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SOLUTION SETS OF BOUNDARY VALUE PROBLEMS FOR NONCONVEX DIFFERENTIAL INCLUSIONS

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(*Submitted by L. Górniewicz*)

Dedicated to the memory of Karol Borsuk

1. Introduction and preliminaries

Topological properties of the solution set of Cauchy problems for differential inclusions have been investigated by several authors [16], [24], [14], [23], [10], [19], [3], [15]. Less attention has been, so far, devoted to analogous questions for boundary value problems.

In the present paper we consider boundary value problems of the type

$$\text{(BV)} \qquad \begin{cases} x''(t) \in F(t, x(t), x'(t)), \\ x(0) = x(1) = 0, \end{cases}$$

where F is a multifunction from $I \times \mathbf{R}^q \times \mathbf{R}^q$, $I = [0,1]$, to the non-empty compact subsets of $\mathbf{R}^q$. If F is Lipschitzean, we prove that the solution set S_F of (BV) is a retract of the Sobolev space $W^{2,1}(I, \mathbf{R}^q)$. In particular, S_F is contractible and hence arcwise connected. Whenever F is convex valued and Lipschitzean, S_F is a retract also of $C^1(I, \mathbf{R}^q)$. Finally, in the nonconvex case, under a continuity assumption on F, it is proved that S_F is non-empty.

To establish the retraction property of S_F, when F is Lipschitzean, we use some recent results due to Ricceri [21] and Bressan, Cellina and Fryszkowski [4], who have studied the existence of a retraction of a Banach space X onto the set of the fixed points of a contractive multifunction from X into itself. Developments and applications of such ideas can be found in Rybiński [22]. The nonemptiness of S_F, when F is continuous, is obtained as in Papageorgiou [18], by a technique based on a selection theorem for decomposable valued multifunctions of Antosiewicz-Cellina type [1], [9], [5].

Unlike the nonconvex case, boundary value problems of the type (BV) with F compact convex valued have been studied by many authors. We mention, among others, Pruszko [20], also for a historical outline and an extensive list of references, and Erbe and Krawcewicz [7] and Frigon [8], who use an approach based on the topological transversality method of Granas, Guenther and Lee [11].

Let X be a metric space with distance d_X. For $x \in X$ and A a non-empty subset of X, we set $d_X(x,A) = \inf_{a\in A} d_X(x,a)$. We denote by $\mathcal{K}(X)$ the space of all non-empty closed bounded subsets of X equipped with the Hausdorff metric

$$D_X(A,B) = \max\left\{\sup_{b\in B} d_X(b,A),\ \sup_{a\in A} d_X(a,B)\right\}, \qquad A,B \in \mathcal{K}(X).$$

Moreover $\mathcal{C}(X)$, where X is a normed space, denotes the space of all non-empty, convex, closed, bounded subsets of X endowed with the Hausdorff metric D_X. By $B_X(x,r)$ (resp. $\widetilde{B}_X(x,r)$) we mean an open (resp. closed) ball in X with center $x \in X$ and radius $r > 0$ (resp. $r \geq 0$).

Let X, Y be metric spaces. A multifunction $F : X \to \mathcal{K}(Y)$ is said to be *Hausdorff lower* (resp. *upper*) *semicontinuous* if, for every $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $F(x_0) \subset \{y \in Y \mid d_Y(y,F(x)) < \varepsilon\}$ (resp. $F(x) \subset \{y \in Y \mid d_Y(y,F(x_0)) < \varepsilon\}$) for every $x \in B_X(x_0,\delta)$. F is called *Hausdorff continuous* if F is Hausdorff lower and upper semicontinuous. A multifunction $F : T \to \mathcal{K}(Y)$, T and interval of $\mathbf{R}$, is said to be *measurable* if for every closed subset C of Y, the set $\{t \in T \mid F(t) \subset C\}$ is Lebesgue measurable. We refer to Castaing and Valadier [6] for further properties of measurable multifunctions.

To study problem (BV) we introduce the following assumptions about F.

Let $F : I \times \mathbf{R}^q \times \mathbf{R}^q \to \mathcal{K}(\mathbf{R}^q)$, $I = [0,1]$, be a multifunction.

We say that F *satisfies* (L) if:

(i) For every $(x,y) \in \mathbf{R}^q \times \mathbf{R}^q$ the multifunction $t \to F(t,x,y)$ is measurable and satisfies

$$D_{\mathbf{R}^q}(F(t,0,0),\{0\}) \leq m(t) \qquad \text{for} \quad t \in I,$$

where $m : I \to \mathbf{R}$ is non-negative and integrable.

(ii) For every (t,x_1,y_1), $(t,x_2,y_2) \in I \times \mathbf{R}^q \times \mathbf{R}^q$ we have

$$D_{\mathbf{R}^q}(F(t,x_1,y_1),\ F(t,x_2,y_2)) \leq a|x_1 - y_1| + b|x_2 - y_2|,$$

where $a \geq 0$, $b \geq 0$ and $a + b = k < 1$.

We say that F *satisfies* (C) if:

(i) For every $(x, y) \in \mathbf{R}^q \times \mathbf{R}^q$ the multifunction $t \to F(t, x, y)$ is measurable and satisfies

$$D_{\mathbf{R}^q}(F(t,x,y),\{0\}) \leq m(t), \qquad \text{for every} \quad (t,x,y) \in I \times \mathbf{R}^q \times \mathbf{R}^q,$$

where $m : I \to \mathbf{R}$ is non-negative and square integrable.

(ii) For $t \in I$ a.e. the multifunction $(x, y) \to F(t, x, y)$ is Hausdorff continuous.

Suppose that F satisfies (L) or (C). A function $x : I \to \mathbf{R}^q$ is said to be a *solution* of the boundary value problem (BV) if: (j) x is absolutely continuous with $x(0) = x(1) = 0$, (jj) x' is absolutely continuous, and (jjj) $x''(t) \in F(t, x(t), x'(t))$, $t \in I$ a.e. The set of all solutions of (BV) is called the *solution set* of (BV) and denoted by S_F.

The above definition of a solution remains valid when F is, in particular, single valued.

Let $I = [0, 1]$. We denote by $C(I, \mathbf{R}^q)$ (resp. $C^1(I, \mathbf{R}^q)$) the Banach space of all continuous (resp. continuously differentiable functions) $x : I \to \mathbf{R}^q$ endowed with the norm

$$\|x\|_C = \max_{t \in I} |x(t)| \qquad (\text{resp.} \quad \|x\|_{C^1} = \max\{\|x\|_C, \|x'\|_C\}).$$

As usual, $L^1(I, \mathbf{R}^q)$ (resp. $L^2(I, \mathbf{R}^q)$) is the Banach space of all (equivalence classes of) integrable (resp. square integrable) functions $x : I \to \mathbf{R}^q$ equipped with the norm $\|x\|_{L_1} = \int_I |x(t)|dt$ (resp. $\|x\|_{L^2} = \int_I |x(t)|^2 dt$). Furthermore, $W^{2,1}(I, \mathbf{R}^q)$ denotes the Sobolev space of all functions $x : I \to \mathbf{R}^q$ such that x and x' are absolutely continuous (thus, with $x'' \in L^1(I, \mathbf{R}^q)$), endowed with the norm

$$\|x\|_{W^{2,1}} = \|x\|_{L^1} + \|x'\|_{L^1} + \|x''\|_{L^1}.$$

We recall that a set $K \subset L^1(I, \mathbf{R}^q)$ is said to be *decomposable* (see Hiai and Umegaki [12]) if $u\chi_J + v\chi_{I \setminus J} \in K$ whenever $u, v \in K$ and J is any measurable subset of I. Here χ_A stands for the characteristic function of a set $A \subset I$. The family of all non-empty, decomposable, closed, bounded subsets of $L^1(I, \mathbf{R}^q)$ is denoted by $\mathcal{D}_{L^1(I,\mathbf{R}^q)}$.

Let Z be a Hausdorff topological space. A subspace X of Z is said to be a *retract* of Z if there is a continuous map $r : Z \to X$ satisfying $r(x) = x$ for every $x \in X$. Any such map r is called *retraction* of Z onto X. Clearly, if X is a retract of Z, then X is closed in Z. A metrizable space X is said to be an *absolute retract* (for metrizable spaces) if for every homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a retract of Y. We recall that every retract of a convex set of a normed space is an absolute retract (see Borsuk [2], p. 85).

2. Topological properties of S_F

THEOREM 1. *Let $F : I \times \mathbf{R}^q \times \mathbf{R}^q \to \mathcal{K}(\mathbf{R}^q)$ satisfy* (L). *Then the solution set S_F of the boundary value problem* (BV) *is a retract of $W^{2,1}(I, \mathbf{R}^q)$.*

PROOF. For $u \in L^1(I, \mathbf{R}^q)$ we denote by $x(u) : I \to \mathbf{R}^q$ the solution of the boundary value problem

$$(P_u) \qquad \begin{cases} x''(t) = u(t) \\ x(0) = x(1) = 0. \end{cases}$$

This solution exists, is unique, and is given by

$$(2.1) \qquad x(u)(t) = \int_0^t \left(\int_0^\tau su(s)\, ds - \int_\tau^1 (1-s)u(s)\, ds \right) d\tau, \quad t \in I.$$

For $u \in L^1(I, \mathbf{R}^q)$, set

$$(2.2) \qquad \mathcal{U}(u) = \{\sigma \in L^1(I, \mathbf{R}^1) \mid \sigma(t) \in F(t, x(u)(t), x'(u)(t)),\ t \in I \text{ a.e.}\}.$$

Clearly $\mathcal{U}(u)$ is a non-empty decomposable closed subset of $L^1(I, \mathbf{R}^q)$. From $F(t, x(u)(t),\ x'(u)(t)) \subset F(t, 0, 0) + \widetilde{B}_{\mathbf{R}^q}(0, D_{\mathbf{R}^q}(F(t, x(u)(t),\ x'(u)(t)),\ F(t, 0, 0)))$ and assumption (L), it follows that $\mathcal{U}(u)$ is bounded in $L^1(I, \mathbf{R}^q)$. Thus (2.2) defines a multifunction $\mathcal{U} : L^1(I, \mathbf{R}^q) \to \mathcal{D}_{L^1(I, \mathbf{R}^q)}$.

For every $u_1, u_2 \in L^1(I, \mathbf{R}^q)$ we have

$$(2.3) \qquad D_{L^1}(\mathcal{U}(u_1), \mathcal{U}(u_2)) \leq k\|u_1 - u_2\|_{L^1},$$

where k is the constant occurring in (L). Indeed, let $u_1, u_2 \in L^1(I, \mathbf{R}^q)$. Let $x(u_1)$ and $x(u_2)$ be the solutions of (P_{u_1}) and (P_{u_2}), respectively. From (2.1) we have

$$(2.4) \quad \|x(u_1) - x(u_2)\|_C \leq \|u_1 - u_2\|_{L^1}, \qquad \|x'(u_1) - x'(u_2)\|_C \leq \|u_1 - u_2\|_{L^1}.$$

Let $\sigma_1 \in \mathcal{U}(u_1)$ be arbitrary. Since the multifunction $\Phi : I \to \mathcal{K}(\mathbf{R}^q)$ given by

$$\Phi(t) = F(t, x(u_2)(t), x'(u_2)(t)) \cap \widetilde{B}_{\mathbf{R}^q}(\sigma_1(t), d_{\mathbf{R}^q}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t))),$$

for $t \in I$ is measurable, there exists $\sigma_2 \in \mathcal{U}(u_2)$ satisfying

$$(2.5) \qquad |\sigma_1(t) - \sigma_2(t)| = d_{\mathbf{R}^q}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t))), \quad t \in I \text{ a.e.}$$

By virtue of (2.5), assumption (L) (ii), and (2.4) we have:

$$\begin{aligned}\|\sigma_1-\sigma_2\| &= \int_I d_{\mathbf{R}^q}(\sigma_1(t), F(t, x(u_2)(t), x'(u_2)(t)))dt\\ &\le \int_I D_{\mathbf{R}^q}(F(t, x(u_1)(t), x'(u_1)(t)), F(t, x(u_2)(t), x'(u_2)(t)))dt\\ &\le \int_I (a|x(u_1)(t)-x(u_2)(t)| + b|x'(u_1)(t)-x'(u_2)(t)|)dt\\ &\le k\|u_1-u_2\|_{L^1}\end{aligned}$$

Hence $d_{L^1}(\sigma_1, \mathcal{U}(u_2)) \le k\|u_1-u_2\|_{L^1}$ and thus, as $\sigma_1 \in \mathcal{U}(u_1)$ is arbitrary,

$$\sup_{\sigma_1\in\mathcal{U}(u_1)} d_{L^1}(\sigma_1, \mathcal{U}(u_2)) \le k\|u_1-u_2\|_{L^1}.$$

Combining this with the analogous inequality obtained by interchanging the roles of u_1 and u_2 gives (2.3).

Put $\mathrm{Fix}(\mathcal{U}) = \{u \in L^1(I, \mathbf{R}^q) \mid u \in \mathcal{U}(u)\}$. By a result of Nadler [17], $\mathrm{Fix}(\mathcal{U})$ is a non-empty closed subset of $L^1(I, \mathbf{R}^q)$. By a theorem of Bressan, Cellina and Fryszkowski [4] the set $\mathrm{Fix}(\mathcal{U})$ is a retract of $L^1(I, \mathbf{R}^q)$. Hence there exists a continuous map $r : L^1(I, \mathbf{R}^q) \to \mathrm{Fix}(\mathcal{U})$ satisfying $r(u) = u$ for every $u \in \mathrm{Fix}(\mathcal{U})$. For $x \in W^{2,1}(I, \mathbf{R}^q)$ define $Rx : I \to \mathbf{R}^q$ by

$$(Rx)(t) = \int_0^t \left(\int_0^\tau s r(x'')(s)\,ds - \int_\tau^1 (1-s)r(x'')(s)\,ds\right) d\tau, \qquad t \in I. \tag{2.6}$$

Clearly, Rx coincides with the solution of the boundary value problem

$$\begin{cases} y''(t) = r(x'')(t)\\ y(0) = y(1) = 0.\end{cases}$$

As $r(x'') \in \mathrm{Fix}(\mathcal{U})$, we have $r(x'') \in \mathcal{U}(r(x''))$ and thus

$$(Rx)''(t) = r(x'')(t) \in F(t, (Rx)(t), (Rx)'(t)), \qquad t \in I \text{ a.e.}$$

Since, in addition, Rx and $(Rx)'$ are absolutely continuous and $(Rx)(0) = (Rx)(1) = 0$, it follows that $Rx \in S_F$. Thus, denoting by R the map which associates with each $x \in W^{2,1}(I, \mathbf{R}^q)$ the function Rx given by (2.6), we have:

$$R : W^{2,1}(I, \mathbf{R}^q) \to S_F.$$

The map R is continuous. In fact, let $x_0, x \in W^{2,1}(I, \mathbf{R}^q)$ and $\varepsilon > 0$ be arbitrary. From (2.6), by simple calculations, we have

$$\|Rx - Rx_0\|_{W^{2,1}} \le 3\|r(x'') - r(x_0'')\|_{L^1}.$$

Take $\delta > 0$ so that $\|r(u) - r(x_0'')\|_{L^1} < \varepsilon/3$ for every $u \in B_{L^1}(x_0'', \delta)$. Let $x \in B_{W^{2,1}}(x_0, \delta)$ be arbitrary. As $x'' \in B_{L^1}(x_0'', \delta)$ we have $\|r(x'') - r(x_0'')\|_{L^1} < \varepsilon/3$, and thus $\|Rx - Rx_0\|_{W^{2,1}} < \varepsilon$. Hence R is continuous.

For each $x \in S_F$ we have $Rx = x$. Indeed; let $x \in S_F$ be arbitrary. Put $u = x''$. Denoting by $y(u)$ the solution of (P_u) we have $y(u) = x$, and so $u(t) = x''(t) \in F(t, x(t), x'(t)) = F(t, y(u)(t), y'(u)(t)), t \in I$ a.e. Hence $u \in \mathcal{U}(u)$, which implies $r(u) = u$ and thus, $r(x'') = x''$. Consequently, for each $t \in I$,

$$(Rx)(t) = \int_0^t \left(\int_0^\tau sr(x'')(s)\,ds - \int_\tau^1 (1-s)r(x'')(s)\,ds \right) d\tau$$
$$= \int_0^t \left(\int_0^\tau sx''(s)\,ds - \int_\tau^1 (1-s)x''(s)\,ds \right) d\tau = x(t),$$

that is $Rx = x$. It follows that R is a retraction of $W^{2,1}(I, \mathbf{R}^q)$ onto S_F. This completes the proof.

3. Continuation

THEOREM 2. *Let $F : I \times \mathbf{R}^q \times \mathbf{R}^q \to \mathcal{C}(\mathbf{R}^q)$ satisfy* (L), *where the function $m : I \to \mathbf{R}$ is square integrable. Then the solution set S_F of the boundary value problem* (BV) *is a retract of $C^1(I, \mathbf{R}^q)$.*

PROOF. For $y \in C^1(I, \mathbf{R}^q)$ we set

$$\mathcal{U}(y) = \{u \in L^1(I, \mathbf{R}^q) \mid u(t) \in F(t, y(t), y'(t)), \quad t \in I \text{ a.e.}\}. \tag{3.1}$$

Clearly $\mathcal{U}(y)$ is a non-empty, convex, closed and bounded subset of $L^1(I, \mathbf{R}^q)$. For $y \in C^1(I, \mathbf{R}^q)$ we define

$$\mathcal{F}(y) = \{x(u) \mid u \in \mathcal{U}(y)\}. \tag{3.2}$$

Here, for $u \in \mathcal{U}(y), x(u)$ denotes the solution of (P_u).

$\mathcal{F}(y)$ is a non-empty, convex and compact subset of $C^1(I, \mathbf{R}^q)$. It is evident that $\mathcal{F}(y)$ is non-empty and convex. To show that $\mathcal{F}(y)$ is compact, consider an arbitrary sequence $\{z_n\} \subset \mathcal{F}(y)$. Let $\{u_n\} \subset \mathcal{U}(y)$ be such that $z_n = x(u_n), n \in \mathbb{N}$. Since, for $t \in I$ a.e.,

$$u_n(t) \in F(t,0,0) + \widetilde{B}_{\mathbf{R}^q}(0, D_{\mathbf{R}^q}(F(t,0,0), F(t, y(t), y'(t))))$$
$$\subset \widetilde{B}_{\mathbf{R}^q}(0, m(t) + a|y(t)| + b|y'(t)|),$$

where the function $t \to m(t) + a|y(t)| + b|y'(t)|$ is square integrable, there exists a subsequence, say $\{u_n\}$, which converges weakly in $L^2(I, \mathbf{R}^q)$ to some $u \in L^2(I, \mathbf{R}^q)$. Clearly, $u \in L^1(I, \mathbf{R}^q)$ and $\{u_n\}$ converges to u weakly in $L^1(I, \mathbf{R}^q)$. By Mazur's theorem [13] it is easy to see that $u \in \mathcal{U}(y)$. Now, for $n \in \mathbb{N}$ and $t \in I$, we have:

$$x(u_n)(t) - x(u)(t) = \int_0^t \left(\int_0^\tau s(u_n(s) - u(s))\,ds \right) d\tau - \int_0^t \left(\int_\tau^1 (1-s)(u_n(s) - u(s))\,ds \right) d\tau \tag{3.3}$$

$$x'(u_n)(t) - x'(u)(t) = \int_0^t s(u_n(s) - u(s))\,ds - \int_t^1 (1-s)(u_n(s) - u(s))\,ds. \tag{3.4}$$

Since $\{u_n\}$ converges to u weakly in $L^1(I, \mathbf{R}^q)$, from (3.3) and (3.4) it follows that $\{x(u_n)\}$ and $\{x'(u_n)\}$ converge in $C(I, \mathbf{R}^q)$ to $x(u)$ and $x'(u)$, respectively. Hence $\{x(u_n)\}$ converges to $x(u)$ in $C^1(I, \mathbf{R}^q)$. As $x(u) \in \mathcal{F}(y)$, the set $\mathcal{F}(y)$ is compact. Thus (3.2) defines a non-empty, convex, compact valued multifunction

$$\mathcal{F} : C^1(I, \mathbf{R}^q) \to \mathcal{C}(C^1(I, \mathbf{R}^q)).$$

For every $y_1, y_2 \in C^1(I, \mathbf{R}^q)$ we have

$$D_{C^1}(\mathcal{F}(y_1), \mathcal{F}(y_2)) \le k\|y_1 - y_2\|_{C^1}, \tag{3.5}$$

where k is the constant occurring in (L). Indeed, let $y_1, y_2 \in C^1(I, \mathbf{R}^q)$. Let $z_1 \in \mathcal{F}(y_1)$ be arbitrary, thus $z_1 = x(u_1)$ for some $u_1 \in \mathcal{U}(y_1)$. As in the proof of Theorem 1, take $u_2 \in \mathcal{U}(y_2)$ satisfying

$$|u_1(t) - u_2(t)| = d_{\mathbf{R}^q}(u_1(t), F(t, y_2(t), y_2'(t))), \qquad t \in I \text{ a.e.}, \tag{3.6}$$

and set $z_2 = x(u_2)$. Clearly, $z_2 \in \mathcal{F}(y_2)$. Using the representation of $x(u_1)$ and $x(u_2)$ given by (2.1), by simple calculations, for every $t \in I$, we have:

$$\begin{aligned} &|x(u_1)(t) - x(u_2)(t)| \\ &\quad = \left| (t-1) \int_0^t s(u_1(s) - u_2(s))\,ds - t \int_t^1 (1-s)(u_1(s) - u_2(s))\,ds \right| \\ &\quad \le \int_0^t |u_1(s) - u_2(s)|\,ds + \int_t^1 |u_1(s) - u_2(s)|\,ds = \int_I |u_1(s) - u_2(s)|\,ds. \end{aligned}$$

From this, using (3.6) and assumption (L) (ii), for every $t \in I$ we obtain:

$$\begin{aligned} |x(u_1)(t) - x(u_2)(t)| &\le \int_I d_{\mathbf{R}^q}(u_1(t), F(t, y_2(t), y_2'(t)))dt \\ &\le \int_I D_{\mathbf{R}^q}(F(t, y_1(t), y_1'(t)), F(t, y_2(t), y_2'(t)))dt \\ &\le \int_I (a|y_1(t) - y_2(t)| + b|y_1'(t) - y_2'(t)|)dt \\ &\le k\|y_1 - y_2\|_{C^1}. \end{aligned}$$

Consequently $\|z_1 - z_2\|_C \leq k\|y_1 - y_2\|_{C^1}$. Likewise one can show that $\|z_1' - z_2'\|_C \leq k\|y_1 - y_2\|_{C^1}$. Hence, $\|z_1 - z_2\|_{C^1} \leq k\|y_1 - y_2\|_{C^1}$. A fortiori, $d_{C^1}(z_1, \mathcal{F}(y_2)) \leq k\|y_1 - y_2\|_{C^1}$ and thus, as $z_1 \in \mathcal{F}(y_1)$ is arbitrary,

$$\sup_{z_1 \in \mathcal{F}(y_1)} d_{C^1}(z_1, \mathcal{F}(y_2)) \leq k\|y_1 - y_2\|_{C^1}.$$

From this and the analogous inequality obtained by interchanging the roles of y_1 and y_2 we obtain (3.5).

Put $\mathrm{Fix}(\mathcal{F}) = \{y \in C^1(I, \mathbf{R}^q) \mid y \in \mathcal{F}(y)\}$, and observe that $\mathrm{Fix}(\mathcal{F})$ is a non-empty closed subset of $C^1(I, \mathbf{R}^q)$. By a result of Ricceri [21] $\mathcal{F}(y)$ is a retract of $C^1(I, \mathbf{R}^q)$. It is routine to show that $\mathrm{Fix}(\mathcal{F}) = S_F$. Hence S_F is a retract of $C^1(I, \mathbf{R}^q)$ and the proof of the theorem is complete.

REMARK 1. By Theorem 1 (resp. Theorem 2), the space S_F with the $W^{2,1}(I, \mathbf{R}^q)$ (resp. $C^1(I, \mathbf{R}^q)$) metric is an absolute retract.

REMARK 2. Theorem 2 is no longer true if F is not convex valued. To see this, denote by S the solution set of the boundary value problem

$$\begin{cases} x''(t) \in \{-1, 1\}, \\ x(0) = x(1) = 0. \end{cases} \tag{3.7}$$

Since S is not closed in $C^1(I, \mathbf{R})$, the set S cannot be a retract of $C^1(I, \mathbf{R})$. On the other hand, from Theorem 1, S is a retract of $W^{2,1}(I, \mathbf{R})$ and so S is closed in $W^{2,1}(I, \mathbf{R})$.

4. An existence result

THEOREM 3. *Let $F : I \times \mathbf{R}^q \times \mathbf{R}^q \to \mathcal{K}(\mathbf{R}^q)$ satisfy* (C). *Then the solution set S_F of the boundary value problem* (BV) *is non-empty.*

PROOF. For $u \in L^1(I, \mathbf{R}^q)$ denote by $y(u)$ the solution of (P_u). Set

$$\Omega = \{y \in C^1(I, \mathbf{R}^q) | y = y(u) \text{ for some measurable } u \text{ with } |u(t)| \leq m(t),\ t \in I \text{ a.e.}\}.$$

Clearly Ω is non-empty and convex. Moreover Ω, endowed with the $C^1(I, \mathbf{R}^q)$ metric, is a compact space. To see this, let $\{y(u_n)\} \subset \Omega$ be an arbitrary sequence where, for each $n \in \mathbb{N}$, $u_n : I \to \mathbf{R}^q$ is measurable and $|u_n(t)| \leq m(t)$, $t \in I$ a.e. As m is square integrable, there is a subsequence, say $\{u_n\}$, which converges weakly to some u in $L^2(I, \mathbf{R}^q)$ and so also in $L^1(I, \mathbf{R}^q)$. By the Mazur theorem [13] one has $|u(t)| \le m(t)$, $t \in I$ a.e., and so $y(u) \in \Omega$. By using the representation of the solution of (P_u) furnished by (2.1), it follows that $\{y(u_n)\}$ converges to $y(u)$ in $C^1(I, \mathbf{R}^q)$, proving that Ω is compact.

For $y \in \Omega$, let $\mathcal{U}(y)$ be given by (3.1). As $\mathcal{U}(y)$ is a non-empty, decomposable, closed, bounded subset of $L^1(I, \mathbf{R}^q)$, (3.1) defines a multifunction $\mathcal{U} : \Omega \to \mathcal{D}_{L^1(I,\mathbb{R}^q)}$. It is routine to verify that $\mathcal{U}$ is Hausdorff lower semicontinuous. By virtue of Theorem 3 of Bressan and Colombo [5], there exists a continuous function $\sigma : \Omega \to L^1(I, \mathbf{R}^q)$ satisfying

$$\sigma(y) \in \mathcal{U}(y) \qquad \text{for every } y \in \Omega. \tag{4.1}$$

For $y \in \Omega$, let $x(y) : I \to \mathbf{R}^q$ denote the solution of the boundary value problem

$$\begin{cases} x''(t) = \sigma(y)(t) \\ x(0) = x(1) = 0. \end{cases}$$

This solution exists, is unique, and is given by

$$x(y)(t) = \int_0^t \left(\int_0^r s\sigma(y)(s)\, ds - \int_r^1 (1-s)\sigma(y)(s)\, ds \right) dr, \qquad t \in I. \tag{4.2}$$

Clearly $x(y) \in \Omega$. Denote by $T : \Omega \to \Omega$ the map defined by $Ty = x(y)$, $y \in \Omega$. T is continuous. Indeed, let $y_0, y \in \Omega$. From (4.2), we have

$$\|Ty - Ty_0\|_C \le \|\sigma(y) - \sigma(y_0)\|_{L^1},$$
$$\|(Ty)' - (Ty_0)'\|_C \le \|\sigma(y) - \sigma(y_0)\|_{L^1}.$$

Hence

$$\|Ty - Ty_0\|_{C^1} \le \|\sigma(y) - \sigma(y_0)\|_{L^1},$$

which implies that T is continuous, for $\sigma : \Omega \to L^1(I, \mathbf{R}^q)$, is so. By Schauder's fixed point theorem, there exists $y \in \Omega$ such that $y = Ty$, thus

$$y(t) = \int_0^t \left(\int_0^r s\sigma(y)(s)\, ds - \int_r^1 (1-s)\sigma(y)(s)\, ds \right) dr, \qquad t \in I.$$

Since y and y' are absolutely continuous, $y(0) = y(1) = 0$ and, by virtue of (4.1) and (3.1),

$$y''(t) = \sigma(y)(t) \in F(t, y(t), y'(t)), \qquad t \in I \text{ a.e.},$$

it follows that y is a solution of the boundary value problem (BV). Thus S_F is non-empty, completing the proof.

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