right)dx.\end{aligned}$$

Simple calculations lead to $\frac{\log E\left[\exp\left(\xi n^{1-2H}\sup_{0\le r\le 1} B^{H_i}(r)\right)\right]}{n^{2-2H}} \to 0$ as $n\to\infty$, which implies $\frac{\log E\left[\exp\left(\xi n^{1-2H}(W_n^* - W_n)\right)\right]}{n^{2-2H}} \to 0$ as well. Therefore Hypothesis 2.3 is satisfied and the proposition follows [6, Corollary 2.3]. □

According to Proposition 2.1, there exists a constant K_0, such that,

$$K_0 = \inf\left\{u : \frac{\log P\left(Q^*(0) > u\right)}{u^{2-2H}} \le -\frac{\theta}{2}\right\}. \tag{15}$$

To prove Theorem 3.1, the following two technical lemmas are needed.

Lemma 4.1. *Let β be defined in (11). Then for all $t \ge e$, $p \in (1,\infty)$ and $K = \max\{K_0, (4/\theta)^\beta\}$ where K_0 is defined in (15),*

$$\int_{3K}^\infty t y^{p-1} P\left(Q^*(0) > \frac{y}{3}(\log t)^\beta\right)dy < \infty. \tag{16}$$

Proof. Rewrite $P\left(Q^*(0) > \frac{y}{3}(\log t)^\beta\right)$ as

$$P\left(Q^*(0) > \frac{y}{3}(\log t)^\beta\right) = \exp\left(\left(\frac{y}{3}\right)^{1/\beta} \log t \; \frac{\log P\left(Q^*(0) > (y/3)(\log t)^\beta\right)}{(y/3)^{1/\beta}\log t}\right).$$

Since $y/3 \geq K \geq K_0$ and $\log t \geq 1$, from (15), it follows that $P\left(Q^*(0) > \frac{y}{3}(\log t)^\beta\right) \leq \exp\left(-\frac{\theta}{2}\left(\frac{y}{3}\right)^{1/\beta}\log t\right)$. Then

$$\int_{3K}^\infty ty^{p-1}P\left(Q^*(0) > \frac{y}{3}(\log t)^\beta\right)dy \leq \int_{3K}^\infty y^{p-1}\exp\left(-\frac{\theta}{2}\left(\frac{y}{3}\right)^{\frac{1}{\beta}}\log t + \log t\right)dy.$$

From $K \geq (4/\theta)^\beta$ and $y \geq 3K$, it follows that $-\frac{\theta}{2}\left(\frac{y}{3}\right)^{1/\beta} + 1 \leq -\frac{\theta}{4}\left(\frac{y}{3}\right)^{1/\beta}$. By substitution, it follows that

$$\int_{3K}^\infty y^{p-1}\exp\left(-\frac{\theta}{2}\left(\frac{y}{3}\right)^{\frac{1}{\beta}}\log t + \log t\right)dy \leq \frac{3^p \cdot 4^{\beta p}\beta}{(\theta \log t)^{\beta p}}\Gamma(\beta p) < \infty.$$

□

Lemma 4.2. *Let β be the constant defined in (11). Then for $t \geq e$, $p \in (1,\infty)$ and $K = \max\left\{K_0, \sqrt{8/C_{G,\gamma}}\right\}$ where $C_{G,\gamma}$ is the constant in (3),*

$$\int_{3K}^\infty ty^{p-1}P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{6}(\log t)^\beta\right)dy < \infty.$$

Proof. From (3), it can be obtained that

$$P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{6}(\log t)^\beta\right) \leq 4\exp\left(-\frac{C_{G,\gamma}y^2}{36}(\log t)^{2\beta}\right).$$

So

$$\begin{aligned}&\int_{3K}^\infty ty^{p-1}P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{6}(\log t)^\beta\right)dy\\ &\leq \int_{3K}^\infty 4y^{p-1}\exp\left(-\frac{C_{G,\gamma}y^2}{36}(\log t)^{2\beta} + \log t\right)dy. \qquad (17)\end{aligned}$$

Since $\beta > 1/2$ and $t \geq e$, from (17), it follows that

$$\begin{aligned}&\int_{3K}^\infty ty^{p-1}P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{y}{6}(\log t)^\beta\right)dy\\ &\leq \int_{3K}^\infty 4y^{p-1}\exp\left(-(\log t)\left(\frac{C_{G,\gamma}y^2}{36} - 1\right)\right)dy.\end{aligned}$$

Since $y \geq 3K \geq 3\sqrt{8/C_{G,\gamma}}$, then $\frac{C_{G,\gamma}y^2}{36} - 1 \geq \frac{C_{G,\gamma}y^2}{72}$. Let $z = y^2$, by substitution, it can be verified that

$$\int_{3K}^\infty 4y^{p-1}\exp\left(-\frac{C_{G,\gamma}y^2}{72}\log t\right)dy \leq 2\left(\frac{C_{G,\gamma}(\log t)}{72}\right)^{-\frac{p}{2}}\Gamma\left(\frac{p}{2}\right) < \infty.$$

□

Proof of Theorem 3.1. The proof mainly follows the arguments of [23] in which the self-similarity of a fractional Brownian motion is used implicitly. Here the input process $Y(t)$, which is an aggregation of independent fractional Brownian motions, is not self-similar, so the Slepian inequality, [20, Theorem C.1], is applied to solve the problem. It is first shown that $\lim_{t\to\infty}\frac{M^*(t)}{(\log t)^\beta}=\left(\frac{1}{\theta}\right)^\beta$ in L^p for each $p\in[1,\infty)$, then the result can be extended to $M(t)$ naturally. The proof consists of three steps. The following expressions, (18) and (19), are proved in **Step I** and **II**, respectively. For a fixed $\delta\in(0,1)$,

$$\lim_{t\to\infty} P\left(M^*(t)\geq\left(\frac{1-\delta}{\theta}\log t\right)^\beta\right)=1, \tag{18}$$

$$\lim_{t\to\infty} P\left(M^*(t)\geq\left(\frac{1+\delta}{\theta}\log t\right)^\beta\right)=0. \tag{19}$$

From (18) and (19), it follows that $\lim_{t\to\infty}\frac{M^*(t)}{(\log t)^\beta}=\left(\frac{1}{\theta}\right)^\beta$ in probability. In **Step III**, the uniform integrability of the random variables $\left(\frac{M^*(t)}{(\log t)^\beta}\right)^p$ is proved for $t\geq e$ and $p\in[1,\infty)$.

Step I Let $\delta\in(0,1)$ be fixed. For brevity, let $\alpha(t)=\left(\frac{1-\delta}{\theta}\log t\right)^\beta$. Fix $\Delta\in(0,t)$, from the definition of Q^*, it follows that

$$\begin{aligned} Q^*(t) &\geq Y(t)-ct-\inf_{0\leq s\leq t}(Y(s)-cs)\\ &\geq Y(t)-ct-Y(t-\Delta)+c(t-\Delta)\\ &=Y(t)-Y(t-\Delta)-c\Delta. \end{aligned} \tag{20}$$

Consequently,

$$\begin{aligned} P\left(M^*(t)\geq\alpha(t)\right) &= P\left(\sup_{0\leq s\leq t}Q^*(s)\geq\alpha(t)\right)\\ &\geq P\left(\sup_{1\leq k\leq\lfloor t/\Delta\rfloor}Y(k\Delta)-Y\left(k\Delta-\Delta\right)\geq\alpha(t)+c\Delta\right). \end{aligned} \tag{21}$$

Let $v^2(t)=E\left[Y^2(t)\right]=\sum_{i=1}^N\sigma_i^2t^{2H_i}$ be the variance function of the process Y. For $j=1,2,...$, let

$$Z_j^\Delta=\frac{Y\left(j\Delta\right)-Y\left(j\Delta-\Delta\right)}{v(\Delta)}. \tag{22}$$

Since Y has stationary increments, $\left\{Z_j^\Delta\right\}$ is a sequence of stationary standard normal random variables. From (21), it can be obtained that

$$P\left(M^*(t)\geq\alpha(t)\right)\geq P\left(\sup_{1\leq j\leq\lfloor t/\Delta\rfloor}Z_j^\Delta\geq\frac{\alpha(t)+c\Delta}{v(\Delta)}\right). \tag{23}$$

Choose $\varepsilon\in(0,\delta]$ and let Δ be dependent on t such that

$$\Delta_t=\left(\frac{2\sigma^2(1-\varepsilon)}{c^2}H^2\log t\right)^\beta, \tag{24}$$

where H is the largest Hurst parameter and σ is as in (2). Then (23) can be written as

$$P\left(M^*(t) \geq \alpha(t)\right) \geq P\left(\sup_{1\leq j\leq \lfloor t/\Delta_t \rfloor} Z_j^{\Delta_t} \geq \frac{\alpha(t)+c\Delta_t}{v(\Delta_t)}\right). \tag{25}$$

Consider the covariance of $\{Z_j^{\Delta_t}\}$, for all $t \geq 0$, $j = 1, 2, ...$ and $k = 0, 1, ...$

$$cov(Z_j^{\Delta_t}, Z_{j+k}^{\Delta_t}) = \frac{\sum_{i=1}^N \sigma_i^2 \Delta_t^{2H_i}\left[(k+1)^{2H_i} - 2k^{2H_i} + (k-1)^{2H_i}\right]}{2\sum_{i=1}^N \sigma_i^2 \Delta_t^{2H_i}}$$

For $H \geq 1/2$, since for $1 \leq i \leq N$, $(k+1)^{2H_i} - 2k^{2H_i} + (k-1)^{2H_i} \leq (k+1)^{2H} - 2k^{2H} + (k-1)^{2H}$, then it is obtained that $cov(Z_j^{\Delta_t}, Z_{j+k}^{\Delta_t}) \leq f(k)$ where $f(k) = \frac{1}{2}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right]$. For $H < 1/2$, let $f(0) = 1$ and $f(k) = 0$ for $k = 1, 2, 3, ...$, then it can be verified that $cov(Z_j^{\Delta_t}, Z_{j+k}^{\Delta_t}) \leq f(k)$. Let $\{\tilde{Z}_j, j = 1, 2, ...\}$ be a sequence of stationary standard normal random variables such that the covariance of $\tilde{Z}_j$ is determined by $f(k)$. By the Slepian inequality, e.g. [20, Theorem C.1], it follows from (25) that

$$P\left(\sup_{1\leq j\leq \lfloor \frac{t}{\Delta_t} \rfloor} Z_j^{\Delta_t} \geq \frac{\alpha(t)+c\Delta_t}{v(\Delta_t)}\right) \geq P\left(\sup_{1\leq j\leq \lfloor \frac{t}{\Delta_t} \rfloor} \tilde{Z}_j \geq \frac{\alpha(t)+c\Delta_t}{v(\Delta_t)}\right). \tag{26}$$

From the definitions of $v^2(\cdot)$ and Δ_t, it can be verified that

$$\frac{\alpha(t)+c\Delta_t}{v(\Delta_t)} = \frac{\alpha(t)+c\Delta_t}{\sqrt{\sum_{i=1}^N \sigma_i^2 \Delta_t^{2H_i}}} \leq \frac{\alpha(t)+c\Delta_t}{\sigma\Delta_t^H} \leq \sqrt{2(1-\varepsilon)\log t}.$$

Let $n = \lfloor \frac{t}{\Delta_t} \rfloor$ and $t_n = \left\{t : \frac{t}{\Delta_t} = n\right\}$. Note that $\lfloor \frac{t}{\Delta_t} \rfloor = n$ if and only if $t_n \leq t < t_{n+1}$. Then for sufficiently large $t \in [t_n, t_{n+1})$, the following inequalities are obtained

$$\frac{\alpha(t)+c\Delta_t}{v(\Delta_t)} \leq \sqrt{2\left(1-\varepsilon\right)\log t} \leq \sqrt{2\left(1-\varepsilon\right)\log t_{n+1}}.$$

Let u_n be defined as

$$u_n = \sqrt{2\left(1-\varepsilon\right)\log t_{n+1}}. \tag{27}$$

So from (26) and (27), it follows that

$$P\left(\sup_{1\leq j\leq \lfloor \frac{t}{\Delta_t} \rfloor} \tilde{Z}_j \geq \frac{\alpha(t)+c\Delta_t}{v(\Delta_t)}\right) \geq P\left(\sup_{1\leq j\leq n} \tilde{Z}_j \geq u_n\right). \tag{28}$$

Following [14, Theorem 4.3.3], it is sufficient to prove that $n(1-\Phi(u_n)) \to \infty$ as $n \to \infty$. Recall that for $x \geq 0$, $1-\Phi(x) \geq \frac{x}{\sqrt{2\pi}(1+x^2)} e^{-x^2/2}$, so

$$n(1-\Phi(u_n)) \geq n\frac{u_n}{\sqrt{2\pi}(1+u_n^2)} \exp\left(-\frac{u_n^2}{2}\right).$$

From (27), it follows that $e^{-u_n^2/2} = t_{n+1}^{-1+\varepsilon}$. Since $\lim_{n\to\infty} u_n \to \infty$, there exists an n_0 such that for all $n > n_0$, $u_n \geq 1$. Then for $n > n_0$, $n(1-\Phi(u_n)) \geq \frac{n}{2\sqrt{2\pi}u_n}\exp\left(-\frac{u_n^2}{2}\right)$. Thus,

$$n(1-\Phi(u_n)) \geq \frac{n}{2\sqrt{2\pi}u_n} t_{n+1}^{-1+\varepsilon} = \frac{n}{2\sqrt{2\pi}(n+1)} \frac{t_{n+1}^{\varepsilon}}{\Delta_{t_{n+1}} u_n}.$$

From (24) and (27), it can be observed that $\Delta_{t_{n+1}} = C_1(\log t_{n+1})^{\beta}$ and $u_n = C_2(\log t_{n+1})^{1/2}$ for some positive constants C_1 and C_2, respectively. Then as $n \to \infty$, $\frac{n}{2\sqrt{2\pi}(n+1)} \frac{t_{n+1}^{\varepsilon}}{\Delta_{t_{n+1}} u_n} \to \infty$. Thus the expression (18) is verified.

Step II Let $V_i = \sup_{i-1\leq s<i} Q^*(s)$, then $M^*(t) \leq \max_{1\leq i\leq t} V_i$. By the stationarity of Q^*, it follows that

$$P\left(M^*(t) \geq \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \leq tP\left(V_1 \geq \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right).$$

To verify (19), it is necessary to show that the right hand side of the above inequality approaches to 0, that is, $\lim_{t\to\infty} tP\left(V_1 \geq \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) = 0$. Since

$$\begin{aligned} V_1 &\leq Q^*(0) + \sup_{0\leq s\leq 1}\left((Y(s)-cs) - \inf_{0\leq r\leq s}(Y(r)-cr)\right) \\ &\leq Q^*(0) + \sup_{0\leq s\leq 1}(Y(s)-cs) - \inf_{0\leq s\leq 1}(Y(s)-cs) \end{aligned}$$

and $(1+\delta)^{\beta} \geq (1+\delta/2)^{\beta} + \delta/10$ for $\beta > 1/2$, $0<\delta<1$, then

$$\begin{aligned} &P\left(V_1 \geq \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \\ &\leq P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^{\beta}\right) + P\left(\sup_{0\leq s\leq 1}(Y(s)-cs) \geq \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta}\right) \\ &\quad + P\left(-\inf_{0\leq s\leq 1}(Y(s)-cs) \geq \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta}\right) \\ &\leq P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^{\beta}\right) + 2P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta} - c\right) \end{aligned}$$

It can be observed that for a fixed δ, there exists t_0 such that for $t \geq t_0$, $\frac{\delta}{20}\left(\frac{\log t}{\theta}\right)^{\beta} - c \geq \frac{\delta}{40}\left(\frac{\log t}{\theta}\right)^{\beta}$. So

$$\begin{aligned} &P\left(V_1 \geq \left(\frac{1+\delta}{\theta}\log t\right)^{\beta}\right) \\ &\leq \underbrace{P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^{\beta}\right)}_{L_1} + 2\underbrace{P\left(\sup_{0\leq s\leq 1} Y(s) \geq \frac{\delta}{40}\left(\frac{\log t}{\theta}\right)^{\beta}\right)}_{L_2} \end{aligned}$$

It is necessary to show that $\lim_{t\to\infty} tL_i = 0$ for $i = 1, 2$. For $i = 1$, it is equivalent to show that $\log t + \log P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right) \to -\infty$. Following Proposition 2.1, it is obtained that as $t \to \infty$,

$$\log t\left[1 + \frac{\log P\left(Q^*(0) \geq \left(\frac{1+\delta/2}{\theta}\log t\right)^\beta\right)}{\log t}\right] \sim \log t\left[1 + \left(-1 - \frac{\delta}{2}\right)\right] \to -\infty.$$

For $i = 2$, from (3), it follows that $\lim_{t\to\infty} tL_2 = 0$.

Step III In this step, the uniform integrability of the random variables $\left(\frac{M^*(t)}{(\log t)^\beta}\right)^p$ for $t \geq e$ is proved. It is sufficient to show that for each $p \in (1, \infty)$,

$$\sup_{t\geq e} E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p < \infty.$$

Let $y = x^{1/p}$, for $t \geq e$,

$$\begin{aligned} E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p &= \int_0^\infty P\left(\left(\frac{M^*(t)}{(\log t)^\beta}\right)^p > x\right)dx \\ &= \int_0^\infty P\left(M^*(t) > y(\log t)^\beta\right)py^{p-1}dy. \end{aligned}$$

Let $K = \max\left\{K_0, (4/\theta)^\beta), \sqrt{8/C_{G,\gamma}}\right\}$, then $K < \infty$ and

$$\begin{aligned} &E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p \\ &= \int_0^{3K} P\left(M^*(t) > y(\log t)^\beta\right)py^{p-1}dy + \int_{3K}^\infty P\left(M^*(t) > y(\log t)^\beta\right)py^{p-1}dy \\ &\leq (3K)^p + \underbrace{\int_{3K}^\infty P\left(M^*(t) > y(\log t)^\beta\right)py^{p-1}dy}_{L_3}. \end{aligned}$$

Similar to the arguments in **Step II**, it can be verified that for all $x > 0$,

$$P\left(M^*(t) > x\right) \leq tP\left(Q^*(0) + \max_{0\leq s\leq 1}(Y(s) - cs) - \min_{0\leq s\leq 1}(Y(s) - cs) > x\right).$$

Then

$$\begin{aligned} P\left(M^*(t) > x\right) &\leq tP\left(Q^*(0) > \frac{x}{3}\right) + tP\left(\max_{0\leq s\leq 1}(Y(s) - cs) > \frac{x}{3}\right) \\ &\quad + tP\left(-\min_{0\leq s\leq 1}(Y(s) - cs) > \frac{x}{3}\right) \\ &\leq tP\left(Q^*(0) > \frac{x}{3}\right) + 2tP\left(\max_{0\leq s\leq 1}(Y(s)) > \frac{x}{3} - c\right). \end{aligned}$$

It is obtained that

$$L_3 \le t \int_{3K}^{\infty} P\left(Q^*(0) > \frac{y(\log t)^\beta}{3}\right) p y^{p-1} dy + 2t \int_{3K}^{\infty} P\left(\max_{0\le s\le 1} Y(s) > \frac{y(\log t)^\beta}{3} - c\right) p y^{p-1} dy.$$

From the choice of K, it follows that $\frac{y(\log t)^\beta}{3} - c \ge \frac{y(\log t)^\beta}{6}$. So

$$L_3 \le \underbrace{t \int_{3K}^{\infty} P\left(Q^*(0) > \frac{y(\log t)^\beta}{3}\right) p y^{p-1} dy}_{L_{3,1}} + \underbrace{2\, t \int_{3K}^{\infty} P\left(\max_{0\le s\le 1} Y(s) > \frac{y(\log t)^\beta}{6}\right) p y^{p-1} dy}_{L_{3,2}}.$$

It is shown in Lemma 4.1 and 4.2 that $L_{3,1} < \infty$ and $L_{3,2} < \infty$ with the choice of K, respectively. Therefore it is obtained that $\sup_{t\ge e} E\left[\frac{M^*(t)}{(\log t)^\beta}\right]^p < \infty$. Combining **Step I, II** and **III**, it is obtained that $\lim_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta$ in L^p for each $p \in [1,\infty)$.

In the following, the result is extended to $M(t)$. Recall that for all $t \ge 0$, $Q(t) \le Q^*(t)$. Consequently, $M(t) \le M^*(t)$ for all $t \ge 0$. In **Step I**, replacing (20) with $Q(t) = Y(t) - ct - \inf_{0\le s\le t}(Y(s) - cs)$, changing $M^*(t)$ and $Q^*(t)$ to $M(t)$ and $Q(t)$, respectively, the rest remains unchanged. For **Step II** and **III**, since $M(t) \le M^*(t)$ for all $t \ge 0$, it is obtained that $\lim_{t\to\infty} P\left(M(t) \ge \left(\frac{1+\delta}{\theta}\log t\right)^\beta\right) = 0$ and $\sup_{t\ge e} E\left[\frac{M(t)}{(\log t)^\beta}\right]^p < \infty$. Thus the proof is complete. □

4.2. Fractional Brownian queueing model

The L^p convergence stated in Theorem 3.2 has been shown in the proof of Theorem 3.1. In the following, the almost sure convergence stated in Theorem 3.2 is proved. An upper bound and a lower bound are derived in Proposition 4.1 and 4.2, respectively. From these two propositions, $\lim_{t\to\infty} \frac{M^*(t)}{(\log t)^\beta} = \left(\frac{1}{\theta}\right)^\beta$ a.s. is concluded, then the proof is extended to $M(t)$. The following two lemmas are needed to prove Proposition 4.1.

Lemma 4.3. *Let θ and β be given in (10) and (11), respectively. Let $\delta \in (0,1)$ be fixed, then for almost every $\omega \in \Omega$, there exists a $K(\omega) < \infty$ such that for $n \ge K(\omega)$,*

$$\frac{Q^*(n)}{(\log n)^\beta} < \left(\frac{1+\delta}{\theta}\right)^\beta \quad \text{a.s.} \tag{29}$$

Proof. Recall that $Q^*(n) \overset{d}{=} Q(\infty)$ for all n. By the Borel-Cantelli lemma, it is sufficient to prove that $\sum_{n=1}^{\infty} P\left(Q^*(n) \ge \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right) < \infty$. Choose $\varepsilon \in \left(0, \frac{\delta}{2(1+\delta)}\theta\right)$, by Proposition 2.1, there exists $N < \infty$ such that $\frac{\log P\left(Q^*(n) \ge \left(\frac{1+\delta}{\theta}\log n\right)^\beta\right)}{\frac{1+\delta}{\theta}\log n} < -\theta + \varepsilon$

for $n \geq N$. Since $\varepsilon < \frac{\delta}{2(1+\delta)}\theta$, then it can be verified that

$$\sum_{n=1}^{\infty} P\left(Q^*(n) \geq \left(\frac{1+\delta}{\theta}\log n\right)^{\beta}\right) \leq N + \sum_{N+1}^{\infty} e^{-\left(1+\frac{\delta}{2}\right)\log n} < \infty.$$

□

Fix an $\omega \in \Omega$ for which (29) holds, then there exists a $K(\omega)$ such that

$$\begin{aligned}\frac{\max_{0\leq n\leq \lfloor t\rfloor} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^{\beta}} &\leq \frac{\max_{0\leq n\leq K(\omega)} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^{\beta}} + \max_{K(\omega)\leq n\leq \lfloor t\rfloor} \frac{Q^*(n,\omega)}{(\log n)^{\beta}} \\ &\leq \frac{\max_{0\leq n\leq K(\omega)} Q^*(n,\omega)}{(\log\lfloor t\rfloor)^{\beta}} + \left(\frac{1+\delta}{\theta}\right)^{\beta}.\end{aligned}$$

Let $t \to \infty$ and δ be arbitrarily small, then

$$\limsup_{t\to\infty} \frac{\max_{0\leq n\leq \lfloor t\rfloor} Q^*(n)}{(\log\lfloor t\rfloor)^{\beta}} \leq \left(\frac{1}{\theta}\right)^{\beta} \quad \text{a.s.} \tag{30}$$

Lemma 4.4. *Suppose that $\sigma > 0$ and $\{B^H(t), t \geq 0\}$ is a standard fractional Brownian motion with $H \in (0,1)$. Let β be defined as in (11). Then*

$$\lim_{t\to\infty} \frac{\max_{0\leq n\leq \lfloor t\rfloor}\left(\sup_{n\leq s\leq n+1} \sigma B^H(s) - \sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}} = 0 \quad \text{a.s.}$$

Proof. It is sufficient to show that for any $\varepsilon > 0$,

$$\sum_{\lfloor t\rfloor=0}^{\infty} P\left(\frac{\max_{0\leq n\leq \lfloor t\rfloor}\left(\sup_{n\leq s\leq n+1} \sigma B^H(s) - \sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}} > \varepsilon\right) < \infty.$$

Since $\sup_{n\leq s\leq n+1} \sigma B^H(s) - \sigma B^H(n) \overset{d}{=} \sup_{0\leq s\leq 1} \sigma B^H(s)$, it follows that

$$\begin{aligned}&\sum_{\lfloor t\rfloor=0}^{\infty} P\left(\frac{\max_{0\leq n\leq \lfloor t\rfloor}\left(\sup_{n\leq s\leq n+1} \sigma B^H(s) - \sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}} > \varepsilon\right) \\ &\leq \sum_{\lfloor t\rfloor=0}^{\infty} (\lfloor t\rfloor + 1) P\left(\sup_{0\leq s\leq 1} \sigma B^H(s) > \varepsilon(\log\lfloor t\rfloor)^{\beta}\right).\end{aligned} \tag{31}$$

From (3), it follows that

$$\begin{aligned}&\sum_{\lfloor t\rfloor=0}^{\infty} (\lfloor t\rfloor + 1) P\left(\sup_{0\leq s\leq 1} \sigma B^H(s) > \varepsilon(\log\lfloor t\rfloor)^{\beta}\right) \\ &\leq \sum_{\lfloor t\rfloor=0}^{\infty} 4\exp\left(-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta} + \log\left(\lfloor t\rfloor + 1\right)\right).\end{aligned} \tag{32}$$

Since $\beta > 1/2$, there exists M such that for all $\lfloor t\rfloor > M$,

$$-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta} + \log\left(\lfloor t\rfloor + 1\right) \leq -2\beta\log\lfloor t\rfloor.$$

From (31) and (32), it can be obtained that

$$\sum_{\lfloor t\rfloor=0}^{\infty} P\left(\frac{\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\sigma B^H(s)-\sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}}>\varepsilon\right)$$
$$\le \sum_{\lfloor t\rfloor=0}^{M} 4\exp\left(-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta}+\log\left(\lfloor t\rfloor+1\right)\right)+\sum_{\lfloor t\rfloor=M}^{\infty} 4\exp\left(-2\beta\log\lfloor t\rfloor\right)$$
$$\le \sum_{\lfloor t\rfloor=0}^{M} 4\exp\left(-\frac{C_{G,\gamma}\varepsilon^2}{\sigma^2}(\log\lfloor t\rfloor)^{2\beta}+\log\left(\lfloor t\rfloor+1\right)\right)+\sum_{\lfloor t\rfloor=M}^{\infty} 4\lfloor t\rfloor^{-2\beta}<\infty.$$

□

Proposition 4.1. *Let $M^*(t)$ be defined in (8). Let θ and β be given in (10) and (11), respectively. Then*

$$\limsup_{t\to\infty}\frac{M^*(t)}{(\log t)^{\beta}}\le\left(\frac{1}{\theta}\right)^{\beta}\quad\text{a.s.}$$

Proof. Since $M^*(t)=\sup_{0\le s\le t}Q^*(s)$, then $M^*(t)\le\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}Q^*(s)\right)$. From (5), it follows that for $s\in[n,n+1]$,

$$Q^*(s)=\sigma B^H(s)-cs$$
$$+\max\left(\sup_{n\le r\le s}\left(-\sigma B^H(r)+cr\right),Q^*(n)-\left(\sigma B^H(n)-cn\right)\right).$$

Then it can be obtained that

$$\sup_{n\le s\le n+1}Q^*(s)\le Q^*(n)+\sup_{n\le s\le n+1}(\sigma B^H(s)-cs)+\sup_{n\le s\le n+1}(-\sigma B^H(s)+cs).$$

So

$$M^*(t)\le\max_{0\le n\le\lfloor t\rfloor}Q^*(n)+\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\sigma B^H(s)-\sigma B^H(n)\right)$$
$$+\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\left(-\sigma B^H(s)+\sigma B^H(n)\right)\right)+c.$$

Thus

$$\frac{M^*(t)}{(\log t)^{\beta}}\le\frac{\max_{0\le n\le\lfloor t\rfloor}Q^*(n)}{(\log\lfloor t\rfloor)^{\beta}}+\frac{\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\sigma B^H(s)-\sigma B^H(n)\right)}{(\log\lfloor t\rfloor)^{\beta}}$$
$$+\frac{\max_{0\le n\le\lfloor t\rfloor}\left(\sup_{n\le s\le n+1}\left(-\sigma B^H(s)+\sigma B^H(n)\right)\right)}{(\log\lfloor t\rfloor)^{\beta}}+\frac{c}{(\log\lfloor t\rfloor)^{\beta}},$$

as $t\to\infty$, from (30) and Lemma (4.4), the proposition follows. □

The following lemma is needed to prove Proposition 4.2.

Lemma 4.5. *Let $M^*(t)$ be defined in (8) and θ, β be given in (10) and (11), respectively. Let $\delta\in(0,1)$ be fixed, then for almost all $\omega\in\Omega$, there exists an $n_0(\omega)$ such that for $n\ge n_0(\omega)$, $\frac{M^*(n,\omega)}{(\log n)^{\beta}}>\left(\frac{1-\delta}{\theta}\right)^{\beta}$.*

Proof. It is sufficient to check that $\sum_{n=1}^{\infty} P\left(M^*(n) \le \left(\frac{1-\delta}{\theta}\log n\right)^{\beta}\right) < \infty$. For a fractional Brownian queueing model, it is known from [10, Equation (23)] that there exists $t_0 < \infty$ such that for $t \ge t_0$,

$$P\left(M^*(t) \le u(t)\right) \le \exp\left(-\frac{c_2 t\left(u(t)\right)^h}{2} e^{-\theta(u(t))^{2-2H}}\right), \tag{33}$$

where $u(t)$ is a function in terms of t, $h = \frac{2(1-H)^2}{H} - 1$ and c_2 is a positive constant in terms of c, H. Then from (33), for the fixed δ and a sufficiently large n, that is, for all $n \ge \lfloor t_0 \rfloor + 1$, $P\left(M^*(n) \le \left(\frac{1-\delta}{\theta}\log n\right)^{\beta}\right) \le \exp\left(-\frac{c_2}{2}\left(\frac{1-\delta}{\theta}\right)^{\beta h} n^{\delta}\left(\log n\right)^{\beta h}\right)$. Thus it can be obtained that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(M^*(n) \le \left(\frac{1-\delta}{\theta}\log n\right)^{\beta}\right)\\
&\le \left(\lfloor t_0 \rfloor + 1\right) + \sum_{n=\lfloor t_0\rfloor+1}^{\infty} P\left(M^*(n) \le \left(\frac{1-\delta}{\theta}\log n\right)^{\beta}\right)\\
&\le \left(\lfloor t_0 \rfloor + 1\right) + \sum_{n=\lfloor t_0\rfloor+1}^{\infty} \exp\left(-\frac{c_2}{2}\left(\frac{1-\delta}{\theta}\right)^{\beta h} n^{\delta}\left(\log n\right)^{\beta h}\right) < \infty.
\end{aligned}$$

□

Proposition 4.2. *Let $M^*(t)$ be defined in (8) and θ, β be given in (10) and (11), respectively. Then* $\liminf_{t\to\infty} \frac{M^*(t)}{(\log t)^{\beta}} \ge \left(\frac{1}{\theta}\right)^{\beta}$ *a.s.*

Proof. From the definition of $M^*(t)$, it can be observed that

$$\frac{M^*(t)}{(\log t)^{\beta}} \ge \frac{M^*(\lfloor t\rfloor)}{(\log(\lfloor t\rfloor+1))^{\beta}} = \frac{M^*(\lfloor t\rfloor)}{(\log\lfloor t\rfloor)^{\beta}}\frac{(\log\lfloor t\rfloor)^{\beta}}{(\log(\lfloor t\rfloor+1))^{\beta}}. \tag{34}$$

From Lemma 4.5, it is obtained that for a fixed $\delta \in (0,1)$ and almost all $\omega \in \Omega$, there exists $t_0(\omega)$ such that for $t \ge \lfloor t_0(\omega)\rfloor$,

$$\frac{M^*(\lfloor t\rfloor)}{(\log\lfloor t\rfloor)^{\beta}} > \left(\frac{1-\delta}{\theta}\right)^{\beta}. \tag{35}$$

Let $t \to \infty$ and δ be arbitrarily small, from (34) and (35), the proposition follows.

□

Proof of Theorem 3.2. From Proposition 4.1 and 4.2, it follows that

$$\lim_{t\to\infty}\frac{M^*(t)}{(\log t)^{\beta}} = \left(\frac{1}{\theta}\right)^{\beta} \quad \text{a.s.}$$

In the following, the proof is extended to $M(t)$. For the upper bound, recall that for all $t \ge 0$, $Q(t) \le Q^*(t)$, which implies that $M(t) \le M^*(t)$ for $t \ge 0$, then

$$\limsup_{t\to\infty}\frac{M(t)}{(\log t)^{\beta}} \le \left(\frac{1}{\theta}\right)^{\beta} \quad \text{a.s.} \tag{36}$$

For the lower bound, rewrite $M^*(t) = \max_{0\le s\le t} Q^*(s)$ as

$$M^*(t) = \max_{0\le s\le t}\left\{\max\left(Q^*(0) + \sigma B^H(s) - cs, Q(s)\right)\right\}$$
$$= \max\left(Q^*(0) + \max_{0\le s\le t}\left(\sigma B^H(s) - cs\right), M(t)\right),$$

it can be observed that $M(t) \ge M^*(t) - Q^*(0) - \max_{s\ge 0}\left(\sigma B^H(s) - cs\right)$. Since $Q^*(0) \overset{d}{=} \max_{s\ge 0}\left(\sigma B^H(s) - cs\right) < \infty$ a.s, then it can be derived that

$$\liminf_{t\to\infty} \frac{M(t)}{(\log t)^\beta} \ge \left(\frac{1}{\theta}\right)^\beta \quad \text{a.s.} \tag{37}$$

Therefore from (36) and (37), the almost sure convergence of (14) follows. □

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