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THE LIMITING CASE FOR STRONGLY INDEFINITE FUNCTIONALS

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(Submitted by J. Mawhin)

Dedicated to the memory of Karol Borsuk

0. Introduction

After the pioneering work of Pucci and Serrin [4], motivated by the forced pendulum equation, many papers were devoted to the limiting case in critical point theory. Let us recall that, in the limiting case,

$$\sup_Y \varphi = \inf_F \varphi$$

where F and Y link. Some very general results are contained in the paper of Ghoussoub ([3]). For strongly indefinite functionals, the only results we know are due to Silva ([6]). However the geometrical assumptions of Silva are more restrictive than the usual ones. In the present paper, we consider the limiting case for strongly indefinite functionals under the usual assumptions of the saddle-point theorem and the generalized mountain pass theorem. The basic tools are the limit relative category defined by Fournier, Lupo, Ramos and Willem [1] and the quantitative deformation lemma proved in [7].

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1. Preliminaries

By a *map* between topological spaces we mean a continuous function. Let (X, A) be a topological pair; a *deformation* $h_t : A \to X$ is a map $h : [0,1] \times A \to X$ such that $h_0(x) = x$ for every $x \in A$.

Let A, B, Y be closed subsets of the topological space X. Then, by definition, $A \prec_Y B$ in X if $Y \subset A \cap B$ and if there exists a deformation $h_t : A \to X$ such that $h_1(A) \subset B$ and $h_t(Y) \subset Y$ for every $t \in [0,1]$.

Definition 1.1. *Let $Y \subset A$ be closed subsets of a topological space X. The relative category of A in X relative to Y is the least integer n such that there exist $n+1$ closed subsets $A_0, A_1, \dots, A_n$ of X satisfying*

(a) $A = \bigcup_{i=0}^{n} A_i$,

(b) $A_1, \dots, A_n$ *are contractible in* X,

(c) $A_0 \prec_Y Y$ *in* X.

When no such integer exists, the category of A in X relative to Y is infinite. The relative category is denoted by $\operatorname{cat}_{X,Y}(A)$.

Remark. The above definition is a variant, due to Szulkin, of the notion of relative category introduced in [5] and [2].

We consider now a topological space X together with a sequence (X_n) of closed subsets of X. We assume that there exists, for every $n \in \mathbb{N}$, a retraction $r_n : X \to X_n$. If A is any subspace of X, denote by A_n the set $A \cap X_n$.

Definition 1.2. (cf. [1]) *Let $Y \subset A$ be closed subsets of X. The limit relative category of A in X relative to Y, with respect to (X_n), is defined by* $\operatorname{cat}^{\infty}_{X,Y}(A) := \overline{\lim}_{n\to\infty} \operatorname{cat}_{X_n,Y_n}(A_n)$.

Let us now recall some notations of critical point theory. Let $\varphi \in C^1(E, \mathbb{R})$ where E is a Banach space and let $S \subset E$, $d \in \mathbb{R}$, $\delta > 0$. Then we set

$$
\begin{aligned}
K_c &:= \{u \in E \ : \ \varphi(u) = c,\ \varphi'(u) = 0\},\\
\varphi^d &:= \{u \in E \ : \ \varphi(u) \le d\},\\
S_\delta &:= \{u \in E \ : \ \operatorname{dist}(u, S) \le \delta\}.
\end{aligned}
$$

We shall use the following quantitative deformation lemma. We only sketch the proof.

Lemma 1.3. (cf. [7]) *Let E be a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon, \delta > 0$ be such that*

$$
(\forall u \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S)_{2\delta}) : \|\varphi'(u)\| \ge 4\varepsilon/\delta. \tag{1.1}
$$

Then there exist $\eta \in C([0,1] \times X, X)$ *such that*

(i) $\eta(0,u) = u, \quad \forall u \in X,$

(ii) $\eta(t,.)$ *is a homeomorphism of* X *for every* $t \in [0,1]$,

(iii) $\eta(t,u) = u, \forall u \notin \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap S_{2\delta}, \forall t \in [0,1],$

(iv) $\|\eta(t,u) - u\| \leq \delta, \forall u \in X, \forall t \in [0,1],$

(v) $\varphi(\eta(.,u))$ *is non-increasing,* $\forall u \in X$,

(vi) $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$.

PROOF. Denote

$$A := S_{2\delta} \cap \varphi^{-1}([c-2\varepsilon, c+2\varepsilon]),$$
$$B := S_{\delta} \cap \varphi^{-1}([c-\varepsilon, c+\varepsilon]),$$

and let $\psi : E \to [0,1]$ be the locally Lipschitz continuous function given by

$$\psi(u) := \frac{\operatorname{dist}(u, X \setminus A)}{\operatorname{dist}(u, X \setminus A) + \operatorname{dist}(u, B)}.$$

Choose a pseudo-gradient vector field g for φ on $\{u \in E : \varphi'(u) \neq 0\}$ and define

$$f(u) := \begin{cases} -\dfrac{\psi(u)}{\|g(u)\|} g(u), & u \in A, \\ 0, & u \in X \setminus A. \end{cases}$$

Then f is a bounded locally Lipschitz continuous vector field on E. Thus the corresponding Cauchy problem

$$\dot{\sigma}(t,u) = f(\sigma(t,u))$$
$$\sigma(0,u) = u$$

has a unique solution $\sigma(.,u)$ defined on $\mathbb{R}$ for any $u \in E$. It is easy to check that

$$\eta(t,u) := \sigma(\delta t, u)$$

satisfies the desired properties. □

2. Location and the limiting case

Like in [3], we first prove a result without assuming a Palais-Smale condition. But we use the quantitative deformation lemma instead of the Ekeland variational principle.

Let X be a Banach space and consider a sequence of closed subspaces

$$X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$$

We assume that each X_n has a closed complement. The limit relative category is always computed with respect to (X_n). For every $\varphi : X \to \mathbb{R}$ we denote by φ_n the function φ restricted to X_n. Let us recall that, for any $A \subset X$, $A_n := A \cap X_n$.

LEMMA 2.1. *Let $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ and let F, Y be non-empty closed subsets of X. Define*

$$c := \inf_{A \in \mathcal{A}} \sup_{u \in A} \varphi(u)$$

where

$$\mathcal{A} := \{A \subset X : A \text{ is closed}, \ Y \subset A, \ \operatorname{cat}^{\infty}_{X,Y}(A) = 1\}.$$

Assume that

(A1) *for any n, for any closed subset B_n of X_n such that $Y_n \subset B_n$ and $B_n \cap F = \emptyset$, we have*

$$\operatorname{cat}_{X_n,Y_n}(B_n) = 0,$$

(A2) $\operatorname{dist}(F, Y) > 0$,

(A3) $-\infty < c = \inf_F \varphi$.

Then, for every $j \in \mathbb{N}$, $\varepsilon > 0$, $\delta \in]0, \operatorname{dist}(F, Y)/2[$, $A \in \mathcal{A}$ such that

$$\sup_A \varphi < c + \varepsilon,$$

there exists $n \geq j$ and $u \in X_n$ such that

a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,

b) $\operatorname{dist}(u, F_n \cap A_\delta) \leq 2\delta$,

c) $\|\varphi_n'(u)\| \leq 4\varepsilon/\delta$.

PROOF. Let j, ε, δ and A satisfy the assumption of the lemma and suppose that the conclusion of the lemma is false. According to Lemma 1.3, applied to $E := X_n$, $\psi := -\varphi$, $S := F_n \cap A_\delta$, for every $n \geq j$ there exists a deformation η_n satisfying (i) to (vi).

Define, for $n \geq j$,

$$B_n := \{u \in X_n \ : \ \eta_n(1, u) \in A_n\}.$$

It follows from (iii) that $\eta_n(1, y) = y$ for any $y \in Y_n$. Proposition 2.6 in [1] and (ii) imply that

$$\operatorname{cat}_{X_n,Y_n}(A_n) = \operatorname{cat}_{X_n,Y_n}(B_n).$$

If, for every $n \geq j$, $B_n \cap F = \emptyset$, assumption (A1) implies that

$$\begin{aligned} 1 = \operatorname{cat}^{\infty}_{X,Y}(A) &= \overline{\lim_{n\to\infty}} \operatorname{cat}_{X_n,Y_n}(A_n) \\ &= \overline{\lim_{n\to\infty}} \operatorname{cat}_{X_n,Y_n}(B_n) = 0, \end{aligned}$$

a contradiction.

Hence there exists $n \geq j$ such that $B_n \cap F = \emptyset$. By definition, there exists $u \in F_n$ such that $\eta_n(1,u) \in A_n$. We obtain from (A3) $u \in F \subset \psi^{-c}$ and from (iv) $\operatorname{dist}(u, A_n) \leq \delta$. Thus $u \in S \cap \psi^{-c}$ and it follows from (vi) that

$$c + \varepsilon \leq \varphi_n(\eta_n(1,u)) \leq \sup_{A_n} \varphi_n < c + \varepsilon,$$

a contradiction. □

We shall use the following conditions.

DEFINITION 2.2. *Let $c \in \mathbb{R}$ and $\varphi \in C^1(X, \mathbb{R})$. The function φ satisfies the* $(\mathrm{PS})_c^*$ *condition if every sequence $(u_{n_j}) \subset X$ satisfying*

$$n_j \to \infty, \quad u_{n_j} \in X_{n_j} \quad \varphi(u_{n_j}) \to c, \quad \varphi'_{n_j}(u_{n_j}) \to 0$$

possesses a subsequence which converges in X to a critical point of φ.

DEFINITION 2.3. *Let $c \in \mathbb{R}$, $\varphi \in C^1(X, \mathbb{R})$ and F be non-empty closed subset of X. The function φ satisfies the* $(\mathrm{PS})_{F,c}^*$ *condition if every sequence $(u_{n_k}) \subset X$ satisfying*

$$n_k \to \infty, \quad u_{n_k} \in X_{n_k}, \quad \operatorname{dist}(u_{n_k}, F_{n_k}) \to 0, \quad \varphi(u_{n_k}) \to c, \quad \varphi'_{n_k}(u_{n_k}) \to 0,$$

possesses a subsequence which converges in X to a critical point of φ.

THEOREM 2.4. *Let $\varphi \in C^1(X, \mathbb{R})$ and let F, Y be non-empty closed subsets of X. Assume that φ, F, Y satisfy* (A1)–(A3) *and the* $(\mathrm{PS})_{F,c}^*$ *condition, where c is as in Lemma* 2.1. *Then F contains a critical point of φ with critical value c.*

THEOREM 2.5. *Let F, Y be non-empty closed subsets of X satisfying* (A1)–(A2). *Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the following assumptions:*

(A4) $\sup_Y \varphi \leq \inf_F \varphi := b$,

(A5) *There is $A \subset X$, A closed, $A \supset Y$, $\operatorname{cat}_{X,Y}^{\infty}(A) = 1$ such that* $\sup_A \varphi < \infty$,

(A6) *φ satisfies the* $(\mathrm{PS})_c^*$ *condition, where c is as in Lemma* 2.1.

Then we have:

(i) $c \geq b$,

(ii) $K_c \neq \emptyset$,

(iii) $K_c \cap F \neq \emptyset$ *if* $c = b$.

PROOF. It is clear that $c \geq b$ from (A1). If $c > b$, then we have by (A4) and (A5) $\sup_Y \varphi < c < \infty$ and then we deduce from Theorem 6.1 in [1] that $K_c \neq \emptyset$. If $c = b$, clearly it suffices to prove $K_c \cap F \neq \emptyset$. Assumption (A6) implies that φ satisfies the $(\mathrm{PS})_{F,c}^*$ condition and the conclusion follows from Theorem 2.4. □

3. Some applications

We consider now the *generalized Saddle Point Theorem.* Let $X = W \oplus Z$ be a Banach space and $X_n = W_n \oplus Z_n$ be a sequence of closed subspaces with $Z_0 \subset Z_1 \subset \ldots \subset Z$, $W_0 \subset W_1 \subset \ldots \subset W$, $1 \leq \dim W_n < \infty$.

THEOREM 3.1. *Let $\varphi \in C^1(X, \mathbb{R})$. Assume that there exists $r > 0$ such that, with $Y := \{w \in W \ : \ \|w\| = r\}$:*

a) $\sup_Y \varphi \leq \inf_Z \varphi$,

b) *φ is bounded from above on $A := \{w \in W \ : \ \|e\| \leq r\}$,*

c) *φ satisfies the $(\mathrm{PS})_c^*$, where c is defined as in Lemma* 2.1.

Then c is a critical value of φ. Moreover if $c = \inf_Z \varphi$, then $K_c \cap Z \neq \emptyset$.

PROOF. We apply Theorem 2.5 with $F := Z$. It is easy to see the properties (A2), (A4)–(A6) are satisfied. Now let B_n be a closed subset of X_n such that $Y_n \subset B_n$ and $B_n \cap Z = \emptyset$. So the deformation $h_n : [0,1] \times B_n \to X_n$ given by

$$h_n(t, w+z) = \left((1-t) + \frac{tr}{\|w\|}\right) w + (1-t)z$$

is well-defined and shows that $\mathrm{cat}_{X_n, Y_n}(B_n) = 0$. The proof is complete, since (A1) is satisfied. □

We consider now the *generalized linking theorem.* Let X, $X_n = W_n \oplus Z_n$ be as above. Let $R > 0$, $\rho > 0$, $r \in]0, R[$ and suppose $e \in \bigcap_{n=0}^{\infty} Z_n$, $\|e\| = 1$. Define

$$\begin{aligned} Q :=&\{w \in W : \|w\| \leq \rho\} \oplus \{\lambda e : 0 \leq \lambda \leq R\}, \\ Y = \partial Q :=&\{w \in W : \|w\| = \rho\} \oplus \{\lambda e : 0 \leq \lambda \leq R\} \\ &\cup \{w \in W : \|w\| \leq \rho\} \oplus \{0, Re\}, \\ F :=&\{z \in Z : \|z\| = r\}. \end{aligned}$$

THEOREM 3.2. *Let $\varphi \in C^1(X, \mathbb{R})$ such that*

a) $\sup_Y \varphi \leq \inf_F \varphi$,

b) *φ is bounded from above on Q,*

c) *φ satisfies the $(\mathrm{PS})_c^*$, where c is defined as in Lemma* 2.1. *Then c is a critical value of φ. Moreover if $c = \inf_F \varphi$ then $K_c \cap F \neq \emptyset$.*

PROOF. We apply Theorem 2.5. It is easy to see that (A2), (A4)–(A6) are satisfied. Let B_n be a closed subset of X_n such that $Y_n \subset B_n$ and $B_n \cap F = \emptyset$. Let $\theta_n : W_n \oplus \{\lambda e : \lambda \in \mathbb{R}\} \setminus \{re\} \to Y_n$ be a retraction. Then the deformation $h_n : [0,1] \times B_n \to X_n$ given by

$$h_n(t, w+z) = (1-t)(w+z) + t\theta_n(w + \|z\|e)$$

is well defined and shows that $\text{cat}_{X_n,Y_n}(B_n) = 0$. The proof is complete, since (A1) is satisfied. □

REMARK. The above theorems are well-known when $\sup_Y \varphi < \inf_F \varphi$ (see [1]).

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