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# ON UNBOUNDED SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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*Dedicated to the memory of Karol Borsuk*

## 1. Introduction

In this paper we consider the oscillatory and nonoscilatory behavior of solutions of the functional differential equation

$$(\mathrm{E}_1) \qquad x'' + q(t)F(x(g(t)), x'(h(t))) = 0 \qquad \Big(\, ' = \frac{d}{dt}\Big)$$

Throughout by a solution of $(\mathrm{E}_1)$ we shall mean a twice continuously differentiable function which exists on some half-line $[t_x, +\infty)$, satisfies $(\mathrm{E}_1)$ and does not eventually vanish.

As usual a solution of $(\mathrm{E}_1)$ is said to be oscillatory or nonoscillatory according to whether it does or does not have arbitrarily large zeros. A nonoscillatory solution $x$ of $(\mathrm{E}_1)$ is said to be weakly oscillatory if $x'$ changes sign for arbitrarily large values of $t$ (see for example, [14], [15]).

In the study of the qualitative behavior of solutions of functional differential equations, it is often assumed that the solutions under consideration are continuable to the right for large $t$ and the oscillatory character of those solutions is obtained by means of integral inequalities and/or integral averaging techniques. It is clear

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that such an approach assures that all the continuable solutions are oscillatory, but does not guarantee their existence.

Another important tool in studying such problems is based on fixed point theorems in Banach or Fréchet spaces. Such a topological method has the feature of assuring also the continuability of solutions under consideration, but, as far as we know, it seems not to have wide applicability in the study of oscillations. On the other hand it is often usefully employed in proving the existence of nonoscillatory solutions with a fixed asymptotic behavior.

Furthermore such a topological approach often requires an appropriate analysis due essentially to the fact that to apply fixed point theorem one needs a suitable topology which makes the fixed point operator continuous with some additional properties (such as compactness, for instance). In fact the operators associated with asymptotic boundary value problems in noncompact intervals may be often discontinuous and noncompact if regarded in their whole domain and not merely in an appropriate bounded subset. A study of continuity and compactness of such operators is given in [3], [4].

In this paper we are interested in the existence of positive increasing unbounded solutions of ($E_1$), which are defined in some neighbourhood of infinity ($+\infty$). A topological approach based on the Tychonov fixed point theorem will be used. In fact nonoscillatory solutions are obtained as fixed points of a suitable operator $\mathcal{M}$ defined by the Schauder linearization device which acts from $C^1(J,\mathbb{R})$ into $C^1(J,\mathbb{R})$, $J=[t_1,\infty)$, $t_1 \geq 0$.

Before stating the main result, we find it convenient to give a brief survey of the corresponding case without deviating argument. Equation ($E_1$) and the corresponding ordinary differential equation

$$(E_2) \qquad x'' + q(t)F(x(t), x'(t)) = 0$$

have been deeply studied. We refer in particular to [6], [9], [16], [17] and to the references contained therein. For the linear equation

$$(E_3) \qquad x'' + q(t)x(t) = 0$$

where $q$ is positive and continuous for $t > 0$, the assumption

$$(1) \qquad \int_a^{+\infty} tq(t)\,dt = \lim_{T\to+\infty} \int_a^T tq(t)\,dt < +\infty$$

guarantees that ($E_3$) is nonoscillatory, i.e. that every solution of ($E_3$) is nonoscillatory (see, e.g., [12], Ch. XI). Since the Sturm Theorem fails in the nonlinear case, it is obvious that in general the same situation does not happen for eq. ($E_2$). In fact in this case oscillatory and nonoscillatory solutions may coexist. Indeed it is sufficient to consider the nonlinear equation

$$x'' + t^{-3}x^3 = 0 \qquad (t > 0)$$

which has simultaneously both oscillatory and nonoscillatory solutions (see, e.g., [22]), while the corresponding linear equation is nonoscillatory.

Another difference between the linear and nonlinear case is connected to the existence of nonoscillatory unbounded solutions, as the following example shows. The linear equation

$$x'' + t^{-2}x = 0 \qquad (t > 0)$$

is nonoscillatory and has unbounded solutions ([20]), while all the solutions of the corresponding nonlinear equation

$$x'' + t^{-2}x^3 = 0 \qquad (t > 0)$$

are bounded ([22]). For the nonlinear equation

$$(E_4) \qquad x'' + q(t)x^{2n-1} = 0 \qquad (n > 1)$$

where $q$ is positive and continuous for $t > 0$, a classical oscillation result given by Atkinson ([1]) states that every solution of $(E_4)$ is oscillatory if and only if

$$(2) \qquad \int_a^{+\infty} tq(t)\,dt = +\infty.$$

Furhermore the argument employed in [1] guarantees also that, if assumption (1) holds, then $(E_4)$ has at least a bounded nonoscillatory solution.

The above result has been extended to more general equations by many authors under suitable monotonicity conditions on the nonlinear term: we recall here only the papers [18], [19] and the excellent survey [22].

The deviating argument may generate some difference between the ordinary and functional cases especially as regards to the existence of unbounded nonoscillatory solutions, as the following example shows. For the functional differential equation

$$(E_5) \qquad x'' + \frac{1}{4}t^{-9/5}\left[x(t^{1/5})\right]^3 = 0 \qquad (t > 0)$$

the function $x(t) = t^{1/2}$ is an unbounded nonoscillatory solution, while from the quoted Atkinson's result all the solutions of the corresponding ordinary equation

$$(E_6) \qquad x'' + \frac{1}{4}t^{-9/5}x^3 = 0 \qquad (t > 0)$$

are oscillatory.

In this paper we extend to functional differential equation $(E_1)$ the quoted results in [1], [18], [19] concerning the ordinary case.

Concerning the functional differential equation $(E_1)$, assuming that the deviating argument $g$ is such that $g(t) < t$ and $\lim_{t\to+\infty} g'(t) = +\infty$, in [14], Ch. IV.5, it is proved that all the solutions of $(E_1)$ are oscillatory if and only if assumption (2) holds. Furthermore in [14], Ch. IV.5 sufficient conditions in order that $(E_1)$ has bounded solutions are given.

In this paper the oscillatory character of all the bounded solutions is also considered. It is shown that additional conditions on the deviating argument $g$ different from $g(+\infty) = +\infty$ are not required in order that assumption (2) is again necessary and sufficient for the oscillatory behavior of every bounded solution of $(E_1)$. Then such a result extends the quoted one in [14]. The necessity is obtained as a consequence of a recent result of the authors [5]; the sufficiency is obtained by means of a wide analysis of the asymptotic behavior of solutions of $(E_1)$.

Related results on the asymptotic behavior of the bounded solutions of functional differential equations are in [10], [11], [13], [21].

## 2. Unbounded solutions

Consider the equation

$$x'' + q(t)F(x(g(t)), x'(h(t))) = 0 \tag{$E_1$}$$

where

(i) $q \in C([0, +\infty), \mathbb{R})$, $q(t) > 0$;
(ii) $g \in C([0, +\infty), \mathbb{R})$, $g(t) > 0$, $g(+\infty) = +\infty$;
(iii) $h \in C([0, +\infty), \mathbb{R})$, $h(t) > 0$, $h(+\infty) = +\infty$;
(iv) $F \in C(\mathbb{R}^2, \mathbb{R})$, $u \cdot F(u, v) > 0$ for $u \neq 0$.

We now prove that if $F(g(\cdot), h(\cdot))$ satisfies a suitable growth condition, then there exists an unbounded nonoscillatory solution of the functional differential equation $(E_1)$.

THEOREM 1. *Let $F(\cdot, v)$, $F(u, \cdot)$ be nondecreasing and let*

$$\int_0^{+\infty} q(s)F(g(s), h(s))ds < +\infty. \tag{3}$$

*Then there exists at least one an eventually positive increasing unbounded solution of the equation* $(E_1)$.

PROOF. Let $t_0 \geq 0$ be sufficiently large such that

$$\int_{t_0}^{+\infty} q(s)F(g(s), h(s))\, ds < \frac{1}{2}, \qquad h(t) \geq 1 \text{ for } t \geq t_0 \tag{4}$$

and let $t_1 \geq t_0$ be such that

$$g(t) \geq t_0, \quad h(t) \geq t_0 \quad \text{for } t \in [t_1, +\infty).$$

Consider now the nonempty, closed, bounded, convex subset $\Omega$ of $C^1([t_0,+\infty),\mathbb{R})$ given by

$$\Omega = \{u \in C^1([t_0,+\infty),\mathbb{R}) : 0 \le u(t) \le t, 0 \le u'(t) \le 1\}$$

and for every $u \in \Omega$ define the operator $\mathcal{M}u$:

$$\begin{aligned}(\mathcal{M}u)(t) =&\left(\int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2}\right)(t-t_0)\\ &- \int_{t_1}^{t}\int_{t_1}^{r} q(s)F(u(g(s)),u'(h(s)))\,ds\,dr\end{aligned}$$

for $t > t_1$ and

$$(\mathcal{M}u)(t) = \alpha(t) \qquad \text{for } t \le t \le t_1$$

where $\alpha = \alpha(t)$ is a suitable $C^1$-function such that

$$\begin{aligned}&\alpha(t_1) = (t_1-t_0)\left(\int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds\right) + \frac{1}{2}(t_1-t_0),\\ &\alpha'(t_1) = \frac{1}{2} + \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds,\\ &0 \le \alpha(t) \le t, \quad 0 \le \alpha'(t) \le 1 \qquad \text{for } t \in [t_0,t_1].\end{aligned}$$

Let us prove that $\mathcal{M}(\Omega) \subset \Omega$. This assertion follows easily from the following four claims a), b), c), d).

CLAIM a): $(\mathcal{M}u)(t) \le t$.

For $t \in [t_0,t_1]$ the assertion follows from the definition of the function $\alpha$. For $t > t_1$ we have

$$(\mathcal{M}u)(t) \le \left(\int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2}\right)(t-t_0) \le t-t_0 \le t.$$

CLAIM b): $(\mathcal{M}u)(t) \ge 0$.

(5)

$$
\begin{aligned}
(\mathcal{M}u)(t) =& \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds \right)(t_1 - t_0) \\
&+ \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds \right)(t - t_1) \\
&+ \frac{1}{2}(t - t_0) - \int_{t_1}^{t} \left( \int_{t_1}^{r} q(s)F(u(g(s)),u'(h(s)))\,ds \right) dr \\
=& \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds \right)(t_1 - t_0) + \frac{1}{2}(t - t_0) \\
&+ \int_{t_1}^{t} \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds - \int_{t_1}^{r} q(s)F(u(g(s)),u'(h(s)))\,ds \right) dr.
\end{aligned}
$$

Since $u \in \Omega$, using (4) we get for $s \geq t_1$

$$
\begin{aligned}
&u(g(s)) \leq g(s) \\
&u'(h(s)) \leq 1 \leq h(s).
\end{aligned} \tag{6}
$$

From (5) and (6), taking into account that $F(u,v)$ is nondecreasing with respect to $u$ and $v$, we obtain

$$
\begin{aligned}
(\mathcal{M}u)(t) \geq& \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds \right)(t_1 - t_0) + \frac{1}{2}(t - t_0) \\
&+ \int_{t_1}^{t} \left( \int_{r}^{+\infty} q(s)F(g(s),h(s))\,ds \right) dr > 0.
\end{aligned}
$$

For $t \in [t_0, t_1]$ the assertion follows as in claim a).

CLAIM c): $\frac{d}{dt}(\mathcal{M}u)(t) \leq 1$.

For $t > t_1$, taking into account (4), we obtain

$$
\begin{aligned}
\frac{d}{dt}(\mathcal{M}u)(t) &= \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} - \int_{t_1}^{t} q(s)F(u(g(s)),u'(h(s)))\,ds \\
&\leq \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} \leq 1.
\end{aligned}
$$

For $t \in [t_0, t_1]$ the assertion follows as in claim a).

CLAIM d): $\frac{d}{dt}(\mathcal{M}u)(t) \geq 0$.

For $t > t_1$ reasoning as in claim a) we obtain

$$\tag{7} \begin{aligned} \frac{d}{dt}(\mathcal{M}u)(t) &\geq \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} - \int_{t_1}^{t} q(s)F(g(s),h(s))\,ds \\ &= \int_{t}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} > 0. \end{aligned}$$

Finally for $t \in [t_0, t_1]$ the assertion follows as in claim a).

In order to apply to operator $\mathcal{M}$ the Tychonov fixed point theorem, it is sufficient now to prove that $\mathcal{M}$ is continuous in $\Omega$ endowed with $C^1$-topology and that $\mathcal{M}(\Omega)$ is relatively compact in $C^1([t_0,+\infty),\mathbb{R})$.

Let $\{u_n\} \subset \Omega$, $u_n \xrightarrow{C^1} u$, $u \in \Omega$: we need to prove that $\mathcal{M}u_n \xrightarrow{C^1} \mathcal{M}u$, e.g. the sequences $\{\mathcal{M}u_n\}$ and $\{\frac{d}{dt}(\mathcal{M}u_n)\}$ tend, uniformly on every compact set of $[t_0,+\infty)$, to $\mathcal{M}u$ and $\frac{d}{dt}(\mathcal{M}u)$ respectively.

For $t > t_1$ we have

$$\begin{aligned} \left|(\mathcal{M}u_n)(t) - (\mathcal{M}u)(t)\right| &= \left| \int_{t_1}^{t} \left( \int_{t_1}^{r} q(s)F(u_n(g(s)),u'(h(s)))\,ds \right) \right. \\ &\quad \left. - \int_{t_1}^{t} \left( \int_{t_1}^{r} q(s)F(u(g(s)),u'(h(s)))\,ds \right) dr \right| \\ &\leq \int_{t_1}^{t} \left( \int_{t_1}^{r} q(s)\left|F(u_n(g(s)),u_n'(h(s))) - F(u(g(s)),u'(h(s)))\right|\,ds \right) dr. \end{aligned}$$

Hence from the continuity of the functions $f,g,h$ we get that the sequence $\{\mathcal{M}u_n\}$ tends, uniformly on every compact set of $[t_0,+\infty)$, to $\mathcal{M}u$. Further we have

$$\begin{aligned} &\left| \frac{d}{dt}(\mathcal{M}u_n)(t) - \frac{d}{dt}(\mathcal{M}u)(t) \right| \\ &\qquad \leq \int_{t_1}^{t} q(s)|F(u_n(g(s)),u_n'(h(s))) - F(u(g(s)),u'(h(s)))|\,ds \end{aligned}$$

so that also the sequence $\{\frac{d}{dt}(\mathcal{M}u_n)\}$ tends, uniformly on every compact set of $[t_0,+\infty)$, to $\frac{d}{dt}(\mathcal{M}u)$. For $t \in [t_0,t_1]$ the above assertions follow from the definition of the operator $\mathcal{M}$.

Concerning the compactness of $\mathcal{M}(\Omega)$ in $C^1([t_0,+\infty),\mathbb{R})$ it is sufficient to show that if $\{u_n\} \subset \Omega$, then the sequences $\{\mathcal{M}u_n\}$ and $\{\frac{d}{dt}(\mathcal{M}u_n)\}$ are equibounded and equicontinuous on every compact set of $[t_0,+\infty)$.

The equiboundedness easily follows taking into account that $\mathcal{M}(\Omega) \subset \Omega$ and $\Omega$ is a bounded subset of $C^1([t_0,+\infty),\mathbb{R})$. The equicontinuity of the sequence $\{\mathcal{M}u_n\}$ follows from the equiboundedness of the sequence $\{\frac{d}{dt}(\mathcal{M}u_n)\}$. Let us prove the equicontinuity of the sequence $\{\frac{d}{dt}(\mathcal{M}u_n)\}$.

For $r_1, r_2 \in [t_1, +\infty)$ we have

$$\left|\frac{d}{dt}(\mathcal{M}u_n)(r_2) - \frac{d}{dt}(\mathcal{M}u_n)(r_1)\right| \leq \left|\int_{r_1}^{r_2} q(s)F(u_n(g(s)), u_n'(h(s)))\,ds\right|.$$

Since $u_n$ belongs to $\Omega$, taking into account (6), for $t > t_1$ we get $u_n(g(t)) \leq g(t)$, $u_n'(h(t)) \leq h(t)$; hence we obtain

$$\left|\frac{d}{dt}(\mathcal{M}u_n)(r_2) - \frac{d}{dt}(\mathcal{M}u_n)(r_1)\right| \leq \left|\int_{r_1}^{r_2} q(s)F(g(s), h(s))\,ds\right|.$$

so that equicontinuity of the sequence $\left\{\frac{d}{dt}(\mathcal{M}u_n)\right\}$ in $[t_1, +\infty)$ is proved. For $t \in [t_0, t_1]$ the assertion follows from the definition of the operator $\mathcal{M}$. For $r_1 \in [t_0, t_1]$, $r_2 \in [t_1, +\infty)$, or vice versa, in view of

$$\begin{aligned}\frac{d}{dt}(\mathcal{M}u_n)(r_2) &- \frac{d}{dt}(\mathcal{M}u_n)(r_1) \\ &= \frac{d}{dt}(\mathcal{M}u_n)(r_2) - \frac{d}{dt}(\mathcal{M}u_n)(t_1) + \frac{d}{dt}(\mathcal{M}u_n)(t_1) - \frac{d}{dt}(\mathcal{M}u_n)(r_1),\end{aligned}$$

the assertion easily follows reasoning as in the above cases. Hence $\mathcal{M}$ is completely continuous in $\Omega$. From the Tychonov Theorem there exists $x \in \Omega$ such that $x = \mathcal{M}x$, i.e. for $t > t_1$

$$\begin{aligned}(8)\quad x(t) = &\left(\int_{t_1}^{+\infty} q(s)F(g(s), h(s))\,ds + \frac{1}{2}\right)(t - t_0) \\ &- \int_{t_1}^{t}\left(\int_{t_1}^{r} q(s)F(x(g(s)), x'(h(s)))\,ds\right)dr.\end{aligned}$$

Clearly $x$ is, for $t > t_1$, a positive solution of the equation ($\mathrm{E}_1$). From (7) we have also that $x$ is eventually increasing. In order to complete the proof it remains only to prove that $x$ is unbounded as $t \to +\infty$. To this end from (8) we obtain ($t > t_1$)

$$\begin{aligned}x(t) = &\left(\int_{t_1}^{+\infty} q(s)F(g(s), h(s))\,ds + \frac{1}{2}\right)(t_1 - t_0) + \frac{1}{2}(t - t_0) \\ &+ \int_{t_1}^{t}\left(\int_{t_1}^{+\infty} q(s)F(g(s), h(s))\,ds \right.\\ &\left. - \int_{t_1}^{r} q(s)F(x(g(s)), x'(h(s)))\,ds\right)dr.\end{aligned}$$

Then

$$\begin{aligned}
x(t) \geq & \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} \right)(t_1 - t_0) + \frac{1}{2}(t - t_0) \\
& + \int_{t_1}^{t} \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds - \int_{t_1}^{r} q(s)F(g(s),h(s))\,ds \right) dr \\
= & \int_{t_1}^{t} \left( \int_{r}^{+\infty} q(s)F(g(s),h(s))\,ds \right) dr \\
& + \left( \int_{t_1}^{+\infty} q(s)F(g(s),h(s))\,ds + \frac{1}{2} \right)(t_1 - t_0) + \frac{1}{2}(t - t_0) \\
\geq & \frac{1}{2}(t - t_0)
\end{aligned}$$

and for $t \to +\infty$ we get the assertion. The proof is now complete. □

REMARK. Concerning the existence of eventually negative decreasing unbounded solutions of ($E_1$), a similar result holds. It is sufficient to consider the subset $\Omega$ given by

$$\Omega = \{u \in C^1([0,+\infty),\mathbb{R}) : -t \leq u(t) \leq 0, -1 \leq u'(t) \leq 0\}.$$

The argument is similar to that of Theorem 1 and hence is omitted.

In Theorem 1 deviating type conditions are not assumed and so such a result may hold for ordinary, retarded, advanced and mixed type differential equations.

The functional differential equation ($E_5$) illustrates Theorem 1. In fact, since

$$\int_a^{+\infty} \frac{1}{4} s^{-9/5} F(g(s),h(s))\,ds = \int_a^{+\infty} \frac{1}{4} s^{-6/5}\,ds < +\infty \qquad (a > 0)$$

the equation ($E_5$) has unbounded increasing solutions (namely $x(t) = t^{1/2}$ is a solution) while, from the quoted Atkinson's result [1] all the solutions of the corresponding ordinary equation ($E_6$) are oscillatory.

If $F(u,v) \equiv u$ and $g(t) \equiv t$, from Theorem 1 we obtain the following well-known result concerning the linear differential equation ($E_3$) (see, e.g., [7]).

COROLLARY 1. *If the assumption* (1) *is satisfied, then* ($E_3$) *has eventually unbounded increasing [decreasing] solutions.*

We remark that Theorem 1 extends to functional differential equation ($E_1$) some results in established for particular types of ordinary differential equations established in [2], [8], [18].

## 3. Bounded oscillatory solutions

Section 2 was concerned with the existence of unbounded nonoscillatory solutions. Concerning the bounded solutions the following existence result holds.

Theorem 2. *If the assumption*

$$(*) \qquad \int_a^{+\infty} tq(t)dt < +\infty \qquad (a \geq 0)$$

*is satisfied, then* ($E_1$) *has at least a bounded eventually positive nondecreasing solution.*

For the proof of Theorem 2, which makes use of a general result [3] on compactness and continuity of operators in locally convex spaces, we refer to [5].

Theorem 2 states that if all the bounded solutions of ($E_1$) are oscillatory, then (2) holds. The following result shows that condition (2) is also sufficient for the oscillation of any bounded solution of ($E_1$).

Theorem 3. *All the bounded solutions of* ($E_1$) *are oscillatory if and only if*

$$(**) \qquad \int_a^{+\infty} tq(t)dt = +\infty \qquad (a \geq 0).$$

Proof. The necessity follows immediately from Theorem 2. Assume now that $x$ is a bounded nonoscillatory solution of ($E_1$). There is no loss of generality in assuming that $x$ is eventually positive. Because $q$ is positive, $x$ is not either weakly oscillatory [6] or eventually constant; then there are two cases:

I) $x$ is eventually decreasing;

II) $x$ is eventually increasing and bounded.

Case I). Let $t_1 > 0$ be such that for $x(t) > 0$, $x(g(t)) > 0$ for $t > t_1$ and $x'(t_1) < 0$. Then for $t > t_1$

$$x'' = -q(t)F(x(g(t)), x'(h(t))) < 0$$

and so

$$x'(t) < x'(t_1).$$

Integrating in $(t_1, t)$ we obtain

$$x(t) < x(t_1) + x'(t_1)(t - t_1)$$

which gives a contradiction as $t \to +\infty$.

Case II). Let $t_1 > 0$ be such that $x(t) > 0$, $x(g(t)) > 0$, $x'(t) > 0$ for $t > t_1$ and $\lim_{t\to+\infty} x(t) = \ell_x$, $\ell_x < +\infty$. Since $x''$ is eventually negative, then $\lim_{t\to+\infty} x'(t)$

$= 0$. Let $\varepsilon > 0$ and let $t_2 > t_1$ be such that $0 < x'(h(t)) < \varepsilon$, $\frac{\ell}{2} < x(g(t)) < \ell$ for $t > t_2$. Integrating ($\mathrm{E}_1$) in $(t, T)$, $t_2 < t < T$, we obtain

$$x'(t) - x'(T) = \int_t^T q(s)F(x(g(s)), x'(h(s)))ds \geq M \int_t^T q(s)ds$$

where $M = \min_{(u,v)\in[\ell/2,\ell]\times[0,\varepsilon]} F(u,v)$. Since $M > 0$, as $T \to +\infty$ we get

$$x'(t) \geq M \int_t^{+\infty} q(s)ds. \tag{9}$$

Thus $\int_1^{+\infty} q(s)ds < +\infty$. Integrating (9) in $(t_2, t)$ we obtain

$$\begin{aligned} x(t) - x(t_2) \geq & \int_{t_2}^t \left( \int_r^{+\infty} q(s)ds \right) dr \\ =& M \int_{t_2}^t (s - t_2)q(s)ds + M(t - t_2) \left( \int_t^{+\infty} q(s)ds \right) \\ \geq & M \int_{t_2}^t sq(s)ds - Mt_2 \left( \int_t^{+\infty} q(s)ds \right). \end{aligned}$$

which gives a contradiction as $t \to +\infty$. □

In Theorem 3 additional assumptions on the deviating argument different from $g(+\infty) = +\infty$ are not assumed. Hence our result extends previous ones in [14], Ch. IV.5, established for delay differential equations. Theorem 3 extends also to functional differential equations some quoted results in [1], [19].

Related criteria on the asymptotic behavior of oscillatory bounded solutions are in [10], [11], [13], [21].

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