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TOPOLOGICAL DEGREE OF SYMMETRIC PRODUCT MAPS ON SPHERES

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(Submitted by Lech Górniewicz)

Dedicated to the memory of Karol Borsuk

0. Introduction

In 1946 S. Eilenberg and D. Montgomery [5] made the important observation that several results of fixed point theory for single-valued mappings can be carried over to the case of multivalued acyclic mappings. In 1957 B. O'Neill [13] introduced a class of continuous non-acyclic mappings for which the images of points consist of one or m acyclic components (with m fixed) and he proved the Lefschetz fixed point theorem for such mappings. In the case when components of values are points, O'Neill's $\{1, m\}$-valued mappings may be considered as symmetric product mappings which map a space into its symmetric product with respect to the m-th symmetric group, (for $m = 2$ it is an equivalent approach). In 1957 C. N. Maxwell [10] proved the Lefschetz fixed point theorem for symmetric product mappings on compact polyhedra. Results of B. O'Neill and C. N. Maxwell were generalized in two directions: i) the enlargement of the class of spaces ([4], [7], [8]) and ii) the localization, i.e. the fixed point index theory ([4], [9], [14]). Some results in the Nielsen fixed point theory of symmetric product mappings were obtained recently ([9], [12], [16]). In 1990 H. Schirmer [15] proved a property of $\{1, 2\}$-valued mappings which is crucial in the Nielsen theory: the removability of isolated fixed points of index zero. The proof of this result is based on the homotopy classification of symmetric product mappings on spheres obtained for symmetric products with respect to the second symmetric group. Our goal is to get such a classification for symmetric products with respect to any group of permutations. For this purpose we introduce the topological degree (which multiplied by a constant is a generalization of the degree studied by C. N. Maxwell [11] in 1977). An application of the main result (4.2) of this paper – a condition equivalent to the removability of isolated fixed points of symmetric product mappings – is the subject of the next paper, now in preparation.

I wish to thank Professor Lech Górniewicz who suggested this subject for many valuable discussions.

1. Symmetric products

Let M be a finite set and G be a subgroup of the group of permutations of M. Group G acts on the cartesian product X^M of a space X by the formula: $xg = x \circ g$ for $x \in X^M$, $g \in G$. The orbit space $\mathrm{SP}^M_G X$ of this action is called the *symmetric product of X with respect to G*. Denote by q (or q^M_G) the projection from X^M onto the orbit space $\mathrm{SP}^M_G X$. If (X, d) is a metric space, then the metrics in the products X^M and $\mathrm{SP}^M_G X$ are defined by the formulas:

$$d(x,y) = \max\{d(x(m), y(m)) : m \in M\},$$
$$d(q(x), q(y)) = \min\{d(x, yg) : g \in G\} \text{ for } x, y \in X^M.$$

For a subset $J \subset M$ denote by $p_J : X^M \to X^J$ the projection: $p_J(x) = x|_J$. In the case $J = \{j\}$ it will be denoted also by $p_j : X^M \to X$. If J is a G-invariant subset of M, then the set $G(J)$ of all permutations of J which are restrictions of permutations from G to J is a group and p_J induces a map $\overline{p}_J : \mathrm{SP}^M_G X \to \mathrm{SP}^J_{G(J)} X$ such that $\overline{p}_J \circ q = q_J \circ p_J$ where $q_J = q^J_{G(J)}$.

By H_* we shall denote the singular homology functor with integer coefficients, and by $\mathbb{S}^k$ the Euclidean k-sphere. We recall that a space is $(k-1)$-connected $(k \geq 1)$ if and only if it is pathwise connected and for every $i < k$ its i-th homotopy group is trivial.

2. Some lemmas

In proving our main result we shall make use of the following lemmas.

LEMMA 2.1. *If X is a $(k-1)$-connected compact metric* ANR, *then $q : X^M \to \mathrm{SP}^M_G X$ induces an epimorphism of k-th homotopy groups.*

Lemma 2.1 will be proved in section 5. Here we use it to deduce

LEMMA 2.2. (cf. [6]) *If X is a $(k-1)$-connected compact metric* ANR, *then so it is the space $\mathrm{SP}^M_G X$. Furthermore, if $k > 1$ or $\pi_1(X)$ is abelian then the homomorphism $q_* : H_k(X^M) \to H_k(\mathrm{SP}^M_G X)$, induced by the standard projection q, is onto.*

The next lemma follows from the naturality of the Künneth homomorphism:

LEMMA 2.3. *If X_i is $(k-1)$-connected, $c_i \in X_i$ and $\psi_j^c : \prod_{i\in M} X_i \to \prod_{i\in M} X_i$ maps any point x to the point y such that $p_i(y) = c_i$ for $i \neq j$ and $p_j(y) = p_j(x)$, then $\sum_{j\in M} \psi^c_{j*} : H_k(\prod_{i\in M} X_i) \to H_k(\prod_{i\in M} X_i)$ is the identity.*

3. The trace homomorphism and the group $H_K(\mathrm{SP}^M_G \mathbb{S}^k)$

Next using Lemma 2.2 we verify the correctness of the following

DEFINITION 3.1. (cf. [11](2), [14](2)) *The unique homomorphism*

$$\mu : H_k(\mathrm{SP}^M_G \mathbb{S}^k) \to H_k(\mathbb{S}^k)$$

such that

$$\mu \circ q_* = \sum_{j\in M} p_{j*}$$

is called the trace homomorphism.

Indeed, q_* is onto, so μ is well defined, if only $\ker(q_*) \subset \ker\left(\sum_{j\in M} p_{j*}\right)$. Let $\tau : H_k(\mathrm{SP}^M_G \mathbb{S}^k) \to H_k((\mathbb{S}^k)^M)$ be the transfer homomorphism for which we have $\tau \circ q_* = \sum_{g\in G} g_*$ (cf. III.7.1 in [1]). Thus $\sum_{i\in M} p_{i*} \circ \tau \circ q_* = \sum_{i\in M} \sum_{g\in G} p_{i*} \circ g_* = \sum_{g\in G} \sum_{i\in M} p_{g(i)*} = |G| \cdot \sum_{j\in M} p_{j*}$ and our inclusion follows.

Proofs of the next two propositions describing the structure of the group $H_k(\mathrm{SP}^M_G \mathbb{S}^k)$ are also based on Lemma 2.2.

PROPOSITION 3.2. *If G acts transitively on M, then the trace homomorphism $\mu : H_k(\mathrm{SP}^M_G \mathbb{S}^k) \to H_k(\mathbb{S}^k)$ is an isomorphism.*

PROOF. Let $M = \{1, \ldots, n\}$, $s_0 \in \mathbb{S}^k$ and $i : \mathbb{S}^k \to \mathrm{SP}^M_G \mathbb{S}^k$ be the map $i(x) = q(x, s_0, \ldots, s_0)$. Take $\psi_j^c : (\mathbb{S}^k)^M \to (\mathbb{S}^k)^M$ from (2.3) with $c_i = s_0$ for

$i, j \in M$. From the transitivity of the action it follows that $i \circ p_j = q \circ \psi_j^c$. Using (2.2), we now prove that i_* is an inverse of μ. In fact:

$$\mu \circ i_* = \mu \circ q_* \circ (\mathrm{id} \times s_0 \times \cdots \times s_0)_* = \sum_{j \in M} p_{j*} \circ (\mathrm{id} \times s_0 \times \cdots \times s_0)_* = \mathrm{id}$$

and

$$i_* \circ \mu \circ q_* = i_* \circ \sum_{j \in M} p_{j*} = q_* \circ \sum_{j \in M} \psi_{j*}^c = q_*.$$

PROPOSITION 3.3. *Let $O(G)$ be the set of all orbits of the G-action on M. Then $(\overline{p}_{J*})_{J \in O(G)} : H_k(\mathrm{SP}_G^M \mathbb{S}^k) \to \bigoplus_{J \in O(G)} H_k(\mathrm{SP}_{G(J)}^J \mathbb{S}^k)$ is an isomorphism.*

PROOF. Since the symmetric products of $\mathbb{S}^k$ are $(k-1)$-connected, one can use the Hurewicz isomorphisms and behaviour of homotopy groups with respect to the cartesian products to write the assertion in the equivalent form:

$$((\overline{p}_J)_{J \in O(G)})_* : H_k(\mathrm{SP}_G^M \mathbb{S}^k) \to H_k\Bigg(\prod_{J \in O(G)} \mathrm{SP}_{G(J)}^J \mathbb{S}^k\Bigg)$$

is an isomorphism. Let $X = \mathbb{S}^k$, $s_0 \in X, c = (s_0, \ldots, s_0) \in X^M = \prod_{I \in O(G)} X^I$. From (2.3) take $\psi_J^c : \prod_I X^I \to \prod_I X^I$ with $c_I = p_I(c)$ and

$$\psi_J^b : \prod_I \mathrm{SP}_{G(I)}^I X \to \prod_I \mathrm{SP}_{G(I)}^I X$$

with $b_I = q_I \circ p_I(c)$ for $I, J \in O(G)$. Let $r_J : \prod_I \mathrm{SP}_{G(I)}^I X \to \mathrm{SP}_{G(J)}^J X$ be the projection and $i_J : X^J \to X^M$ be the unique map such that

$$p_I \circ i_J = \begin{cases} p_I(c) & \text{if } I \neq J, \\ \mathrm{id} & \text{if } I = J, \end{cases}$$

for $I, J \in O(G)$. There is an induced map $\overline{i}_J : \mathrm{SP}_{G(J)}^J X \to \mathrm{SP}_G^M X$ such that $\overline{i}_J \circ q_J = q \circ i_J$. A direct calculation shows that $\psi_J^b = (\overline{p}_I \circ \overline{i}_J \circ r_J)_{I \in O(G)}$ and $\psi_J^c = i_J \circ p_J$. Thus

$$((\overline{p}_I)_{I \in O(G)})_* \circ (\sum_J \overline{i}_{J*} \circ r_{J*}) = \sum_J \psi_{J*}^b = \mathrm{id}$$

and

$$\begin{aligned}(\sum_J \overline{i}_{J*} \circ r_{J*}) \circ ((\overline{p}_I)_{I \in O(G)})_* \circ q_* &= \sum_J \overline{i}_{J*} \circ \overline{p}_{J*} \circ q_* \\ &= \sum_J \overline{i}_{J*} \circ q_{J*} \circ p_{J*} \\ &= \sum_J q_* \circ i_{J*} \circ p_{J*} \\ &= q_* \circ \sum_J \psi_{J*}^c = q_*.\end{aligned}$$

4. Topological degree

Let $O(G)$ be the set of all orbits of the G-action on M and η be a generator of $H_k\mathbb{S}^k$. Let $\mathbb{Z}$ be the set of integers. For $d \in \mathbb{Z}^{O(G)}$ let $\overline{d} : H_k(\mathbb{S}^k) \to \bigoplus_{J\in O(G)} H_k(\mathbb{S}^k)$ be a homomorphism defined by the formula $\overline{d}(\eta) = (d(J)\eta)_{J\in O(G)}$.

DEFINITION 4.1. *The topological degree* $\deg(f)$ *of the symmetric product map* $f : \mathbb{S}^k \to \mathrm{SP}^M_G\mathbb{S}^k$ *is defined to be a function* $d \in \mathbb{Z}^{O(G)}$ *such that the homomorphism*

$$H_k(\mathbb{S}^k)\overset{f_*}{\to}H_k(\mathrm{SP}^M_G\mathbb{S}^k)\overset{(P_{J*})}{\to}\bigoplus_{J\in O(G)} H_k(\mathrm{SP}^J_{G(J)}\mathbb{S}^k)\overset{\oplus\mu_J}{\to}\bigoplus_{J\in O(G)} H_k(\mathbb{S}^k)$$

is equal with $\overline{d}$.

We may now formulate the main result of this note.

THEOREM 4.2. *Symmetric product maps of the same degree are homotopic.*

Proof follows directly from (3.2), (3.3) and the Hurewicz isomorphism theorem.

PROPOSITION 4.3. *If G acts transitively on M, $f_i : S^k \to S^k$ for $i \in M$ and $f = q^M_G \circ (f_i)_{i\in M}$, then* $\deg(f) = \sum_{i\in M}\deg(f_i)$.

Proof of Proposition 4.3 follows immediately from (3.1).

REMARK. C. N. Maxwell [11] defined a degree Deg for symmetric product maps with respect to the n-th symmetric group. By definition, $\mathrm{Deg} = n^{-1}\deg$. Some properties of Deg (see (3.2), (3.3), (3.5), (3.8) [11]) may be easily deduced from our (2.1) and (4.3).

5. Proof of Lemma 2.1

Let $x_0 \in X, t_0 = (x_0, ..., x_0) \in X^M$ and $f : (I^k, \dot{I}^k) \to (\mathrm{SP}^M_G X, q(t_0))$ be a map. We shall define a map $F : (I^k, \dot{I}^k) \to (X^M, t_0)$ such that $f \simeq q \circ F \ \mathrm{rel}(\dot{I}^k)$. The definition of the map F will be prepared in several steps.

1) For every subset H of G and $\delta > 0$, we define the sets:

$$\mathrm{Fix}(H) = \bigcap\{\{x \in X^M : xg = x\} : g \in H\},$$
$$\mathrm{Fix}_\delta(H) = \bigcap\{\{x \in X^M : d(xg, x) \leq 2\delta\} : g \in H\}.$$

The set $\mathrm{Fix}(H)$ is homeomorphic to the cartesian product of X. Take $\varepsilon_k > 0$ such that any two maps $g, h : I^k \to \mathrm{SP}^M_G X$, $d(g, h) < \varepsilon_k$, are homotopic $\mathrm{rel}\{x \in I^k : g(x) = h(x)\}$. For $i = k, \ldots, 1$ choose $\varepsilon_{i-1} < \varepsilon_i/3$ such that, for every

$H \subset G$ any subset of $\mathrm{Fix}(H)$ whose diameter is less than ε_{i-1} is contractible in a subset of $\mathrm{Fix}(H)$ of the diameter less that $\varepsilon_i/3$. For $\varepsilon > 0$ denote by $O_\varepsilon(\cdot)$ the ε-neighbourhood. Choose $0 < \delta_2 < \varepsilon_0/4$ such that $\mathrm{Fix}_{2\delta_2}(H) \subset O_{\varepsilon_0/4}(\mathrm{Fix}(H))$ for every $H \subset G$. Take $\delta_1 > 0$ such that for any $x, y \in I^k$ with $d(x,y) < \delta_1$ we have $d(f(x), f(y)) < \delta_2$ and choose a positive integer number $m > 1$ such that $m^{-1}\sqrt{k} < \delta_1$.

2) We divide I^k into cubes with the length $(4m-2)^{-1}$ of each edge and define some sets and maps related to this subdivision. Let $N_k = \{1, \dots, k\}$. For every $\alpha \in \{0, 1, \dots, 4m-2\}^k$ and $P \subset N_k$, such that $\{i \in N_k : \alpha_i = 4m-2\} \subset P$ we define the set $C(\alpha, P)$ as the product $\prod_{i=1}^k C_i(\alpha, P)$ of sets:

$$C_i(\alpha, P) = \begin{cases} [\alpha_i/(4m-2), (\alpha_i+1)/(4m-2)] & \text{for } i \in N_k \backslash P, \\ \{\alpha_i/(4m-2)\} & \text{for } i \in P. \end{cases}$$

Let $P_s^\alpha = \{i \in P : \alpha_i \equiv s \pmod 4\}$, $I_s^\alpha = \{i \in N_k \backslash P : \alpha_i \equiv s \pmod 4\}$ for $s = 0, 1, 2, 3$. The set $C(\alpha, P)$ is called a *face* or *i-face*, if $I_2^\alpha = I_3^\alpha = P_3^\alpha = \emptyset$ and the set P has $(k-i)$ points. All k-faces having the nonempty intersection form a set called a *white cube*. The white cube has form $\prod_{i=1}^k \left[\frac{4p_i}{4m-2}, \frac{4p_i+2}{4m-2}\right]$, where $0 \le p_i \le m-1$. For every face $C(\alpha, P)$, we define the set $S(\alpha, P)$ as the product $\prod_{i=1}^k S_i(\alpha, P)$ of sets:

$$S_i(\alpha, P) = \begin{cases} [(\alpha_i - 1)/(4m-2), (\alpha_i+1)/(4m-2)] & \text{for } i \in P_1^\alpha, \\ C_i(\alpha, P) & \text{otherwise.} \end{cases}$$

Denote by $\mathrm{VS}(\alpha, P)$ the set of all vertices of a white cube lying in $S(\alpha, P)$.

We say that the faces $C(\alpha, P)$, $C(\beta, P)$ are *conjugate* and write $\alpha P \beta$, if $\alpha_i = \beta_i$ for $i \in I_0^\alpha \cup I_1^\alpha \cup P_1^\alpha$, ($\beta_i = \alpha_i + 2$ or $\beta_i = \alpha_i$) for $i \in P_2^\alpha$ and ($\beta_i = \alpha_i - 2$ or $\beta_i = \alpha_i$) for $i \in P_0^\alpha$.

If $\alpha P \beta$, then denote by $s_{\beta\alpha} : S(\alpha, P) \to S(\beta, P)$ the product map $\prod_{i=1}^k s_i$ where s_i is the identity for $i \in I_0^\alpha \cup I_1^\alpha \cup P_1^\alpha$ and $s_i(\alpha_i/(4m-2)) = \beta_i/(4m-2)$ for $i \in P_0^\alpha \cup P_2^\alpha$.

Let $\Lambda = \{(4p+3)/(4m-2) : p = 0, 1, \dots, m-2\}$. We define $\mathbf{r} : (I \setminus \Lambda)^k \to I^k$ as the product map ρ^k induced by a map $\rho : (I \setminus \Lambda) \to I$, which is defined by the formula:

$$\rho((4p+s)/(4m-2)) = \begin{cases} (4p+s)/(4m-2) & \text{for } s \in [0,2],\ p = 0, \dots, m-1, \\ (4p+2)/(4m-2) & \text{for } s \in [2,3),\ p = 0, \dots, m-2, \\ (4p+4)/(4m-2) & \text{for } s \in (3,4],\ p = 0, \dots, m-2. \end{cases}$$

We shall consider also the *second subdivision* of I^k into the cubes with the length m^{-1} of each edge. We define maps $L, G_t : (I^k, \dot{I}^k) \to (I^k, \dot{I}^k)$ by the formulas:

$L = l^k, G_t = (g_t)^k$ where, for $s \in [0,1]$,

$$l((4p+2s)/(4m-2)) = (p+s)/m \quad \text{for } p = 0, \dots, m-1;$$
$$l((4p-2s)/(4m-2)) = p/m \quad \text{for } p = 1, \dots, m-1;$$

$$g_t((p+s)/m) = \begin{cases} (4p-t+s(2+2t))/(4m-2) & \text{for } p = 1, \dots, m-2, \\ s(2+t)/(4m-2) & \text{for } p = 0,\ t \in [0,1], \\ (4p-t+s(2+t))/(4m-2) & \text{for } p = m-1. \end{cases}$$

One can check the following facts:

(i) The multivalued map $G : I^k \times I \to I^k$ defined by $G(x,t) = G_t(x)$ is upper semi-continuous.

(*) (ii) The map G_1 is single-valued and homotopic to the identity.

(iii) The map G_0 maps open cubes of the second subdivision onto open white cubes and $L \circ G_0$ is the identity.

3) We shall define a map F on the set of vertices of white cubes. If two such vertices w, w' are conjugate as 0-faces, then $L(w) = L(w')$. For each w, choose $\overline{F}(w) \in q^{-1}(f(L(w))$ such that $\overline{F}(w) = \overline{F}(w')$ if w, w' are conjugate. In every white cube choose one of its vertices w and define $F(w)$ as $\overline{F}(w)$. If v is another vertex of the same white cube, then there exists $g(v) \in G$ such that $d(\overline{F}(w),\ \overline{F}(v)g(v)) = d(f(L(w)), f(L(v)))$; we define $g(w)$ to be the identity permutation and $F(v) = \overline{F}(v)g(v)$. Of course, $d(F(w), F(v)) < \delta_2$ and $F(v') = F(v)g(v)^{-1}g(v')$ for any conjugate $v,\ v'$.

4) Let $C^i(\alpha)$ be the sum of all i-faces contained in the face $C(\alpha, \emptyset)$ and W_i be the sum of all i-faces. For every face $C(\alpha, P)$ and $w \in \mathrm{VS}(\alpha, P)$ let $H_w(\alpha, P) = \{g(w)^{-1}g(s_{\beta\alpha}(w))[g(w')^{-1}g(s_{\beta\alpha}(w'))]^{-1} : \beta P \alpha,\ w' \in \mathrm{VS}(\alpha, P)\}$. By induction we shall define a map $F : (W_i, W_i \cap \dot{I}^k) \to (X^M, t_0)$ such that:

(4.i) $\mathrm{diam}(F(C^i(\gamma))) < \varepsilon_i$,

(4.ii) $F(C(\alpha, P)) \subset \mathrm{Fix}(H_w(\alpha, P))$,

(4.iii) $F(s_{\beta\alpha}(x)) = F(x)g(w)^{-1}g(s_{\beta\alpha}(w))$

for any face $C(\gamma, \emptyset)$, i-face $C(\alpha, P)$, $w \in \mathrm{VS}(\alpha, P)$, $x \in C(\alpha, P)$ and $\beta P \alpha$. In any conjugacy class of 0-faces which are not vertices of white cubes choose $z_\alpha = C(\alpha, N_k)$. Choose $w \in \mathrm{VS}(\alpha, N_k)$. If $w' \in \mathrm{VS}(\alpha, N_k)$ and $\beta N_k \alpha$, then $d(F(w), F(w'))$ $< 2\delta_2$ and

$$d(F(w)g(w)^{-1}g(s_{\beta\alpha}(w)), F(w')g(w')^{-1}g(s_{\beta\alpha}(w'))$$
$$= d(F(s_{\beta\alpha}(w)), F(s_{\beta\alpha}(w')) < 2\delta_2,$$

thus $F(w) \in \mathrm{Fix}_{2\delta_2}(H_w(\alpha, N_k)) \subset O_{\varepsilon_0/4}(\mathrm{Fix}(H_w(\alpha, N_k))$. We choose the value $F(z_\alpha) \in \mathrm{Fix}(H_w(\alpha, N_k))$ such that $d(F(w), F(z_\alpha)) < \varepsilon_0/4$. On faces conjugate to z_α we define F by (4.iii). Observe that if $v_1, v_2 \in C^0(\gamma)$, then $d(F(v_1), F(v_2)) < \varepsilon_0/4 + 2\delta_2 + \varepsilon_0/4 < \varepsilon_0$.

For the inductive step let us notice that if $C(\mu, Q)$ is an $(i-1)$-face of an i-face $C(\alpha, P)$, then $\mathrm{VS}(\alpha, P) \subset \mathrm{VS}(\mu, Q)$ and $H_w(\alpha, P) \subset H_w(\mu, Q)$. Denote by $\dot{C}(\alpha, P)$ the sum of all $(i-1)$-faces of the i-face $C(\alpha, P)$. In any conjugacy class of i-faces let us choose an element $C(\alpha, P)$. Choose $w \in \mathrm{VS}(\alpha, P)$. The inductive assumption implies $\mathrm{diam}(F(\dot{C}(\alpha, P)) < \varepsilon_{i-1}$ and $F(\dot{C}(\alpha, P)) \subset Fix(H_w(\alpha, P)$. From the choice of ε_{i-1} it follows that there exists an extension of F on $C(\alpha, P)$ such that (4.ii) holds and $\mathrm{diam}(F(C(\alpha, P))) < \varepsilon_i/3$. On faces conjugate to $C(\alpha, P)$ we define F by (4.iii). Then $\mathrm{diam}(F(C^i(\gamma))) < \varepsilon_i/3 + \varepsilon_{i-1} + \varepsilon_i/3 < \varepsilon_i$.

5) Let $W = W_k$, $U = W_{k-1}$, $\mathrm{Co}(U)$ be the cone $(U \times I)/(U \times \{1\})$. $[U \times \{1\}] = \Omega$, $U \times \{0\} = U$. Since the set $\mathrm{Fix}(H)$ is $(k-1)$-connected for every $H \subset G$, we extend inductively $F : (W, W \cap \dot{I}^k) \to (X^M, t_0)$ to a map

$$\widehat{F} : (W \cup \mathrm{Co}(U), \mathrm{Co}(U \cap \dot{I}^k)) \to (X^M, t_0)$$

such that:

(5.i) $\widehat{F}(\Omega) = t_0$,
(5.ii) $\widehat{F}(\mathrm{Co}(C(\alpha, P))) \subset \mathrm{Fix}(H_w(\alpha, P))$,
(5.iii) $\widehat{F}((s_{\beta\alpha}(x), t)) = \widehat{F}((x, t))g(w)^{-1}g(s_{\beta\alpha}(w))$

for any i-face $C(\alpha, P)$, $i \le k-1$, $(x, t) \in \mathrm{Co}(C(\alpha, P))$, $w \in \mathrm{VS}(\alpha, P)$ and $\beta P \alpha$.

6) For $x \in (I \setminus \Lambda)^k \setminus W$ let $\omega(x) = \mathbf{r}(x) + \xi_0(x - \mathbf{r}(x))$ where

$$\xi_0 = \inf\{\xi \ge 0 : \mathbf{r}(x) + \xi(x - \mathbf{r}(x)) \in I^k \setminus (I \setminus \Lambda)^k\}.$$

We define maps

$$T : (I^k, \dot{I}^k) \to (W \cup \mathrm{Co}(U), \mathrm{Co}(U \cap \dot{I}^k))$$

and

$$F : (I^k, \dot{I}^k) \to (X^M, t_0)$$

by the formulas:

$$T(x) = \begin{cases} \Omega & \text{for } x \in I^k \setminus (I \setminus \Lambda)^k, \\ x & \text{for } x \in W, \\ (\mathbf{r}(x), \|x - \mathbf{r}(x)\| / \|\omega(x) - \mathbf{r}(x)\|) & \text{for } x \in (I \setminus \Lambda)^k \setminus W \end{cases}$$

$$F = \widehat{F} \circ T.$$

7) We are going to prove that $f \simeq q \circ F \ \mathrm{rel}(\dot{I}^k)$. One can check that the map $q \circ F \circ G_t$ is single-valued, so $q \circ F \simeq q \circ F \circ G_1 \simeq q \circ F \circ G_0 \ \mathrm{rel}(\dot{I}^k)$. If

v is a vertex of the second subdivision, then $G_0(v)$ consists of vertices of white cubes, so $q \circ F \circ G_0(v) = f \circ L \circ G_0(v) = f(v)$. From (*) (iii) it follows that $q \circ F \circ G_0 = q \circ F|_W \circ G_0$ maps any cube of the second subdivision into a $q \circ F$-image of a white cube having a diameter less than $2\varepsilon_k/3$. Let x be any point from a cube of the second subdivision and v be a vertex of this cube. Then $d(x,v) < \delta_1, d(q \circ F \circ G_0(x), f(x)) \leq d(q \circ F \circ G_0(x), q \circ F \circ G_0(v)) + d(f(v), f(x)) < 2\varepsilon_k/3 + \delta_2 < \varepsilon_k$. From the choice of ε_k it follows that $q \circ F \circ G_0 \simeq f$ rel$(\dot{I}^k)$.

REMARK. In the case, when G is the n-th symmetric group and $k > 1$, Lemma 2.1 easily follows from [2] and [3].

Indeed, let X be a connected CW-complex, $x_0 \in X, X_n$ be the symmetric product of X with respect to the n-th symmetric group and $X_\infty = \bigcup_{m=1}^{\infty} X_m$ be the infinite symmetric product (see [3] (3.4)). Let $s : X \to X^n$ be the inclusion: $s(x) = (x, x_0, \ldots, x_0)$. From [3] (6.10) it follows that there exists an isomorphism $j : H_k(X) \to \pi_k(X_\infty)$ such that the following diagram

$$\begin{array}{ccc}
\pi_k(X^n) & \xrightarrow{q_\#} & \pi_k(X_n) \\
{\scriptstyle s_\#}\uparrow & & \downarrow{\scriptstyle \beta} \\
\pi_k(X) & \xrightarrow{\alpha} & \pi_k(X_\infty) \\
{\scriptstyle \phi}\downarrow & \nearrow{\scriptstyle j}\,{\scriptstyle \cong} & \\
H_k(X) & &
\end{array}$$

where α, β are homomorphisms induced by inclusions, and ϕ is the Hurewicz homomorphism, is commutative. If $H_i(X) = 0$, for $i = 1, \ldots, k-1$, then β is an isomorphism (see [2](12.30)). If moreover X is $(k-1)$-connected, then ϕ is an isomorphism, so $q_\#$ is an epimorphism.

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