Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 1, 1993, 183–201

NONLINEAR EIGENVALUES AND MOUNTAIN PASS METHODS

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Dedicated to the memory of Karol Borsuk

1. Introduction

Mountain pass methods have proved very helpful in many applications. In the original formulation, Ambrosetti-Rabinowitz [1] considered a C^1 functional $G(u)$ defined on the whole of a Banach space B. It was assumed that there were elements $e_0, e_1 \in B$ such that

$$\max G(e_i) < c := \inf_{\varphi\in\Phi} \max_{0\le s\le 1} G(\varphi(s)) \tag{1.1}$$

where Φ is the set of all continuous maps φ of $[0,1]$ into B such that $\varphi(i) = e_i$, $i = 0, 1$. It was desired to find a point $u \in B$ such that

$$G'(u) = 0, \qquad u \ne e_i, i = 0, 1. \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 47H15, 58E05, 49B27.
[1]Research supported in part by an NSF grant.

Assumption (1.1) is not sufficient for such a point to exist, but it does imply that there is a sequence $\{u_k\} \subset B$ such that

$$G(u_k) \to c, \qquad G'(u_k) \to 0. \tag{1.3}$$

If, in addition, G satisfies the Palais-Smale (PS)-condition, then indeed one does obtain a solution of (1.2). (The (PS)-condition states that (1.3) implies that the sequence $\{u_k\}$ has a convergent subsequence). In order to conclude that (1.2) has a solution (or at least that (1.3) holds), it is necessary to allow the paths in Φ complete freedom to roam over the entire space B. As a result, one might obtain $\|u_k\| \to \infty$.

This situation is common for various mountain pass geometries which we do not study here. For further references on mountain pass theorems we address the reader to extensive bibliographies in [2] and [4].

In some previous publications [5–9] the authors considered the situation when one restricts the paths in Φ to remain in a fixed region of B. If the competing paths touch the boundary, one generally does not obtain a solution of (1.2), but rather the Lagrange multiplier relation similar to one for a constrained minimum.

In the present paper we give a general analysis of what happens when one restricts the paths in Φ to fixed regions. We consider regions of the form

$$B_R := \{u \in H \mid F(u) \le R\} \tag{1.4}$$

where $G(u)$, $F(u)$ are C^1 functionals on a Hilbert space H. Assuming that B_R is path connected for $R = R_0$ and $e_0, e_1 \in B_{R_0}$, we let Φ_R denote the continuous maps φ from $[0,1]$ to B_R such that $\varphi(i) = e_i$, $i = 0, 1$. Then we define

$$c(R) := \inf_{\varphi \in \Phi_R} \max_{0 \le s \le 1} G(\varphi(s)). \tag{1.5}$$

We still have to impose some convergence conditions, but they do not amount to (PS). In typical situations in applications one obtains in our setting a bounded approximation sequence on which the gradient of the functional tends to zero and which has weak limit points. Assumptions then are needed to establish those weak limit points as critical points. The case when competing paths stay away from the boundary ∂B_R is a "good" one: one has a point u where $G'(u) = 0$, $G(u) = c(R)$ and $F(u) \le R$. Our attention is to the "bad" case, when approximation paths do not stay away from the boundary for any R. In this case there is a solution of

$$G'(u) = -\alpha F'(u), \qquad u \in \partial B_R. \tag{1.6}$$

Moreover,

$$D^+ c(R) \le -\lambda_0(R), \qquad D_- c(R) \ge -\mu_0(R), \tag{1.7}$$

where $\mu_0(R)$ (resp. $\lambda_0(R)$) is the upper (resp. lower) bound of the set of all α satisfying (1.6).

Finally we prove that

$$\liminf_{R\to\infty} \lambda_0(R) = 0. \tag{1.8}$$

This provides a "qualified" approximation of a solution to (1.2). As it was mentioned in [9], relations (1.6)–(1.8) have an advantage over (1.3), since they often allow a uniform a priori bound for $\{u_k\}$ and, as result, convergence of u_k to a solution of (1.2).

The following example in $H = \mathbf{R}^2$ illustrates our main results. Let

$$G(x,y) = (1-x^2)e^{-y^2},\ F(x,y) = x^2 + y^2,\ e_1 = (2,0),\ e_2 = (-2,0). \tag{1.9}$$

Then G possesses a mountain pass geometry in any B_R, $R > 4$, $c(R) = e^{-R}$, the Palais-Smale condition is not satisfied and G has no critical point corresponding to a critical value in $[0, e^{-4}]$. However, (1.3) holds with $u_j = (0, \pm R_j^{\frac{1}{2}})$, $\alpha_j = -e^{-R_j}$ for any sequence $R_j \to \infty$.

In Section 2 we prove a "mountain pass alternative", namely, we study sequences approximating the critical value $c(R)$ on B_R or, if possible, on ∂B_R. In Section 3 we associate the rate of decrease of $c(R)$ with would-be eigenvalues of (1.7). In Section 4 we discuss convergence of approximating sequences to critical points. In Section 5 we prove two technical lemmas used in Sections 2 and 3.

2. The Mountain Pass Alternative

In this section we generalize the alternative proved in [8]. Let $F(u)$, $G(u)$ be C^1 functionals on a Hilbert space H, and assume that

$$B_R := \{u \in H \mid F(u) \le R\} \tag{2.1}$$

is path connected for each $R \ge R_0$, with some $R_0 \in \mathbf{R}$. Let e_0, e_1 be fixed elements in B_{R_0} and define

$$\Phi_R := \{\Phi \in C([0,1]), B_R) \mid \varphi(j) = e_j,\ j = 0,1\}. \tag{2.2}$$

We assume that $G(u)$ has mountain pass geometry in B_R relative to the e_j. This means that

$$\max_{j=0,1} G(e_j) < c(R) := \inf_{\varphi\in\Phi_R} \max_{s\in[0,1]} G(\varphi(s)). \tag{2.3}$$

Let

$$\nu(u) := (F'(u), G'(u)), \tag{2.4}$$

$$\tau(u) := \nu(u)/\|F'(u)\|\,\|G'(u)\| \quad \text{for} \quad \|F'(u)\| \neq 0,\ \|G'(u)\| \neq 0, \tag{2.5}$$

and let Ψ denote the set of those positive non-increasing functions $\psi(t)$ on $[0,\infty)$ such that

$$\int_1^\infty \psi(t)dt = \infty. \tag{2.6}$$

Our first result is

THEOREM 2.1. *Under the above hypotheses, the following alternative holds: either*

(a) *for each $\psi \in \Psi$ there is a sequence $\{u_k\} \subset B_R$ such that*

$$G(u_k) \to c(R), \quad G'(u_k)/\psi(\|u_k\|) \to 0, \tag{2.7}$$

or

(b) *there is a sequence $\{u_k\} \subset \partial B_R$ such that*

$$G(u_k) \to c(R), \quad \nu(u_k) < 0 \tag{2.8}$$

and

$$\frac{G'(u_k)}{\|G'(u_k)\|} + \frac{F'(u_k)}{\|F'(u_k)\|} \to 0. \tag{2.9}$$

In proving this theorem we shall make use of

LEMMA 2.2. *In addition to the above hypotheses, assume that there are constants $\varepsilon_0 > 0$, $\theta < 1$ such that $F'(u) \neq 0$ and*

$$\nu(u) + \theta\|G'(u)\|\,\|F'(u)\| \geq 0 \tag{2.10}$$

holds for all $u \in \partial B_R$ satisfying

$$|G(u) - c(R)| \leq 3\varepsilon_0. \tag{2.11}$$

Then for every $\psi \in \Psi$ there is a sequence $\{u_k\} \subset B_R$ such that (2.7) *holds.*

Before proving Lemma 2.2 we shall show it implies Theorem 2.1. Assume that option (b) of Theorem 2.1 does not hold. Then there are positive constants ε_0, a such that

$$\left\| \frac{G'(u)}{\|G'(u)\|} + \frac{F'(u)}{\|F'(u)\|} \right\| \geq a \tag{2.12}$$

holds whenever $u \in \partial B_R$ satisfies

$$|G(u) - c(R)| \leq 3\varepsilon_0, \quad \nu(u) < 0. \tag{2.13}$$

But (2.12) is equivalent to

$$\nu(u) + \left(1 - \frac{1}{2}a^2\right)\|F'(u)\|\,\|G'(u)\| \geq 0 \tag{2.14}$$

and this holds trivially when $\nu(u) \geq 0$ provided we take $a^2 < 2$. This implies that (2.10) holds whenever $u \in \partial B_R$ satisfies (2.11). Lemma 2.2 now implies that option (a) of Theorem 2.1 holds. □

Our proof of Lemma 2.2 will depend upon the following lemma to be proved in Section 5.

LEMMA 2.3. *Suppose $X(u)$, $Y(u)$ are continuous mappings from a subset B of a Hilbert space H into H. Let $\widehat{B}$ be the set of those $u \in B$ such that $X(u) \neq 0$, and assume that $Y(u) \neq 0$ for all u in a closed subset Q_0 of $\widehat{B}$. Assume also that there is a $\theta < 1$ such that*

$$(X(u), Y(u)) \leq \theta\|X(u)\|\,\|Y(u)\|, \quad u \in Q_0. \tag{2.15}$$

Then for each $\alpha < 1 - \theta$ there is a locally Lipschitz map $Z(u)$ of $\widehat{B}$ into H such that

$$\|Z(u)\| \leq 1, \quad u \in \widehat{B}, \tag{2.16}$$

$$(X(u), Z(u)) \geq \alpha\|X(u)\|, \quad u \in \widehat{B} \tag{2.17}$$

and

$$(Y(u), Z(u)) < 0, \quad u \in Q_0. \tag{2.18}$$

Using this lemma we give the

PROOF OF LEMMA 2.2. If the conclusion were not true, there would be a $\psi \in \Psi$ and a positive constant ε such that

$$\|G'(u)\| \geq \psi(\|u\|) \tag{2.19}$$

holds for all $u \in B_R$ satisfying

$$|G(u) - c(R)| \leq 3\varepsilon. \tag{2.20}$$

We may take $\varepsilon \leq \varepsilon_0$ and $3\varepsilon < c(R) - \max G(e_j)$. Let

$$Q = \{u \in B_R \mid |G(u) - c(R)| \leq 2\varepsilon\}, \tag{2.21}$$

$$Q_1 = \{u \in B_R \mid |G(u) - c(R)| \leq \varepsilon\}, \tag{2.22}$$

$Q_2 = B_R \setminus Q$ and

$$\eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].$$

Then $\eta(u)$ is Lipschitz continuous on H, vanishes on $\overline{Q}_2$ and equals one on Q_1. Take

$$Q_0 = \{u \in \partial B_R \mid \ |G(u) - c(R)| \le 3\varepsilon\}, \tag{2.23}$$

$Y(u) = -F'(u)$, $X(u) = G'(u)$, $B = B_R$ in Lemma 2.3. Note that $Y(u) \ne 0$ on Q_0 and that (2.10) implies (2.15). By Lemma 2.3 there is a locally Lipschitz map $Z(u)$ of $\widetilde{B}_R$ into H such that (2.16)–(2.18) hold. Hence

$$(G'(u), Z(u)) \ge \alpha \|G'(u)\|, \quad u \in \widehat{B}_R \tag{2.24}$$

and

$$(F'(u), Z(u)) > 0, \quad u \in Q_0. \tag{2.25}$$

Let $W(u) := -\eta(u)Z(u)$. We can solve

$$d\sigma(t)/dt = W(\sigma(t)), \quad \sigma(0) = u \tag{2.26}$$

uniquely in $[0, \infty)$ for each $u \in B_R$ provided $\sigma(t)$ does not exit B_R. (Note that $W(u)$ is locally Lipschitz and bounded on the whole of B_R since $\eta(u)$ vanishes outside Q, which is a closed subset of $\widehat{B}_R$). But indeed the solution $\sigma(t, u)$ of (2.26) does not exit B_R for $t \ge 0$. To see this note that if $u_1 \in Q_0$, then $(Z(u), F'(u)) > 0$ in a neighbourhood of u_1. If $u_1 \in \partial B_R \setminus Q_0$, then $\eta(u)$ vanishes in a neighbourhood of u_1. Since

$$(W(\sigma), F'(\sigma)) = -\eta(\sigma)(Z(\sigma), F'(\sigma)),$$

any solutions of (2.26) would either be constant or directed into B_R at a point of ∂B_R. Since

$$(G'(u), W(u)) \le -\alpha\eta(u)\|G'(u)\| \, \|W(u)\| \le \eta(u)$$

for $u \in B_R$, we have

$$\|\sigma(t, u) - u\| \le t, \qquad t \ge 0, \tag{2.27}$$

and

$$dG(\sigma(t, u))/dt = (G'(\sigma(t, u)), W(\sigma(t, u))) \le -\alpha\eta(\sigma(t, u))\|G'(\sigma(t, u))\|. \tag{2.28}$$

Thus

$$G(\sigma(t_2, u)) \le G(\sigma(t_1, u)), \qquad t_1 < t_2. \tag{2.29}$$

By the definition (2.3) of $c(R)$, there is a $\varphi \in \Phi_R$ such that

$$G(\varphi(s)) < c(R) + \varepsilon, \qquad 0 \le s \le 1. \tag{2.30}$$

Pick T so that

$$2\varepsilon < \alpha \int_M^{M+T} \psi(t)dt, \tag{2.31}$$

where

$$M = \max_{0 \le s \le 1} \|\varphi(s)\|. \tag{2.32}$$

This can be done by (2.6). If for some s and $t_1 < T$, $\sigma(t_1, \varphi(s)) \notin Q_1$, then we must have

$$G(\sigma(t_1, \varphi(s))) < c(R) - \varepsilon \tag{2.33}$$

since (2.29) and (2.30) exclude $G(\sigma(t_1, \varphi(s))) > c(R) + \varepsilon$. Hence by (2.29)

$$G(\sigma(T, \varphi(s))) < c(R) - \varepsilon. \tag{2.34}$$

On the other hand, if for a particular s, $\sigma(t, \varphi(s)) \in Q_1$ for all t satisfying $0 \le t \le T$, then we have

$$\begin{aligned} G(\sigma(T, \varphi(s))) - G(\varphi(s)) &\le -\alpha \int_0^T \|G'(\sigma(t, \varphi(s)))\| dt \\ &\le -\alpha \int_0^T \psi(\|\sigma(t, \varphi(s))\|) dt \\ &\le -\alpha \int_0^T \psi(\|\varphi(s)\| + t) dt \\ &\le -\alpha \int_0^T \psi(M + t) dt = -\alpha \int_M^{M+T} \psi(\tau) d\tau < -2\varepsilon \end{aligned}$$

by (2.28), (2.21), (2.27), (2.32) and (2.31). This combined with (2.30) shows that (2.34) holds in this case as well. Moreover η vanishes in the neighbourhood of e_i. Hence $\sigma(T, e_i) = e_i$ and $\varphi_T(s) := \sigma(T, \varphi(s))$ is in Φ_R. But then (2.34) contradicts (2.3). The conclusion of the lemma must be valid. □

3. Estimate of Remainder

In this section we continue the analysis of Section 2. We add the assumption that for each $R \ge R_0$ there is a $\delta > 0$ such that $\|F'(u)\|$ is bounded away from 0 on the set

$$\{u \in H \mid F(u) \le R + \delta,\ |G(u) - c(R)| \le \delta\}.$$

For $\delta > 0$ we define

$$Q_\delta(R) := \{u \in H \mid |F(u) - R| \le \delta,\ |G(u) - c(R)| \le \delta,\ \tau(u) \le \delta - 1\}. \tag{3.1}$$

When $Q_\delta(R) \neq \emptyset$, we define

$$\lambda_\delta(R) := \inf_{Q_\delta(R)} \|G'(u)\|/\|F'(u)\|, \tag{3.2}$$

$$\mu_\delta(R) := \sup_{Q_\delta(R)} \|G'(u)\|/\|F'(u)\|, \tag{3.3}$$

$$\lambda(R) := \lim_{\delta\to 0} \lambda_\delta(R), \tag{3.4}$$

$$\mu(R) := \lim_{\delta\to 0} \mu_\delta(R). \tag{3.5}$$

We have

THEOREM 3.1. *Under the above hypotheses the following alternative holds for each $R \geq R_0$: either*

(a) *there is a sequence $\{u_k\} \subset H$ such that*

$$G(u_k) \to c(R), \qquad G'(u_k) \to 0 \tag{3.6}$$

and

$$\lim F(u_k) \leq R, \tag{3.7}$$

or

(b) *$Q_\delta(R) \neq \emptyset$ for each $\delta > 0$ and*

$$c(R+T) \leq c(R) + (\delta - 1)\lambda_\delta(R)T, \qquad c(R-T) \leq c(R) + (1+\delta)\mu_\delta(R)T \tag{3.8}$$

for $T > 0$ sufficiently small depending on δ.

In proving Theorem 3.1 we shall make use of the following lemmas.

LEMMA 3.2. *Let V be a closed subset of a real Hilbert space H, and let X_i, $i = 1, \dots, k$ be continuous maps from V to H. Assume that there is a continuous map Y from V to H and constants $\alpha_i \in \mathbb{R}$ such that*

$$(X_i, Y) \leq \alpha_i \quad \text{and} \quad \|Y\| \leq M \quad \text{on } V,\ i = 1, \dots, k. \tag{3.9}$$

Then for each $\varepsilon > 0$ there is a locally Lipschitz continuous map Z from V to H such that

$$(X_i, Z) \leq \alpha_i + \varepsilon \quad \text{and} \quad \|Z\| \leq M \quad \text{on } V,\ i = 1, \dots, k. \tag{3.10}$$

Lemma 3.2 will be proved in Section 5. Here we use it to prove

LEMMA 3.3. *Assume that there are positive constants a, δ, such that*

$$\|G'(u)\| \geq a \tag{3.11}$$

whenever

$$F(u) \leq R + 3\delta, \qquad |G(u) - c(R)| \leq 3\delta. \tag{3.12}$$

Then $Q_\rho(R) \neq \emptyset$ for all $\rho > 0$, and for each $\varepsilon > 0$ there is a locally Lipschitz continuous map Z from

$$W := \{u \in H \mid F(u) \leq R + 2\delta,\ |G(u) - c(R)| \leq 2\delta\} \tag{3.13}$$

to H such that

$$\|Z(u)\| \leq M, \qquad u \in W, \tag{3.14}$$

$$(G'(u), Z(u)) \leq (\varepsilon + 3\delta - 1)\lambda_{3\delta}(R), \qquad u \in W, \tag{3.15}$$

$$(F'(u), Z(u)) \leq 1 + \varepsilon, \quad u \in W,\ F(u) \geq R - \delta, \tag{3.16}$$

$$(F'(u), Z(u)) \leq c, \quad u \in W. \tag{3.17}$$

PROOF. If $Q_\rho(R)$ were empty for some $\rho > 0$, then the hypothesis of Lemma 2.2 would be satisfied for $\varepsilon_0 = \rho$, $\theta = 1 - \rho$. In virtue of that lemma, there would be a sequence $\{u_k\} \subset B_R$ satisfying (3.6). This contradicts (3.11). By Lemma 3.2 it suffices to find a continuous map which satisfies (3.14)–(3.17). Let

$$\begin{aligned}
Q &= \{u \in H \mid |F(u) - R| < 3\delta,\ |G(u) - c(R)| < 3\delta\},\\
Q_1 &= \{u \in Q \mid \tau(u) < 3\delta - 1\},\\
Q_2 &= \{u \in Q \mid |\tau(u)| < 1 - \delta\},\\
Q_3 &= \{u \in Q \mid \tau(u) > 1 - 3\delta\},\\
Q_4 &= \{u \in H \mid F(u) < R - \delta,\ |G(u) - c(R)| < 3\delta\},
\end{aligned}$$

where we assumed that $\delta < \frac{1}{3}$. The sets Q_j are open, and their union contains W. Let $\{\psi_k\}$ be a partition of unity subordinate to this covering. Let

$$Z_1(u) = F'(u)/\|F'(u)\|^2, \qquad u \in Q_1.$$

Then

$$(Z_1(u), F'(u)) = 1$$

and

$$(Z_1(u), G'(u)) = \nu(u)/\|F'(u)\|^2 = \tau(u)\|G'(u)\|/\|F'(u)\| \leq (3\delta - 1)\lambda_{3\delta}(R)$$

for $u \in Q_1$. Let

$$Z_2(u) = \lambda_{3\delta}(R)[\nu(u)F'(u) - \|F'(u)\|^2 G'(u)]/\|F'(u)\|^2\|G'(u)\|^2(1 - \tau(u)^2),\ u \in Q_2.$$

Then

$$(Z_2(u), F'(u)) = 0$$

and

$$(Z_2(u), G'(u)) = -\lambda_{3\delta}(R).$$

Let

$$Z_3(u) = -\lambda_{3\delta}(R)G'(u)/\|G'(u)\|^2, \qquad u \in Q_3.$$

Then

$$(Z_3(u), F'(u)) = -\lambda_{3\delta}(R)\nu(u)/\|G'(u)\|^2 \leq (3\delta - 1)\lambda_{3\delta}(R)\|F'(u)\|/\|G'(u)\| \leq 0$$

and

$$(Z_3(u), G'(u)) = -\lambda_{3\delta}(R).$$

Finally, let

$$Z_4(u) = -\lambda_{3\delta}(R)G'(u)/\|G'(u)\|^2, \qquad u \in Q_4.$$

Then

$$(Z_4(u), F'(u)) \leq \lambda_{3\delta}(R)\|F'(u)\|/\|G'(u)\| \leq C_0$$

and

$$(Z_4(u), G'(u)) = -\lambda_{3\delta}(R).$$

Thus we have

$$(Z_k, F') \leq 1, \quad (Z_k, G') \leq (3\delta - 1)\lambda_{3\delta}(R) \quad \text{in} \quad Q_k,\ k = 1, 2, 3,$$

$$(Z_4, F') \leq C_0, \quad (Z_4, G') = -\lambda_{3\delta}(R) \quad \text{in} \quad Q_4.$$

Let

$$Z(u) = \sum_{k=1}^{4} \psi_k(u) Z_k(u).$$

This map is defined and continuous on the whole of W. Clearly it satisfies (3.14)–(3.17). Application of Lemma 3.2 completes the proof. □

LEMMA 3.4. *Under the hypotheses of Lemma 3.3, for each $\varepsilon > 0$ there is a locally Lipschitz continuous map Z of W into H such that* (3.14) *and* (3.17) *hold and*

$$(G'(u), Z(u)) \leq (1 + \varepsilon)\mu_{3\delta}(R), \qquad u \in W, \tag{3.18}$$

$$(F'(u), X(u)) \leq \varepsilon - 1, \qquad u \in W, \quad F(u) \geq R - \delta. \tag{3.19}$$

PROOF. As in the preceding proof, we cover W with the Q_i. On Q_1 we define

$$Z_1(u) = -F'(u)/\|F'(u)\|^2, \qquad u \in Q_1.$$

Then

$$(Z_1(u), F'(u)) = -1$$

and

$$(Z_1(u), G'(u)) = -\tau(u)\|G'(u)\|/\|F'(u)\| \leq \mu_{3\delta}(R).$$

On Q_2 we define

$$Z_2(u) = [\nu(u)G'(u) - \|G'(u)\|^2 F'(u)]/\|F'(u)\|^2\|G'(u)\|^2(1 - \tau(u)^2).$$

Then

$$(Z_2(u), F'(u)) = -1$$

and

$$(Z_2(u), G'(u)) = 0.$$

Next take

$$Z_3(u) = -F'(u)\|/F'(u)\|^2, \qquad u \in Q_3.$$

Then

$$(Z_3(u), F'(u)) = -1$$

and

$$(Z_3(u), G'(u)) = -\tau(u)\|G'(u)\|/\|F'(u)\| \leq 0.$$

Finally set

$$Z_4(u) = -G'(u)/\|G'(u)\| \, \|F'(u)\|, \qquad u \in Q_4.$$

Then

$$(Z_4(u), F'(u)) = -\tau(u) \leq 1$$

and

$$(Z_4(u), G'(u)) = -\|G'(u)\|/\|F'(u)\| \leq 0.$$

As before we take

$$Z(u) = \sum_{k=1}^{4} \psi_k(u) Z_k(u).$$

This is clearly continuous on the whole of W and satisfies (3.14), (3.17)–(3.19). Application of Lemma 3.2 completes the proof. □

PROOF OF THEOREM 3.1. If option (a) does not apply, then there are positive constants a, δ such that the hypothesis of Lemma 3.3 holds. By the lemma there is a locally Lipschitz mapping Z from W to H such that (3.14)–(3.17) hold with $\varepsilon = \delta_1 < \delta$. Let

$$\begin{aligned}
Q &= \{u \in H \mid |F(u) - R| \leq 2\delta, \ |G(u) - c(R)| \leq 2\delta\}, \\
W_1 &= \{u \in W \mid F(u) \leq R + \delta, \ |G(u) - c(R)| \leq \delta\}, \\
W_2 &= H \setminus W,
\end{aligned}$$

and

$$\eta(u) = d(u, W_2)/[d(u, W_1) + d(u, W_2)].$$

For each $u \in W$, let $\sigma(t, u)$ be the unique solution of

$$d\sigma(t)/dt = \eta(\sigma(t))Z(\sigma(t)), \qquad \sigma(0) = u. \tag{3.20}$$

Since $\eta(u)Z(u)$ is locally Lipschitz continuous and bounded on the whole of H, $\sigma(t, u)$ will exist for all real t. Now

$$dF(\sigma)/dt = (F'(\sigma), \sigma') = \eta(\sigma)(F'(\sigma), Z(\sigma)). \tag{3.21}$$

If $F(u) < R - \delta$ and $T < \delta/C$, then (3.17) implies that

$$F(\sigma(t, u)) - F(u) < \delta, \qquad 0 \le t \le T. \tag{3.22}$$

On the other hand, if $F(u) \ge R - \delta$, then (3.16) implies

$$F(\sigma(t, u)) - F(u) \le (1 + \delta_1)T, \qquad 0 \le t \le T. \tag{3.23}$$

Since

$$dG(\sigma)/dt = (G', \sigma') = \eta(\sigma)(G'(\sigma), Z(\sigma)), \tag{3.24}$$

we have

$$G(\sigma(t, u)) - G(u) \le t\eta(\sigma)(\delta_1 + 3\delta - 1)\lambda_{3\delta}(R) \tag{3.25}$$

by (3.15). Let $P \in \Phi_R$ be a path such that

$$\max_P G < c(R) + \delta_2, \qquad \text{where} \quad \delta_2 < \delta. \tag{3.26}$$

If $u \in P$ and there is a $t < T$ such that $\sigma(t, u) \notin W_1$, then

$$G(\sigma(T, u)) < c(R) - \delta \tag{3.27}$$

or

$$F(\sigma(T, u)) > R + \delta. \tag{3.28}$$

But (3.28) is excluded by (3.23) and the size of T. Moreover, if $\sigma(t, u) \in W_1$ for $0 \le t \le T$, then (3.25) gives

$$G(\sigma(T, u)) \le c(R) + \delta_2 + (\delta_1 + 3\delta - 1)\lambda_{3\delta}(R)T. \tag{3.29}$$

If we take $(1 - \delta_1 - 3\delta)\lambda_{3\delta}(R)T \le 2\delta$, then (3.22), (3.23), (3.27) and (3.29) imply

$$c(R + (1 + \delta_1)T) \le c(R) + \delta_2 + (\delta_1 + 3\delta - 1)\lambda_{3\delta}(R)T. \tag{3.30}$$

Letting $\delta_1, \delta_2 \to 0$, we obtain

$$c(R + T) \le c(R) + (3\delta - 1)\lambda_{3\delta}(R)T.$$

This implies the first inequality in (3.8) if we replace 3δ by δ. To obtain the second, we use Lemma 3.4. This time we take $T < \delta/(C+1)$. If $F(u) \geq R-\delta$, then (3.19) gives

$$F(\sigma(T,u)) - F(u) \leq (\delta_1 - 1)T \tag{3.31}$$

where we take $\varepsilon = \delta_1 < \delta$. If $F(u) < R-\delta$, then (3.17) gives

$$F(\sigma(T,u)) - F(u) \leq CT.$$

These imply

$$F(\sigma(T,u)) \leq R + (\delta_1 - 1)T.$$

On the other hand, (3.18) and (3.24) imply

$$G(\sigma(T,u)) - G(u) \leq (1+\delta_1)\mu_{3\delta}(R)T. \tag{3.32}$$

If P satisfies (3.26) and $u \in P$, then (3.31), (3.32) imply

$$c(R + (\delta_1 - 1)T) \leq c(R) + \delta_2 + (1+\delta_1)\mu_{3\delta}(R)T,$$

which implies

$$c(R-T) \leq c(R) + (1+3\delta)\mu_{3\delta}(R)T.$$

This gives the second inequality in (3.8). □

COROLLARY 3.5. *If option* (b) *of Theorem* 3.1 *holds, then*

$$D_{-}c(R) \geq -\mu(R), \qquad D^{+}c(R) \leq -\lambda(R). \tag{3.33}$$

COROLLARY 3.6. *If there is a* $\delta > 0$ *such that*

$$c(R+\delta) = c(R) \tag{3.34}$$

then option (a) *of Theorem* 3.1 *holds.*

PROOF. If option (a) did not hold, then the hypothesis of Lemma 3.3 would be satisfied. Thus (3.2), (3.4), (3.1) and the assumptions on $\|F'(u)\|$ would imply that $\lambda(R) > 0$. Corollary 3.5 would then imply that $c(R+\delta) < c(R)$ for every $\delta > 0$. □

COROLLARY 3.7. *If option* (b) *of Theorem* 3.1 *holds, then*

$$\int_{R_0}^{R} \lambda(r)dr \leq c(R_0) - c(R). \tag{3.35}$$

Hence

$$\inf_{R_0 \leq r \leq R} \lambda(r) \leq [c(R_0) - c(R)]/(R-R_0).$$

COROLLARY 3.8. *If option* (b) *of Theorem* 3.1 *holds and* $\mu(R) < \infty$, *then* $c(r)$ *is continuous from the left at* $r = R$. *If* $\mu(r)$ *is bounded for* r *near* R, *then* $c(r)$ *is continuous at* $r = R$.

PROOF. By (3.8) for each $\delta > 0$

$$0 \leq c(R - T) - c(R) \leq (1 + \delta)\mu_\delta(R)T.$$

Thus $c(r)$ is continuous from the left. Moreover for T sufficiently small

$$0 \leq c(R) - c(R + T) \leq (1 + \delta)\mu_\delta(R + T)T.$$

When $\mu(R + T)$ is bounded for T small, this gives continuity from the right. □

4. An Absolute Continuity Condition

In this section we introduce a compactness criterion which will help us locate solutions of

$$G'(u) = 0 \tag{4.1}$$

and

$$G'(u) = \alpha F'(u). \tag{4.2}$$

We define

$$\beta(u) := \nu(u)/\|F'(u)\|^2. \tag{4.3}$$

Our compactness assumption is

I. A sequence $\{u_k\} \subset H$ satisfying

$$G(u_k) \to c(R), \qquad \lim F(u_k) \leq R \tag{4.4}$$

and

$$\text{either} \quad G'(u_k) \to 0 \quad \text{or} \quad \tau(u_k) \to -1, \tag{4.5}$$

has a convergent subsequence.

We have

THEOREM 4.1. *In addition to the hypotheses of Theorem* 3.1, *assume compactness condition* I. *Then the following alternative holds: either*

(a) *there is a solution* u *of* (4.1) *in* B_R *satisfying*

$$G(u) = c(R) \tag{4.6}$$

or

(b) *there is a solution of* (4.2) *on* ∂B_R *satisfying* (4.6) *and*

$$\alpha = -\lambda_0(R), \tag{4.7}$$

and a solution satisfying (4.6) *and*

$$\alpha = -\mu_0(R). \tag{4.8}$$

Moreover,

$$D^+c(R) \leq -\lambda_0(R), \qquad D_-c(R) \geq -\mu_0(R), \tag{4.9}$$

where

$$\lambda_0(R) = \inf \left\{ \begin{array}{l} -\alpha \mid \alpha < 0 \textit{ satisfies } (4.2) \textit{ for some } u \in \partial B_R \\ \textit{with } (4.6) \textit{ holding} \end{array} \right\} \tag{4.10}$$

and

$$\mu_0(R) = \sup \left\{ \begin{array}{l} -\alpha \mid \alpha < 0 \textit{ satisfies } (4.2) \textit{ for some } u \in \partial B_R \\ \textit{with } (4.6) \textit{ holding} \end{array} \right\}. \tag{4.11}$$

PROOF. We apply Theorem 3.1. If option (a) holds, then $G'(u_k) \to 0$. Thus $\{u_k\}$ has a renamed subsequence converging to an element $u \in H$. By (3.6) and (3.7), $u \in B_R$ and satisfies (4.1) and (4.6). This gives option (a) of our theorem. If option (b) of Theorem 3.1 holds, then $\lambda_0(R) > 0$ and there is a sequence $\{u_k\} \subset H$ such that

$$\lambda_{\delta_k}(R) \leq \|G'(u_k)\|/\|F'(u_k)\| \leq \lambda_{\delta_k}(R) + \delta_k$$

and

$$|G(u) - c(R)| \leq \delta_k, \qquad |F(u_k) - R| \leq \delta_k, \qquad \tau(u_k) \leq \delta_k - 1, \tag{4.12x}$$

where $\delta_k \to 0$. Then there is a renamed subsequence $u_k \to u$ in H. Thus u satisfies (4.6) and

$$\tau(u) = -1, \quad \|G'(u)\|/\|F'(u)\| = \lambda(R), \quad \beta(u) = -\lambda(R), \quad F(u) = R.$$

Also

$$\left\| \frac{G'(u)}{\|G'(u)\|} + \frac{F'(u)}{\|F'(u)\|} \right\|^2 = 2 + 2\tau(u) = 0.$$

Hence u satisfies (4.2) with $\alpha = \beta(u) = -\lambda(R)$. Similarly, we can obtain a sequence satisfying (4.12) and

$$\|G'(u)\|/\|F'(u_k)\| \to \mu(R).$$

In this case $\beta(u_k) \to -\mu(R)$ and we again have a convergent subsequence. This time the limit satisfies (4.2) with $\alpha = -\mu(R)$. If α satisfies (4.2) for some $u \in \partial B_R$ with (4.6) holding, then $\alpha = \beta(u)$ and

$$|\alpha| = \|G'(u)\|/\|F'(u)\|. \tag{4.13}$$

Then

$$\lambda(R) \leq |\alpha| \leq \mu(R) \tag{4.14}$$

if $\tau(u) = -1$, i.e., if $\alpha < 0$. Since we have found such α which satisfy $|\alpha| = \lambda(R)$ and $|\alpha| = \mu(R)$, we see that $\lambda_0(R) = \lambda(R)$, $\mu_0(R) = \mu(R)$. We now apply (3.8). □

REMARK 4.2. Without further assumptions we cannot tell if $\lambda_0(R) \neq \mu_0(R)$.

Let

$$c := \lim_{R\to\infty} c(R). \tag{4.15}$$

Then

$$c \geq \max G(e_i) > -\infty. \tag{4.16}$$

We have

THEOREM 4.3. *Under the hypotheses of Theorem* 4.1 *assume that* (4.1) *has no solution satisfying*

$$c(R_0) \geq G(u) \geq c. \tag{4.17}$$

Then (4.9) *holds for each* $R > R_0$ *and*

$$\liminf_{R\to\infty} \lambda_0(R) = 0. \tag{4.18}$$

PROOF. By hypothesis, option (a) of Theorem 4.1 does not hold for any $R > R_0$. Thus (4.9) must hold by option (b):

$$D^+c(R) \leq -\lambda_0(R), \qquad R > R_0. \tag{4.19}$$

If there were an $m > 0$ such that $\lambda_0(R) \geq m$ for $R \geq R_1$, then we would have

$$c(R) - c(R_1) \leq -\int_{R_1}^{R} m\,dr \to -\infty \quad \text{as} \quad R \to \infty \tag{4.20}$$

contradicting (4.16). Thus (4.18) holds. □

THEOREM 4.4. *Under the hypotheses of Theorem* 4.1 *assume that* (4.1) *has no solution satisfying*

$$G(u) \geq c(R_1) \tag{4.21}$$

for some $R_1 > R_0$, *and that* $\|G'(u)\|$ *is bounded for all* u *in* B_{R_1}, *satisfying* (4.21). *Then* $c(R)$ *is continuous in the interval* $[R_0, R_1]$.

PROOF. By Theorem 4.1, $0 \geq D_-c(R) > -\infty$ for $R \leq R_1$, since $\mu_0(R)$ has a uniform bound for $R \in [R_0, R_1]$ under assumptions on $\|G'\|$ and $\|F'\|$. □

THEOREM 4.5. *Under the hypotheses of Theorem* 4.1, *if there is a* $\delta > 0$ *such that* (3.34) *holds, then* (4.1) *has a solution in* B_R *satisfying* (4.6).

PROOF. By Corollary 3.6, option (a) of Theorem 3.1 holds. Under the compactness condition I this implies option (a) of Theorem 4.1.

5. Types of Pseudogradients

In this section we shall prove Lemmas 2.3 and 3.2. Lemma 2.3 was essentially proved in [5–7]. We give the proof here for completeness. First we prove

LEMMA 5.1. *Let* α, θ *satisfy* $0 \leq \alpha < 1 - \theta < 1$. *Then for any elements* $u \neq 0$, $v \neq 0$ *satisfying* $(u, v) \leq \theta\|u\|\,\|v\|$ *there is an element* h *such that* $(u, h) \geq \alpha\|u\|\,\|h\|$ *and* $(h, v) < 0$.

PROOF. We may assume that u and v are unit vectors. We take $h = u - \beta v$ with $\beta \geq 0$. Then $\|h\| \leq 1 + \beta$, $(h, u) = 1 - \beta(v, u) \geq 1 - \beta\theta$ and $(h, v) \leq \theta - \beta$. We take $\beta > \theta$ such that $\alpha(1 + \beta) \leq 1 - \beta\theta$. This can be done by the assumptions on θ, α. This gives the desired inequalities. □

PROOF OF LEMMA 2.3. Let α' satisfy $\alpha < \alpha' < 1 - \theta$. For each $u \in \widehat{B} \setminus Q_0$ let $h(u) = X(u)/\|X(u)\|$ and for each $u \in Q_0$ let $h(u)$ satisfy

$$\|h(u)\| = 1, \quad (X(u), h(u)) \geq \alpha\|X(u)\|, \quad u \in \widehat{B}, \quad (Y(u), h(u)) < 0, \quad u \in Q_0.$$

By continuity, for each $u \in \widehat{B}$ there is a neighbourhood $N(u)$ such that

$$(X(g), h(u)) \geq \alpha\|X(g)\|, \qquad g \in N(u), \tag{5.1}$$

and if $u \in Q_0$, then

$$(Y(g), h(u)) < 0, \qquad g \in N(u). \tag{5.2}$$

If $u \notin Q_0$, we reduce $N(u)$ so that $N(u) \cap Q_0 = \emptyset$. The collection $\{N(u)\}$ is an open cover of $\widehat{B}$. Since $\widehat{B}$ is a metric space, it is paracompact. Thus $\{N(u)\}$ has a locally finite refinement $\{N_\tau\}$. Let $\{\psi_\tau\}$ be a locally Lipschitz continuous partition subordinate to this refinement. For each τ, let u_τ be an element for which $N_\tau \subset N(u)$. Write

$$Z(g) = \sum_\tau \psi_\tau(g) h(u_\tau).$$

Since u_τ is fixed on the support of ψ_τ, $Z(g)$ is locally Lipschitz continuous. By (5.1) and (5.2)

$$(X(g), Z(g)) = \sum \psi_\tau(g)(X(g), h(u_\tau)) \geq \alpha \sum \psi_\tau(g)\|X(g)\| = \alpha\|X(g)\|, \quad g \in \widetilde{B}$$

and

$$(Y(g), X(g)) = \sum \psi_\tau(g)(Y(g), h(u_\tau)) < 0, \qquad g \in Q_0,$$

since $u_\tau \notin Q_0$ implies $g \notin N_\tau$. Also

$$\|Z(g)\| \leq \sum \psi_\tau(g)\|h(u_\tau)\| = \sum \psi_\tau(g) = 1.$$

This proves the lemma. □

PROOF OF LEMMA 3.2. Let $\varepsilon > 0$ be given. For each $w \in V$ there is a relatively open neighbourhood V_w of w such that

$$(X_i(u), Y(x)) \leq \alpha_i + \varepsilon, \qquad i \in 1, \dots, k, \; u \in V_w.$$

Since V is a metric space, the collection $\{V_w\}$ has a locally finite refinement $\{V_\tau\}$. Let $\{\psi_\tau\}$ be a locally Lipschitz continuous partition of unity subordinate to this refinement. For each τ let w_τ be an element for which $V_\tau \subset V_w$. Let

$$Z(u) = \sum_\tau \psi_\tau(u) Y(w_\tau).$$

Then

$$\begin{aligned}(X_i(u), Z(u)) &= \sum_\tau \psi_\tau(u)(X_i(u), Y(w_\tau)) \\ &\leq (\alpha_i + \varepsilon) \sum_\tau \psi_\tau(u) = \alpha_i + \varepsilon\end{aligned}$$

Finally,

$$\|Z(u)\| \leq \sum_\tau \psi_\tau(u)\|Y(x_\tau)\| \leq M \sum_\tau \psi_\tau(u) = M.$$

□

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Manuscript received August 25, 1992

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