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ON THE STRUCTURE OF NILPOTENT GROUPS OF A CERTAIN TYPE

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Dedicated to the memory of Karol Borsuk

1. Introduction

Let $\mathcal{N}$ be the class of nilpotent groups. Then, given any group N in $\mathcal{N}$ and any family of primes P we may construct the P-localization N_p of N (see [4]). Thus N_p is *P-local*, meaning that it admits unique q^{th} roots for q outside P, and there is a homomorphism $e : N \to N_p$ which is universal for homomorphisms from N into P-local nilpotent groups.

Now let $N \in \mathcal{N}$ be finitely generated (fg). Mislin [5] defined the *genus* of N to be the set $\mathcal{G}(N)$ of isomorphism classes of fg nilpotent groups M such that $M_p \cong N_p$, for all primes p. He showed that the genus is, in general, non-trivial, but gave no means of calculating it in this generality. He also demonstrated its relevance for the discussion of genus in the collection of homotopy types of nilpotent polyhedra of finite type (see [4]).

Let $\mathcal{N}_0$ be the class of finitely generated, but not finite, nilpotent groups with finite commutator subgroup $[N, N]$. Then for any N in $\mathcal{N}_0$ the genus $CalG(N)$ (see [5, 3]) has the structure of a finite abelian group. This *genus-group* was calculated in [1] in the case that N belongs to a certain subclass $\mathcal{N}_1$ of $\mathcal{N}_)$.

To explain $\mathcal{N}_1$, we consider the short exact sequence (valid for any nilpotent group N)

$$TN \rightarrowtail N \twoheadrightarrow FN \tag{1.1}$$

where TN is the torsion subgroup of N, and FN is the torsionfree quotient. Then $N \in \mathcal{N}_0$ if and only if TN is finite and FN is free abelian of finite rank. We say that $N \in \mathcal{N}_1 \subset \mathcal{N}_0$ if, additionally,

(a) TN is commutative;
(b) (1.1) splits on the right, so that N is the semidirect product for an action $\omega : FN \to \operatorname{Aut} TN$;
(c) the action ω satisfies $\omega(FN) \subseteq Z(\operatorname{Aut} TN)$, where Z is the center.

Note that, in the presence of (a), (c) is equivalent to requiring that, for each $\xi \in FN$, there exists an integer u, depending on ξ, such that $\xi.a = ua$ for all $a \in TN$ (written additively).

Now let t be a *height* of $\ker \omega$ in FN; here the height of a (non-trivial) subgroup R of a free abelian group F is the largest positive integer h such that $R \subseteq hF$. Then it is proven in [1] that

$$\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\} \quad \text{if} \quad N \in \mathcal{N}_1. \tag{1.2}$$

We will prove the following structure theorem for groups in $\mathcal{N}_1$.

THEOREM 1.1. *Let $N \in \mathcal{N}_1$. Then* (i) $t = 1$ *or* 2, *or* (ii) *FN is cyclic.*

We will also show that there are groups N in $\mathcal{N}_1$ such that $t = 1$ and FN is not cyclic; $t = 2$ and FN is not cyclic; and FN is cyclic but $t \neq 1, 2$. As a consequence of Theorem 1.1 and (1.2), we have (with N^k the k^{th} direct power of N)

COROLLARY 1.2. *Let $N \in \mathcal{N}_1$ with FN not cyclic. Then $\mathcal{G}(N^k) = 0$, $k \geq 1$.*

For $\mathcal{G}(N) = 0$ by (1.2), since $t = 1$ or 2; and for any $N \in \mathcal{N}_0$ there is, by Theorem 4.1 of [1], a surjection $\mathcal{G}(N) \twoheadrightarrow \mathcal{G}(N^k)$, $k \geq 2$, given by $M \mapsto M \times N^{k-1}$, $M \in \mathcal{G}(N)$. It is relevant to allow $k \geq 2$ in Corollary 1.2, for, although $\mathcal{N}_0$ is closed under direct products, one may show that N^k, $k \geq 2$, is in $\mathcal{N}_1$ only if N is itself commutative.

If FN is cyclic, the situation is utterly different. It is clear from calculations in [2] that t can take any value. If N is such that $\mathcal{G}(N) \cong (\mathbb{Z}/t)^*/\{\pm 1\}$, $N \in \mathcal{N}_1$, then $\mathcal{G}(N^k)$, $k \geq 2$, is independent of k and is obtained from $\mathcal{G}(N)$ by factoring out a certain explicitly described elementary abelian 2-group.

The results of this paper have important implications in homotopy theory. Indeed, it was the study of the genus of a nilpotent space of finite type which gave rise to the purely group-theoretical studies reported in [1, 2, 3, 5]. In particular, given a group N in the class $\mathcal{N}_i$, we can construct a torus-bundle X over a space M such that (i) M depends only on the genus of N, (ii) N is the group of free homotopy classes of maps of S^1 into ΩX; and (iii) corresponding to any group $\tilde{N}$ in the genus of N we may construct $\widetilde{X}$ (as a torus-bundle over M) in the genus of X. Then Theorem 1.1 implies that, in order to obtain an interesting genus set $\mathcal{G}(X)$, we must have FN cyclic, so that X is a circle-bundle. It is then not difficult to prove that each of $\widetilde{X}$, X is a covering space of the other.

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2. Some Preliminary Lemmas

Let $N \in \mathcal{N}_1$ and let us write FN additively; then

$$FN = \langle \xi_1, \xi_2, \dots, \xi_r \rangle, \qquad \ker \omega = \langle t_1\xi_1, t_2\xi_2, \dots, t_r\xi_r \rangle, \tag{2.1}$$

where $t = t_1 | t_2 | \dots | t_r$. Let

$$\xi_i.a = u_i a, \quad a \in TN, \tag{2.2}$$

and let $\exp TN = n = p_1^{m_1} p_2^{m_2} \dots p_\lambda^{m_\lambda}$, where $m_i \geq 1$ and $p_1 < p_2 < \dots < p_\lambda$ are the prime factors of n. Then u_i is of order $t_i \bmod n$. We prove

LEMMA 2.1. $p_1 p_2 \dots p_\lambda | (u_i - 1)$, $i = 1, 2, \dots, r$.

PROOF. In fact, we show that, with FN, TN commutative and (1.1) split, the condition given is the necessary and sufficient condition for N to be nilpotent. For it is not difficult to see that Γ^q is the q^{th} term of the lower central series of N ($\Gamma^0 = N$), the $\Gamma^1 = \langle a^{u_1 - 1}, a \in TN, 1 \leq i \leq r \rangle$, $\Gamma^2 = \langle a^{(u_i-1)(u_j-1)}, a \in TN, 1 \leq i, j \leq r \rangle$, etc. Thus $\Gamma^q = \{1\}$ for q sufficiently large if, and only if, $n | (u_{i_1} - 1)(u_{i_2} - 1) \dots (u_{i_q} - 1)$, q sufficiently large, for i_ρ such that $1 \leq i_\rho \leq r$. But this is plainly equivalent to the given condition. □

Our second lemma is number-theoretical.

LEMMA 2.2. *Let* u_0, u_1, u_2 *be elements of finite order in a group* G, *such that* $u_1, u_2 \in \langle u_0 \rangle$ *and* $|u_1| \, | \, |u_2|$. *Then* $u_1 \in \langle u_2 \rangle$.

(Here $|u|$ is the order of u and $\langle u \rangle$ is the subgroup generated by u.)

PROOF. Let $u_i = u_0^{q_i}$, $i = 1, 2$ and let $|u_0| = s$. Then $|u_i| = \frac{s}{(s,q_i)}$, $i = 1, 2$, so that $\frac{s}{(s,q_1)} \mid \frac{s}{(s,q_2)}$, whence

$$(s, q_2) \mid (s, q_1) | q_1.$$

This, however, is the condition that we can solve, in integers, the equation

$$as + bq_2 = q_1. \tag{2.3}$$

From (2.3) we infer that $u_0^{bq_2} = u_0^{q_1}$, or $u_1 = u_2^b$. □

3. Proof of Theorem 1.1

We assume $r \geq 2$ in (2.1), we assume (2.2) holds and we set

$$n = p_1^{m_1} p_2^{m_2} \dots p_\lambda^{m_\lambda}, \tag{3.1}$$

as in Section 2. We say that we are in Case 2 (the *exceptional* case) if $p_1 = 2$ and $m_1 \geq 3$. Otherwise, we say we are in Case 1 (the *general* case). We deal first with Case 1.

Case 1: Set $u_0 = 1 + p_1 p_2 \dots p_\lambda$. We regard u_i as an element of $(\mathbb{Z}/n)^*$, $i = 0, 1, \dots, r$. Then, according to [2] (see also (3.3) of [1]) the order of $u_0 \bmod n$ is $p_1^{m_1-1} p_2^{m_2-1} \dots p_\lambda^{m_\lambda-1}$. Now the number of distinct residues $u \bmod n$ satisfying $p_1 p_2 \dots p_\lambda | (u-1)$ is also $p_1^{m_1-1} p_2^{m_2-1} \dots p_\lambda^{m_\lambda-1}$; and every power of u_0 is such a residue. thus the powers of u_0 completely exhaust all the residues $u \bmod n$ satisfying $p_1 p_2 \dots p_\lambda | (u-1)$. It therefore follows from Lemma 2.1 that $u_1, u_2 \in \langle u_0 \rangle$. Moreover, $|u_1| = t_1$, $|u_2| = t_2$, and $t_1 | t_2$. Thus, by Lemma 2.2, $u_1 \in \langle u_2 \rangle$. We can therefore solve for d the congruence

$$u_1 u_2^d \equiv 1 \bmod n. \tag{3.2}$$

But (3.2) implies that (in additive notation) $\xi_1 + d\xi_2 \in \ker \omega$. Comparison with (2.1), recalling that $t = t_1$, shows that $t = 1$.

Case 2: We now have $n = 2^{m_1} p_2^{m_2} \dots p_\lambda^{m_\lambda}$, $m_1 \geq 3$. We set $u_0 = 1 + 4p_2 \dots p_\lambda$. Then, again according to [2] or (3.3) of [1], the order of $u_0 \bmod n$ is $2^{m_1-2} p_2^{m_2-1} \dots p_\lambda^{m_\lambda-1}$ and a similar counting argument shows that any residue $u \bmod n$ satisfying $4p_2 \dots p_\lambda | (u-1)$ must belong to $\langle u_0 \rangle$. By Lemma 2.1 we know that, for any u_i in (2.2), $2p_2 \dots p_\lambda | (u_i - 1)$, so that $4p_2 \dots p_\lambda | (u_i^2 - 1)$. Thus

$$u_1^2, u_2^2 \in \langle u_0 \rangle. \tag{3.3}$$

Now $|u_i^2| = \frac{t_i}{(t_i,2)}$; and it is plain that if $t_1 | t_2$, then $\frac{t_1}{(t_1,s)} \big| \frac{t_2}{(t_2,s)}$, for any s. Thus $|u_1^2| \mid |u_2^2|$, so by (3.3) and Lemma 2.2, $u_1^2 \in \langle u_2^2 \rangle \subseteq \langle u_2 \rangle$. Now if $t (= t_1)$ is odd, then

$u_1 \in \langle u_1^2 \rangle$ so that $u_1 \in \langle u_2 \rangle$. Thus if t is odd, we infer as in Case 1 that $t = 1$. If, on the other hand, t is even, we infer that we can solve for d the congruence

$$u_1^2 u_2^d \equiv 1 \bmod n. \tag{3.4}$$

But (3.3) implies that $2\xi_1 + d\xi_2 \in \ker \omega$ so that, by (2.1), $t = 2$. This completes the proof.

REMARK. We have already pointed out that, if we take FN cyclic, we can achieve any value of t by suitably choosing N in $\mathcal{N}_1$. It was also shown in [1] that if $N_1, N_2 \in \mathcal{N}_1$ with $\exp TN_1$, $\exp TN_2$ mutually coprime, then $N_1 \times N_2 \in \mathcal{N}_1$ and $t(N_1 \times N_2) = 1$; of course $F(N_1 \times N_2) = FN_1 \times FN_2$ and so is non-cyclic. To obtain an example of a group N in $\mathcal{N}_1$ with FN not cyclic and $t = 2$, we set $TN = \mathbb{Z}/8 = \langle a \rangle$, $FN = \mathbb{Z} \oplus \mathbb{Z} = \langle \xi_1, \xi_2 \rangle$ with $\xi_1.a = 3a$, $\xi_2.a = 5a$. Then $\ker \omega = \langle 2\xi_1, 2\xi_2 \rangle$ and $t = 2$.

In fact, the case $t = 2$ is truly exceptional. Its presence arises from the fact that $(\mathbb{Z}/n)^*$ is not cyclic if $n = 2^m$, $m \geq 3$, being $\mathbb{Z}/2 \oplus \mathbb{Z}/2^{m-2}$ (additively). Indeed, one easily shows

PROPOSITION 3.1. *Let $N \in \mathcal{N}_1$ with $t = 2$. Then $r = \operatorname{rank}(FN) = 1$ or 2. Moreover, even if $r = 2$, N cannot be the direct product of two members of $\mathcal{N}_1$.*

Of course, if $N \in \mathcal{N}_1$ with $t = 1$, then r can take any value.

REFERENCES

[1] C. CASACUBERTA AND P. HILTON, *Calculating the Mislin genus for a certain family of nilpotent groups*, Comm. in Alg. **19(7)** (1991), 2051–2069.

[2] P. HILTON, *Non-cancellation properties for certain finitely presented groups*, Quaestiones Math. **9** (1986), 281–292.

[3] P. HILTON AND G. MISLIN, *On the genus of a nilpotent group with finite commutator subgroup*, Math. Z. **146** (1976), 201–211.

[4] P. HILTON, G. MISLIN AND J. ROITBERG, *Localization of nilpotent groups and spaces*, North Holland Mathematics Studies **15** (1975).

[5] G. MUSLIN, *Nilpotent groups with finite commutator subgroups*, Lecture Notes in Math. **418** (1974), Springer-Verlag, 103–120.

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