Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 1, 1993, 231–245

THE TOPOLOGICAL VERSIONS OF KKM THEOREM AND FAN'S MATCHING THEOREM WITH APPLICATIONS

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(Submitted by Ky Fan)

Dedicated to the memory of Karol Borsuk

Recently, some versions of the KKM theorem in topological spaces without linear structures have been considered by Bardaro and Ceppitelli [1], [2], Chang and Ma [5], [6], Park [13], Bielawski [3], Horvath [11], Shin and Tan [14] and Cheng and Lin [7]. The purpose of this paper is to obtain some more general topological types of the KKM theorem and Fan's matching theorem. As applications, we shall utilize our main theorems to study the minimax problems and the existence of solutions for a class of generalized variational inequalities. Our results not only generalize the corresponding results of Ky Fan [8], [9], Park [13], Bardaro-Ceppitelli [1], [2], Bielawski [3] and Browder [4] but also contain the main results of Lin et al. [10], [12] and Wu [15].

Throughout this paper, we assume that topological spaces X and Y are Hausdorff spaces.

DEFINITION 1. *Let X be a topological space, $\{C_A\}$ be a family of nonempty connected subsets of X, indexed by all finite subsets of X, such that $A \subset C_A$. The ordered pair $(X, \{C_A\})$ is called a W-space.*

Note that Hausdorff topological linear spaces, convex spaces, contractible spaces and connected spaces are special cases of W-spaces. Moreover, if $(X, \{\Gamma_A\})$ is a H-space with $A \subset \Gamma_A$ ([1], [2]), then $(X, \{\Gamma_A\})$ is also a W-space.

Let $C(X,Y) = \{S : X \to Y : S$ is a continuous mapping$\}$ and $C^*(X,Y) = \{S \in C(X,Y) : S^{-1}$ maps each connected subset of Y into a such subset of $X\}$.

DEFINITION 2. *Let $(X, \{C_A\})$ be a W-space.*

(1) *$F : X \to 2^Y$ is called a* W-KKM *mapping if for any $x_1, x_2 \in X$*

$$F(C_{\{x_1,x_2\}}) \subset \bigcup_{i=1}^{2} F(x_i).$$

(2) *A subset D of X is called a W-convex set with respect to a subset C of X if for a finite subset A of C, $C_A \subset D$. In particular, if $C = D$, then D is said to be W-convex.*

(3) *A subset L of X is said to be W-compact if for any finite subset A of X, there exists a compact W-convex set L_A such that $L \cup A \subset L_A$.*

(4) *$F : X \to 2^Y$ is called a generalized* W-KKM *mapping (briefly, a* GW-KKM*), if there exists a $S \in C^*(X,Y)$ such that $S^{-1}F : X \to 2^X$ is a* W-KKM *mapping.*

1. Main results

Now, we give the topological versions of the KKM theorem and the matching theorems:

THEOREM 1. *Let $(X, \{C_A\})$ be a W-space, Y be a topological space and $F : X \to 2^Y$ be a* W-KKM *mapping. If the following conditions are satisfied:*

(i) *$F(x)$ is nonempty open (or closed) for each $x \in X$,*

(ii) *$F^{-1}(y)$ is open for each $y \in Y$,*

(iii) *for any finite subset A of X, $\bigcap_{x \in A} F(x)$ is connected,*

then

(1) *the family $\{F(x) : x \in X\}$ of sets $F(x)$ has the finite intersection property.*

(2) *If for each $x \in X$, $F(x)$ is closed and there exists an $x_0 \in X$ such that $F(x_0)$ is compact, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

PROOF. We prove the conclusion (1) by induction. By the condition (i), $F(x) \neq \emptyset$ for all $x \in X$. Suppose that for any n elements of $\{F(x) : x \in X\}, n \geq 2$, their intersection is nonempty. Now, we shall prove that for any $n+1$ elements of $\{F(x) : x \in X\}$, their intersection is also nonempty. Suppose the contrary, then there exists some subset $\{x_1, \ldots, x_n, x_{n+1}\}$ in X such that

$$\bigcap_{i=1}^{n+1} F(x_i) = \emptyset.$$

Letting $H = \bigcap_{i=3}^{n+1} F(x_i)$, by the assumption of induction and the condition (iii), we see that $H \cap F(x_i), i = 1, 2$, is nonempty connected and

$$(H \cap F(x_1)) \cap (H \cap F(x_2)) = \emptyset. \tag{1.1}$$

Since F is a W-KKM mapping, for the set $\{x_1, x_2\}$, we have

$$F(C_{\{x_1,x_2\}}) \subset F(x_1) \cup F(x_2).$$

Therefore we have

$$H \cap F(C_{\{x_1,x_2\}}) \subset (H \cap F(x_1)) \cup (H \cap F(x_2)).$$

Let

$$E_1 = \{x \in C_{\{x_1,x_2\}} : (H \cap F(x)) \subset (H \cap F(x_1)\},$$
$$E_2 = \{x \in C_{\{x_1,x_2\}} : (H \cap F(x)) \subset (H \cap F(x_2)\}.$$

Since $x_1 \in E_1$ and $x_2 \in E_2$, E_1 and E_2 both are nonempty. By the conditions (i) and (1.1), we know that $C_{\{x_1,x_2\}} = E_1 \cup E_2$. Because $C_{\{x_1,x_2\}}$ is connected and $E_1 \cap E_2 = \emptyset$, we know that either E_1 or E_2 must not be a closed set. Without loss of generality, we can assume that E_2 is not closed. Taking $x_0 \in (\overline{E_2} \setminus E_2) \cap E_1$, then there exists a net $\{x_\alpha\}_{\alpha \in I}$ in E_2 such that $x_\alpha \to x_0$. Since $x_0 \in E_1$,

$$H \cap F(x_0) \subset H \cap F(x_1). \tag{1.2}$$

Since $x_\alpha \in E_2$,

$$H \cap F(x_\alpha) \subset H \cap F(x_2) \tag{1.3}$$

for all $\alpha \in I$. It follows from the induction hypothesis that $H \cap F(x_0) \neq \emptyset$. Take $y_0 \in H \cap F(x_0)$. By (1.2), $y_0 \in H \cap F(x_1)$. On the other hand, from (1.1) we know $y_0 \notin H \cap F(x_2)$ and hence $y_0 \notin F(x_2)$. It follows from (1.3) that $y_0 \notin F(x_\alpha)$ for all $\alpha \in I$. Hence we have

$$\{x_\alpha\}_{\alpha \in I} \subset X \setminus F^{-1}(y_0).$$

In view of the condition (ii), we know that $X \setminus F^{-1}(y_0)$ is closed and $x_\alpha \to x_0$, and hence $x_0 \in X \setminus F^{-1}(y_0)$, i.e., $y_0 \notin F(x_0)$. This contradicts $y_0 \in H \cap F(x_0)$. This implies that $\{F(x) : x \in X\}$ has the finite intersection property.

Next we prove the conclusion (2). In fact, by the assumption, for each $x \in X$, $F(x)$ is nonempty closed and there exists $x_0 \in X$ such that $F(x_0)$ is compact. Therefore $\{F(x) \cap F(x_0) : x \in X\}$ is a family of closed sets in $F(x_0)$ which has the finite intersection property. Consequently, $\{F(x) \cap F(x_0) : x \in X\}$ has the nonempty intersection property. Thus we have

$$\emptyset \neq \bigcap_{x \in X} (F(x) \cap F(x_0)) = \bigcap_{x \in X} F(x)$$

This completes the proof.

Recall that a set $U \subset X$ is compactly open (resp., compactly closed) if for any compact set $K \subset X$, $U \cap K$ is open (resp., closed).

THEOREM 2. *Let $(X, \{C_A\})$ be a compact W-space, Y be a topological space, $G : X \to 2^Y$ be a mapping with compactly open values, and for all $x \in X$, $G(x) \neq Y$. Suppose further that the following conditions are satisfied:*

(i) *for each $y \in Y, G^{-1}(y)$ is closed,*
(ii) *for any finite subset A of X, $Y \setminus \bigcup_{x \in A} G(x)$ is connected,*
(iii) *$G(X) = Y$.*

Then for any $S \in C^*(X, Y)$, there exist $x_1, x_2 \in X$, $x_0 \in C_{\{x_1, x_2\}}$ and $z_0 \in X \setminus S^{-1}G(x_0)$ such that

$$S(z_0) \in \bigcap_{i=1}^{2} G(x_i).$$

PROOF. Let $F(x) = Y \setminus G(x)$, $x \in X$. Then $F : X \to 2^Y$ is a mapping with nonempty compactly closed values. If the conclusion of this theorem does not hold, then there exists $S \in C^*(X, Y)$ such that for any $x_1, x_2 \in X$,

$$S(z) \notin \bigcap_{i=1}^{2} G(x_i) = Y \setminus \bigcup_{i=1}^{2} F(x_i)$$

for all $x \in C_{\{x_1, x_2\}}$ and $z \in X \setminus S^{-1}G(x)$. Hence $z \notin X \setminus \bigcup_{i=1}^{2} S^{-1}F(x_i)$, i.e., $z \in \bigcup_{i=1}^{2} S^{-1}F(x_i)$. Therefore for all $x \in C_{\{x_1, x_2\}}$, we have

$$X \setminus S^{-1}G(x) \subset \bigcup_{i=1}^{2} S^{-1}F(x_i),$$

$$\text{or} \quad S^{-1}(Y \setminus G(x)) = S^{-1}F(x) \subset \bigcup_{i=1}^{2} S^{-1}F(x_i)$$

for all $x \in C_{\{x_1, x_2\}}$. This implies that $S^{-1}F : X \to 2^X$ is a W-KKM mapping.

On the other hand, since S is continuous and $F : X \to 2^Y$ is a mapping with nonempty compactly closed values, $S^{-1}F : X \to 2^X$ is a mapping with nonempty closed values. By the condition (ii) we know that the mapping $S^{-1}F$ satisfies the condition (iii) in Theorem 1. On the other hand, since

$$\begin{aligned}F^{-1}(y) &= \{x \in X : y \in F(x)\}\\ &= \{x \in X : y \notin G(x)\}\\ &= \{x \in X : x \notin G^{-1}(y)\}\\ &= X \setminus G^{-1}(y)\end{aligned}$$

for all $y \in Y$, by the condition (i), we know that $F^{-1}(y)$ is an open set for all $y \in Y$. Therefore $(S^{-1}F)^{-1}(z) = F^{-1}(S(z))$ is open for each $z \in X$, and so the mapping $S^{-1}F : X \to 2^X$ satisfies the condition (ii) in Theorem 1. Thus by Theorem 1, we know that $\bigcap_{x\in X} S^{-1}F(x) \neq \emptyset$ and so $\bigcap_{x\in X} F(x) \neq \emptyset$, i.e., $G(X) \neq Y$. This contradicts the condition (iii). Therefore, the conclusion of Theorem 2 is proved.

By the same way as in Theorem 2, we can prove the following :

THEOREM 3. *Let $(X, \{C_A\})$ be a compact W-space, Y be a topological space, $G : X \to 2^Y$ be a mapping with nonempty compactly open values and for all $x \in X$, $G(x) \neq Y$. Suppose further that the following conditions are satisfied:*

(i) *for all $y \in Y, G^{-1}(y)$ is closed,*
(ii) *for any finite set $A \subset X$, $X \setminus \bigcap_{x\in A} S^{-1}G(x)$ is connected, where $S \in C(X,Y)$ is a given mapping,*
(iii) *$G(X) = Y$.*

Then there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1,x_2\}}$, $z_0 \in X \setminus S^{-1}G(x_0)$ such that

$$S(z_0) \in \bigcap_{i=1}^{2} F(x_i).$$

THEOREM 4. *Let $(X, \{C_A\})$ be a W-space, Y be a topological space, $G : X \to 2^Y$ be a mapping with nonempty compactly open values and $S \in C(X,Y)$. If the following conditions are satisfied :*

(i) *$G^{-1}(y)$ is closed for all $y \in Y$,*
(ii) *$G(X) = Y$,*
(iii) *there exist a W-compact set $L \subset X$ and a compact set $K \subset Y$ such that $Y \setminus G(L) \subset K$ and for any compact W-convex subset X_0 with $L \subset X_0 \subset X$, and for any finite subset $A \subset X_0$, $X_0 \setminus \bigcup_{x\in A} S^{-1}G(x)$ is connected,*

then there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1,x_2\}}$, and $z_0 \in X \setminus S^{-1}G(x_0)$ such that

$$S(z_0) \in \bigcap_{i=1}^{2} G(x_i).$$

PROOF. First, if $G(L) = Y$, then by the W-compactness of L, there exists a compact W-convex set L_1 with $L \subset L_1$. Therefore we have $G(L_1) = Y$. By the W-convexity of L_1, we know that $(L_1, \{C_{A \cap L_1}\})$ is a compact W-space. Thus, by Theorem 3, the conclusion is proved.

Next, if $G(L) \neq Y$, then $Y \setminus G(L) \neq \emptyset$. By the assumption, we know that K is compact,

$$Y \setminus G(L) \subset K \subset Y = G(X) \tag{1.4}$$

and for any $x \in X$, $G(x)$ is compactly open. Hence there exists $\{x'_1, \ldots, x'_m\} \subset X$ such that $K \subset \bigcup_{i=1}^m G(x'_i)$. It follows from (1.4) that there exists $\{x_1, \ldots, x_n\} \subset X \setminus L$ such that

$$Y \setminus G(L) \subset \bigcup_{i=1}^n G(x_i).$$

Let $M = \{x_1, \ldots, x_n\}$. By the W-compactness of L, there exists a compact W-convex set $L_M \supset L \cup M$ such that $G(L_M) = Y$ and $(L_M, \{C_{A \cap L_M}\})$ is a compact W-space. Hence by Theorem 3, the conclusion of Theorem 4 is proved.

THEOREM 5. *Let $(X, \{C_A\})$ be a compact W-space, Y be a topological space, $F : X \to 2^Y$ be a mapping with nonempty compactly closed values. If the following conditions are satisfied:*

(i) *$F^{-1}(y)$ is open for each $y \in Y$,*
(ii) *for any finite set $A \subset X$, $\bigcap_{x \in A} F(x)$ is connected.*
(iii) *F is a* GW-KKM *mapping with nonempty values,*

then

$$\bigcap_{x \in X} F(x) \neq \emptyset.$$

PROOF. Suppose that $\bigcap_{x \in X} F(x) = \emptyset$. Let $G(x) = Y \setminus F(x)$ for all $x \in X$. Then $G : X \to 2^Y$ is a mapping with compactly open values and

$$G(X) = \bigcup_{x \in X} G(x) = \bigcup_{x \in X} (Y \setminus F(x)) = Y \setminus \bigcup_{x \in X} F(x) = Y.$$

On the other hand, since for each $y \in Y$

$$G^{-1}(y) = \{x \in X \ : \ y \in G(x)\} = \{x \in X \ : \ y \notin F(x)\} = X \setminus F^{-1}(y),$$

by the condition (i), $G^{-1}(y)$ is closed. Since $F : X \to 2^Y$ is a GW-KKM mapping, there exists a $S \in C^*(X, Y)$ such that $S^{-1}F : X \to 2^X$ is a W-KKM mapping. By the condition (ii), we know that G also satisfies the condition (ii) in Theorem 2.

Thus, by Theorem 2, there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1,x_2\}}$ and $z_0 \in X \setminus S^{-1}G(x_0)$ such that $S(z_0) \in \bigcap_{i=1}^{2} G(x_i)$, i.e.,

$$z_0 \in X \setminus \bigcup_{i=1}^{2} S^{-1}F(x_i).$$

However, it follows from $z_0 \notin S^{-1}G(x_0)$ that $z_0 \notin X \setminus S^{-1}F(x_0)$. Therefore we have

$$z_0 \in S^{-1}F(x_o) \subset S^{-1}F(C_{\{x_1,x_2\}}) \subset \bigcup_{i=1}^{2} S^{-1}F(x_i),$$

which contradicts $z_0 \in X \setminus \bigcup_{i=1}^{2} S^{-1}F(x_i)$. This completes the proof.

REMARK. Theorems 1 ~ 5 are the topological versions of the KKM theorem and Fan's matching theorem without linear structures on the given spaces, which generalize the corresponding results of Ky Fan [8], [9], Bardaro-Ceppitelli [1], [2], Park [13], Bielawski [3] and Horvath [11].

By using Theorem 1, we can obtain the following:

THEOREM 6. *Let $(X, \{C_A\})$ be a W-space, Y be a topological space, $F : X \to 2^Y$ be a* GW-KKM *mapping with nonempty open values. If the following conditions are satisfied:*

(i) *$F^{-1}(y)$ is open for each $y \in Y$,*

(ii) *for any finite set $A \subset X$, $\bigcap_{x \in A} F(x)$ is connected,*

then $\{F(x) : x \in X\}$ has the finite intersection property.

PROOF. Since $F : X \to 2^Y$ is a GW-KKM mapping, there exists a $S \in C^*(X, Y)$ such that $S^{-1}F : X \to 2^X$ is a W-KKM mapping. Since F is open valued, $S^{-1}F$ is also open valued. By the condition (i), $F^{-1}(y)$ is open for all $y \in Y$. Therefore for each $z \in X$, the set $(S^{-1}F)^{-1}(z) = \{x \in X : z \in S^{-1}F(x)\} = F^{-1}(S(z))$ is also open. From condition (ii) and $S \in C^*(X, Y)$, for any finite set $\{s_1, \ldots, x_n\} \subset X$, $\bigcap_{i=1}^{n} S^{-1}F(x_i)$ is connected. By Theorem 1, for any finite set $A \subset X$, $\bigcap_{x \in A} S^{-1}F(x) \neq \emptyset$. Therefore, $\bigcap_{x \in A} F(x) \neq \emptyset$, which implies that $\{F(x) : x \in X\}$ has the finite intersection property.

REMARK. Theorem 6 generalizes and improves the corresponding result of Chang-Ma [4].

In the following, we give two other matching theorems:

THEOREM 7. *Let $(X, \{C_A\})$ be a W-space, Y be a compact topological space, $G : X \to 2^Y$ be a mapping with closed values. If the following conditions are satisfied:*

(i) *$G^{-1}(y)$ is closed for all $y \in Y$ and $G(x) \neq Y$ for all $x \in X$,*

(ii) *for any finite set* $A \subset X$, $Y \setminus \bigcup_{x \in A} G(x)$ *is connected,*

(iii) $G^0(X) \triangleq \bigcup_{x \in X} G^0(x) = Y$, *where* $G^0(x)$ *denotes the interior of* $G(x)$,

then for any $S \in C^*(X,Y)$, *there exist* $x_1, x_2 \in X$, $x_0 \in C_{\{x_1,x_2\}}$ *and* $z_0 \in X \setminus S^{-1}G(x_0)$ *such that*

$$S(z_0) \in \bigcap_{i=1}^{2} G(x_i).$$

PROOF. Letting $F(x) = Y \setminus G(x)$, $x \in X$, then $F : X \to 2^Y$ is a mapping with open values. If the conclusion of the theorem does not hold, then there exists a $S \in C^*(X,Y)$ such that for any $x_1, x_2 \in X$, $x \in C_{\{x_1,x_2\}}$ and $z \in X \setminus S^{-1}G(x)$, the following holds:

$$S(z) \notin \bigcap_{i=1}^{2} G(x_i) = \bigcap_{i=1}^{2} (Y \setminus F(x_i)) = Y \setminus \bigcap_{i=1}^{2} F(x_i).$$

Therefore, $z \notin X \setminus \bigcup_{i=1}^{2} S^{-1}F(x_i)$, i.e., $z \in \bigcup_{i=1}^{2} S^{-1}F(x_i)$. By the same way as stated in Theorem 2, we can prove $S^{-1}F : X \to 2^X$ is a mapping with open values and satisfies all the conditions in Theorem 1. Hence by Theorem 1, the family $\{S^{-1}F(x) : x \in X\}$ of sets has the finite intersection property. Therefore $\{F(x) : x \in X\}$ has the finite intersection property and so $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Since Y is compact, $\bigcap_{x \in X} \overline{F(x)} = \emptyset$. Since

$$\bigcap_{x \in X} \overline{F(x)} = \bigcap_{x \in X} (Y \setminus G^0(x)) = Y \setminus \bigcup_{x \in X} G^0(x) = Y \setminus G^0(X) \neq \emptyset,$$

$G^0(X) \neq Y$. This contradicts the condition (iii) and completes the proof.

Similarly, we can prove the following:

THEOREM 8. *Let* $(X, \{C_A\})$ *be a W-space,* Y *be a compact topological space,* $G : X \to 2^Y$ *be a mapping with closed values and* $S \in C(X,Y)$ *be a given mapping. If the following conditions are satisfied:*

(i) *for any* $y \in Y$, $G^{-1}(y)$ *is closed and* $G(x) \neq Y$ *for all* $x \in X$,

(ii) *for any finite set* $A \subset X$, $X \setminus \bigcup_{x \in A} S^{-1}G(x)$ *is connected,*

(iii) $G^0(X) = Y$,

then there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1, x_2\}}$ *and* $z_0 \in X \setminus S^{-1}G(x_0)$ *such that*

$$S(z_0) \in \bigcap_{i=1}^{2} G(x_i).$$

2. Minimax Theorems of Topological Types

As applications, we shall first use the results presented in section 1 to study Minimax Theorems of topological types.

THEOREM 9. *Let $(X, \{C_A\})$ be a W-space, Y be a topological space, $f, g : X \times Y \to R$ satisfy the following conditions:*

(i) *$y \mapsto g(x, y)$ and $x \mapsto g(x, y)$ are upper semi-continuous and $y \mapsto f(x, y)$ is lower semi-continuous,*

(ii) *$f(x, y) \leq g(x, y)$ for all $(x, y) \in X \times Y$,*

(iii) (a) *for any finite set $A \subset X$ and $r \in R$, $\{y \in Y : g(x, y) < r, \forall x \in A\}$ is connected,*

(b) *for any $\{x_1, x_2\} \subset X$, there exists a connected subset $C_{\{x_1,x_2\}} \subset X$ such that $\{x_1, x_2\} \subset C_{\{x_1,x_2\}}$ and $g(x, y) \geq \min\{g(x_1, y), g(x_2, y)\}$ for all $x \in C_{\{x_1,x_2\}}$ and $y \in Y$,*

(iv) *Y is compact.*

Then

$$\sup_{x \in X} \inf_{y \in Y} g(x, y) \geq \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

PROOF. Letting $\alpha = \sup_{x \in X} \inf_{y \in Y} g(x, y)$, $\beta = \inf_{y \in Y} \sup_{x \in X} f(x, y)$ and $\alpha < \beta$, then there exists $r_0 \in R$ such that $\alpha < r_0 < \beta$.

Now we define two set-valued mappings $F, G : X \to 2^Y$ by

$$F(x) = \{y \in Y : f(x, y) < r_0\}, \quad x \in X,$$
$$G(x) = \{y \in Y : g(x, y) < r_0\}, \quad x \in X.$$

By the condition (ii), we have $G(x) \subset F(x)$ for all $x \in X$. By the condition (i) and the choice of r_0, we know that $G(x)$ is a nonempty open set, which means that G satisfies the condition (i) in Theorem 1.

By the condition (iii)(a), G satisfies the condition (iii) in Theorem 1. By condition (iii)(b), for any $x_1, x_2 \in X$, there exists a connected subset $C_{\{x_1,x_2\}}$ with $\{x_1, x_2\} \subset C_{\{x_1,x_2\}}$ such that

$$g(x, y) \geq \min\{g(x_1, y), g(x_2, y)\} \tag{2.1}$$

for all $x \in C_{\{x_1,x_2\}}$ and $y \in Y$. Hence for any $y_0 \in G(C_{\{x_1,x_2\}})$, there exists $x_0 \in C_{\{x_1,x_2\}}$ such that $y_0 \in G(x_0)$, i.e., $g(x_0, y_0) < r_0$. In view of (2.1), we have

$$r_0 > g(x_1, y_0) \quad \text{or} \quad r_0 > g(x_2, y_0).$$

This implies that $y_0 \in G(x_1) \cup G(x_2)$. Hence we have

$$G(C_{\{x_1,x_2\}}) \subset G(x_1) \cup G(x_2),$$

which means that $G : X \to 2^Y$ is a W-KKM mapping. By the upper semi-continuity of $x \mapsto g(x,y)$, we know that G satisfies the condition (ii) in Theorem 1. Hence by Theorem 1 (1), $\{G(x) : x \in X\}$ has the finite intersection property. Since $G(x) \subset F(x)$ for all $x \in X$, $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Since Y is compact, $\bigcap_{x\in X} \overline{F(x)} \neq \emptyset$. By the lower semi-continuity of $y \mapsto f(x,y)$, we have $\overline{F(x)} \subset \{y \in Y : f(x,y) \le r_0\}$. Hence it follows that

$$\bigcap_{x\in X} \{y \in Y : f(x,y) \le r_0\} \neq \emptyset.$$

Taking $y_0 \in Y$, we have $f(x, y_0) \le r_0$ for all $x \in X$. Hence

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \le r_0,$$

i.e., $\beta \le r_0 < \beta$. This is a contradiction. Therefore we have $\alpha \ge \beta$. This completes the proof.

We would like to point out that Theorem 9 contains the main results of Lin et al [10], [12] and Wu [15] as its special cases, i.e., as an immediate consequence of Theorem 9, we have the following:

COROLLARY 10. (cf. [15]) *Let X be a separable compact space, Y be a pathconnected space. Let $u : X \times Y \to R$ be a function satisfying the following conditions:*

(i) *$x \mapsto u(x,y)$, $y \mapsto u(x,y)$ are continuous,*
(ii) *for any y_0, $y_1 \in Y$, there exists a continuous mapping $S : [0,1] \to Y$ such that $S(0) = y_0$, $S(1) = y_1$, and for any $x \in X$ and $r \in R$, the set $\{t \in [0,1] : u(x, S(t)) \ge r\}$ is connected,*
(iii) *for any finite set $\{y_1, \dots, y_n\} \subset Y$ and for any given $r \in R$, the set $\{x : u(x, y_i) < r,\ i = 1, \dots, n\}$ is connected.*

Then

$$\inf_{x\in X} \sup_{y\in Y} u(x,y) = \sup_{y\in Y} \inf_{x\in X} u(x,y).$$

PROOF. Define a function $h : Y \times X \to R$ by $h(y,x) = u(x,y)$ for all $x \in X$ and $y \in Y$. Taking $f = g = h$, it is easy to see that the conditions (i), (ii), (iii)(a) and (iv) in Theorem 9 are satisfied.

Next we prove that the condition (iii)(b) in Theorem 9 is also satisfied.

In fact, by the condition (i), for any y_0, $y_1 \in Y$, there exists a continuous mapping $S : [0,1] \to Y$ such that $S(0) = y_0$, $S(1) = y_1$ and for any $x \in X$ and $r \in R$,

$$\{t \in [0,1] : u(x, S(t)) \ge r\}$$

is connected. Taking $C_{\{y_0,y_1\}} = S(I)$, where $I = [0,1]$, it is obvious $\{y_0, y_1\} \subset C_{\{y_0,y_1\}}$ and $C_{\{y_0,y_1\}}$ is connected. For any $y = S(t_0) \in S(I)$ and $x \in X$, taking

$$r = \min\{u(x, y_0), u(x, y_1)\},$$

by the condition (ii), we know that the set $\{t \in I : u(x, S(t)) \geq r\} \triangleq A$ is connected. It is obvious that $\{0,1\} \subset A$ and hence $[0,1] \subset A \subset [0,1]$, which implies that $A = I$. Thus for a given $t_0 \in I$, we have

$$u(x, S(t_0)) \geq \min\{u(x, y_0),\ u(x, y_1)\}.$$

By the arbitrariness of $y = S(t_0) \in S(I)$ and $x \in X$, we have

$$u(x, y) \geq \min\{u(x, y_0),\ u(x, y_1)\}$$

for all $y \in S(I)$ and $s \in X$. Therefore we have

$$h(y, x) \geq \min\{h(y_0, x),\ h(y_1, x)\}$$

for all $y \in S(I)$ and $x \in X$, which implies that the function h satisfies the condition (iii)(b) in Theorem 9. Thus, by Theorem 9 we have

$$\sup_{y \in Y} \inf_{x \in X} h(y, x) \geq \inf_{x \in X} \sup_{y \in Y} h(y, x). \tag{2.2}$$

On the other hand, it is obvious that

$$\sup_{y \in Y} \inf_{x \in X} h(y, x) \leq \inf_{x \in X} \sup_{y \in Y} h(y, x). \tag{2.3}$$

Therefore, combining (2.2) and (2.3), we have

$$\sup_{y \in Y} \inf_{x \in X} h(y, x) = \inf_{x \in X} \sup_{y \in Y} h(y, x),$$

$$\sup_{y \in Y} \inf_{x \in X} u(y, x) = \inf_{x \in X} \sup_{y \in Y} u(y, x).$$

REMARK. In the proof of Corollary 10, we didn't use the separability of X and the lower semi-continuity of $u(x, \cdot)$. Therefore Theorem 9 not only contains the main results in Wu [15] as its special case but also sharpens its conditions.

3. Application to Variational Inequalities

As another application, we shall use the matching theorem presented in section 1 to study the existence problem of solutions for a class of generalized variational inequalities.

THEOREM 11. *Let $(X, \{C_A\})$ be a W-space, Y be a topological space, X be a compact space. Let $(E, \mathcal{C})$ be a topological Riesz space, where $\mathcal{C}$ is a closed cone with $\mathcal{C}^o \neq \emptyset$, and $r \in E$. Suppose that $h : X \times Y \to E$ satisfies the following conditions:*

(i) *for any $x \in X$, the set $\{y \in Y : h(x,y) \notin r + \mathcal{C}^o\}$ is nonempty and compactly closed,*
(ii) *for any $y \in Y$, the set $\{x \in X : h(x,y) \in r + \mathcal{C}^o\}$ is compactly closed,*
(iii) *for any finite set $A \subset X$, $\bigcap_{x\in A}\{y \in Y : h(x,y) \notin r + \mathcal{C}^o\}$ is connected,*
(iv) *the set $\{x \in X : h(x,y) \in x = \mathcal{C}^o\}$ is W-convex for each $y \in Y$.*

Then the following generalized variational inequality

$$h(x,y) \notin r + \mathcal{C}^o \quad \text{for all } x \in X \tag{3.1}$$

has a solution in Y.

PROOF. For any $x \in X$, we define a mapping $F : X \to 2^Y$ by

$$F(x) = \{y \in Y : h(x,y) \in r + \mathcal{C}^o\}.$$

By the condition (i), F is a mapping with compactly open values, and for any $x \in X$, $F(x) \neq Y$. By the condition (ii), $F^{-1}(y)$ is a closed set for any $y \in Y$. By the condition (iii), F satisfies the condition (ii) in Theorem 2.

If the conclusion of Theorem 11 does not hold, then for any $y \in Y$, there exists $x \in X$ such that $h(x,y) \in r + \mathcal{C}^o$, which means $y \in F(x)$. Hence we have $F(X) = Y$. This implies that F satisfies all the conditions in Theorem 2. Hence by Theorem 2, for any $S \in C^*(X,Y)$, there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1,x_2\}}$ and $z_0 \in X \setminus S^{-1}F(x_0)$ such that $(z_0) \in \bigcap_{i=1}^2 F(x_i)$, i.e., $\{x_1, x_2\} \subset F^{-1}(S(z_0))$. It follows from the condition (iv) that $C_{\{x_1,x_2\}} \subset F^{-1}(S(z_0))$. Since $x_0 \in C_{\{x_1,x_2\}}$, $S(z_0) \in F(x_0)$.

On the other hand, it follows from $z_0 \in X \setminus S^{-1}F(x_0)$ that $z_0 \notin S^{-1}F(x_0)$, i.e., $S(z_0) \notin F(x_0)$. This is a contraction. Therefore, the variational inequality (3.1) has a solution in Y.

Similarly, we can prove the following:

THEOREM 12. *Let $(X, \{C_A\})$ be a compact W-space, Y be a topological space, E be a topological Riesz space and $r \in E$. Suppose that the mapping $h : X \times Y \to E$ satisfies the following conditions:*

(i) *for each $x \in X$, the set $\{y \in Y : h(x,y) \leq r\}$ is nonempty compactly closed,*
(ii) *for each $y \in Y$, the set $\{x \in X : h(x,y) \nleq r\}$ is compactly closed,*
(iii) *for any finite set $A \subset X$, $\bigcap_{x\in A}\{y \in Y : h(x,y) \leq r\}$ is connected,*
(iv) *for each $y \in Y$, the set $\{x \in X : h(x,y) \nleq r\}$ is W-convex.*

Then the following variational inequality

$$h(x,y) \le r \quad \text{for all } x \in X \tag{3.2}$$

has a solution in Y.

By using Theorem 7, we have the following:

THEOREM 13. *Let* $(X, \{C_A\})$ *be a W-space,* Y *be a compact topological space,* $h : X \times Y \to R$ *and* $r \in R$. *If the following conditions are satisfied:*

(i) $y \mapsto h(x,y)$ *is continuous and* $\{y \in Y : h(x,y) < r\} \neq \emptyset$,
(ii) $x \mapsto h(x,y)$ *is upper semi-continuous,*
(iii) *for any finite set* $A \subset Z$, $\bigcap_{x \in A}\{y \in Y : h(x,y) < r\}$ *is connected,*
(iv) *for each* $y \in Y$, $\{x \in X : h(x,y) \ge r\}$ *is W-convex,*

then the following variational inequality

$$h(x,y) \le r \quad \text{for all } x \in X \tag{3.3}$$

has a solution in Y.

PROOF. Letting $G(x) = \{y \in Y : h(x,y) \ge r\}$, $x \in X$, by the condition (i) for each $x \in X$, $G(x)$ is a closed set, $G(x) \neq Y$ and

$$G^o(x) = \{y \in Y : h(x,y) > r\}.$$

In view of the condition (ii), for each $y \in Y$, $G^{-1}(y)$ is closed. By the condition (iii), G satisfies the condition (ii) in Theorem 7.

If the conclusion of Theorem 13 does not hold, then for any $y \in Y$, there exists $x \in X$ such that $h(x,y) > r$, i.e., $y \in G^o(x)$. Hence $G^o(X) = Y$. This implies that G satisfies the condition (iii) in Theorem 7. Therefore by Theorem 7, for any $S \in C^*(X,Y)$, there exist $\{x_1, x_2\} \subset X$, $x_0 \in C_{\{x_1,x_2\}}$ and $z_0 \in X \setminus S^{-1}G(x_0)$ such that $S(z_0) \in \bigcap_{i=1}^2 G(x_i)$. Hence we have

$$\{x_1, x_2\} \subset G^{-1}(S(z_0)) = \{x \in X : h(x, S(z_0)) \ge r\}.$$

In view of the condition (iv), we have $C_{\{x_1,x_2\}} \subset G^{-1}(S(z_0))$. Since $x_0 \in C_{\{x_1,x_2\}}$, $x_0 \in G^{-1}(S(z_0))$, i.e., $S(z_0) \in G(x_0)$.

On the other hand, since $z_0 \notin S^{-1}G(x_0)$, $S(z_0) \notin G(x_0)$. This is a contraction. Therefore the variational inequality (3.3) has a solution in Y. This completes the proof.

As an immediate consequence of Theorem 13, we can obtain the following:

COROLLARY 14. (cf. [4]) *Let* E *be a locally convex topological linear space,* $K \subset E$ *be a compact convex subset,* $T : K \to E^*$ *be a continuous mapping, where*

E^* is the dual of E. Then the following variational inequality

$$\langle Ty, x - y\rangle \geq 0 \quad \text{for all } x \in X \tag{3.4}$$

has a solution in K.

PROOF. Taking $X = Y = K$, $h(x,y) = \langle Tx, y - x\rangle$, $y \in K$, and $r = 1/n$, $n = 1, 2, \dots$, in Theorem 13, it is easy to see that h satisfies all the conditions in Theorem 13. Thus, by Theorem 13, for each $n = 1, 2, \dots$, there exists a $y_n \in X$ such that

$$\langle Tx, y_n - x\rangle \leq \frac{1}{n} \quad \text{for all} \quad x \in K.$$

Since K is compact, without loss of generality, we can assume that $y_n \to y_0 \in K$. Hence, for all $x \in K$, we have

$$\langle Tx, y_0 - x\rangle \leq 0 \quad \text{for all} \quad x \in K. \tag{3.5}$$

For any $w \in K$, letting $v = tw + (1-t)y_0$, $t \in (0,1)$, then $v \in K$. In (3.5), taking $x = v$, we have $\langle Tv, t(y_0 - w)\rangle \leq 0$ for all $t \in (0,1)$. Thus we have

$$\langle Tv, w - y_0\rangle \geq 0. \tag{3.6}$$

Letting $t \to 0$ and so $v \to y_0$, from (3.6), we have

$$\langle Ty_0, w - y_0\rangle \geq 0 \quad \text{for all} \quad w \in K.$$

This completes the proof.

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Manuscript received February 19, 1993

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