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EQUIVARIANT DEGREE FOR ABELIAN ACTIONS PART I: EQUIVARIANT HOMOTOPY GROUPS

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Dedicated to the memory of Juliusz Schauder

0. Introduction

The classical degree theories and their Leray-Schauder extensions to infinite dimensions have been very useful in the study of nonlinear problems. The presence of symmetries in such problems, which restricts the class of maps and sets, gives a richer structure to the possible degrees. In our previous papers [6] and [7] we have defined a degree theory for such maps. In particular, we have studied and applied the degree for the case of a S^1-action.

Let E and F be two Banach spaces and Γ be a compact Lie group acting linearly, via isometries, on both of them. Let Ω be a bounded open invariant subset of E and f be an equivariant map defined on $\overline{\Omega}$ with values in F, that is, $f(\gamma x) = \widetilde{\gamma} f(x)$, for all x in $\overline{\Omega}$ and γ in $\Gamma, \widetilde{\gamma}$ representing the action on F.

If $f(x) \neq 0$ on $\partial\Omega$, then the Γ-degree is constructed as follows: take a large ball B centered at the origin and containing Ω and let $\widetilde{f} : B \to F$ be a Γ-equivariant continuous extension of f, with the usual compactness properties. Let N be a

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bounded, invariant, open neighborhood of $\partial\Omega$ such that $\widetilde{f}(x) \neq 0$ on $\overline{N}$. Let $\phi : B \to [0,1]$ be an invariant Urysohn function such that $\phi(x) = 1$ if x is outside $\Omega \cup N$ and $\phi(x) = 0$ in $\overline{\Omega}$. Let $\widetilde{F} : [0,1] \times B \to \mathbb{R} \times F$ be the map defined by

$$\widetilde{F}(t,x) = (2t + 2\phi(x) - 1, \widetilde{f}(x)),$$

where Γ acts trivially on both [0,1] and $\mathbb{R}$.

Since $\widetilde{F}(t,x)$ is zero only if $f(x) = 0$, x in Ω, and $t = 1/2$, $\widetilde{F}$ can be regarded as a Γ-equivariant map from S^E, the boundary of $[0,1] \times B$, into $S^F \simeq \mathbb{R} \times F \backslash \{0\}$.

The Γ-degree of f, $\deg_\Gamma(f;\Omega)$, is defined to be the Γ-equivariant homotopy class $[\widetilde{F}]_\Gamma$ of $\widetilde{F}$ considered as an element of the Γ-equivariant homotopy group of spheres $\Pi^\Gamma_{S^E}(S^F)$.

As we proved in [6], this degree has all the properties of the Brouwer-Leray-Schauder degree and reduces to it when Γ is trivial. In the infinite dimensional case we shall assume that $E = V \times \widetilde{E}$ and $F = W \times \widetilde{E}$, where V, W are finite dimensional Γ-spaces, and $f(x,y) = (f_1(x,y), y - f_2(x,y))$, where f_2 is compact. In this case $\Pi^\Gamma_{S^E}(S^F)$ is the inductive stable limit of $\Pi^\Gamma_{S^{V\times\widetilde{V}}}(S^{W\times\widetilde{V}})$, with $\widetilde{V}$ any finite dimensional invariant subspace of $\widetilde{E}$.

The present paper is devoted to the study of these equivariant homotopy groups of spheres in the case of an abelian group Γ. In part II we shall compute the degree and apply it to different examples: bifurcation, existence of solutions with different symmetry type and symmetry breaking.

In order to study $\Pi^\Gamma_{S^V}(S^W)$, we show that for each isotropy subgroup H, there is a fundamental cell in B^H, of dimension $\dim V^H - \dim(\Gamma/H)$, which generates B^H and where the action is free. If this dimension is less than $\dim W^H$ then we show, in Section 3, that any Γ-map has an equivariant extension to B^H, while if the dimensions are equal, then there is an obstruction, an integer which is the degree of an extension of the map to the boundary of the cell.

If Γ preserves orientations, we prove in Section 4 that this obstruction is unique and well defined. We give, under a suspension hypothesis, conditions under which the obstruction is independent of the previous extensions.

In Section 5, we decompose $\Pi^\Gamma_{S^V}(S^W)$ into subgroups corresponding to Weyl groups of fixed dimension k (Theorems 5.1, 5.2 and 5.3). If for each such isotropy subgroup H, the dimension of its fundamental cell is less than or equal to $\dim W^H$, then (Theorem 5.1) the subgroup is a product of $\mathbb{Z}$'s, one for each H where one has equality. This result enables us to characterize the Γ-degree if $\dim V^H \leq \dim W^H + 1$, for all H.

By using subgroups of Γ, we study in Section 6 the relationship between the different possible equivariant degrees and we give an explicit relation, if $\dim V^H =$

$\dim W^H$, between these equivariant degrees and the usual degrees of restrictions of the map to isotropy subspaces.

In Section 7, we give explicit generators, in terms of invariant polynomials, for the different pieces of the equivariant homotopy groups.

Section 8 is devoted to a complete characterization of $\Pi^{\Gamma}_{SV}(S^W)$ when $k = 1$. In this case the torsion part of this group is a finite group generated by explicit maps and relations.

The final section gives an equivariant suspension theorem which is necessary for the extension to infinite dimensions. We also prove that any element in $\Pi^{\Gamma}_{SV}(S^W)$ is the Γ-degree of a map on $\Omega \subset V$, in the cases $k = 0$ or 1 or $\dim \Gamma/H = k$.

The ideas behind our paper are those of equivariant obstruction theory and equivariant extension of maps. There are many other papers which have exploited these ideas, usually with a strong use of algebraic topology. Our approach is direct and based on explicit computations. If $\dim V^H = \dim W^H$, then we recover the results of [16], and, for the case $V = W$, those of [9], [14] and [4]. For this case we give "mod-p" or Borsuk-Ulam type results for the computation of the usual degree for equivariant maps (see [1], [13], [10] and [15]). The idea of a fundamental cell is also used in [3] in the context of equivariant extensions.

If $V = W \times \mathbb{R}^k$, Gęba and coworkers have defined, in [5] in the case of a general Γ, a degree which corresponds to the "free part" of $\Pi^{\Gamma}_{SV}(S^W)$, that is, to the copies of $\mathbb{Z}$'s. This degree is defined first for perturbations, normal maps, which have their zeros of a fixed isotropy type, with $\dim N(H)/H = k$. We show, in Section 5, that this degree is included in ours.

1. Irreducible Representations of Abelian Groups

Let Γ be an abelian group, hence $\Gamma \cong T^n \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_k}$ generated by $(\varphi_1, \ldots, \varphi_n) \in T^n$ and γ_j a cyclic element of order m_j (see [16, p. 25]). Then it is known that the irreducible real representations of Γ are either two dimensional, with an induced complex structure, or one dimensional with an action of $\mathbb{Z}_2$ (see [16, Prop. 8.8, p. 110]).

Now, let V be an irreducible representation of Γ and let $X \in V\backslash\{0\}$. Let $\Gamma_X = H$ be its isotropy subgroup and $W(H) = \Gamma/H$ be its Weyl group. Then, $W(H)$ acts freely on $V\backslash\{0\} = V^H\backslash\{0\}$ (see [2, p. 90]). Hence, from [2, Thm. 8.5, p. 153], either $W(H) \cong S^1$ if $\dim W(H) > 0$, since $W(H)$ is abelian, or (see [2, Thm. 8.2, p. 149]) $W(H) \cong \mathbb{Z}_p$. In this preliminary section we would like to derive these results directly since they will be used later on.

The action of Γ on V is of the form $\exp i(\sum n_j\varphi_j + 2\pi \sum k_j l_j/m_j)$, where $\varphi_j \in [0, 2\pi]$, $n_j \in \mathbb{Z}$, $0 \leq k_j \leq m_j - 1$ is fixed and l_j varies from 0 to $m_j - 1$.

If $\dim V = 1$, then $n_j = 0$, and $\sum 2k_j l_j/m_j$ is an integer, for all l_j's. Thus, if we take $l_j = 0$ except for one index j_0 and $l_{j_0} = 1$, then $2k_j/m_j$ is an integer, m_j is even and k_j is either 0 or $m_j/2$. The action of Γ reduces to $(-1)^{\Sigma l_j}$, where the sum is over those j's such that $k_j = m_j/2$.

Let $H = \Gamma_X$. Since $\dim V = 1$, it follows that H contains T^n, $\mathbb{Z}_{m_j}$ if m_j is odd or $k_j \neq m_j/2$, and all the elements with $\sum l_j$ even.

If $\dim V = 2$, then $\gamma \in H$ if $\sum n_j \varphi_j + 2\pi \sum k_j l_j/m_j = 2k\pi$, for some integer k. There are then three cases: either $n_j = 0$ for all j, or $n_j \neq 0$ for some j and $k_j = 0$, and the general case.

a) If $n_j = 0$ for all j's, hence $H > T^n$, then $\sum \tilde{k}_j l_j/\tilde{m}_j$ is an integer, where $\tilde{k}_j$ and $\tilde{m}_j$ are relatively prime. Let $\tilde{m}$ be the least common multiple (l.c.m.) of the $\tilde{m}_j$'s, with $\tilde{m}_j = \tilde{m}/p_j$. Then $\sum \tilde{k}_j l_j/\tilde{m}_j = \sum \tilde{k}_j l_j p_j/\tilde{m}$.

LEMMA 1.1. *There are* $(l_1^0, \dots, l_k^0) = l_0$ *such that* $\sum \tilde{k}_j l_j^0/\tilde{m}_j \equiv 1/\tilde{m}\,[2\pi]$ *and any other element of* $W(H)$ *gives an action of the form* $\alpha/\tilde{m}$ *for some* $\alpha \in \{0, \tilde{m} - 1\}$.

PROOF. If $k = 1$, then $\tilde{k}l/\tilde{m}$ is an integer if and only if l is a multiple of $\tilde{m}$, $e^{2\pi i \tilde{k} l/\tilde{m}}$ are $\tilde{m}$ roots of unity, hence the result is clear.

Assuming the result true for $(l_1, \dots, l_k)$, let $\hat{m}$ be the l.c.m. of $(\tilde{m}_1, \dots, \tilde{m}_k)$ and $\tilde{m}$ be the l.c.m. of $\hat{m}$ and $\tilde{m}_{k+1}$,

$$\sum_{j=1}^{k} \tilde{k}_j l_j/\tilde{m}_j + \tilde{k}l/\tilde{m}_{k+1} \equiv \alpha_0/\hat{m} + \tilde{k}l/\tilde{m}_{k+1},$$

where α_0 is given by the induction hypothesis in such a way that

$$\sum_{j=1}^{k} \tilde{k}_j l_j^0/\tilde{m}_j \equiv 1/\hat{m} \qquad \text{and} \qquad l_j = \alpha_0 l_j^0.$$

One is then reduced to two "modes" l_1 and l_2. From the one mode case, $\tilde{k}_j l_j/\tilde{m}_j \equiv \tilde{l}_j/\tilde{m}_j$ with $0 \leq \tilde{l}_j < \tilde{m}_j - 1$. Thus, one has to consider $\tilde{l}_1/\tilde{m}_1 + \tilde{l}_2/\tilde{m}_2$. Now, $\tilde{m} = p_1\tilde{m}_1 = p_2\tilde{m}_2$, with p_1 and p_2 relatively prime. Thus, there are integers α_1, α_2 such that $\alpha_1 p_1 + \alpha_2 p_2 = 1$, where α_1 and α_2 have opposite signs. Assume that $\alpha_1 > 0$.

If $\alpha_1 \geq \tilde{m}_1$, divide α_1 by $\tilde{m}_1$ and get $\alpha_1 = k_1\tilde{m}_1 + \tilde{l}_1$, $0 \leq \tilde{l}_1 < \tilde{m}_1$; then $p_1\tilde{l}_1 + \alpha_2 p_2 = 1 - k_1\tilde{m}$. Similarly, $-\alpha_2 = k_2\tilde{m}_2 + \beta_2$, $0 \leq k_2$, $0 \leq \beta_2 < \tilde{m}_2$; $-\alpha_2 = (k_2 + 1)\tilde{m}_2 - \tilde{l}_2$, $1 \leq \tilde{l}_2 \leq \tilde{m}_2$ and $p_1\tilde{l}_1 + p_2\tilde{l}_2 = 1 + (k_2 + 1 - k_1)\tilde{m}$, defining $\tilde{l}_1^0$ and $\tilde{l}_2^0$ (if $\beta_2 = 0$, take $\tilde{l}_2 = 0$).

For any other pair $(\tilde{l}_1, \tilde{l}_2)$, we have $\tilde{l}_1/\tilde{m}_1 + \tilde{l}_2/\tilde{m}_2 = (p_1\tilde{l}_1 + p_2\tilde{l}_2)/\tilde{m} \equiv (p_1\tilde{l}_1 + p_2\tilde{l}_2)(\tilde{l}_1^0/\tilde{m}_1 + \tilde{l}_2^0/\tilde{m}_2)$. Hence, $W(H) \cong \mathbb{Z}_{\tilde{m}}$. □

b) For the action of T^n given by $\sum n_j\varphi_j$, let n_0 be the largest common divisor of the $|n_j|$'s, let $\widetilde{n}_j = n_j/n_0$, and define $\psi = (\sum n_j\varphi_j)/n_0$. Then ψ goes from $-2\pi\sum_1|\widetilde{n}_j|$ to $2\pi\sum_2|\widetilde{n}_j|$, where $\sum_1$ is the sum over all negative n_j's and $\sum_2$ the sum over all positive n_j's. One may also change φ_j to $2\pi - \varphi_j$ whenever n_j is negative and assume $n_j \geq 0$. If $N = \sum|n_j|$, the congruence $\sum n_j\varphi_j \equiv 0\,[2\pi]$ gives N hyperplanes in T^n and $H \cong (T^{n-1}\times\mathbb{Z}_N)\times\mathbb{Z}_{m_1}\times\cdots\times\mathbb{Z}_{m_k}$ with $W(H)\cong S^1 = T/\mathbb{Z}_N$. If $n_0\psi = N\Phi$, then Φ, belonging to $(0, 2\pi/N)$, generates $W(H)$. □

c) Finally, in general, one may write $\sum n_j\varphi_j + 2\pi\sum k_j l_j/m_j$ as $n_0\psi + 2\pi\alpha/\widetilde{m}$, with $0 \leq \alpha < \widetilde{m}$, $n_0\psi = N\Phi$ and Φ belonging to $[0, 2\pi/N]$. The relation $N\Phi + 2\pi\alpha/\widetilde{m} = 2k\pi$ will give $\Phi = (\sum n_j\varphi_j)/N = 2k\pi/N + 2\pi\alpha/(\widetilde{m}N)$, which represents $\widetilde{m}N$ different parallel hyperplanes in T^n. Thus, $H \cong T^{n-1}\times\mathbb{Z}_{\widetilde{m}N}$ and $W(H)\cong S^1 = T/Z_{\widetilde{m}N}$. □

In the rest of the paper, we shall assume that the representation V of Γ has a fixed orthogonal decomposition into irreducible representations (real and complex) in such a way that any X in V can be written as $X = \sum x_j e_j$, where $x_j \in \mathbb{C}$ if $W(\Gamma_{e_j}) \cong \mathbb{Z}_p$ or S^1, $p > 2$, or $x_i \in \mathbb{R}$ if $W(\Gamma_{e_i}) = \{e\}$ or $\mathbb{Z}_2$.

Then $\gamma X = \sum x_j\gamma e_j$ and $\gamma X = X$ gives $\gamma e_j = e_j$ if $x_j \neq 0$. ($\gamma e_j = e^{iN_j\Phi + 2\pi\alpha_j/(\widetilde{m}_j N_j)}e_j$ for the complex case, or $\pm e_j$ in the real case.) Hence, $\Gamma_X = \bigcap\Gamma_{e_j}$, where the intersection is over those j's such that $x_j \neq 0$. Thus, $W(\Gamma_{e_j}) < W(\Gamma_X)$.

In particular, if $\dim W(\Gamma_X) = 0$, then $W(\Gamma_{e_j})$ is a finite group and Γ_{e_j} contains T^n. In this case Γ_X also contains T^n, that is, X belongs to V^{T^n}. Conversely, if X is fixed by T^n, then $W(\Gamma_X)$ is a factor of $\mathbb{Z}_{m_1}\times\cdots\times\mathbb{Z}_{m_k}$ and hence is finite.

We have seen that the set $\{X \in V \mid W(\Gamma_X) < \infty\}$ is the subspace V^{T^n}.

2. The Fundamental Cell Lemma

Let V be a representation of Γ and H_0 be a subgroup of Γ. Decompose V as $\bigoplus V_j$, with $V_j \cong \mathbb{C}$ or $\mathbb{R}$ and generated by e_j with isotropy subgroup H_j. Write X in V as $\sum_1^n x_j e_j$. Consider $C = \{X \in V \mid |x_j| = 1\}$, a torus in V.

Now, $\widetilde{H}_{j-1} \equiv H_0 \cap H_1 \cap \ldots \cap H_{j-1}$ acts on $V_j\backslash\{0\}$ with isotropy subgroup $\widetilde{H}_j \equiv \widetilde{H}_{j-1}\cap H_j$ and $\widetilde{H}_{j-1}/\widetilde{H}_j$ acts freely on $V_j\backslash\{0\}$. Thus, as in the first part, this Weyl group is isomorphic either to S^1, to $\{e\}$, or to $\mathbb{Z}_p$, $p \geq 2$. Let k_j be the cardinality of this Weyl group. ($k_j = \infty$ means that the group is S^1, while $k_j = 1$ means that $\widetilde{H}_{j-1} = \widetilde{H}_j$. If $k_j = 2$ and V_j is complex, then V_j splits into two real representations of $\widetilde{H}_{j-1}/\widetilde{H}_j \cong \mathbb{Z}_2$. If V_j is real then $k_j = 1$ or 2.) Let $H = H_1 \cap H_2 \cap \ldots \cap H_n$ be the isotropy type of C.

with $k_j > 1$ and z_j complex, or $\operatorname{Arg} z_j = 0$ or $2\pi/k_j$ when $1 < k_j < \infty$ and z_j complex. Now, if $x_j = 0$ for some j, then from the minimality of the intersection, the corresponding isotropy subgroup strictly contains H and, for $x_j = 0$ and by Gleason's lemma for $|x_j| \leq \epsilon(1 - 1/k_j)$, one has the given equivariant extension of F.

The piece of ∂C corresponding to $\operatorname{Arg} z_j = 0$ for all j's (there are $\widehat{s} \leq \widetilde{s}$ of these) has dimension $r + 2\widetilde{s} - \widehat{s} + k < \dim W^H$, if $\widehat{s} > 0$. Thus, one has a continuous non-zero extension on it. If $\widehat{s} = 0$, then there is no need to consider that piece of ∂C.

It remains to consider the rest of ∂C, that is, when $\varphi_j \equiv \operatorname{Arg} z_j$ is $2\pi/k_j$ with $1 < k_j < \infty$.

Let $\Delta_{\widehat{s}}$ be the $\widehat{s}$-torus $\{\varphi_j \in [0, 2\pi/k_j] \equiv I_j,\ j$ such that $1 < k_j < \infty,\ z_j$ complex$\}$. The extension to $\partial\Delta_{\widehat{s}}$ and, eventually, to $\Delta_{\widehat{s}}$ is by induction on $\widehat{s}$. If $\widehat{s} = 1$, then $\Delta_{\widehat{s}} = I_n$ and one already has an extension to $\varphi_n = 0$. The group $\widetilde{H}_{n-1}/\widetilde{H}_n$ acts on z_n and there is a unique γ_n in it which sends $e^{2\pi i/k_n} z_n$ onto z_n and which either leaves untouched x_j (with $\operatorname{Arg} x_j = 0$) or moves x_j on ∂C. For $e^{2\pi i/k_n}|z_n|$ define $F(x_1, \dots, e^{2\pi i/k_n}|z_n|, \dots) = \gamma_n F(x_1, \dots, |z_n|, \dots)$. Furthermore, if $r + 2\widetilde{s} + k < \dim W^H$, then one may choose any continuous non-zero extension of the preceding F to φ_n in I_n, obtaining a non-zero extension to $\overline{C}$, with the right symmetry property on ∂C. By letting Γ/H act on $\overline{C}$ one obtains a Γ-equivariant extension to B^H.

Assume now that one has obtained an extension for the last $\widehat{s} - i$ phases in $\Delta_{\widehat{s}}$, that is, for $\Delta_{\widehat{s}-i} \equiv \{\varphi_j = 0$ for $j = 1, \dots, i;\ \varphi_j \in I_j$ for $j = i+1, \dots, \widehat{s}\}$. (There is a slight abuse of notation here: x_j are not necessarily consecutive variables, but the phases φ_j in $\Delta_{\widehat{s}}$ are ordered. Recall that the action of $\widetilde{H}_j$ preserves the variables in C which have zero argument and moves harmlessly those for which $k_i = 1$.)

The induction step will be for $\varphi_j = 0$, $j = 1, \dots, i-1$, $\varphi_i \in I_i$ and $\varphi_j \in I_j$, $j > i$. Now, if $\varphi_i = 2\pi/k_i$, then $\widetilde{H}_{i-1}/\widetilde{H}_i$ leaves invariant φ_j, for $j \leq i-1$, and one has γ_i in this group which takes $|z_i|e^{2\pi i/k_i}$ into $|z_i|$ and $(\varphi_{i+1}, \dots, \varphi_{\widehat{s}})$ into another element of that face of $\Delta_{\widehat{s}}$. Extend F for $\operatorname{Arg} z_i = 2\pi/k_i$ by using the action of γ_i, obtaining a non-zero map on the front face ($\varphi_i = 2\pi/k_i$) from the back face ($\varphi_i = 0$) of $\Delta_{\widehat{s}-i+1}$. Recall that $\Delta_{\widehat{s}-i+1}$ gives an $(r + k + 2\widetilde{s} - (i-1))$-dimensional ball in ∂C, for i going from $\widehat{s}+1$, for $\Delta_0 = \{\varphi_j = 0$ for all j's$\}$, to $i = 1$ for $\Delta_{\widehat{s}}$, giving C. Thus, except for the last step $i = 1$, one always has a non-zero continuous extension for a map defined on the boundary of a ball to the ball itself. The extension has to preserve the symmetry imposed by the action of Γ/H on ∂C.

For the face $\varphi_{i+1} = 0$, one starts with the edge $\varphi_i \in I_i$, $\varphi_j = 0$, $j > i$ and any continuous extension of the previous map. The map for the edge with $\varphi_{\widehat{s}} = 2\pi/k_{\widehat{s}}$ is given by the action of $\widetilde{H}_{\widehat{s}-1}/\widetilde{H}_{\widehat{s}}$ (which leaves fixed all the other phases).

The dimension argument will give a non-zero continuous extension to the two-dimensional torus $\varphi_i \in I_i$, $\varphi_{\widehat{s}} \in I_{\widehat{s}}$. This extension is reproduced by the action of $\widetilde{H}_{\widehat{s}-2}/\widetilde{H}_{\widehat{s}-1}$ to the set $\varphi_i \in I_i$, $\varphi_{\widehat{s}} \in I_{\widehat{s}}$, $\varphi_{\widehat{s}-1} = 2\pi/k_{\widehat{s}-1}$. The dimension argument will permit extending to $\varphi_i \in I_i$, $\varphi_{\widehat{s}-1} \in I_{\widehat{s}-1}$, $\varphi_{\widehat{s}} \in I_{\widehat{s}}$. A new induction argument will give the extension to the face $\varphi_{i+1} = 0$, which respects the symmetry on its boundary. The extension to $\varphi_{i+1} = 2\pi/k_{i+1}$ is given by the action of $\widetilde{H}_i/\widetilde{H}_{i+1}$. Then one proceeds to the face $\varphi_{i+2} = 0$ and so on. The complete argument requires a sequence of inductions similar to the one given in [7, p. 32] and is left to the reader.

In order to complete the induction argument, one needs to see what happens when adding a new variable x_{n+1} in such a way that $H_1 \cap \ldots \cap H_n = H$ and $H_1 \cap \ldots \cap H_{n+1} = H$ (one is still in V^H). Thus, $\widetilde{H}_{n+1} = \widetilde{H}_n$, $k_{n+1} = 1$ and $C_{n+1} = C_n \times \{|x_{n+1}| \leq R\}$. On ∂C_{n+1} one has either $|x_{n+1}| = R$, with the original map F, or X in ∂C_n and $|x_{n+1}| \leq R$. On ∂C_n one has to consider first what happens if $x_j = 0$, where the minimality argument is replaced by the induction hypothesis, while for $\operatorname{Arg} z_j = 0$ one has exactly the same extension steps as before, since x_{n+1} plays no role in that argument.

Thus, if $\dim V^H < \dim W^H - k$, one may go all the way and obtain an extension to C which respects the action of Γ/H on ∂C and this extension is reproduced by Γ/H to give a Γ-equivariant map on B^H. While if $\dim V^H = \dim W^H - k$, one has the same extension to ∂C and, given any continuous extension to $\overline{C}$, with maybe zeros, one obtains a Γ-equivariant map on B^H which is non-zero on $\Gamma(\partial C)$. The possibility of an extension (continuous and non-zero) to C will be determined by the Brouwer degree of this map from ∂C into $W^H \backslash \{0\}$. □

4. The Extension Degree

In this section we shall keep the notation of the preceding one and we shall prove that when $\dim V^H - \dim \Gamma/H = \dim W^H$, then the obstruction to extension, the degree of F on C, is independent of the previous extensions under the hypothesis below. This integer will be called the *extension degree* of F and denoted by $\deg_E(F)$. (Later on we shall specify the dependence on H.)

Let $\gamma \in H$ be represented by the diagonal matrices γ on V^H and $\widetilde{\gamma}$ on W^H (see the first section). Assume the following:

(H) For all γ in H, $\det \gamma$ and $\det \widetilde{\gamma}$ have the same sign.

Let $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$ be the subspaces of V^H and W^H generated by the real representations of Γ. It is clear that $\operatorname{sign} \det \gamma$ depends only on the behavior of γ on $V_{\mathbb{R}}$ and in fact on $V'_{\mathbb{R}}$, the orthogonal complement of V^Γ in $V_{\mathbb{R}}$. (The complex representations give a positive determinant.)

Thus, (H) is satisfied if $V'_{\mathbb{R}} = W'_{\mathbb{R}}$, for example if $V_{\mathbb{R}} = \mathbb{R}^k \times W_{\mathbb{R}}$. In this case γ reverses the sign of the same number of variables in $V'_{\mathbb{R}}$ and $W'_{\mathbb{R}}$, for example an odd map. But one may also have the case of an action which reverses a different number of signs, provided the difference is even, for example an even map on an even dimensional $V'_{\mathbb{R}}$.

Recall that $k_j = |\widetilde{H}_{j-1}/\widetilde{H}_j|$ and that there are k of them which are infinite. Let B_k be defined as the intersection of B^H with the set of X's which have all their z_j's, with $k_j = \infty$, real and non-negative. B_k is an $(r+2\widetilde{s}+k)$-dimensional ball, i.e. the dimension of C and of W^H. Note that any map F which has a Γ-extension to B^K, with $K > H$, has a continuous extension to ∂B_k, either from this property or from the dimension hypothesis. Let $\widetilde{F}$ be this extension. One then has the following result:

THEOREM 4.1. *If* (H) *is satisfied, then the extension degree of F depends only on the Γ-homotopy class of F restricted to ∂B^H and on the extension $\widetilde{F}$ to $\bigcup B^K$, $K > H$. Moreover, $\deg_E(F)$ is independent of the extension to ∂C and of the choice of C itself. Furthermore,*

$$\deg(\widetilde{F}; B_k) = (\prod k_j) \deg_E(F),$$

where the product is taken over all j's with k_j finite.

PROOF. As it is easy to see, one may take ϵ to be 0 in the definition of C. We claim that B_k is generated by $\prod k_j$ disjoint images of C and that, on each, $\widetilde{F}$ has the same degree.

As a matter of fact, let X_j be a point on ∂C such that all its components are non-zero and its j-th component has $\operatorname{Arg} x_j = 2\pi/k_j$, with $1 < k_j < \infty$ (if x_j is real, this means x_j is negative).

From the Fundamental Cell Lemma, there is a unique γ_j in $\widetilde{H}_{j-1}/\widetilde{H}_j$ such that $\gamma_j^{-1}X_j$ belongs to C and the argument of the j-th component of $\gamma_j^{-1}X_j$ is 0. This implies that γ_j^{-1} leaves invariant the arguments of the components of X_j which correspond to $k_i = \infty$ or $k_i = 2$ and x_i real, $i \neq j$ (if not, $\gamma_j^{-1}X_j$ would not belong to C), i.e. γ_j belongs to the isotropy subgroup of the corresponding x_i's.

Thus, $\gamma_j^p(C)$, $p = 0, \ldots, k_j - 1$, are k_j disjoint cells, contained in B_k, with $y_i \geq 0$ for $i \neq j$ and $k_i = 2$, with the same volume. Moreover, $\overline{C}$ and $\gamma_j\overline{C}$ have the face $\operatorname{Arg} x_j = 2\pi/k_j$ in common.

Note that, if $\operatorname{Arg} x_i$ belongs to $[0, 2\pi/k_i)$, $i \neq j$ and $1 < k_i < \infty$, then $\operatorname{Arg} \gamma_j x_i$ belongs to an interval of length $2\pi/k_i$ which intersects the previous one, since this is the case for $\gamma_j^{-1}X_j$ and X_j. Furthermore, since γ_j belongs to $\widetilde{H}_{j-1}$, x_i is unchanged if $i < j$.

Suppose then that there is an X in B_k which belongs to $\gamma_i^p(C^\circ) \cap \gamma_j^q(C^\circ)$ for some p, $1 \leq p < k_i - 1$, and q, $1 \leq q \leq k_j - 1$. If γ_i corresponds to $k_i = 2$ and a real

representation, then γ_j preserves y_i and one has an empty intersection. Thus, the only possibility is for complex z_i and z_j, with $\operatorname{Arg} z_i$ in $(p2\pi/k_i, (p+1)2\pi/k_i)$ and $\operatorname{Arg} z_j$ in $(q2\pi/k_j, (q+1)2\pi/k_j)$. But, if $i < j$, then γ_j fixes z_i and hence, $\operatorname{Arg} z_i$ must belong to $(0, 2\pi/k_i)$ and $p = 0$. But then $C^\circ \cap \gamma_j^q C^\circ = \emptyset$ unless $q = 0$.

Thus, the $\prod k_j$ images of C° do not intersect and, since B_k may also be decomposed in the same number of cells, with the same volume, these images cover properly B_k.

Now, recall that F was extended to ∂C in such a way that $\widetilde{F}(\gamma X) = \widetilde{\gamma}\widetilde{F}(X)$, whenever X and γX belonged to ∂C, and that $\widetilde{F}$ was also extended to $\Gamma(\partial C)$ with the same property.

Thus, if one takes, on $\gamma_j^p(\partial C)$, $0 \leq p < k_j$, the map $F_{j,p}(X)$ defined as $F_{j,p}(X) \equiv \widetilde{\gamma}_j^p \widetilde{F}(\gamma_j^{-p} X)$, then $F_{j,p}(X) = \widetilde{F}(X)$ is continuous and non-zero on the boundary of that cell and coincides with $\widetilde{F}$ on the boundaries of the adjoining cells. Note that, if X is in ∂C with $\operatorname{Arg} x_j = 2\pi/k_j$, then $\gamma_j^{-1} X$ has its j-th component with zero argument but may not belong to C. But there is a γ in $\widetilde{H}_j$ such that $\gamma\gamma_j^{-1} X$ is in C. Now, $\widetilde{F}(X)$ was defined as $\widetilde{\gamma}_j \widetilde{\gamma}^{-1} F(\gamma\gamma_j^{-1} X)$ and $\widetilde{F}(\gamma^{-1} Y) = \widetilde{\gamma}^{-1} F(Y)$ for γ in $\widetilde{H}_j$ and Y on the back face of C, i.e. with $\operatorname{Arg} Y_j = 0$. Hence,

$$\deg(\widetilde{F}; B_k) = \sum \deg(F_{j,p}; \gamma_j^p(C)).$$

Now, $\deg(F_{j,p}; \gamma_j^p(C)) = \operatorname{sign}\det(\gamma_j^{-p}) \operatorname{sign}\det(\widetilde{\gamma}_j^p) \deg(F; C)$. From (H), one has $\deg(F_{j,p}; \gamma_j^p(C)) = \deg(F; C)$ and thus, $\deg(\widetilde{F}; B_k) = \prod k_j \deg(F; C)$.

At this stage of the proof, we have shown that the extension degree depends only on the extension of F to ∂B_k.

Let z_i be such that $k_i = \infty$, set $V_i^H = V^H \cap \{X \text{ with } z_i = 0\}$ and let B_i^H be the corresponding ball with dimension equal to $\dim W^H + k - 2$. If the isotropy group of B_i^H is bigger than H, then F has an extension by hypothesis. However, if it is H, assume that for a given F, one has two equivariant extensions F_0 and F_1 to B_i^H. On the boundary of $[0,1] \times B_i^H$, define an equivariant map to be F_0 for $t = 0$, F_1 for $t = 1$ and F for $[0,1] \times (\partial B_i^H \cup B_i^K)$. From the first part of Theorem 3.1, one obtains an equivariant extension to $[0,1] \times B_i^H$, that is, a Γ-equivariant homotopy from F_0 to F_1.

It is clear that, by starting from $\bigcap_i B_i^H$ and going up in dimension, one may extend this homotopy to a Γ-homotopy on $\bigcup B_i^H$ and, by restriction, a plain homotopy on ∂B_k and prove that $\deg_E(F)$ depends only on F on ∂B^H and its extension $\widetilde{F}$ on $\bigcup B^K$. Note that, in fact, $\deg_E(F)$ depends only on the homotopy class of $(F, \widetilde{F})$ on $(\partial B^H) \cup B^K$, hence not on C, since in the above argument one may put the Γ-homotopy between two elements of $[F, \widetilde{F}]$ on $[0,1] \times (\partial B_i^H \cup B_i^K)$ and obtain an extension of the Γ-homotopy to $\bigcup B_i^H$ and, by gluing the Γ-homotopy of F on

∂B^H with the restriction of the Γ-homotopies on $\bigcup B_i^H$ to ∂B_k, a homotopy on ∂B_k and the same extension degree. □

REMARK 4.1. B_k and C give a global "Poincaré section" since one takes $\epsilon \leq z_i \leq R$ for all i's with $k_j = \infty$. By writing $\Gamma/H = \prod_1(\widetilde{H}_{i-1}/\widetilde{H}_j)\prod_2(\widetilde{H}_{i-1}/\widetilde{H}_i)$, where the first product corresponds to $k_i = \infty$ (here the action of each component is not related to the order of the coordinates), one has $\Gamma/H \cong T^k \times \widetilde{H}$, where T^k is a maximal torus and $\widetilde{H}$ is a finite group of order $\prod k_j$. Note that, if one makes a permutation of the coordinates, one obtains naturally a new map F' and a new group $\widetilde{H}'$. For a simple example, take (z_1, z_2) with S^1-action $(e^{2i\varphi}z_1, e^{3i\varphi}z_2)$; it is easy to see that $|\widetilde{H}| \neq |\widetilde{H}'|$, but we shall prove later on, with the explicit form of the generators given in Section 7, that $\deg_E(F) = \eta \deg_E(F')$ where η is the signature of the permutation.

Note also that we have only used the fact that $\operatorname{sign}\det\gamma_j = \operatorname{sign}\det\widetilde{\gamma}_j$ for the $|\widetilde{H}|\,\gamma_j$'s needed in the proof and not the full strength of hypothesis (H). We have seen that γ_j changes only the sign of y_j if $k_j = 2$ and leaves invariant the other y_i's with $k_i = 2$, while γ_j leaves invariant all y_i's with $k_i = 2$, if z_j is complex. However, γ_j may act on those y_i's with $k_i = 1$, hence one may not ask only that $\widetilde{\gamma}_j$ reverse an odd number of real components for y_j real and $k_j = 2$.

Finally, note that if $\det\gamma_j$ and $\det\widetilde{\gamma}_j$ have opposite signs, then $\deg(F(\gamma_j); B_k) = \operatorname{sign}(\det\gamma_j)\deg(F; B_k) = \operatorname{sign}(\det\widetilde{\gamma}_j)\deg(F; B_k)$. Thus, $\deg(F; B_k) = 0$.

Let $\Pi(H)$ denote the subset of $\Pi^\Gamma_{S^{\overline{V}}}(S^{\overline{W}})$ consisting of those elements F which have a non-zero Γ-equivariant extension to $\bigcup_{K>H} B^K$. Here $\overline{V}$ and $\overline{W}$ stand for V^H and W^H. Note that, if F_0 and F_1 are Γ-homotopic on ∂B^H and F_0 has a Γ-extension, $\widetilde{F}_0$, to $\bigcup B^K$, then F_1 also has a Γ-extension, $\widetilde{F}_1$, and $(F_0, \widetilde{F}_0)$ is Γ-homotopic to $(F_1, \widetilde{F}_1)$ on $\partial B^H \cup B^K$ (use the Γ-equivariant Borsuk extension theorem given in [6, 1.7]).

Denote by $\Pi(H, K)$ the set of Γ-homotopy classes of maps $[F, \widetilde{F}], F : \partial B^H \to W^H\backslash\{0\}$, $\widetilde{F} : \bigcup B^K \to \bigcup(W^K\backslash\{0\})$, $\widetilde{F}$ a Γ-extension of F to all B^K with $K > H$. Let Π be the assignment $[F, \widetilde{F}] \to [F]$, from $\Pi(H, K)$ into $\Pi(H)$. We have the following:

THEOREM 4.2. *$\Pi(H)$ is a subgroup of $\Pi^\Gamma_{S^{\overline{V}}}(S^{\overline{W}})$. Furthermore, $\Pi(H, K)$ is an abelian group, which is, if not trivial, isomorphic to $\mathbb{Z}$ via the extension degree. Π is a morphism onto $\Pi(H)$, with $\ker\Pi = \{[(1, 0), \widetilde{F}]\}$, for all possible extensions of the map $(1, 0)$.*

PROOF. As in [6, Appendix A], we shall write X as (t, X) where t is the invariant variable on which the addition of $\Pi^\Gamma_{S^{\overline{V}}}(S^{\overline{W}})$ is defined. If $H = \Gamma$, then the result is trivial. Thus, assume that H is a proper subgroup of Γ.

Let $A = \{(t, X) \mid t = 0 \text{ or } 1, \text{ or } X \text{ is in } B^K,\ K > H\}$. If $(F, \widetilde{F})$ belongs to $\Pi(H, K)$, then the Γ-homotopy $(F(t, \tau X), \widetilde{F}(t, \tau X))$, $\tau \in [0, 1]$, is admissible on A, since, if (t, X) belongs to A, then $(t, \tau X)$ also belongs to A and these maps are non-zero on A ($t = 0$ or $t = 1$ belongs to ∂B^H). Furthermore, $(t, 0) \in B^\Gamma$, hence $\widetilde{F}(t, 0) \neq 0$ and $(F(t, 0), \widetilde{F}(t, 0))$ is deformable to $(F(0, 0), F(0, 0))$, since H is a proper subgroup of Γ. This last map is in turn deformable to $((1, 0), (1, 0))$; if W^Γ is one dimensional, the admissibility of F requires that $F(0, 0) > 0$.

Thus $(F, \widetilde{F})$ is Γ-homotopic to $((1, 0), (1, 0))$ on A. The Borsuk equivariant extension theorem implies that $(F, \widetilde{F})$ is Γ-homotopic, on $\partial B^H \cup B^K$, to a map $(F_0, (1, 0))$ (see [6, Prop. A.1]). Hence, one may assume that $(F, \widetilde{F})$ is of the form $(F_0, (1, 0))$ on A. As in [6, p. 486], this implies that one may define a group structure on $\Pi(H, K)$. Following the proof of [6, Proposition A.4], the fact that $\Pi(H, K)$ is abelian requires that $\dim V^\Gamma \geq 2$. If $\dim V^\Gamma = 1$, i.e. V^Γ is reduced to t, the commutativity will come from the rest of the proof.

Note that, by reducing A to the set ($t = 0$ or $t = 1$), one sees that $\Pi(H)$ is a subgroup of $\Pi^\Gamma_{S^{\overline{W}}}(S^{\overline{W}})$, abelian if $\dim V^\Gamma \geq 2$. Furthermore, it is clear that Π is a morphism, onto and $\ker \Pi = \{[(1, 0), \widetilde{F}]\}$.

Note also that, by taking $A = \{(t, X) \mid t = 0 \text{ or } t = 1, \text{ or } X \in B^K, \text{ or } X \in B^H_i = B^H \cap \{z_i = 0\}, \text{ with } k_i = \infty\}$, we have seen that $(F, \widetilde{F})$ has an extension F_i to B^H_i. On A, consider also the homotopy $F_i(t, \tau X)$. Then, as before, one finds that $(F, \widetilde{F}, F_i)$ is Γ-homotopic, on $\partial B^H \bigcup B^K \bigcup B^H_i$, to a map F_0 which has value (1,0) on $\bigcup B^K \bigcup B^H_i$ with the same extension degree, $\deg_E(F, \widetilde{F})$.

Finally, the assignment $[F, \widetilde{F}] \to \deg_E(F, \widetilde{F})$ is one-to-one (from Theorem 3.1) and clearly a morphism into $\mathbb{Z}$. Thus, $\Pi(H, K)$ is abelian and isomorphic to a subgroup of $\mathbb{Z}$ that is either 0 or the subgroup generated by the single element $\deg_E(F_0, \widetilde{F}_0)$. Thus, $\deg_E$ will be onto $\mathbb{Z}$ if there is a $(F_0, \widetilde{F}_0)$ with extension degree 1. This will be proved later in particular cases. □

It may happen that Π is not one-to-one: this is the case if $\Gamma = S^1$, $\dim V^\Gamma = \dim W^\Gamma - 1$ (see [7, Lemma 2.3]).

However, our interest in this paper is to give a complete description of $\Pi^\Gamma_{S^V}(S^W)$, thus, we shall try to avoid the case when $\ker \Pi$ is not trivial. This will happen under one of the following two hypotheses.

(H1) For all K's, $K > H$, $\dim V^K \leq \dim W^K + \dim \Gamma/K - 2$.

It is not difficult to check that, for $\Gamma = S^1$, the hypothesis of [7, Theorem 3.1] implies (H1) if $\dim V^\Gamma = \dim W^\Gamma + 1 - 2p$, with $p > 1$ ($p = 1$ is the case where

$\deg_E$ depends on $\widetilde{F}$). For $K = \Gamma$, (H1) implies that $\dim V^\Gamma \leq \dim W^\Gamma - 2$.

(H2) $\begin{cases} \text{For each minimal } K,\ K > H, \text{ there is a non-zero } \Gamma\text{-equivariant map} \\ F^\perp : \partial(B^K)^\perp \to (W^K)^\perp \backslash \{0\}, \text{ where } (B^K)^\perp \text{ and } (W^K)^\perp \text{ are the} \\ \text{orthogonal complements of } B^K \text{ and } W^K \text{ in } B^H \text{ and } W^H, \text{ respectively.} \end{cases}$

Minimal means that on adding a variable to B^K, the isotropy subgroup of the new space is H.

As we shall see later on, from Borsuk-Ulam type theorems, (H2) implies that $\dim(V^K)^\perp \leq \dim(W^K)^\perp$. For the moment it suffices to note that (H2) is satisfied if $V = \mathbb{R}^k \times W$, with $F^\perp$ being the identity, or if $\Gamma = S^1$, $V^\Gamma = \mathbb{R} \times W^\Gamma$ and the conditions of [7, Theorem 3.2] hold.

THEOREM 4.3. (a) *If* (H1) *holds, then* $\Pi^\Gamma_{S^{\overline{V}}}(S^{\overline{W}})$ *is* 0 *or* $\mathbb{Z}$.
(b) *If* (H2) *holds, then* $\Pi(H,K) \cong \Pi(H)$, *and* $\deg_E(F)$ *depends only on* $[F]$ *in* $\Pi(H)$.

PROOF. If (H1) holds then, by Theorem 3.1, F has an extension to B^K, for any $K > H$. Replacing B^K by $I \times B^K$, F by a first extension for $\tau = 0$, by a second extension for $\tau = 1$ and by F itself on $I \times \partial B^K$, we see that any two extensions are Γ-homotopic on B^K relative to ∂B^K. By starting from Γ and working on intersections $B^{K_1} \cap B^{K_2}$, it is easy to see that one obtains extensions on $\bigcup B^K$, $K > H$, which are Γ-homotopic relative to ∂B^H. Thus, $\deg_E(F, \widetilde{F})$ is independent of $\widetilde{F}$ and, if $F = (1,0)$, then $\ker \Pi = \{0\}$. Hence $\Pi(H,K) = \Pi(H) = \Pi^\Gamma_{S^{\overline{V}}}(S^{\overline{W}})$.

If (H2) holds, assume that $(1,0)$ is extended by $\widetilde{F}(X_K) : B^K \to W^K \backslash \{0\}$. Write X in V^H as $(X_K, X_\perp)$ and define

$$\widehat{F}(X) = ((1 - \|X_\perp\|)\widetilde{F}(X_K) + \|X_\perp\|(1,0), (1 - \|X\|)t(1-t)\|X_\perp\| F^\perp(X_\perp / \|X_\perp\|)).$$

Recall that $\|X\| = \sup |x_i|$ and that $B^H = I \times \{X \mid \|X\| \leq 1\}$.

It is easy to see that $\widehat{F}$ is Γ-equivariant, $\widehat{F}$ and $\widetilde{F}$ coincide on B^K and $\widehat{F} = (1,0)$ on ∂B^H. Furthermore $\widehat{F}(X) \neq 0$, thus $[(1,0), \widehat{F}] = 0$ in $\Pi(H,K)$, or else $[(1,0), \widetilde{F}] - [(1,0), \widehat{F}] = [(1,0), \widetilde{F}]$. If

$$G(t,X) = \begin{cases} \widetilde{F}(2t, X), & 0 \leq t \leq 1/2, \\ \widehat{F}(2 - 2t, X), & 1/2 \leq t \leq 1, \end{cases}$$

(recall that all maps have value $(1,0)$ for $t = 0$ or $t = 1$), then $[(1,0), G(t,X)] = [(1,0), \widetilde{F}]$, G has value $(1,0)$ on ∂B^H and G is Γ-homotopic on B^K (relative to its boundary) to $(1,0)$. From the equivariant Borsuk extension theorem applied to $\partial B^H \cup B^K$, $((1,0), \widetilde{F})$ is Γ-homotopic to a map $((1,0), \widetilde{F}_0)$ with value $(1,0)$ on B^K.

Now suppose that $\widetilde{F}(X_K) = (1,0)$ for all $X = (X_K, X_\perp)$ in $B^{K'}$, for some $K' > H$ (thus, X_K has at least one component zero, from the minimality of K).

On $A = B^K \cup B^{K'} \cup \partial B^H$ define a Γ-homotopy $\widehat{F}(\tau, X)$ by multiplying $F^{\perp}$ by τ in the definition of $\widehat{F}$. Then $\widehat{F}(1, X) = \widehat{F}$, $\widehat{F}(\tau, X) = (1, 0)$ on ∂B^H, $\widehat{F}(0, X)$ is $(1, 0)$ on $B^{K'}$ and $\widetilde{F}$ on B^K. Thus, one may extend the Γ-homotopy on $\partial B^H \cup B^{K''}$, $K'' > H$, and assume that $\widetilde{F}$ is (1,0) on $B^K \cup B^{K'}$. By induction, one finds that $\widetilde{F}$ is Γ-homotopic, relative to ∂B^H, to (1,0) on $\bigcup B^K$. Hence $\deg_E(F, \widetilde{F}) = \deg_E((1, 0), (1, 0)) = 0$, since the later extensions to $\bigcup B_i^H$ may be taken to be $(1, 0)$. We have proved that $\ker \Pi = \{0\}$ and $\deg_E$ is independent of $\widetilde{F}$. □

REMARK 4.2. In the proofs of Theorems 4.2 and 4.3 (b), it is easy to isolate the points where the hypothesis $\dim V^H - \dim \Gamma/H = \dim W^H$ was used, which was only when the extension degree was computed. In general one has the following result:

THEOREM 4.4. *$\Pi(H)$ is a subgroup of $\Pi^{\Gamma}_{S^{\overline{V}}}(S^{\overline{W}})$, $\Pi(H, K)$ is a group (abelian if $\dim V^{\Gamma} > 1$). Π is a morphism onto $\Pi(H)$, with $\ker \Pi = \{[(1, 0), \widetilde{F}]\}$, for all possible extensions of the map $(1, 0)$. Furthermore, if (H2) holds then $\ker \Pi = \{0\}$, in particular given $[F]$ in $\Pi(H)$, then $[F, \widetilde{F}_1]$ and $[F, \widetilde{F}_2]$ are Γ-homotopic on $\partial B^H \bigcup B^K$, for any two Γ-extensions $\widetilde{F}_1$ and $\widetilde{F}_2$ of F to $\bigcup B^K$ and Γ-homotopic to a map $[F_0, (1, 0)]$.*

5. Homotopy Groups of Γ-Maps

In this section, we shall begin our computations of the Γ-equivariant homotopy groups of spheres, from our previous results on the extension degree. Consider the set of H's, isotropy subgroups, with $\dim \Gamma/H = k$ fixed, and look at maps

$$F : \bigcup \partial B^H \to \bigcup W^H \backslash \{0\}$$

which have Γ-extensions, $\widetilde{F}$, to $\bigcup B^K \to \bigcup W^K \backslash \{0\}$, for all K's with $\dim \Gamma/K \leq k - 1$. Define

$$\Pi(k) = \{[F]_{\Gamma}\} \qquad \text{and} \qquad \Pi(k, k-1) = \{[F, \widetilde{F}]_{\Gamma}\}$$

of maps F (and extensions $\widetilde{F}$) as above.

If $F \in \Pi(k)$ and F is Γ-homotopic to G on $\bigcup \partial B^H$, then G also has an extension $\widetilde{G}$, with $(F, \widetilde{F})$ Γ-homotopic to $(G, \widetilde{G})$ on $\bigcup \partial B^H \bigcup B^K$. Thus, $\Pi(k)$ and $\Pi(k, k-1)$ depend on homotopy classes.

As before, one may deform F so that it is (1,0) on $\{t = 0,\ t = 1,\ \bigcup B^K\}$. One may define group structures on $\Pi(k)$ and $\Pi(k, k-1)$ which are abelian if $\dim V^{\Gamma} > 1$.

Let $\Pi : \Pi(k, k-1) \to \Pi(k)$ be the restriction. Π is a morphism onto $\Pi(k)$. As in Theorem 4.2, one has

LEMMA 5.1. *$\Pi(k, k-1)$ and $\Pi(k)$ are groups (abelian if $\dim V^\Gamma > 1$), Π is onto and $\ker \Pi = \{[(1,0), \widetilde{F}]\}$, where $\widetilde{F}$ is any Γ-extension of $(1,0)$ to $\bigcup B^K$, $\dim \Gamma/K \leq k-1$.*

Assume now the following hypothesis:

$$(\mathrm{H2})' \begin{cases} \text{(a) For every } H, \text{ with } \dim \Gamma/H = k, \text{ and minimal } K > H, \text{ there is} \\ \quad \text{a non-zero equivariant map } F_K^\perp : \partial(B^K)^\perp \to (W^K)^\perp \backslash \{0\}, \text{ where} \\ \quad (B^K)^\perp \text{ and } (W^K)^\perp \text{ are the orthogonal complements of } B^K \text{ and} \\ \quad W^K \text{ in } B^H \text{ and } W^H, \text{ respectively.} \\ \text{(b) For every } H, \text{ with } \dim \Gamma/H = k, \text{ there is a non-zero equivariant} \\ \quad \text{map } F_H^\perp : \partial(B^H)^\perp \to (W^H)^\perp \backslash \{0\}. \end{cases}$$

Notice that part (a) is just (H2), while part (b) will give an extension to ∂B.

LEMMA 5.2. *If (H2)$'$ holds, then $\Pi(k, k-1) \cong \Pi(k)$. Thus, $[F, F_1]$ and $[F, F_2]$ are Γ-homotopic on $\bigcup \partial B^H \bigcup B^K$ for any two extensions F_1 and F_2 to $\bigcup B^K$ and also Γ-homotopic to $[F_0, (1,0)]$.*

PROOF. Let H, with $\dim \Gamma/H = k$, be such that any K, $K > H$, satisfies $\dim \Gamma/K \leq k-1$. Then for minimal K, the proof of Theorem 4.3 implies that one has a Γ-homotopy of $((1,0), \widetilde{F})$, an element of $\ker \Pi$, to $((1,0), F_H)$, where $F_H = (1,0)$ on B^H. If $X = X_K \oplus X_1 \oplus X_2$, with X_1 in $(V^K)^{\perp_H} \equiv (V^K)^\perp \cap V^H$ and X_2 in $(V^H)^\perp$, then the map $\widehat{F}$ of Theorem 4.3 has to be replaced by

$$\begin{aligned} \widehat{F}(X) = ((1 - \|X_\perp\|)\widetilde{F}(X_K) + \|X_\perp\|(1,0), \\ (1 - \|X\|)t(1-t)\|X_1\| F_K^\perp(X_1/\|X_1\|), \\ (1 - \|X\|)t(1-t)\|X_2\| F_H^\perp(X_2/\|X_2\|)), \end{aligned}$$

where $X_\perp = X_1 \oplus X_2$. This is where part (b) of (H2)$'$ enters, the map $\widehat{F}$ is then restricted to $\bigcup \partial B^H$. The induction argument on H, so that one has compatible extensions on intersections of B^H's, is then similar to the proof of Theorem 4.3. □

THEOREM 5.1. *Assume (H2)$'$. Then*

$$\Pi(k) \cong \prod_H \Pi(H)$$

for all H's with $\dim \Gamma/H = k$. If furthermore (H) holds and $\dim V^H \leq \dim W^H + \dim \Gamma/H$, for all H's with $\dim \Gamma/H = k$, then $\Pi(k) \cong \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, where

one has one $\mathbb{Z}$ *for each* H_i *such that* $\dim \Gamma/H_i = k$, $\dim V^{H_i} = \dim W^{H_i} + k$ *and* $\Pi(H_i) \neq 0$.

PROOF. Let $[F]$ be an element of $\Pi(k)$ and H_i be a minimal isotropy subgroup, that is, $\dim \Gamma/H_i = k$, but $\dim \Gamma/K \leq k-1$ for any $K > H_i$. Then $F^{H_i} : \partial B^{H_i} \to W^{H_i} \backslash \{0\}$ is an element of $\Pi(H_i)$. In the particular case, i.e. the second part of the statement of the theorem, $\deg_E(F^{H_i}) = d_i \alpha_i$ and, from Theorem 4.3, there is an element F_i in $\Pi(H_i)$, with minimal extension degree α_i, such that $[F^{H_i}] = d_i[F_i]$ in $\Pi(H_i)$. As we have seen in Lemma 5.2 one may assume that F and F_i are $(1,0)$ on $\bigcup B^K$, $K > H_i$, and on $t = 0$ or $t = 1$.

For any element G of $\Pi(H_i)$, consider the suspension operation defined by $\widetilde{G}(X) = (G(X_i), t(1-t)\|X_i^{\perp}\| F_i^{\perp}(X_i^{\perp}/\|X_i^{\perp}\|))$, as an extension of G to V, where X is written as $X_i \oplus X_i^{\perp}$. Then $[F] - [\widetilde{F}^{H_i}] = [\widehat{F}_i]$ has an extension from B^{H_i} into $W^{H_i} \backslash \{0\}$.

One may do the same procedure for each such minimal H_i since we know that on $V^{H_i} \cap V^{H_j}$, one has an isotropy subgroup K with $\dim \Gamma/K < k$, and there F is assumed to be $(1,0)$. Let $[F_1] = [F] - \sum [\widetilde{F}^{H_i}]$ where the sum is taken over all minimal H_i's. Note that, at this stage, d_i are uniquely determined by F^{H_i}. Note also that, from the analogue of Lemma 5.2, one may assume that $F_1 = (1,0)$ on $\bigcup B^{H_i}$ and that the homotopy type on $\bigcup \partial B^H \bigcup B^K \bigcup B^{H_i}$ is unchanged. Take then H, with $\dim \Gamma/H = k$, $\dim \Gamma/K < k$ or $K = H_i$ for some of the preceding H_i's. The map F_1^H defines an element of $\Pi(H)$, which is $d_H[\widetilde{F}_H]$ in the particular case where F_H has minimal extension degree and $\widetilde{F}_H$ is constructed from F_H as above. Clearly, $[F_1] - [\widetilde{F}_1^H]$ is extendable to B^H. One may perform the same construction for all H's with the same characteristics, to conclude that $[F_1] - \sum [\widetilde{F}_1^H]$ is extendable to $\bigcup B^H$. Note again that given the d_i's, the d_H's are unique and, in the general case, that the $[\widetilde{F}_1^H]$'s are completely and uniquely determined by $[F]$.

One may go on to the next stage of isotropy subgroups and prove that $[F] - \sum [\widetilde{F}_j^H] = 0$ in $\Pi(k)$, with $[\widetilde{F}_j^H] = d_H[\widetilde{F}_H]$ in the particular case when F_H has a minimal extension degree, that is, one has an extension to $\bigcup B^H$, H with $\dim \Gamma/H = k$. The sets of $[\widetilde{F}_j^H]$'s and of d_H's are uniquely determined by F (for the first generation F_1) and $\widetilde{F}_H$.

Let now Π be the morphism $\prod_H \Pi(H) \to \Pi(k)$ defined as

$$\Pi(F_1^{H_1}, F_2^{H_2}, \ldots, F_j^H, \ldots) = \sum [\widetilde{F}_j^H],$$

and in the particular case $\sum d_H[\widetilde{F}_H]$ where $\widetilde{F}_H$ is given above and $d_H \in \Pi(H) \cong \mathbb{Z}$ (or 0).

From the previous argument, Π is onto and one-to-one, since from the equality $[F] = \sum [\widetilde{F}_j^H]$ one has an inverse to Π, since $\{\widetilde{F}_j^H\}$ and $\{d_H\}$ are uniquely determined by $[F]$. Thus, Π is an isomorphism and the proof is complete. □

Let $\Pi_{k-1} = \{[F]_\Gamma \mid F : \bigcup \partial B^K \to \bigcup W^K \backslash \{0\}$, for all K with $\dim \Gamma/K \leq k-1\}$. It is clear that Π_{k-1} is a group (abelian if $\dim V^\Gamma > 1$).

THEOREM 5.2. *Assume* (H2)′ *holds for all* H*'s with* $\dim \Gamma/H = k$. *Then*
(a) $\Pi_k \cong \widetilde{\Pi}_{k-1} \times \Pi(k)$, *where* $\widetilde{\Pi}_{k-1}$ *is the suspension of* Π_{k-1} *in* Π_k.
(b) *If, moreover,* $\dim V^L \leq \dim W^L + \dim \Gamma/L - 1$, *for all* L*'s with* $\dim \Gamma/L > k$, *then* $\Pi_k \cong \Pi^\Gamma_{S^V}(S^W)$.

Note that these hypotheses are satisfied if $V = \mathbb{R}^k \times W$. In this case the free part $\Pi(k)$ of Π_k is also computed in [5].

PROOF. Let $P_* : \Pi_k \to \Pi_{k-1}$ be the restriction map. We shall show that P_* is onto: in fact, let $[F]$ be an element of Π_{k-1}. Take a minimal K with $\dim \Gamma/K = k-1$ (i.e. if $H < K$, then $\dim \Gamma/H = k$). Obviously, F^K maps ∂B^K into $W^K \backslash \{0\}$. Choose any equivariant extension to B^K (it may have zeros) and call it again F^K. Write $X = X_K \oplus X_K^\perp$ and define the suspension $\widetilde{F}_K(X) = (F^K(X_K), \|X_K^\perp\| F_K^\perp(X_K^\perp/\|X_K^\perp\|))$, where $F_K^\perp$ is given by (H2)′. Clearly, $\widetilde{F}_K|\partial B^K = F^K$, hence $[F] - P_*[\widetilde{F}_K]$ is deformable to $(1,0)$ on ∂B^K. Thus, from the equivariant Borsuk extension theorem, the above difference is Γ-homotopic in Π_{k-1} to a map $\widetilde{F}$ which has value (1,0) on ∂B^K and which may be extended as (1,0) to B^K.

Let K' be another minimal isotropy subgroup, with $\dim \Gamma/K' = k-1$. As above let $\widetilde{F}_{K'}(X) = (\widetilde{F}^{K'}(X), \|X_{K'}^\perp\| F_{K'}^\perp(X_{K'}^\perp/\|X_{K'}^\perp\|))$. Then $\widetilde{F}_{K'} = \widetilde{F}^{K'}$ on $B^{K'}$ and $\widetilde{F}_{K'}(X) = ((1,0), \|X_{K'}^\perp\| F_{K'}^\perp)$ for X in B^K, since then $\widetilde{F}(X) = \widetilde{F}^{K'}(X) = (1,0)$. Thus, $\widetilde{F}_{K'}(X)$ is deformable to $(1,0)$ on B^K, $[\widetilde{F}] - P_*[\widetilde{F}_{K'}]$ is deformable to (1,0) on $\partial B^K \bigcup \partial B^{K'}$ and this difference may be replaced by a map with this value on these two spheres.

By performing this operation on all minimal K's we shall arrive at $[F] - \sum P_*[\widetilde{F}_K]$, which is deformable to (1,0) on $\bigcup \partial B^K$, for all K's, hence zero in Π_{k-1}. That is,

$$[F] = P_*\Big(\sum [\widetilde{F}_K]\Big),$$

or else, from the equivariant Borsuk extension theorem, F has an extension $\widetilde{F}$ with $[\widetilde{F}] = \sum [\widetilde{F}_K]$. Note that in this sum the choice of the $\widetilde{F}_K$'s may vary, according to the choices of the order in the sequence of the minimal K's and of the complementing maps $F_K^\perp$. However, for a given choice it is easy to see, from the equivalent of Lemma 5.2, that if F and G are Γ-homotopic then this is also true for $\sum \widetilde{F}_K$ and $\sum \widetilde{G}_K$, that $\{\widetilde{F}_K\}$ are uniquely determined by this choice and that the assignment from $[F]$ to $[\widetilde{F}] = \sum [\widetilde{F}_K]$ is a morphism. The morphism from $[F]$ to $[\widetilde{F}]$ will generate the suspension $\widetilde{\Pi}_{k-1}$ of Π_{k-1} in Π_k.

Furthermore, it is clear that if $P_*[F] = 0$, for $[F]$ in Π_k, then F is extendable to $\bigcup B^K$, for K with $\dim \Gamma/K \leq k-1$, that is, $[F]$ belongs to $\Pi(k)$. Moreover, if F

belongs to $\Pi(k)$, one may assume, from Theorem 5.1, that F restricted to $\bigcup B^K$, $\dim \Gamma/K \leq k-1$, is (1,0) and the explicit construction of the corresponding $\widetilde{F}_K$'s implies that $[\widetilde{F}_K] = 0$ for each K. Hence $\Pi(k) = \ker P_*$. Thus, $[F] = \sum d_H[\widetilde{F}_H]$, with $\dim \Gamma/H = k$, $\widetilde{F}_H$ are the generators of $\Pi(H)$, as in Theorem 5.1, and d_H are the extension degrees in the case that $\dim V^H \leq \dim W^H + \dim \Gamma/H$.

In general, if $[F]$ is an element of Π_k, then $P_*[F] = \sum P_*[\widetilde{F}_K]$, and as above, $[F] - \sum[\widetilde{F}_K]$ is an element of $\Pi(k)$, which is equal to $\sum d_H[\widetilde{F}_H]$. Thus, $[F] = \sum[\widetilde{F}_K] + \sum d_H[\widetilde{F}_H]$. Thus, given F and $\widetilde{F}$, the d_H's are unique. Hence, one gets the required isomorphism.

Under the hypothesis (b), let $[F]$ be an element of $\Pi^\Gamma_{SV}(S^W)$ and let $P_*[F]$ be the class of F in Π_k. Then $[F] - \sum[\widetilde{F}_K] - \sum d_H[\widetilde{F}_H]$ is 0 in Π_k, that is, it belongs to $\Pi(k+1)$. From the dimension hypothesis, one has an extension to B^L, for any L with $\dim \Gamma/L = k+1$, but $\dim \Gamma/H \leq k$, for $L < H$. This extension can then be pursued, step by step, to all of B, giving the trivial element in $\Pi^\Gamma_{SV}(S^W)$. □

COROLLARY 5.1. (a) *If* (H) *and* (H2)′ *hold for* $k = 0$ *and* $\dim V^H \leq \dim W^H$ *for any isotropy subgroup* H *of* Γ, *for example if* $V = W$, *then*

$$\Pi^\Gamma_{SV}(S^W) \cong \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z} \cong \Pi^{\Gamma'}_{SV'}(S^{W'}),$$

with one $\mathbb{Z}$ *for each* H *with* $\dim V^H = \dim W^H$, $\dim \Gamma/H = 0$ *and* $\Pi(H) \neq 0$. *Here* $\Gamma' = \Gamma/T^n$, $V' = V^{T^n}$, $W' = W^{T^n}$.

(b) *If* (H) *and* (H2)′ *hold for* $k = 1$ *and* $\dim V^H \leq \dim W^H + 1$ *for any isotropy subgroup* H *of* Γ, *for example if* $V = \mathbb{R} \times W$, *then*

$$\Pi^\Gamma_{SV}(S^W) \cong \Pi^{\Gamma'}_{SV'}(S^{W'}) \times \mathbb{Z} \times \ldots \times \mathbb{Z},$$

with one $\mathbb{Z}$ *for each* H *with* $\dim V^H = \dim W^H + 1$, $\dim \Gamma/H = 1$ *and* $\Pi(H) \neq 0$ *and* $V' = V^{T^n}$, $W' = W^{T^n}$, $\Gamma' = \Gamma/T^n$.

PROOF. (a) is an immediate consequence of Theorem 5.2, while, for (b), one needs to recall that $\bigcup B^K$, $\dim \Gamma/K = 0$, is in this case B^{Γ/T^n} and that K contains T^n, thus Π_0 is the first group. □

Note that in (a), we recover well known results, mentioned in [6], and which will be given more explicitly in the next section.

REMARK 5.1. Let P_k be the map, from $\Pi^\Gamma_{SV}(S^W)$ into Π_k, induced by the restriction to $\bigcup \partial B^H$, $\dim \Gamma/H \leq k$. Assume that for each minimal H, one has a Γ-map $F^\perp_H : \partial(B^H)^\perp \to (W^H)^\perp \backslash \{0\}$. As in the proof of Theorem 5.2, for any $[F]$ in Π_k, there are maps $\widetilde{F}_H : \partial B \to W \backslash \{0\}$, constructed iteratively from $(F^H, F^\perp_H)$ and from differences of such maps such that $[F] = \sum P_k[\widetilde{F}_H]$, that is, P_k is onto.

Furthermore, for any $[F]$ in $\Pi^\Gamma_{SV}(S^W)$, we deduce that $[F] - \sum[\widetilde{F}_H]$ belongs to $\ker P_k$. Finally, if (H2)′ holds for k, then, from Theorem 5.2, $\sum[\widetilde{F}_H] = \sum[\widetilde{F}_K] +$

$\sum d_i[\widetilde{F}_i]$, where $\widetilde{F}_K$ depends on F and corresponds to K's with $\dim \Gamma/K \leq k-1$ and $[\widetilde{F}_i]$ are the generators for $\Pi(k)$.

Putting together Theorems 5.1, 5.2 and Remark 5.1, one has then the following result:

THEOREM 5.3. *If* (H2)′ *holds for all* k*'s then*

$$\Pi^{\Gamma}_{SV}(S^W) \cong \prod_H \widetilde{\Pi}(H)$$

for all isotropy subgroups H *of* Γ, *where* $\widetilde{\Pi}(H)$ *stands for the suspension of* $\Pi(H)$ *by* $F^{\perp}_H$.

Note that the construction of this isomorphism is involved and requires a step-by-step extension process on the spaces V^H for decreasing H's. However, one may use this construction in order to give a computational formula for the class of F. We shall give below such a formula which is similar to the one given, for the S^1 case, in [7, p. 78] and close to the idea of normal maps given in [5] for a general Lie group.

Decompose V as $V^H \oplus V^{\perp_H}$, W as $W^H \oplus W^{\perp_H}$ and write $X = X^H \oplus X^{\perp_H}$, $F = (F^H, F^{\perp_H})$. Assume (H2)′ holds for all k's and let $F^{\perp}_H$ be the "suspension map" from $V^{\perp_H}$ into $W^{\perp_H}\backslash\{0\}$.

Suppose that $(F^{\perp}_H)^{\perp_K} = (F^{\perp}_K)^{\perp_H}$ for all K, H, that is, the suspensions are compatible. This will be the case if hypothesis (H3), given in Section 7, holds, in particular if $F^{\perp}_H$ is the identity.

Let $\psi_H : V^{\perp_H} \to \mathbb{R}$ be defined as a non-increasing function of $\|X^{\perp_H}\|$, with value 1 if $\|X^{\perp_H}\| \leq \epsilon$ and value 0 if $\|X^{\perp_H}\| \geq 2\epsilon$. Since $F^{\perp_H} = 0$ on V^H and $F^H \neq 0$ on ∂B^H, it is easy to see that F is Γ-homotopic, on ∂B, to $(F^H, (1-\psi_H)F^{\perp_H} + \psi_H F^{\perp}_H)$ (replacing ψ_H by $\tau\psi_H$ and taking ϵ small enough).

Arrange the isotropy subgroups in the usual decreasing sequence, $H_0 = \Gamma, H_1, \ldots$ and let

$$F_{j+1} = (F_j^{j+1}, (1-\psi_{j+1})F_j^{\perp_{j+1}} + \psi_{j+1}F^{\perp}_{H_{j+1}}).$$

Here the subscript H has been omitted for clarity, F_j^{j+1} stands for $F_j^{H_{j+1}}$, ψ_{j+1} for $\psi_{H_{j+1}}$, and so on. It is clear that F_{j+1} is Γ-homotopic to F_j and, by induction, to F.

Assume, by induction, that, if $i \leq j$, then $F_j^{\perp_i} = F^{\perp}_{H_i}$ whenever $\|X_i^{\perp}\| \leq \epsilon$. Then for such small $X_i^{\perp}$,

$$\begin{aligned} F_{j+1}^{\perp_i} &= ((F_j^{j+1})^{\perp_i}, (1-\psi_{j+1})(F_j^{\perp_{j+1}})^{\perp_i} + \psi_{j+1}(F^{\perp}_{H_{j+1}})^{\perp_i}) \\ &= ((F^{\perp}_{H_i})^{j+1}, (1-\psi_{j+1})(F^{\perp}_{H_i})^{\perp_{j+1}} + \psi_{j+1}(F^{\perp}_{H_i})^{\perp_{j+1}}) = F^{\perp}_{H_i}, \end{aligned}$$

where one has used the hypothesis on the suspensions, the induction hypothesis and the facts that $(F^H)^{\perp_K} = (F^{\perp_K})^H$ and $(F^{\perp_H})^{\perp_K} = (F^{\perp_K})^{\perp_H}$ coming from the projections.

Thus, F is Γ-homotopic to a map $\widetilde{F}$, a normal map in the terminology of [5], such that $\widetilde{F}^{\perp_H} = F_H^{\perp}$ if $\|X^{\perp_H}\| \leq \epsilon$. One then has the following result:

THEOREM 5.4. *Under the above hypothesis,* $[F]$ *is, up to one suspension, the sum of* $\Sigma_H \deg_\Gamma(\widetilde{F}^H; B^H \backslash \bigcup_{K>H} B_\epsilon^K)$, *where* B_ϵ^K *is the closed* ϵ*-neighborhood of* B^K *in* V^H *and* Σ_H *is the class suspended by* $F_H^{\perp}$ *in* $\widetilde{\Pi}(H)$.

PROOF. It is enough to note that $\deg_\Gamma(\widetilde{F}; B) = \Sigma_0[\widetilde{F}]$ and that $\deg_\Gamma$ is additive up to one suspension [6, pp. 444 and 445]. Furthermore, the sets $(B^H \backslash \bigcup B_\epsilon^K) \times \{X^{\perp_H} \mid \|X^{\perp_H}\| < \epsilon\}$ are disjoint and cover B. $\widetilde{F}$ is non-zero on the boundary of these sets and $F_H^{\perp}$ acts as a suspension. Finally, $(2t + 2\phi(X) - 1, \widetilde{F}^H)$ is non-zero on B^K, $K > H$, where ϕ enters in the definition of the Γ-degree on the set $B^H \backslash \bigcup B_\epsilon^H$, hence the above pair defines an element of the suspension of $\Pi(H)$. □

Note that the same result holds for the Γ-degree of f with respect to an open invariant set Ω, instead of B, as in [7, Prop. 4.3].

6. Relationship between Homotopy Groups

Let H_0 be a subgroup of Γ. Let V_0 and W_0 denote V^{H_0} and W^{H_0} respectively and assume that there is a Γ-map

$$F_0^{\perp} : V_0^{\perp} \backslash \{0\} \to W_0^{\perp} \backslash \{0\}.$$

Let P_0 be the map, from $\Pi_{SV}^{\Gamma}(S^W)$ into $\Pi_{SV_0}^{\Gamma}(S^{W_0})$, induced by the restriction to V_0, F giving F^{H_0}. From the existence of $F_0^{\perp}$ it is easy to see that P_0 is onto.

Assume that (H2)′ holds for some k, both for V and V_0. If $H > H_0$ one may complement F^H by a non-zero map on the orthogonal complement of V^H in V_0 and then by $F_0^{\perp}$.

In the commutative diagram

$$\begin{array}{ccc} \Pi_{SV}^{\Gamma}(S^W) & \xrightarrow{P_0} & \Pi_{SV_0}^{\Gamma}(S^{W_0}) \\ {\scriptstyle P_k}\downarrow & & \downarrow{\scriptstyle P_k} \\ \Pi_k & \xrightarrow{P_0} & \Pi_k(V_0) \end{array}$$

each map is onto, from Remark 5.1. Now, if $[F]$ is in $\Pi_{SV}^{\Gamma}(S^W)$, then, from Theorem 5.2, it follows that $P_k[F] = \sum P_k[\widetilde{F}_K] + \sum d_H P_k[\widetilde{F}_H]$, where $\widetilde{F}_K$ and $\widetilde{F}_H$ are of the form $(F^K, F_K^{\perp})$. Thus, if H_0 is not a subgroup of H or K then, from the construction of $\widetilde{F}_K$, we deduce that $\widetilde{F}_K^{H_0}$ is homotopic to $(1, 0)$. That is, in this case, $P_0[\widetilde{F}_K] = 0$.

Hence, $P_k[F^{H_0}] = \sum_0 P_k[\widetilde{F}_K^{H_0}] + \sum_0 d_H P_k[\widetilde{F}_H^{H_0}]$, where the sum is taken over the K's and H's which contain H_0.

Similarly, if $\widetilde{F}^{H_0} = (F^{H_0}, F_0^{\perp})$, then

$$[\widetilde{F}^{H_0}] - \sum\nolimits_0 [\widetilde{F}_K] - \sum\nolimits_0 d_H [\widetilde{F}_H]$$

belongs to $\ker P_k P_0 = \ker P_0 P_k$, i.e. this map extends to $\bigcup B^H$, for $H > H_0$, $\dim \Gamma/H \leq k$. Moreover, if H_0 is not a subgroup of H, with $\dim \Gamma/H \leq k$, then the extension to $\bigcup B^K, K > H_0$, $\dim \Gamma/H \leq k$, complemented by $F_0^{\perp}$, is not zero on B^H (a zero should be in $V^H \cap V_0$ from the definition of $F_0^{\perp}$). Thus, the above difference belongs to $\ker P_k$.

Let us order all the isotropy subgroups $H_1, \ldots, H_n$ with $\dim V^{H_i} = \dim W^{H_i} + k$ and $\dim \Gamma/H_i = k$ so that if $H_i > H_j$ then $i < j$. For a given $[F]$ in $\Pi_{SV}^{\Gamma}(S^W)$, we have

$$P_k[F] = \sum P_k[\widetilde{F}_{K_j}] + \sum d_i P_k[\widetilde{F}_i],$$
$$P_k[\widetilde{F}^{H_l}] = \sum\nolimits_l P_k[\widetilde{F}_{K_j}] + \sum\nolimits_l d_i P_k[\widetilde{F}_i],$$

where $\widetilde{F}_{K_j}, j = 1, \ldots, m$, depends on F and gives an element in Π_{k-1}, while F_i are the generators for $\Pi(k)$ and $\sum_l$ denotes the sum over K's and H's which contain H_l.

The above relations may be written in matrix form:

$$P_k[\widetilde{F}] = \epsilon' P_k[\widetilde{F}_K] + \eta d,$$

where

$$P_k[\widetilde{F}] = (P_k[\widetilde{F}^{H_1}], P_k[\widetilde{F}^{H_2}], \ldots, P_k[\widetilde{F}^{H_n}])^T,$$
$$\widetilde{F}_K = (\widetilde{F}_{K_1}, \ldots, \widetilde{F}_{K_m})^T, \qquad d = (d_1, \ldots, d_n)^T,$$
$$\epsilon'_{ij} = 1 \qquad \text{if } K_i < K_j,\ 0 \text{ otherwise, } i = 1, \ldots, n;\ j = 1, \ldots, m,$$
$$\eta \text{ is a lower triangular matrix, } \eta_{ij} = \epsilon_{ij}[\widetilde{F}_j] \text{ with}$$
$$\epsilon_{ij} = 1 \qquad \text{if } H_i < H_j,\ 0 \text{ otherwise, } i = 1, \ldots, n;\ j = 1, \ldots, m.$$

As an application, let us take $k = 0, V' = V^{T^n}, W' = W^{T^n}$ and $\dim V' = \dim W'$. Hence, $\dim(V^{H_i})^{\perp} = \dim(W^{H_i})^{\perp}$. Let $\Gamma' = \Gamma/T^n$, $H_i' = H_i/T^n$, $k_i' = |\Gamma'/H_i'|$, where H_i' are the isotropy subgroups for V'. Let $F_i : V'^{H_i} \to W'^{H_i}$ be the generator of $\Pi(H_i')$, with minimal extension degree α_i. Let β_i be the Brouwer degree of $F_i^{\perp}$. Then, by using the product formula for the Brouwer degree, and using Theorems 4.1 and 5.2 we get

$$\deg(\widetilde{F}_i) = \beta_i \deg(F^{H_i}) = \sum \epsilon_{ij} d_j \beta_j \alpha_j k_j'.$$

Since $\epsilon_{jj} = 1$, d_j can be computed from the set $\{\deg(F^{H_i})\}$.

Note that d_j is an arbitrary integer, by taking $d_j[F_j]$ and using the fact that all the maps are morphisms.

If one complements F_j to the orthogonal complement of V^{H_j} in V^{H_i} first and then by $F_i^{\perp}$, if $H_i < H_j$, then $\beta_j = \beta_i\beta_{ij}$, where β_{ij} is the Brouwer degree of the first map. Thus, $\beta_n = 1$ and $\epsilon_{nj} = 1$, for the isotropy group H_n of V'. The following result holds:

THEOREM 6.1.

$$\deg(F^{H_i}) = \sum_{j \le i} \epsilon_{ij}\beta_{ij}\alpha_j d_j |\Gamma/H_j|;$$

in particular,

$$\deg(F) = \sum \beta_j \alpha_j d_j |\Gamma/H_j|.$$

For instance, if Γ' acts semi-freely on V', i.e. $H_1 = \Gamma, H_2 = \{e\}$ and $\Gamma' \cong \mathbb{Z}_n$ on $(V'^{\Gamma})^{\perp}$ (see Lemma 1.1), with $\dim V'^{\Gamma} = \dim W'^{\Gamma}$, $\dim V' = \dim W'$, then

$$\deg(F^{\Gamma'}) = d_1,$$
$$\deg(F) = \beta_1 d_1 + \alpha_2 n d_2.$$

We have $\alpha_1 = 1$, since there is a map from V'^{Γ} to W'^{Γ} of degree 1 and we shall prove that $\alpha_2 = 1$.

Note that, if $F^{\perp} : \partial(V'^{\Gamma})^{\perp} \to (W'^{\Gamma})^{\perp}\backslash\{0\}$ is any Γ'-map, then $\deg(F(X_0), F^{\perp}) = \deg(F^{\perp})$, by complementing with any map $F(X_0)$ of degree 1. In this case, $d_1 = 1$ and $\deg(F^{\perp}) = \beta_1 + \alpha_2 n d_2$. (The passage through V'^{Γ}, for a free action on $(V'^{\Gamma})^{\perp}$, is due to the fact that an invariant part is needed in our computation of these groups.)

Let $\mathbb{Z}_n$ act freely on $V = \{(z_1, \dots, z_m)\}$ and act on $W = \{(\xi_1, \dots, \xi_m)\}$, with $W^{\Gamma} = \{0\}$.

Thus, if γ generates $\mathbb{Z}_n$, then $\gamma z_j = e^{2\pi i m_j/n} z_j$, with $1 \le m_j < n$, m_i and n relatively prime, and $\widetilde{\gamma}\xi_j = e^{2\pi i n_j/n}\xi_j$, with $1 \le n_j < n$, $n = r_j\widetilde{n}_j$, $n_j = r_j k_j$, $\widetilde{n}_j$ and k_j relatively prime. Thus, r_j is the order of the isotropy subgroup of ξ_j.

Now, there is a unique p_j, $1 \le p_j < n$, such that $p_j m_j \equiv 1\,[n]$, thus $\gamma^{p_j} z_j = e^{2\pi i/n} z_j$ and p_j and n are relatively prime. Similarly, there is a unique $q_j, 1 \le q_j < \widetilde{n}_j$, with $q_j k_j \equiv 1\,[\widetilde{n}_j]$, i.e. $q_j n_j \equiv r_j\,[n]$. Let $[p_j n_j]$ be the residue class of $p_j n_j$ modulo n. It is clear that $[p_j n_j] \ne 0$ and $[p_j n_j] = 1$ if $n_j = m_j$.

THEOREM 6.2. *Any Γ-equivariant map $F : S^V \to W\backslash\{0\}$ has a degree,* $\deg(F) = \beta + nd(F)$, *where $0 \le \beta < n$ is fixed and $d(F)$ may be any integer,* $\beta \equiv \prod[p_j n_j]\,[n]$ *and* $\beta \prod q_j m_j \equiv \prod r_i\,[n]$, *thus $\beta \ne 0$ if $\prod r_i \not\equiv 0\,[n]$.*

PROOF. We shall add the variable t and, for notational purposes, take $S^V = \{(z_1, \dots, z_m) \text{ with } |z_j| = 2 \text{ for some } j\}$. Let

$$\widetilde{F}_0(t, z_1, \dots, z_m) = (2t - 1, z_1^{[p_1 n_1]}, \dots, z_m^{[p_m n_m]}).$$

Since $(\gamma z_j)^{p_j n_j} = \widetilde{\gamma} z_j^{p_j n_j}$, the map $\widetilde{F}_0$ is Γ-equivariant. Then $d_0(\widetilde{F}_0) = 1$, $\deg(\widetilde{F}_0) = \prod [p_j n_j] = \beta_1$, by choosing this complementing map. Let

$$\widetilde{F}_1(t, z_1, \dots, z_m) = \big(2t + 1 - 2\prod |z_j|, z_1^{p_1 n_1}(z_1^n - 1), \\ z_2^{p_2 n_2}(\overline{z}_1^{p_1 m_2} z_2 - 1), \dots, z_m^{p_m n_m}(\overline{z}_1^{p_1 m_m} z_m - 1)\big).$$

Since the terms $\overline{z}_1^{p_1 m_j} z_j$ are invariant, $\widetilde{F}_1$ is Γ-equivariant and it has n zeros in $I \times B$, $I = [0,1]$, with $|z_j| = 1$, $t = \frac{1}{2}$, each of index 1 (use standard deformations). Thus, $d_0(\widetilde{F}_1) = 0$, $d_1(\widetilde{F}_1) = n$. Hence $\alpha_2 = 1$, $d_2(\widetilde{F}_1) = 1$. Then, given any map F, for $\widetilde{F} = (2t - 1, F)$ one has $d_1(\widetilde{F}) = 1$, $\deg(\widetilde{F}) = \deg(F) = \beta_1 + n d_2(\widetilde{F})$. Taking $\beta = [\beta_1]$ one has $\deg(F) = \beta + n d(F)$. Note that $\beta \neq 0$ if the action on W is also free, for example if $n_j = m_j$ in which case $\beta = 1$, recovering the usual result.

It remains to prove that any d can be achieved. Consider the map

$$F(z_1, \dots, z_m) = (z_1^{p_1 n_1}(z_1^n - 1), z_2^{p_2 n_2}(\overline{z}_1^{p_1 m_2} z_2 - 1), \\ z_3^{p_3 n_3}(\overline{z}_2^{p_2 m_3} z_3 - 1), \dots, z_m^{p_m m_m}((z_{m-1}^{p_{m-1} m_m} \overline{z}_m)^N - 1)).$$

The zeros of F in B are $(0, \dots, 0)$ with index equal to $\prod p_j n_j = \beta + ln$, $(e^{2\pi i k/n}, 0, \dots, 0)$ of index equal to $\prod_{j \geq 2} p_j n_j$ (there are n of them), $(e^{2\pi i k/n}, e^{2\pi i k p_1 m_2/n}, 0, \dots, 0)$ of index equal to $\prod_{j \geq 3} p_j n_j$ (also n of them), $\dots, nN$ zeros with $|z_j| = 1$ for all j's, with index -1. Thus,

$$\deg(F) = \beta + n\left(l + \sum_{k \geq 2} \prod_{j \geq k} (p_j n_j) - N\right).$$

One may choose N such that $\deg(F) = \beta + nd$, for any integer d. □

Note that one may have $\beta = 0$. For example if $n = 6$ and $m_1 = m_2 = 1$, $n_1 = 2 = r_1$, $n_2 = 3 = r_2$. The map (z_1^2, z_2^3) has degree 6, and the map $(z_1^2(z_1^6 - 1), z_2^3(z_1^4 \overline{z}_2^4 - 1))$ has degree 0.

As an application, we shall verify the assertion made in Section 4, on the consequence of hypothesis (H2); let $K > H$ be minimal, i.e. $K \cap H_j = H$ for any $H_j = \Gamma_{x_j}$ with x_j in $(V^K)^\perp$. Then, if $(V^K)^\perp$ is spanned by $\{X = (x_1, \dots, x_l)\}$, we have $\Gamma_X \cap K = H$ for $X \neq 0$, and K/H acts freely on that space. If $(W^K)^\perp = \{\xi = (\xi_1, \dots, \xi_m)\}$, then K/H acts on that space with no fixed points. If $l > m$, take any m-dimensional subspace V_m of $(V^K)^\perp$. Then, if $F^\perp$ is a non-zero Γ-map from $\partial(B^K)^\perp$ into $(W^K)^\perp \backslash \{0\}$, then $\deg(F^\perp | V_m)$ is 0 by deforming the map through the remaining variables.

However, if $K/H \cong S^1$, with free action of x_j, given by $e^{i\varphi}$, and action on ξ_j given by $e^{in_j\varphi}, n_j \neq 0$, then from [6, Theorem 4.4], $\deg(F^\perp|V_m) = \prod n_j$. While, if $K/H \cong \mathbb{Z}_n$ with the actions given in Theorem 6.2, then $\deg(F^\perp|V_m) = \prod p_j n_j + dn$, which is non-zero if $\prod p_j n_j \not\equiv 0\,[n]$.

Note that, if $K/H \cong S^1$, then one may take $\mathbb{Z}_n$ as a subgroup of S^1, for any n which is relatively prime to all n_j's, for example a large prime. In this case, $p_j = 1$, $\deg(F^\perp|V_m) = \prod n_j + dn$, for any such n, hence $d = 0$ and $\deg(F^\perp|V_m) = \prod n_j$. We have proved the following result:

COROLLARY 6.1. *If Γ acts freely on V and $W^\Gamma = \{0\}$ and if there is a Γ-map $\partial B \to W\backslash\{0\}$, then $\dim V \leq \dim W$, provided $\prod p_j n_j \not\equiv 0\,[|\Gamma|]$, when Γ is a finite group.*

REMARK 6.1. There is a vast literature on the "mod-p" or Borsuk-Ulam type results (see [10], [15]). In most of these results the symmetry is used in order to compute the ordinary degree of self-maps with the same action on both sides, or with a free or quasi-fixed point free action as in [1], [3], [17] and [13].

The advantage of classifying all equivariant maps is that one has a complete set of possible relations. This is the case of [14], [4] and [9] for self-maps with the same action. Here we give, under hypotheses similar to those of [16], a precise formula for these degrees. It is clear that one may extend most of these results and play with different situations, as for example in the case of $\Gamma = \mathbb{Z}_n$, define a map from $\widetilde{V}$ with a canonical action $e^{2\pi i/n}$ into V, given by $z_j^{m_j}$, and get the same sort of results as in [6] or [1] for the Fuller like maps. However, in order to keep the length of this paper to a reasonable size, we leave these applications to the reader.

7. Generators

From this point until the end of the paper, we shall assume the following hypothesis:

(H3) $V = \mathbb{R}^k \times U$ with $\dim U^H = \dim W^H$ for all isotropy subgroups H of U.

Thus, $V^H = \mathbb{R}^k \times U^H$ and (H3) will be satisfied if $U = W$.

Denote by $(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n)$ the elements of V, and by $(\xi_1, \ldots, \xi_n)$ the elements of W.

LEMMA 7.1. *If* (H3) *is satisfied, then so are* (H) *and* (H2), *for all H, and there are integers $l_j, 1 \leq l_j < m_j$, such that the map*

$$(x_1, \ldots, x_n) \to (x_1^{l_1}, \ldots, x_n^{l_n}) \textit{is } \Gamma\textit{-equivariant.}$$

PROOF. (H3) implies that $\dim(U^K)^{\perp_H} = \dim(W^K)^{\perp_H}$ for all $K > H$. If $\Gamma/H_j \cong \mathbb{Z}_2$ and γ in $\mathbb{Z}_2$ acts as $-I$ on $(V^\Gamma)^{\perp_{H_j}}$, then on $(W^\Gamma)^{\perp_{H_j}}$, $\widetilde{\gamma}$ must also act as $-I$, since if not one would violate the above equality of dimensions. Hence (H) is satisfied.

We shall identify U^Γ and W^Γ and take $l_j = 1$ for these components.

Let H be maximal in $\{H_j = \Gamma_{x_j}\}$. Then $\Gamma/H \cong \mathbb{Z}_n$ or S^1 and acts freely on $(U^\Gamma)^{\perp_{H_j}}$ and without fixed points on $(W^\Gamma)^{\perp_{H_j}}$. As in the proof of Theorem 6.2, either $l_j = [p_j n_j] \bmod n$, or $l_j = n_j$ in the case of a S^1-action will give the answer. Note that n_j in this last case may be negative. In that case we shall redefine W by taking conjugates on the corresponding ξ_j and maintain n_j strictly positive. This will be an implicit assumption in the paper, which is taken for notational convenience.

If H is submaximal, i.e. $H = H_j < H_i$, then let $\langle x_1, \dots, x_r \rangle$ be such that $\Gamma_{x_k} = H_i$, with the corresponding equivariant map $(x_1^{l_1}, \dots, x_r^{l_r})$ onto W^{H_i}. The space $(U^{H_i})^{\perp_{H_j}}$ is spanned by $(x_{r+1}, \dots, x_{r+s})$, while, from (H3), the space $(W^{H_i})^{\perp_{H_j}}$ is spanned by $(\xi_{r+1}, \dots, \xi_{r+s})$. The group Γ/H_j acts freely on the first space and without fixed points on the second, since H_i/H_j acts without fixed points and $\Gamma/H_j > H_i/H_j$ with $W^\Gamma \cap (W^{H_i})^{\perp_{H_j}} = 0$. The same construction as before will give an equivariant map, upon taking conjugates when necessary.

Now, if $H = H_j < H_{i_1} \cap \cdots \cap H_{i_k} \equiv \widetilde{H}$, where one has constructed an equivariant map from $U^{\widetilde{H}}$ onto $W^{\widetilde{H}}$, then on $(U^{\widetilde{H}})^{\perp_H}$, the group Γ/H acts freely while Γ/H acts without fixed points on $(W^{\widetilde{H}})^{\perp_H}$ and one may repeat the above construction. It is also clear that this map gives $F_K^\perp$ from $(V_K)^{\perp_H}$ into $(W_K)^{\perp_H}$, satisfying (H2).

Note that if $m_j = 2$, then $l_j = 1$ and the map on the real representations is just the identity. □

In order to construct the generators we shall exhibit some invariant polynomials. Let $\{x_1, \dots, x_s\}$ be coordinates in U, with $H_j = \Gamma_{x_j}$. Let H_0 be a subgroup of Γ and define $\widetilde{H}_j = H_0 \cap H_1 \cap \cdots \cap H_j$. Let $k_j = |\widetilde{H}_{j-1}/\widetilde{H}_j|$.

LEMMA 7.2. *If $k_j < \infty$, for $j = 1, \dots, s$, then there are integers $\alpha_1, \dots, \alpha_s = k_s$ such that $x_j^{\alpha_j} \dots x_s^{k_s}$ is $\widetilde{H}_{j-1}$-invariant. (If α is negative, then x^α means $\overline{x}^{|\alpha|}$.)*

PROOF. Since $H_0/\widetilde{H}_s = (H_0/\widetilde{H}_1)(\widetilde{H}_1/\widetilde{H}_2) \cdots (\widetilde{H}_{s-1}/\widetilde{H}_s)$ has finite order $(\prod k_j)$, it is a finite group. Furthermore, $H_0/\widetilde{H}_s = (H_0/H_j \cap H_0)((H_j \cap H_0)/\widetilde{H}_s)$, hence $H_0/H_j \cap H_0$ is a finite group which acts freely on x_j and there is a γ_j in H_0 with $\gamma_j x_j = e^{2\pi i/n_j} x_j$, with $n_j = |H_0/H_j \cap H_0|$.

The proof of the lemma will be by induction on j. If $j = s$, then $\widetilde{H}_{s-1}/\widetilde{H}_s$ acts freely on x_s and any γ in $\widetilde{H}_{s-1}$ can be written as $\gamma = \beta_s^\alpha \delta$, with $\beta_s x_s = e^{2\pi i/k_s} x_s$ and δ in $\widetilde{H}_s$. Hence, $(\gamma x_s)^{k_s} = \beta_s^{\alpha k_s} x_s^{k_s} = x_s^{k_s}$ is $\widetilde{H}_s$-invariant.

Assume that $P(x_{j+1},\dots,x_s) \equiv x_{j+1}^{\alpha_{j+1}}\cdots x_s^{\alpha_s}$ is $\widetilde{H}_j$-invariant, for some $j \geq 1$. Any γ of $\widetilde{H}_{j-1}$ is written as $\gamma = \beta_j^{\alpha}\delta$, with β_j generating $\widetilde{H}_{j-1}/\widetilde{H}_j$ and acting as $e^{2\pi i/k_j}$ on x_j, $0 \leq \alpha < k_j$, and δ in $\widetilde{H}_j$. Then $P(\gamma x_{j+1},\dots,\gamma x_s) = \beta_j^{\alpha\alpha_{j+1}}(\delta x_{j+1})^{\alpha_{j+1}}\cdots\beta_j^{\alpha\alpha_s}(\delta x_s)^{\alpha_s}$. Now, as before, $\beta_j = \beta_k^{\epsilon_k}\eta_k$, where β_k generates $\widetilde{H}_{j-1}/H_k \cap \widetilde{H}_{j-1}$, $\beta_k x_k = e^{2\pi i/n_k}x_k$, n_k is the order of this group, $0 \leq \epsilon_k < n_k$, and η_k is in $H_k \cap \widetilde{H}_{j-1}$. Thus, $\beta_j^{\alpha\alpha_k}(\delta x_k)^{\alpha_k} = e^{2\pi i\alpha\alpha_k\epsilon_k/n_k}(\delta x_k)^{\alpha_k}$. Hence, $P(\gamma x_{j+1},\dots,\gamma x_s) = e^{2\pi i\epsilon}P(\delta x_{j+1},\dots,\delta x_s) = e^{2\pi i\epsilon}P(x_{j+1},\dots,x_s)$, with $\epsilon = \alpha(\sum_{k=j+1}^{s}\alpha_k\epsilon_k/n_k)$.

Now, if $\gamma = \beta_j^{k_j}$, i.e. $\alpha = k_j$, then this γ belongs to $\widetilde{H}_{j-1}\cap H_j = \widetilde{H}_j$ and the corresponding $\epsilon = k_j(\sum_{k=j+1}^{s}\alpha_k\epsilon_k/n_k)$ must be an integer, which we will call $-\alpha_j$.

If $P(x_j,\dots,x_s) = x_j^{\alpha_j}x_{j+1}^{\alpha_{j+1}}\cdots x_s^{\alpha_s}$, then

$$\begin{aligned} P(\gamma x_j,\dots,\gamma x_s) &= (\beta_j^{\alpha}x_j)^{\alpha_j}e^{\alpha\pi i\epsilon}P(x_{j+1},\dots,x_s) \\ &= e^{2\pi i\alpha\alpha_j/k}e^{2\pi i\epsilon}P(x_j,\dots,x_s) = P(x_j,\dots,x_s). \end{aligned}$$

□

Note that one may take for α_j the product of k_j by the non-integer part of $\sum\alpha_k\epsilon_k/n_k$. In particular, if $k_j = 1$, one may omit the term x_j. If x_j is real, then $k_j = 1$ or 2 and the argument goes through.

LEMMA 7.3. *The conclusion of the previous lemma is valid if some of the k_j's are infinite. If $k_s = \infty$, take $\alpha_s = 0$.*

PROOF. If $k_s = \infty$, one may take $\alpha_s = 0$ and obtain a Γ-invariant constant, or replace x_s by $x_s\overline{x}_s$ which is also Γ-invariant. If i is the last index for which k_i is infinite, then, from Lemma 7.2, one has $x_{i+1}^{\alpha_{i+1}}\cdots x_s^{\alpha_s}$ which is $\widetilde{H}_i$-invariant. Now, $\widetilde{H}_{i-1}/\widetilde{H}_i \cong S^1$ acts freely on x_i and acts as $e^{in_j\varphi}$ on x_j, for $j = i,\dots,s$, with $n_i = 1$. Hence, $e^{i\varphi}(x_{i+1}^{\alpha_{i+1}}\cdots x_s^{\alpha_s}) = e^{\sum n_j\alpha_j}x_{i+1}^{\alpha_{i+1}}\cdots x_s^{\alpha_s}$. One may then choose $\alpha_i = -\sum n_j\alpha_j$ and $x_i^{\alpha_i}\dots x_s^{\alpha_s}$ will be, by the proof of the preceding lemma, $\widetilde{H}_{i-1}$-invariant. If $k_{i-1} < \infty$, one applies again Lemma 7.2, while if $k_{i-1} = \infty$, one has to repeat the above argument. □

It is easy to see that for $H_0 = S^1$, one recovers the invariants which appear in the generators of [7].

THEOREM 7.1. *If* (H3) *holds, then, for each H with* $\dim\Gamma/H = k$, *there are Γ-equivariant maps with any given extension degree. In particular,* $\Pi_{SV}^{\Gamma}(S^W) \cong \Pi_{k-1}\times\mathbb{Z}\times\cdots\times\mathbb{Z}$, *with one $\mathbb{Z}$ for each H with* $\dim\Gamma/H = k$.

PROOF. The second part follows from Theorems 5.1 and 5.2. It is then enough to exhibit a map of extension degree d, for any integer d. Recall that $\dim W^{\Gamma} \geq 1$. Let

U^Γ be spanned by $(t, X_0), (U^\Gamma)^{\perp_H}$ by $(x_1, \dots, x_s)$ and $(U^H)^\perp$ by $(x_{s+1}, \dots, x_m)$. Denote by $z_1, \dots, z_k$ the x_j's for which $k_j = \infty$. Let $B = \{t \in [0,1], \|X_0\| < 2, |x_j| < 2\}$.

Define

$$
\begin{aligned}
F_d(t, X_0, x_1, \dots, x_m) = \\
&\Big(2t + 1 - 2\prod |x_j|^2, X_0, (\lambda_1 + i(|z_2|^2 - 1))z_1^{l_1}, \\
&(\lambda_2 + i(|z_3|^2 - 1))z_2^{l_2}, \dots, (\lambda_{k-1} + i(|z_k|^2 - 1))z_{k-1}^{l_{k-1}}, (\lambda_k + i(2t-1))^d z_k^{l_k}, \\
&\{(P_j(x_1, \dots, x_j) - 1)x_j^{l_j}\}_{j=1,\dots,s, x_j \neq z_1, \dots, z_k; x_j \in \mathbb{C}}, \\
&(Q_j(y_j) - 1)y_j, \{x_j^{l_j}\}_{j=s+1,\dots,m}\Big).
\end{aligned}
$$

The product in the first factor is over the j's such that x_j is complex or x_j real and $k_j = 2$. Here $P_j(x_1, \dots, x_j)$ is the Γ-invariant monomial given in Lemma 7.3, ending with $x_j^{k_j}$, if x_j is complex. Also, $Q_j(y_j) = y_j^2$ if y_j is real and $k_j = 2$, $Q_j(y_j) = 2$ if y_j is real and $k_j = 1$. Note that, if y_j is real and $k_j = 1$, then y_j does not appear in the monomials P_l and that $l_j = 1$ if y_j is real. The order of the components has been chosen so that the notation is lighter; however, the terms in z_j and y_j should appear in their corresponding place for W.

If $k = 0$ and there is at least one complex x_j, replace P_j by P_j^d. If $k = 0$ and all x_j's are real, then if $s \geq 2$, take two y's, say y_1 and y_2, with $k_1 = k_2 = 2$ and replace $Q_1 - 1$ and $Q_2 - 1$ by $\mathrm{Re}(y_1^2 - 1 + i(y_2^2 - 1))^d$ and $\mathrm{Im}(y_1^2 - 1 + i(y_2^2 - 1))^d$. If $k_j = 1$ for $j > 1$, i.e. $\Gamma/H \equiv \mathbb{Z}_2$, replace $Q_1 - 1$ and $Q_2 - 1$ by $\mathrm{Re}(y_1^2 - 1 + iy_2^2)^d$ and $\mathrm{Im}(y_1^2 - 1 + iy_2^2)^d$. Finally, if $s = 1$, replace X_0 by a map of index d at the origin or replace the first factor by $\mathrm{Re}(2t + 1 - 2y^2 + i(y^2 - 1))^d$ and the second by $\mathrm{Im}(2t + 1 - 2y^2 + i(y^2 - 1))^d y$. Clearly, F_d restricted to V^H will lose the terms $x_j^{l_j}$, $j = s+1, \dots, m$.

The zeros of F_d will be for $X_0 = 0$, $\lambda_j = 0$, $|z_j| = 1$, $|x_j| = 1$ if x_j is complex or if x_j is real and $k_j = 2$, $y_j = 0$ if y_j is real and $k_j = 1$ and $t = 1/2$, i.e. with isotropy type H. In particular, $F_d | V^K \neq 0$ for all $K > H$. By replacing, as in [7, Theorem 3.1], the 1, in the term $P_j(x_1, \dots, x_j) - 1$, by ϵ_j with $|\epsilon_j| = 1$, one may choose ϵ_j such that for $z_j = 1$, one has no zeros on ∂C, where C is the corresponding fundamental cell. Then, on $B_k = B \cap \{z_j > 0, j = 1, \dots, k\}$, there will be exactly $\prod k_j$ zeros, as one can easily see from the form of P_j; for given $x_1, \dots, x_{j-1}$, one has k_j possible values for x_j.

As in [7, Theorem 3.2], one may deform F_d, near a zero $\{x_j^0\}$, to $(2t + 1 - 2z_1^2, X_0, \lambda_1 + i(z_2^2 - 1), \dots, (\lambda_k + i(2t-1))^d, \{x_j - x_j^0\})$. The first term is deformable to $1 - z_1$, and the index of this map at $\{x_j^0\}$ will be $-d\epsilon$, where ϵ is the orientation factor coming from the map $(t, \lambda_1, \dots, \lambda_k, X_0, z_1, \dots, z_k) \to (z_1, X_0, \lambda_1, z_2, \dots, \lambda_k, t)$. It is easy to see that one needs $k(k+1)/2 + kN$ permutations to get to the last

arrangement, thus $\epsilon = (-1)^{k((k+1)/2+N)}$. Thus, $\deg(F_d; B_k) = -(\prod k_j)\epsilon d$ and, from Theorem 4.1, $\deg_E(F_d) = -\epsilon d$.

Recall also that, from Theorem 4.1, all the zeros have the same index. For the case $k = 0$, it is easy to see that one has a map of extension degree d. □

8. The One Parameter Case, $k = 1$

Let $V = \mathbb{R} \times U$ and assume that hypothesis (H3) holds. Then, from Corollary 5.1 (b) and Theorem 7.1, $\Pi^{\Gamma}_{SV}(S^W) \cong \Pi^{\Gamma'}_{SV'}(S^{W'}) \times \mathbb{Z} \times \cdots \times \mathbb{Z}$, with one $\mathbb{Z}$ for each isotropy subgroup H with $\dim \Gamma/H = 1$. Furthermore, from Theorem 5.3, $\Pi^{\Gamma'}_{SV'}(S^{W'}) \cong \prod \widetilde{\Pi}(H)$, over H with $\dim \Gamma/H = 0$. In this section we shall compute these groups, recalling that, from Theorem 4.4, $\Pi(H)$ is an abelian group, the class of $[F]$ does not depend on the extension to $\bigcup B^K$, $K > H$, and that F may be chosen, if necessary, such that it has a trivial extension to this union. We shall then assume that $\Gamma = \Gamma'$ is finite.

If $H = \Gamma$, $\dim U^{\Gamma} = d$, then $\widetilde{\Pi}(H) = \Sigma_H \Pi_d(S^{d-1})$, where Σ_H is the suspension by the map given in Lemma 7.1; Σ_H is an isomorphism if $U = W$ and $d > 3$, $\widetilde{\Pi}(H) = \Sigma_H(\mathbb{Z}_2)$ if $d \geq 3$ and Γ acts non-trivially on U, while this group is 0 if $d \leq 2$. This part of the Γ-equivariant class of F is given by $[F^{\Gamma}]$.

As before we shall write any element of V^H as $(t, \mu, X_0, y_1, \ldots, y_s, z_1, \ldots, z_r)$ with y_j in $\mathbb{R}$ with $\Gamma/H_j \cong \mathbb{Z}_2$ and z_j in $\mathbb{C}$ with $\Gamma/H_j \cong \mathbb{Z}_{m_j}$. Define $B^H = \{0 \leq t \leq 1, |\mu|, \|X_0\|, |y_j|, |z_j| \leq 2\}$. Set $\lambda = 2t - 1 + i\mu$.

If $\Gamma/H \cong \mathbb{Z}_2$, then one may assume that $\Gamma/H = \Gamma/H_1$, $k_1 = 2$, $k_j = 1$ for $j > 1$, the fundamental cell is given by $y_1 > 0$ and Γ acts as $-\mathrm{Id}$ on y_j (the other variables are fixed by Γ/H). Let $[\widetilde{F}]_{\Gamma} = [F]_{\Gamma} - [F^{\Gamma}]_{\Gamma}$; then we may assume that $\widetilde{F}$ is (1,0) on V^{Γ}. Let $s = \dim V^H - \dim V^{\Gamma}$.

If $s = 1$, then there is an obstruction to extension to the set $B^H \cap \{y_1 > 0\}$ with boundary S^d_+, given by its class in $\Pi_{d+1}(S^d_+)$, generated by the suspension $\widetilde{\eta}$ of the Hopf map. For $d \geq 2$, let $X_0 = (x_0, \widetilde{X}_0)$ and let η_1 be the map

$$((1/4 - (y_1 - 1)^2 - x_0^2)(1/4 - (y_1 + 1)^2 - x_0^2), \widetilde{X}_0, \\ \mathrm{Re}(\lambda(y_1^2 - 1 + ix_0), y_1), y_1 \mathrm{Im}(\lambda(y_1^2 - 1 + ix_0), y_j)).$$

It is clear that η_1 is equivariant, $\eta_1 | S^d_+$ is homotopic to the suspension of the Hopf map and that $\eta_1 | (\partial B^H)$ has the class of $2\widetilde{\eta}$ in $\Pi_{d+1}(S^d)$. Hence, there is a d_1, in $\mathbb{Z}$ if $d = 2$ and in $\mathbb{Z}_2$ if $d > 2$, such that $\widetilde{F}$ and $d_1\eta_1$ have the same obstruction on S^d_+. Thus, $[\widetilde{F}]_{\Gamma} - d_1[\eta_1]_{\Gamma} = 0$. From this relation d_1 is uniquely determined and $\Pi(H) \cong \Pi_{d+1}(S^d)$. If $d \leq 1$, then $\Pi_{d+1}(S^d_+) = 0$ and $[F]_{\Gamma} = 0$ ($[F^{\Gamma}]$ is also trivial).

If $s > 1$, then, from Theorem 3.1, one has an equivariant extension to the set $B^H \cap \{y_1 = y_2 = 0\}$ and an obstruction, an integer, to extension to the set

$B^H \cap \{y_1 = 0\}$. Note that on this set the action of Γ does not satisfy hypothesis (H), that is, we cannot guarantee, except if $s = 2$, the uniqueness of this obstruction, given by the class of the extension to $\partial(B^H \cap \{y_1 = 0,\ y_2 > 0\})$. Note that, from Remark 4.1, the degree on $B^H \cap \{y_1 = 0\}$ is 0, i.e. there is no obstruction to non-equivariant extension (one may also deform F on the boundary of this last set, to $F(t, \mu, X_0, 2, y_2)$ which is not 0). However, let

$$d\eta = (1 - y_1^2 - y_2^2, X_0, \lambda^d(y_1 + iy_2), y_j).$$

Then it is easy to see that the degree of $d\eta$ on $B^H \cap \{y_1 = 0,\ y_2 > 0\}$ is $(-1)^{n+1}d$, $n = \dim X_0$. Thus, for some d, $[\widehat{F}]_\Gamma \equiv [\widetilde{F}]_\Gamma - d[\eta]_\Gamma$ has an equivariant extension to $B^H \cap \{y_1 = 0\}$.

Note that if $s > 2$, then, from the fact that

$$\begin{pmatrix} \lambda^{d_1} & 0 \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} \lambda^{d_2} & 0 \\ 0 & I \end{pmatrix}$$

represent the same class in $\Pi_1(GL(\mathbb{R}^s))$ if and only if d_1 and d_2 have the same parity, one may restrict d to be 0 or 1, by deforming equivariantly $d_1\eta$ to $d_2\eta$ by the above argument.

As before, the next obstruction will be the class of $\widehat{F}$ in $\Pi_{n+1+s}(S_+^{n+s})$. Let $d_1\eta_1$ be the equivariant map

$$((4/3)^2(1/4 - (y_1 - 1)^2 - y_1^2y_2^2)(1/4 - (y_1 + 1)^2 - y_1^2y_2^2), \\ X_0y_1^2, \lambda^{d_1}(y_1(y_1^2 - 1) + iy_1^2y_2), y_jy_1^2).$$

Again, it is easy to see that $d_1\eta_1 = (1, 0)$ for $y_1 = 0$ and that its class in $\Pi_{n+s+1}(S_+^{n+s})$ is $d_1[\eta]$ (deform y_1^2 to 1) and $2d_1[\eta]$ in $\Pi_{n+s+1}(S^{n+s})$. Hence, there is a d_1 (in $\mathbb{Z}_2$ if $n + s > 2$, in $\mathbb{Z}$ if $n = 0$, $s = 2$) such that $[\widehat{F}]_\Gamma - d_1[\eta_1]_\Gamma = 0$.

From the fact that $[\widetilde{F}]_\Gamma = d[\eta]_\Gamma + d_1[\eta_1]_\Gamma$, one sees that, if $n + s > 2$, the class of $\widetilde{F}$ in $\Pi_{n+s+1}(S^{n+s})$ is $d[\eta]$, hence the parity of d is uniquely determined by $\widetilde{F}$, thus, the first invariant d (in $\mathbb{Z}$ if $s = 2$, in $\mathbb{Z}_2$ if $s > 2$) is unique. Therefore, from the above formula, d_1 (in $\mathbb{Z}$ if $s = 2$, $n = 0$, in $\mathbb{Z}_2$ if $n + s > 2$) is also unique. We have proved the following:

THEOREM 8.1. *If* $\Gamma/H \cong \mathbb{Z}_2$, *then*

$$\Pi(H) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } s > 2, \\ \mathbb{Z} \times \mathbb{Z}_2 & \text{if } s = 2,\ n > 0, \\ \mathbb{Z} \times \mathbb{Z} & \text{if } s = 2,\ n = 0. \end{cases}$$

Consider now the case of a general isotropy subgroup H. Let $\mathcal{C}$ be the corresponding fundamental cell. As in the proof of Theorem 3.1, we shall extend and modify a given element $[F]_\Gamma$ of $\Pi(H)$ to an equivariant map $\widetilde{F}$ without zeros on

$\partial\mathcal{C}$. There will be obstructions to modifications on each of the faces of $\partial\mathcal{C}$ (i.e. with just one $y_j = 0$ or one z_j with $\operatorname{Arg} z_j = 0$). As seen in the proof of Theorem 3.1, the value of F on an edge $\operatorname{Arg} z_j = 0$, $\operatorname{Arg} z_i = 2\pi/k_i$ may be given by the value on a face $\operatorname{Arg} z_k = 0$ for some $k < i$. Thus, one has to start with the first face, modify F so that the new map will have a non-zero extension on that face and work the way up on the faces. For $\widetilde{F}$ one will have a last obstruction, in $\mathbb{Z}_2$, for the extension to $\mathcal{C}$.

Consider first the face $y_1 = 0$. If the intersection of the H_i's, $i > 1$, strictly contains H, then there is an equivariant (non-zero) extension to that face for any element of $\Pi(H)$. If this intersection is H, then from Theorem 3.1, one has an integer as an obstruction to Γ-extension (note again that (H) is not satisfied) given by the class of an extension to the boundary of the fundamental cell $\mathcal{C}'$ of that face.

Assume $y_1, y_2, \dots, y_s$ have the same isotropy H_1. (If y_1 is the only coordinate with isotropy H_1 and $\bigcap_{j>1} H_j = H$, then one may reorder the coordinates and put y_1 at the end; y_1 will play no role in the new fundamental cell). Assume $s > 1$ and let

$$d\eta_1 = \Big(2t + 1 - 2\prod |x_i|^2(y_1^2 + y_2^2), X_0, \widetilde{\lambda}^d(y_1 + iy_2), \\ (Q_i - 1)y_i, (P_i(y_1 + iy_2, \dots, x_i) - \epsilon_i)x_i^{l_i}\Big)$$

where the product in the first term is over $i \geq 3$ with $k_i \geq 2$. Moreover, Q_i is y_i^2 if $k_i = 2$ and 2 if $k_i = 1$. Finally, P_i is defined, as in Lemma 7.2 with $P_i = 0$ if $k_i = 1$, as an invariant monomial such that, if $y_2 = 1$, $y_1 = 0$, then the set $\{Q_i - 1, P_i - \epsilon_i\}$ has exactly $|H_1/H|$ zeros with $|x_i| = 1$ and just one in $\mathcal{C}'$, and $\widetilde{\lambda} = \mu + i(y_1^2 + y_2^2 - 1)$.

It is easy to see that $\deg(d\eta_1|_{y_1=0}; \mathcal{C}') = (-1)^n d$. Since $(-1)^n$ is an orientation factor, due to the chosen order of the components, we may assume that it is 1, since if not we change $\widetilde{\lambda}$ to its conjugate.

Now, the problem of finding an H_1-equivariant extension for $y_1 = 0$, $y_2 > 0$ (y_2 considered as a parameter), given a Γ-equivariant non-zero extension to $B^H \cap \{y_1 = 0,\ y_2 = 0\}$, is classified by the invariants of Theorem 6.1. Here, since F has an extension to $\bigcup B^K$, $K > H$, one has $\deg(F^K; B^K) = 0$. Thus, $d_K = 0$ for $K > H$ and $\deg_{H_1}(F; B^H \cap \{y_1 = 0,\ y_2 > 0\}) = d_{H_1}$, with $\deg(F; B^H \cap \{y_1 = 0,\ y_2 > 0\}) = |H_1/H| d_{H_1}$. Note that the fundamental cell for this H_1-extension is just $\mathcal{C} \cap \{y_1 = 0,\ y_2 > 0\}$ and that the generator (as seen in Theorem 7.1) is $\eta_1|_{y_1=0}$.

Let $d_1 = d_{H_1}$; then the map $[F]_\Gamma - d_1[\eta_1]_\Gamma$ will have a non-zero extension to the face $y_1 = 0$.

REMARK 8.1. If $s \geq 3$ and the P_i's do not depend on y_1, then one may show that $2[\eta_1]_\Gamma = 0$, as above.

For a face of the form $\operatorname{Arg} z_j = 0$, or for a pair of real variables with $z_j = y_j + iy'_j$, $y_j = 0$, $y'_j > 0$ as above, if one considers $\mathcal{C} \cap \{z_j = 0\}$, then there is always, from Theorem 3.1, an equivariant extension (if the isotropy of the face is $K > H$, then the extension is given a priori). Assume F has been modified to $[F]_\Gamma - \sum_{i<j} d_i[F_i]_\Gamma$, a map without zeros on the faces $\operatorname{Arg} z_i = 0$, $i < j$. In order to get an H_j-equivariant extension to the ball $B^H \cap \{\operatorname{Arg} z_j = 0\}$, one has again that the H_j-class of this map is $\sum d_K[F_K]$, $H < K < H_j$ with the relations, given in Theorem 6.1, between the ordinary degrees and the d_K's. Since the above map has a non-zero H_j-extension to $B^K \cap \{\operatorname{Arg} z_j = 0\}$ it follows that $d_K = 0$ if $K > H$ and d_H is the degree of the above map on the corresponding cell $\mathcal{C}_j$. (Note that $\mathcal{C}_j$ and $\mathcal{C}$ coincide in the variables x_i, $i > j$.) The ordinary degree of this map on $B^H \cap \{\operatorname{Arg} z_j = 0\}$ is $|H_j/H| d_H$.

Now, the set $B^H \cap \{\operatorname{Arg} z_j = 0\}$ is covered by $|H_j/H|$ disjoint replicae of $\mathcal{C}_j$, on each of which the map has the same degree d_H. On the other hand, this set is also covered by the disjoint sectors $2\pi p_i/k_i < \operatorname{Arg} z_i < 2\pi(p_i+1)/k_i$, $i \neq j$, $p_i = 0, \dots, k_{i-1}$, $\operatorname{Arg} z_j = 0$. There are $\prod_{i\neq j} k_i$ such sectors. Notice that $\prod_{i\neq j} k_i = |\Gamma/H|/k_j = |\Gamma/H_j||H_j/H|/k_j = |H_j/H| m_j/k_j$. Now, there are γ_1 in Γ/H_1, γ_2 in $\widetilde{H}_1/\widetilde{H}_2, \dots, \gamma_s$ in $\widetilde{H}_{s-1}/H$ such that such a sector will be sent into $\mathcal{C}$ under $\gamma_1 \dots \gamma_s$ (in fact into the image of $\{\operatorname{Arg} z_j = 0\}$ under $\gamma_1 \dots \gamma_{j-1} = \gamma$, since γ_i fixes this argument for $i > j$). There are m_j/k_j possible such hyperplanes in $\mathcal{C}$. The part of the boundary of the sector corresponding to $\operatorname{Arg} z_i = 2\pi p_i/k_i$ or $2\pi(p_i+1)/k_i$, $i < j$, is sent to $\operatorname{Arg} z_i = 0$ or $2\pi/k_i$ where the modified F has a non-zero extension (hence to the intersection with the m_j/k_j hyperplanes). If $i > j$, that part of the boundary of the sector is sent into the corresponding part of $\partial\mathcal{C}$, without moving the previous z's, including z_j. Hence, from the construction of the modified F, one has a non-zero map on the boundary of each sector with equal degrees, by applying γ and using $\operatorname{Arg} z_j$ as a deformation parameter. Thus, the degree of the modified F on $B^H \cap \{\operatorname{Arg} z_j = 0\}$ is a multiple of $\prod_{i\neq j} k_i$. Since this has to be true for any such map, this implies that d_H is a multiple of m_j/k_j. Let

$$d\eta_j = \left(2t + 1 - 2\prod |x_i|^2, X_0, (Q_i - 1)y_i, \widetilde{\lambda}^d z_j^{l_j}, \{(P_i(x_1, \dots, x_i) - \epsilon_i)x_i^{l_i}\}_{i\neq j}\right)$$

where $\widetilde{\lambda} = \mu + i(|z_j|^2 - 1)$, the product is over all the variables except y_i real with $k_i = 1$, P_i is again the Γ-invariant polynomial such that the set $\{Q_i - 1, P_i - \epsilon_i\}$ has, for $z_j = 1$, $\prod_{i\neq j} k_i$ zeros, with $|x_i| = 1$, and just one of them on the face of $\mathcal{C}$ corresponding to $\operatorname{Arg} z_j = 0$.

The degree of $d\eta_j$ on that face of $\mathcal{C}$ is $(-1)^n d$ (again in order to avoid carrying this orientation factor, one may change $\widetilde{\lambda}$ to its conjugate if n is odd), $d\eta_j$ is trivial when restricted to previous faces of $\mathcal{C}$, $\operatorname{Arg} z_i = 0$, $i < j$. Finally, the ordinary degree of $d\eta_j$, on $B^H \cap \{\operatorname{Arg} z_j = 0\}$, is $(-1)^n d \prod_{i\neq j} k_i$.

Note that $d\eta_j$ is Γ-deformable to the map

$$d\widetilde{\eta}_j = \left(1 - \prod |x_i|^2, X_0, (Q_i - 1)y_i, \lambda^d z_j^{l_j}, \{(P_i - \epsilon_i |z_j|^{\alpha_j}) x_i^{l_i}\}_{i \neq j}\right)$$

where the product is over all x_i ($x_i = y_i + iy_i'$ for real y_i) and α_j is the exponent of z_j in the monomial of Lemma 7.2. Here, $\lambda = \mu + i(2t - 1)$.

Hence, $d\eta_j$ generates all possible obstructions on the face $\mathcal{C} \cap \{\operatorname{Arg} z_j = 0\}$ and does not modify the previous construction. One has an integer d_j such that $[F]_\Gamma - \sum_{i \leq j} d_i [\eta_i]_\Gamma$ is non-zero on the faces of $\mathcal{C}$ with $\operatorname{Arg} z_i = 0$, $i \leq j$ (this extension is then reproduced by the action of Γ on the other faces).

Thus, one may construct a step-by-step modification of F, with integer obstructions d_j. Note that d_j depends only on the extension to $\mathcal{C} \cap \{z_j = 0\}$, from the H_j-extension argument and

$$\left(\prod_{i \neq j} k_i\right) d_j = \deg\left([F]_\Gamma - \sum_{i<j} d_i [\eta_i]_\Gamma; B^H \cap \{\operatorname{Arg} z_j = 0\}\right).$$

REMARK 8.2. Note that at each step F is modified on the subsequent faces. In the above formula $[F]_\Gamma - \sum_{i<j} d_i [\eta_i]_\Gamma$ stands for a Γ-homotopy on ∂B^H and for an extension to the faces $\mathcal{C} \cap \{\operatorname{Arg} z_i = 0\}_{i<j}$. However, the homotopy has not been extended to these faces and, in particular, there is no relationship between the ordinary degree of F on the face $\{\operatorname{Arg} z_j = 0\}$ and the sum of the degrees of $d_i \eta_i$ on that face and even less with respect to the degrees on $B^H \cap \{\operatorname{Arg} z_j = 0\}$, except, as we have seen in the above proof, for the first face for which F has no extension.

In order to extend $[F]_\Gamma - \sum d_i [\eta_i]_\Gamma$ to $\mathcal{C}$ one has a last obstruction, this time in $\mathbb{Z}_2$. A generator for this obstruction is given by the map

$$\widetilde{d\eta} = \left(\epsilon^2 - \prod_{i \neq n} |x_i|^2 |P_n - \epsilon_n|^2, X_0, (Q_i - 1)y_i, (P_i - \epsilon_i) x_i^{l_i}, \lambda^{\widetilde{d}} (P_n - \epsilon_n) x_n^{l_n}\right).$$

The constants ϵ_i, with $|\epsilon_i| = 1$, are chosen such that the set $\{Q_i - 1, P_i - \epsilon_i\}$ has $|\Gamma/H|$ zeros, with $|x_i| = 1$, and just one of them, X^0, in $\mathcal{C}$. Note that k_n may be 1. As before x_n is treated as $y_n + iy_n'$ if $\Gamma/H_n \cong \mathbb{Z}_2$. One chooses ϵ so small that the disc $\|X - X^0\| \leq \epsilon$ is contained in $\mathcal{C}$. Hence, the only zeros of $\widetilde{d\eta}$ in $\mathcal{C}$ are for $x_i = x_i^0$, $\lambda = 0$, $|x_n - x_n^0|^2 = \epsilon^2$ and on $\partial\mathcal{C}$ one may deform $\widetilde{d\eta}$ to the suspension of $(\epsilon^2 - |x_n - x_n^0|^2, \lambda^{\widetilde{d}}(x_n - x_n^0))$, which is $\widetilde{d}$ times the Hopf map.

REMARK 8.3. Note that the non-equivariant class of $\widetilde{d\eta}$ with respect to ∂B^H is $|\Gamma/H|\widetilde{d}$ times the Hopf map, since on each of the replicae of $\mathcal{C}$ one obtains the same class and that, by construction, $2[\widetilde{\eta}]_\Gamma = 0$. This last fact can also be seen

directly if one suspends $\widetilde{\eta}$ by x'_n such that x_n and x'_n have the same isotropy, with $|\Gamma/H_n| = m$. In fact, one may use, in the definition of $2\widetilde{\eta}$, the equivariant deformations $\epsilon^2 - \prod |x_i|^2(|P_n - \epsilon_n|^2 + \tau|x'_n|^2)$ in the first component and then $((1-\tau)\lambda^2(P_n - \epsilon_n)x_n^{l_n} + \tau\lambda x_n'^{l_n}, -\tau\lambda(P_n - \epsilon_n)x_n^{l_n} + (1-\tau)x_n'^{l_n})$. The map obtained for $\tau = 1$ can be further deformed by $((1-\tau)\lambda x_n'^{l_n} + \tau(\overline{x}_n^{m-2}\overline{x}_n)^{l_n}(\overline{P}_n - \overline{\epsilon}_n)$, $\tau(\overline{x}_n^{m-2}\overline{x}'_n)^{l_n} - (1-\tau)\lambda x_n^{l_n}(P_n - \epsilon_n))$, recalling that $\overline{x}_n^{m-2}\overline{x}'_n$ is equivariant (as well as $\overline{x}_n^{m-1}$); by conjugating the second equation it is easy to see that this deformation has no zeros for positive τ, hence that $2[\widetilde{\eta}]_\Gamma = 0$.

Thus, one has obtained

$$[F]_\Gamma = \sum d_j[\eta_j]_\Gamma + \widetilde{d}[\widetilde{\eta}]_\Gamma$$

where $\widetilde{d}$ belongs to $\mathbb{Z}_2$, d_j are obtained by the above construction and given by the formula

$$\left(\prod_{i\neq j} k_i\right) d_j = \deg\left([F]_\Gamma - \sum_{i<j} d_i[\eta_i]_\Gamma; B^H \cap \{\operatorname{Arg} z_j = 0\}\right)$$

($\operatorname{Arg} z_j = 0$ has to be replaced by $y_j = 0$, $y'_j > 0$ for the case of real variables).

Here the only extra hypothesis is the repetition of y_j by y'_j. The set of d_j's is recursively and uniquely determined by the successive extensions to the edges $\{z_j = 0\}$ (or $y_j = y'_j = 0$ in the real case).

By setting a morphism Π from $(\prod \mathbb{Z}) \times \mathbb{Z}_2$ into $\Pi(H)$; given by the above formula, one sees that Π is onto and its kernel is the set of all possible d_j's and $\widetilde{d}$'s corresponding to the trivial element $(1,0)$. In order to make the computations easier we shall assume that for each z_j, with $k_j > 1$, there is a $z'_j = z_{j+1}$ with the same isotropy, in U and W. In the real case this means that there are at least three variables with the same isotropy.

Now, $(1,0)$ can always be extended Γ-equivariantly to the set $\{y_1 = y'_1 = 0\}$ which has a fundamental cell of the form $\{y''_1 > 0\} \times C''$, where C'' corresponds to the same variables as C. On $\{y''_1 = 0\} \times C''$ one may assume, by a dimension argument, that this extension is still $(1,0)$. By taking the extension $(1,0)$ on the set $\{y_1 = 0, y'_1 > 0, y''_1 = 0\}$, the degree of an extension with respect to $B^H \cap \{y_1 = 0, y'_1 > 0\}$ is twice the degree with respect to $B^H \cap \{y_1 = 0, y'_1 > 0, y''_1 > 0\}$, i.e. the corresponding d_1 must be even.

If $[(1,0)]_\Gamma - \sum_{i<j} d_i[\eta_i]_\Gamma$ has been extended to $\{\operatorname{Arg} z_i = 0\}_{i<j}$, then for $z_j = 0$ one has a Γ-equivariant extension, from the dimension. On the edge $(z_j = 0)$ the fundamental cell C' has the same form as the one for C, with z'_j replacing z_j and $|\Gamma/H|$ replicae of C' covering $B^H \cap \{z_j = 0\}$. From the dimension, one may deform the map on $\partial C'$ to $(1,0)$ without changing the homotopy type of the map on C', relative to its boundary. Now, if $d_i = 0$, $i < j$, or if $d_i\eta_i$ is trivial on $\{\operatorname{Arg} z_j = 0\}$, then one obtains a partition of $B^H \cap \{\operatorname{Arg} z_j = 0\}$ by the ball $C' \times \{0 < z_j < 2\}$ and

its replicae, such that on $(\partial C') \cap \{0 < z_j < 2\}$ one may extend the map by $(1,0)$. Hence, the degree of the map with respect to $B^H \cap \{\operatorname{Arg} z_j = 0\}$ is a multiple of $|\Gamma/H|$. Thus, if $d_i = 0$, $i < j$, or if $d_i \eta_i$ is trivial on $\{\operatorname{Arg} z_j = 0\}$ for all $i < j$, since this degree is $d_j \prod_{i \neq j} k_i$, it follows that d_j is a multiple of k_j.

Consider now the map

$$pF_j = \Big(2 - \prod_{i \neq j} |x_i|^2 |z_j'|^2 (1 + |z_j|^2), X_0, \\ (Q_i' - 1) y_i, \lambda^p z_j^{l_j}, (P_j' - \epsilon_j) z_j'^{l_j}, \{(P_i' - \epsilon_i) z_i^{l_i}\}_{i \neq j}\Big)$$

where P_i' is the set of invariant polynomials where z_j has been replaced by z_j', such that the set $\{Q_i' - 1, P_i' - \epsilon_i\}$ has $|\Gamma/H|$ zeros of the form γX^0, with $|x_i^0| = 1$, none of which is on the faces $\operatorname{Arg} z_i = 0$ or for $\operatorname{Arg} z_j' = 0$. The zeros of pF_j are for $X_0 = 0$, $\lambda = 0$, $X = \gamma X^0$, $|z_j| = 1$. For $\operatorname{Arg} z_j = 0$, the degree of this map is $p \prod k_i$, while for $\operatorname{Arg} z_i = 0$, $i \neq j$, the degree is 0 (the map is non-zero there). Hence, in $[pF_j]_\Gamma = \sum d_i [\eta_i]_\Gamma + \widetilde{d}[\widetilde{\eta}]_\Gamma$, one has $d_i = 0$ for $i < j$ and $d_j = pk_j$. Furthermore, one may replace, in the first component, 2 by $1/2 + (1-\tau)3/2$, obtaining an equivariant deformation of the map on ∂B^H to a trivial map, i.e. $[pF_j]_\Gamma = 0$.

Taking $p = 1$, we have obtained the relations, for $j = 1, \ldots, n$,

$$0 = k_j [\eta_j]_\Gamma + \sum_{i > j} d_{ji} [\eta_i]_\Gamma + \widetilde{d}_j [\widetilde{\eta}]_\Gamma,$$
$$0 = 2[\widetilde{\eta}]_\Gamma.$$

From these relations, one finds, using the equation $2[\widetilde{\eta}]_\Gamma = 0$, that $2k_n[\eta_n]_\Gamma = 0$, $2k_{n-1}k_n[\eta_{n-1}]_\Gamma = 0$, $\ldots$, $2\prod k_i [\eta_1]_\Gamma = 0$ and $2|\Gamma/H|[F]_\Gamma = 0$ for any element of $\Pi(H)$.

Finally, if one has a representation of the trivial map, $0 = \sum d_j [\eta_j]_\Gamma + \widetilde{d}[\widetilde{\eta}]_\Gamma$, then we know that d_1 is a multiple of k_1; $d_1 = p_1 k_1$ (d_1 is even for y_1 real). Substituting the first relation in the above sum, one obtains that $d_2 - p_1 d_{12} = p_2 k_2$. Continuing this argument, one concludes

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 & \ldots & 0 \\ d_{12} & k_2 & 0 & \ldots & 0 \\ \vdots & & & & \\ d_{1n} & d_{2n} & d_{3n} & \ldots & k_n \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

together with the relation $\widetilde{d} \equiv \sum p_i \widetilde{d}_i$ [2].

Since, on the other hand, one may take the p_i's to be arbitrary integers, we have proved that $\ker \Pi$ is generated by the above relations.

THEOREM 8.2. *Under the hypothesis of the repetition of the z_i's, $\Pi(H)$ is a finite group with generators $[\eta_j]_\Gamma, [\widetilde{\eta}]_\Gamma$, of order at most $2|\Gamma/H|$, and relations $0 = k_j[\eta_j]_\Gamma + \sum_{i>j} d_{ji}[\eta_i]_\Gamma + \widetilde{d}_j[\widetilde{\eta}]_\Gamma$, $2[\widetilde{\eta}]_\Gamma = 0$.*

It remains to compute d_{ji} and $\widetilde{d}_j$. Since this computation is rather involved we only indicate a way to do it. We begin by calculating $\widetilde{d}_n$. For simplicity assume that $l_i = 1$ for $i > n$. Consider the map

$$F_n = \Big(1 - \prod_{i<n} |x_i|^2[|z_n|^2 + |z_n'|^2], X_0, (Q_i - 1)y_i, \{(P_i - \epsilon_i)x_i^{l_i}\}_{i<n}, \\ \lambda z_n'^{l_n}, \lambda^{k_n-1}(\overline{Q}_n Q_n^{k_n-1} z_n^{(k_n-1)^2})^{l_n}, \{x_i\}_{i>n}\Big).$$

The product is over all i's, with $k_i > 1$ if x_i is real, $P_n = Q_n z_n^{k_n}$. Note that the term $\overline{Q}_n Q_n^{k_n-1} z_n^{(k_n-1)^2} = (P_n)^{k_n-2}\,|Q_n|^2 z_n$ is equivariant. Consider the equivariant deformation

$$((1-\tau)\lambda z_n'^{l_n} - \tau(\overline{Q}_n \overline{z}_n^{k_n-1})^{ln}, \tau(\overline{Q}_n \overline{z}_n'^{k_n-1})^{l_n} + (1-\tau)\lambda^{k_n-1}(\overline{Q}_n Q_n^{k_n-1} z_n^{(k_n-1)^2})^{l_n}).$$

On a zero, conjugate the first equation and take its $(k_n - 1)$-th power. One obtains the system

$$\begin{pmatrix} (1-\tau)^{k_n-1}\overline{\lambda}^{k_n-1} & -\tau^{k_n-1} \\ \tau & (1-\tau)\lambda^{k_n-1} \end{pmatrix} \begin{pmatrix} \overline{z}_n'^{(k_n-1)l_n} \\ (Q_n z_n^{k_n-1})^{(k_n-1)l_n} \end{pmatrix} = 0.$$

The only zero of the deformed map is for $\lambda = 0$, $|x_i| = 1$, $i \neq n$, $|z_n|^2 + |z_n'|^2 = 1$. Furthermore, for $\tau = 1$, the map has no zeros, that is, F_n is trivial. One may also perform the equivariant deformation

$$((1-\tau)\lambda z_n'^{l_n} + \tau\lambda^{k_n}(\overline{Q}_n Q_n^{k_n-1} z_n^{(k_n-1)^2})^{l_n}, \\ -\tau z_n'^{l_n} + (1-\tau)\lambda^{k_n-1}(\overline{Q}_n Q_n^{k_n-1} z_n^{(k_n-1)^2})^{l_n})$$

which deforms F_n to

$$F_n' = \Big(1 - \prod |x_i|^2(|z_n|^2 + |z_n'|^2), X_0, (Q_i - 1)y_i, \\ \{(P_i - \epsilon_i)x_i^{l_i}\}_{i<n}, \lambda^{k_n} P_n^{(k_n-2)l_n} z_n^{l_n}, z_n'^{l_n}, \{x_i\}_{i>n}\Big).$$

This map is non-zero on the faces of C, by choosing ϵ_i appropriately, except for $\operatorname{Arg} z_n = 0$, on which it has degree $l_n k_n$. Hence,

$$0 = [F_n]_\Gamma = l_n k_n [\eta_n] + l_n \widetilde{d}_n [\widetilde{\eta}].$$

One may also rotate λ^{k_n} and obtain the components

$$\Big(1 - \prod |x_i|^2(|z_n|^2 + |z_n'|^2), -\lambda^{k_n} z_n'^{l_n}, P_n^{(k_n-2)l_n} z_n^{l_n}\Big).$$

Replace P_n by $P_n - \epsilon\tau\epsilon_n$, with $0 < \epsilon < 1$, $|\epsilon_n| = 1$. For $\tau = 1$, one has a map with zeros at $|x_i| = 1$, $i < n$, $\lambda = 0$, $z_n = 0$ and $|z_n'| = 1$ or $|z_n| = \epsilon$ and $|z_n'|^2 = 1 - \epsilon^2$. Divide B^H into the two invariant sets $B_1 = B^H \cap \{|z_n| < \epsilon/2\}$ and its complement B_2. One may compute the Γ-degree on each one: $\Sigma_0 \deg_\Gamma(F_n; B^H) = \Sigma_0^2[F_n]_\Gamma = \Sigma_0 \deg_\Gamma(F_n; B_1) + \Sigma_0 \deg_\Gamma(F_n; B_2)$, where Σ_0 is the suspension by a trivial variable (see [6] for these properties).

Now, $\deg_\Gamma(F_n; B_1) = \Sigma_0[F_n']_\Gamma$. On B_1, one may deform P_n to 0, rotate back λ^{k_n} and obtain the map with components $(\lambda^{k_n} z_n^{l_n}, z_n'^{l_n})$ which contributes $k_n l_n [\eta_n]_\Gamma$ to the Γ-degree.

On ∂B_2, one may deform linearly the first component to $(\epsilon^2 - \prod |x_i|^2(|P_n - \epsilon\epsilon_n|^2 + |z_n'|^2))$, replace $P_n - \epsilon\epsilon_n$ by $P_n - \epsilon_n$ and rotate back λ^{k_n} to get the map with components

$$\left(\epsilon^2 - \prod |x_i|^2(|P_n - \epsilon_n|^2 + |z_n'|^2), \lambda^{k_n}(P_n - \epsilon_n)^{(k_n-2)l_n} z_n^{l_n}, z_n'^{l_n}\right),$$

by choosing $\epsilon < 1/2$ so that any disc, with center at a point with $|z_n| = 1$, does not intersect ∂B_2. Deforming $|z_n'|$, in the first component, to 0 one has a map with zeros at $|x_i| = 1$, $\lambda = 0$, $z_n' = 0$, $|P_n - \epsilon_n|^2 = \epsilon^2$. Hence, $\deg_\Gamma(F_n; B_2)$ is the degree of this last map with respect to B^H, that is, the suspension of the class of the map

$$\widetilde{F}_n = \Big(\epsilon^2 - \prod |x_i|^2 |P_n - \epsilon_n|^2, X_0, (Q_i - 1)y_i, (P_i - \epsilon_i)x_i^{l_i}, \lambda^{k_n}(P_n - \epsilon_n)^{(k_n-2)l_n} z_n^{l_n}, z_n'^{l_n}\Big).$$

This map has no zeros on the faces of C, hence its class is a multiple of $[\widetilde{\eta}]_\Gamma$, which is given by its ordinary class with respect to C, where the set $\{Q_i - 1, P_i - \epsilon_i\}$ has just one zero. It is easy to see that this class is $k_n(k_n - 2)l_n^2$ times the Hopf map, in $\mathbb{Z}_2$. Thus, $[\widetilde{F}_n]_\Gamma = k_n l_n [\widetilde{\eta}]_\Gamma$ and $\Sigma_0^2[F_n]_\Gamma = \Sigma_0^2 k_n l_n([\eta_n]_\Gamma + [\widetilde{\eta}]_\Gamma) = 0$.

Now, from the equivariant suspension theorem ([11] and [6, Theorem B]), Σ_0 is an isomorphism if $\dim X_0 \geq 3$ (we shall study the suspension in the next section). Hence,

$$k_n l_n([\eta_n]_\Gamma + [\widetilde{\eta}]_\Gamma) = 0,$$

in particular, if k_n is odd (1 for example), then $\widetilde{d}_n \equiv k_n$ [2].

Construct a new fundamental cell C', corresponding to moving the variables x_j, x_j', and any other with isotropy H_j, at the end of the set $\{x_1, \ldots, x_n\}$. This will give new coefficients k_i', with $|\Gamma/H| = \prod k_i'$, and new polynomials P_i'. In particular, one may have $k_j' = 1$. Construct the map F_j' on the model of F_n above, but with P_i replaced by P_i' and P_n, z_n by P_j', z_j. It is clear that if $k_j' \geq 2$, then $[F_j']_\Gamma = 0$ and that F_j' is deformable to $(1 - \prod_{i \neq j} |x_i|^2(|z_j|^2 + |z_j'|^2), X_0, (Q_i' - 1)y_i, \{(P_i' - \epsilon_i)x_i^{l_i}\}_{i \neq j}, \lambda^{k_j'} P_j'^{(k_j'-2)l_j} z_j^{l_j}, z_j'^{l_j}, \{x_i\}_{i>n})$.

By choosing ϵ_i, one may assume that there are no zeros on the faces of $\mathcal{C}$ corresponding to $\operatorname{Arg} z_i = 0$, $i < j$, while $\deg(F'_j; B^H \cap \{\operatorname{Arg} z_j = 0\}) = l_j \prod k'_i = l_j k_j \prod_{i \neq j} k_i$. Thus, $0 = [F'_j]_\Gamma = l_j k_j [\eta_j]_\Gamma + \sum_{i>j} l_j d_{ji} [\eta_i]_\Gamma + l_j \widetilde{d}_j [\widetilde{\eta}]_\Gamma$, since we may assume that F'_j gives the j-th relation. Let

$$\eta'_j = \Big(1 - \prod_{i \neq j} |x_i|^2 |z_j|^2 |z'_j|^2, X_0, (Q'_i - 1)y_i,$$
$$\{P'_i - \epsilon_i) x_i^{l_i}\}_{i \neq j}, \lambda z_j^{l_j}, (\overline{z}_j z'_j - \epsilon_j |z_j|) z_j'^{l_j}, \{x_i\}_{i>n}\Big).$$

That is, η'_j is the last generator for the cell C' if $k'_j > 1$. Since one may choose ϵ_i such that η'_j is non-zero on the faces of $\mathcal{C}$ with $\operatorname{Arg} z_i = 0$, $i < j$, and

$$\deg(\eta'_j; \mathcal{C} \cap \{\operatorname{Arg} z_j = 0\}) = \prod_{i \neq j} k'_i \Big/ \prod_{i \neq j} k_i = k_j / k'_j,$$

it follows that k'_j divides k_j and $[\eta'_j]_\Gamma = k_j / k'_j [\eta_j]_\Gamma + \Sigma_{i>j} d'_{ji} [\eta_i]_\Gamma + d'_j [\widetilde{\eta}]_\Gamma$.

If $k'_j \geq 2$, following the deformations given for F_n, one will have $[F'_j]_\Gamma = 0 = l_j k'_j ([\eta'_j]_\Gamma + [\widetilde{\eta}']_\Gamma)$.

By using the fact that $\{Q'_i - 1, P'_i - \epsilon_i\}$ has just one zero in $\mathcal{C}$ and in $\mathcal{C}'$, one easily proves that $[\widetilde{\eta}']_\Gamma = [\widetilde{\eta}]_\Gamma$. From the above relations, one sees that $d_{ji} = d'_{ji} k'_j$ and $l_j \widetilde{d}_j \equiv l_j k'_j (d'_j + 1)$. Note that the first equalities could be changed by a multiple of k_i.

If $k'_j = 1$, then $[\eta'_j]_\Gamma$ is a multiple of $[\widetilde{\eta}']_\Gamma$. On C', η'_j is deformable to the suspension of $(1 - |z_j|^2 |z'_j|^2, \lambda z_j^{l_j}, (\overline{z}_j z'_j - \epsilon_j |z_j|) z_j'^{l_j})$. By deforming linearly $|z'_j|$ to 1 and $z_j'^{l_j}$ to $\overline{\epsilon}_j^{l_j}$ and then to 1, one may replace ϵ_j by 0 and get the map $(1 - |z_j|^2, \lambda z_j^{l_j}, \overline{z}_j z'_j)$. Rotating λ and $\overline{z}_j$, one obtains $(1 - |z_j|^2, \lambda z_j^{l_j - 1}, z'_j)$, i.e. $l_j - 1$ times the Hopf map. Hence, $[\eta'_j]_\Gamma = (l_j - 1)[\widetilde{\eta}']_\Gamma = (l_j - 1)[\widetilde{\eta}]_\Gamma = k_j [\eta_j]_\Gamma + \sum_{i>j} d'_{ji} [\eta_i]_\Gamma + d'_j [\widetilde{\eta}]_\Gamma$.

In order to compute d_{ji} one could continue with the permutations of the variables $x_1, \ldots, x_n$ and obtain different relations between the new generators. However, we shall not continue this process, except in particular cases, in Remark 8.4 and Theorem 8.3 below.

REMARK 8.4. We shall give an example of how to use the above ideas and to complete Remark 8.2. Suppose $\Gamma \cong \mathbb{Z}_9$ acts on (z_1, z_2) as $(e^{2\pi i/3} z_1, e^{2\pi i/9} z_2)$. Taking $\mathcal{C}$ as $\{|z_j| \leq 2,\ 0 \leq \operatorname{Arg} z_i < 2\pi/3\}$, we get $k_1 = k_2 = 3$, $\eta_1 = (1 - |z_1|^2 |z_2|^2, \lambda z_1, (z_1^2 z_2^3 - \epsilon_1) z_2)$, $\eta_2 = (1 - |z_1|^2 |z_2|^2, (z_1^3 - 1) z_1, \lambda z_2)$, where $\epsilon_1 = e^{i\varphi}$ is chosen such that $\varphi/2 + k\pi$ does not belong to $[0, 2\pi/3]$.

One has the relations $3\eta_1 + d_1 \eta_2 + \widetilde{d}_1 \widetilde{\eta} = 0$, $3\eta_2 + \widetilde{\eta} = 0$. On the other hand, one may choose $C' = \{|z_2| \leq 2,\ 0 \leq \operatorname{Arg} z_2 < 2\pi/9\}$ and $\eta' = (1 - |z_2|^2, z_1, \lambda z_2)$, with the relation $9\eta' + \widetilde{\eta} = 0$. It is easy to see that $\eta_1 = 2\eta' + d'_1 \widetilde{\eta}$, $\eta_2 = 3\eta' + d'_2 \widetilde{\eta}$

where, in fact, $d_2' = 0$. Using the above relations one also has $d_1 = 1$, $d_1' + \widetilde{d}_1 = 1$. Furthermore, it is easy to see that the homotopy class in $\Pi_5(S^4)$ of η_2, η' and $\widetilde{\eta}$ is the suspension of the Hopf map.

In order to compute the class of η_1 one may rotate λ and z_2 to get $(1 - |z_1 z_2|^2, \lambda z_1 z_2, (z_1 z_2)^2 z_2 - 1)$, deform linearly $(z_1 z_2)^2 z_2 - 1$ to $z_2 - (\overline{z}_1 \overline{z}_2)^2$, and then deform linearly $1 - |z_1 z_2|^2$ to $1 - |z_2|$, next $z_2 - (\overline{z}_1 \overline{z}_2)^2$ to $z_2^3 - \overline{z}_1^2$ and arrive at $(1 - |z_1| - |z_2|, \lambda z_1 z_2, z_2^3 - \overline{z}_1^2)$. Deform linearly the first component to $|\lambda| - 1$; then the class of η_1 will be the suspension of the Hopf map multiplied by the index of the map $(z_1, z_2) \to (z_1 z_2, z_2^3 - \overline{z}_1^2)$. In order to calculate this index, one may deform the first component to $z_1 z_2 - \epsilon |z_1|^2$, with ϵ small, then linearly to $z_2 - \epsilon \overline{z}_1$. The index of $(z_2 - \epsilon \overline{z}_1, z_2^3 - \overline{z}_1^2)$ can be computed by replacing z_2^3 by $((1-\tau) z_2 + \tau \epsilon \overline{z}_1)^3$ and finally deforming ϵ to 0; the index is -2, hence the class of η_1 in $\Pi_5(S^4)$ is 0. Thus, $d_1' = 0$ and $\widetilde{d}_1 = 1$. The relations are $3\eta_1 + \eta_2 + \widetilde{\eta} = 0$, $3\eta_2 + \widetilde{\eta} = 0$. Note that $\deg(\eta_1; \mathcal{C} \cap \{\operatorname{Arg} z_2 = 0\}) = 0$ (from the choice of ϵ_1) and that $\deg(\eta_1; B^H \cap \{\operatorname{Arg} z_2 = 0\}) = 2$, while $d_1 = 1$, i.e. these numbers have no relationship.

REMARK 8.5. Our definition of the generators η_j seems to be more complicated than necessary, since if a variable x_e has k_e equal to 1, this variable should not count. The reason for multiplying $x_e^{l_e}$ by $P_e - \epsilon_e$ is the factor l_e. If $l_e = 1$, one could define η_j', a new generator, by deleting x_j from the product $\prod |x_i|^2$ and not including it in P_i (as seen in Section 6) and leaving x_e as a suspension. In this case $[\eta_j']_\Gamma = [\eta_j]_\Gamma + \sum_{i>j} d_{ji}' [\eta_i]_\Gamma + d_j' [\widetilde{\eta}]_\Gamma$, that is, $\{\eta_j'\}$ can be chosen as generators in place of $\{\eta_j\}$.

This will be the case if $U = W$.

THEOREM 8.3. *Assume $V = \mathbb{R} \times W$ and $k_j = m_j = |\Gamma / H_j|$ for all j's. Then $[F]_\Gamma = \sum d_j [\eta_j]_\Gamma + \widetilde{d} [\widetilde{\eta}]_\Gamma$, with $d_j = \deg(F; B^H \cap \operatorname{Arg} z_j = 0) / \prod_{i \neq j} m_i$ and the relations for $\Pi(H)$ are $m_j([\eta_j]_\Gamma + [\widetilde{\eta}]_\Gamma) = 0$, $2[\widetilde{\eta}]_\Gamma = 0$.*

PROOF. The condition $k_j = m_j$ means that one has the same k_j's regardless of the order in the construction of $\mathcal{C}$. Hence, one may take $P_j = z_j^{m_j}$, d_j will be as stated by putting z_j as the first variable in the cell $\mathcal{C}$ and the relations will come by putting z_j as the last variable. The generators η_j are the same, independently of the order, and $\widetilde{\eta}_j'$, having no zeros in $\partial \mathcal{C}$, is a multiple of $\widetilde{\eta}$, multiple which is easily seen to be 1. □

We end this section by giving another description of $\Pi(H)$. Assume $V = \mathbb{R} \times W$ and $\Gamma / H \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_m}$, generated by $\gamma_1, \ldots, \gamma_m$. Let $X = \{Z_1, Z_1', \ldots, Z_m, Z_m'\}$ be a new space with the following action: $\gamma_j Z_j = e^{2\pi i / p_j} Z_j$, $\gamma_j Z_i = Z_i$, $i \neq j$, Z_j' the duplicate of Z_j. For any F in $\Pi(H)$ take the suspension $\Sigma_X F = (F, \mathrm{Id}) : V^H \times X^H \to W^H \times X^H$.

If one takes $C \times B_X^H$ as fundamental cell, then any Γ-map G from $V^H \times X^H$ into $W^H \times X^H$ which is non-zero on the ball in $(V \times X)^K$, $K > H$, is classified by the formula

$$[G]_\Gamma = \sum d_j [\Sigma_X \eta_j]_\Gamma + \tilde{d}[\Sigma_X \tilde{\eta}]_\Gamma$$

since the suspensions $\Sigma_X \eta_j$ and $\Sigma_X \tilde{\eta}$ are clearly the generators, by Remark 8.5, for the group $\Pi(H)$ corresponding to $V \times X$. This formula also proves that $\Sigma_X \Pi(H) \cong \Pi(H)$.

But one may also choose the cell C' given by $\{0 \leq \operatorname{Arg} Z_j < 2\pi/p_j\}$, which gives the generators

$$d\Sigma_V \eta_j' = \Big(1 - \prod |Z_j|^2, X_0, \{x_i\}, \{(Z_i^{p_i} - \epsilon_i) Z_i\}_{i \neq j}, \lambda^d Z_j\Big),$$

$$d\Sigma_V \tilde{\eta}' = \Big(\epsilon^2 - \prod_{i<m} |Z_i|^2 |Z_m^{p_m} - \epsilon_m|^2, X_0, x_i, \{(Z_i^{p_i} - \epsilon_i) Z_i\}_{i \neq j}, \lambda^d (Z_m^{p_m} - \epsilon_m) Z_m\Big).$$

Then $[G]_\Gamma = \sum d_j' [\Sigma_V \eta_j']_\Gamma + \tilde{d}' [\Sigma_V \tilde{\eta}']_\Gamma$, since, as in Remark 8.4, one may interchange Z_j and Z_m in the definition of $\tilde{\eta}'$. Furthermore, one has the relations $p_j([\Sigma_V \eta_j']_\Gamma + [\Sigma_V \tilde{\eta}']_\Gamma) = 0$, $2[\Sigma_V \tilde{\eta}']_\Gamma = 0$. Since Σ_V is also an isomorphism, we have proved the following

THEOREM 8.4. *If $V = \mathbb{R} \times W$ and $\Gamma/H \simeq \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_m}$, then any F in $\Pi(H)$ is given by $\Sigma_X [F]_\Gamma = \sum d_j' \Sigma_V [\eta_j']_\Gamma + d' \Sigma_V [\tilde{\eta}']_\Gamma$, with the relations $p_j([\eta_j']_\Gamma + [\tilde{\eta}']_\Gamma) = 0$, $2[\tilde{\eta}']_\Gamma = 0$.*

Note that in order to compute d_j' one has to perturb $\Sigma_X F$ so that it has no zeros on the edges of C', that is, for $Z_j = 0$.

One may give a better presentation of the above relations. In fact, let $[\eta_j]_\Gamma = [\eta_j']_\Gamma + [\tilde{\eta}']_\Gamma$, $j = 1, \ldots, m$, $[\eta_0]_\Gamma = [\tilde{\eta}']_\Gamma$; then $\Pi(H)$ is presented by $[\eta_j]_\Gamma$, $j = 0, \ldots, m$, with the relations $p_j [\eta_j]_\Gamma = 0$, $p_0 = 2$. Now, given p_i, p_j, let $p = (p_i : p_j) = \text{g.c.d.}(p_i, p_j)$; then there are k_i, k_j such that $p_i k_i + p_j k_j = p$.

Let $\xi_i = (p_i/p)\eta_i - (p_j/p)\eta_j$, $\xi_j = k_j \eta_i + k_i \eta_j$. Then $(p_i/p)\xi_j = \eta_j + k_j \xi_i$, $(p_j/p)\xi_j = \eta_i - k_i \xi_i$ with $p\xi_i = p_i \eta_i - p_j \eta_j = 0$, $(p_i p_j/p)\xi_j = (p_j k_j/p) p_i \eta_i + (p_j k_i/p) p_j \eta_j = 0$.

Without taking into account the relations, one may express, in the basis ξ_i, ξ_j, these equations in the form

$$\begin{pmatrix} p & 0 \\ 0 & p_i p_j/p \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ p_j k_j/p & p_i k_i/p \end{pmatrix} \begin{pmatrix} p_i & 0 \\ 0 & p_j \end{pmatrix} \begin{pmatrix} k_i & p_j/p \\ -k_j & p_i/p \end{pmatrix}$$

where the non-diagonal matrices have determinant equal to 1, i.e. they are invertible over $\mathbb{Z}$. Thus, one may replace η_i, η_j by ξ_i, ξ_j and p_i, p_j by $(p_i : p_j)$ and the least

common multiple of p_i, p_j. Note that p may be 1 and that if $p = \min(p_i, p_j)$, say $p = p_i$, then one may take $k_i = 1$, $k_j = 0$ and the change of variables does not change the relations.

Continuing this process, it is easy to see that one arrives at a new set of generators $\zeta_0, \dots, \zeta_m$ and relations $q_j \zeta_j = 0$ where $q_0 = (p_0 : p_1 : \cdots : p_m)$, $q_m = \text{l.c.m.}(p_0, \dots, p_m)$, q_j divides q_{j+1} and one has matrices M and N, invertible over $\mathbb{Z}$, such that $Q = MPN$, where $Q = \text{diag}(q_0, \dots, q_m)$, $P = \text{diag}(p_0, \dots, p_m)$; this is the content of the Fundamental Theorem for abelian groups [8, p. 57]. The integers $\{q_j\}$ are called the invariant factors of P. If $h_i(A)$ is the highest common factor of the principal $(i \times i)$-minors of a matrix A with integer entries, then one may prove that $h_i(P) = h_i(Q)$ if $Q = MPN$, with M and N invertible over $\mathbb{Z}$. Furthermore, since q_j divides q_{j+1}, we have $h_i(Q) = \prod_{j<i} q_j$, and $h_i(P)$ is the largest common factor of all possible products of i of the p_j's. Hence, $q_i = h_i(P)/h_{i-1}(P)$. Since the above results are true for arbitrary P and Q, they will also hold for the triangular matrix T of Theorem 8.2, where $h_i(T) = h_i(\text{diag}(2, k_1, \dots, k_n))$.

THEOREM 8.5. *If $V = \mathbb{R} \times W$ and $\Gamma/H \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_m}$, then $\Pi(H) \cong \mathbb{Z}_{q_0} \times \cdots \times \mathbb{Z}_{q_m}$, with $q_0 = (2 : p_i : \cdots : p_m)$, $q_m = \text{l.c.m}(2, p_1, \dots, p_m)$, $q_j = h_j/h_{j-1}$ where h_j is the largest common factor of all possible products of j of the p_i's or of j of the k_i's for any fundamental cell, with corresponding $k_i > 1$ and $k_0 = 2$. In particular, if any two p_i, p_j are relatively prime and odd (or any two k_i, k_j are relatively prime and odd) then $\Pi(H) \cong \mathbb{Z}_{2|\Gamma/H|}$.*

REMARK 8.6. *The case* k > 1. As we have seen in Theorem 5.3, $\Pi^{\Gamma}_{SV}(S^W) = \prod_H \widetilde{\Pi}(H)$ and one has to compute $\Pi(H)$. From our previous discussion it is clear that, for the case $k > 1$, one has to construct, on the boundary of the fundamental cell C, a series of equivariant maps which will modify an element F of $\Pi(H)$ until one gets a trivial element in this group. Thus, for the piece of ∂C given by $\text{Arg}\, x_{i_j} = 0$, $j = 1, \dots, k$, $x_j \neq 0$ for $j = l_i$, $i = 1, \dots, r$, such that $H_{l_1} \cap \cdots \cap H_{l_r} = H$, $l_i \neq i_j$ and the other variables being set to 0, one gets a ball B_{i_j, l_i}. If on its boundary one has the restriction of a Γ-equivariant map which is non-zero, one will need to see if this map has an $H_{i_1} \cap \cdots \cap H_{i_k}$-extension to the ball. Thus, one will have an obstruction in $\Pi^{H_{i_1} \cap \cdots \cap H_{i_k}}_{S_{i_j, l_i}}(S^{W^H})$.

This group has again a decomposition into subgroups $\Pi_{H_{i_1} \cap \cdots \cap H_{i_k}}(K)$ for K's such that $H < K < H_{i_1} \cap \cdots \cap H_{i_k}$. Since the map F is extendable to B^K for $H < K$, the only surviving element will be in $\Pi_{H_{i_1} \cap \cdots \cap H_{i_k}}(H)$.

In order to compute this group, one will have to consider its own fundamental cell and look for obstructions to K-equivariant extensions, i.e. elements in $\Pi^K_{S^K}(S^{W^H})$, where S^K is a sphere in $I \times V^H \cap \{\text{Arg}\, x_{i_j} = 0,\ H_{i_1} \cap \cdots \cap H_{i_l} = K\}$. The process will end when $K = H$ with obstructions in $\Pi^H_{S^n}(S^{W^H})$, i.e. an ordinary

homotopy element, where n is the dimension of the sphere. Since there are at least $\dim \Gamma/H$ elements x_{i_j} needed in order to achieve the equality $K = H$, it follows that $n \leq \dim V^H - \dim \Gamma/H$.

From each one of these elementary obstructions one constructs, from the respective fundamental cells, a Γ-equivariant map F_k such that $[F] = \sum [F_k]$. Of course, one needs to follow a strict order in the construction, as seen in this section and in Section 3, so that the inclusion of a map F_k does not affect the preceding maps. Clearly, this process depends on the choice of the fundamental cells and on the preceding modifications, giving rise to relations between the generators.

9. The Infinite Dimensional Degree

In the infinite dimensional case we shall consider spaces $E = \mathbb{R}^k \times U \times \widetilde{E}$, $F = W \times \widetilde{E}$, where U and W are finite dimensional Γ-spaces, with $\dim U^H = \dim W^H$, where H is any isotropy subgroup for U, and $\widetilde{E}$ is an infinite dimensional Γ-space. We shall look at maps $f(\lambda, x, y) = (f_1(\lambda, x, y), y - f_2(\lambda, x, y))$, where f_2 is compact. Since f_2 can be approximated by finite dimensional Γ-maps, in this case the degree is an element of $\Pi^\Gamma_{S^E}(S^F)$, the inductive stable limit of $\Pi^\Gamma_{S^{V\times\overline{V}}}(S^{W\times\overline{V}})$, with $\overline{V}$ any finite dimensional invariant subspace of $\widetilde{E}$.

The existence of this stable limit requires that the suspension by **any** equivariant representation V_0 contained in $\widetilde{E}$ generates a one-to-one map Σ_{V_0} from $\Pi^\Gamma_{S^{V\times\overline{V}}}(S^{W\times\overline{V}})$ into $\Pi^\Gamma_{S^{V\times\overline{V}\times V_0}}(S^{W\times\overline{V}\times V_0})$ if $\overline{V}$ is large enough. Although there are general results on the equivariant suspension (see [11]), these are too restrictive for our problem (in particular they require that V_0 is already contained as a representation of $V \times \overline{V}$, which is in general not the case in applications).

Let then V_0 be an irreducible representation of Γ, generated by a real or complex variable x with isotropy subgroup H_0 (hence Γ/H_0 is trivial or $\mathbb{Z}_2$ in the first case, $\mathbb{Z}_m, m \geq 3$, or S^1 in the second case). From Theorem 5.3, $\Pi^\Gamma_{S^V}(S^W) = \prod_H \widetilde{\Pi}(H)$ and $\Pi^\Gamma_{S^{V\times V_0}}(S^{W\times V_0}) = \prod_{H'} \widetilde{\Pi}(H')$, where H is an isotropy subgroup for V while H' is an isotropy subgroup for $V \times V_0$. The group H' will be of the form H, with H an isotropy subgroup for V, or $H \cap H_0$. Thus, if H_0 is not an isotropy subgroup for V, there will be more isotropy types for $V \times V_0$ (at least H_0) and the equivariant group for $V \times V_0$ will have more components (unless trivial).

Let $H' = H \cap H_0$; then, if $X \in W^{H'}$, one has $H' < \Gamma_X = \bigcap H_j$, j such that the component x_j of X is non-zero. Hence, $H' < \overline{H} = \bigcap H_j$, j for all possible variables in $W^{H'}$. Thus, $W^{\overline{H}} \subset W^{H'}$. But, from the definition, we have $W^{H'} \subset W^{\overline{H}}$. Hence, $W^{\overline{H}} = W^{H'}$. Furthermore, $W^H \subset W^{\overline{H}}$ and $\overline{H} < H$, hence $\overline{H} \cap H_0 < H'$. Moreover, $H' < \overline{H}$ implies that $H' < \overline{H} \cap H_0$, thus, $H' = \overline{H} \cap H_0$ with $W^{H'} = W^{\overline{H}}$. This

implies that, if F belongs to $\Pi(H')$, where H' is not an isotropy type for V, then F maps $(B^{V\times V_0})^{K'}$ into $(W\times V_0)^{K'}\backslash\{0\}$ for all $K'>H'$, in particular for $K'=\overline{H}$ which is not a subgroup of H_0. Then $(V\times V_0)^{\overline{H}}=V^{\overline{H}}$ and $F^{\overline{H}}=F|_{x=0}\neq 0$. That is, F cannot come from a suspension of a non-trivial element.

On the other hand, if $H'=H$, then $\Pi(H')$ consists of maps from $(V\times V_0)^H$ into $(W\times V_0)^H$ which map $(V\times V_0)^{K'}$ into $(W\times V_0)^{K'}\backslash\{0\}$ for all $K'>H$, with K' equal to K or to $K\cap H_0$, i.e. for $K>H$. Thus, if H is not a subgroup of H_0, then $(V\times V_0)^H=V^H$ and $(V\times V_0)^{K'}=V^K$ (there are no K' of the form $K\cap H_0>H$ in this case), and any element of $\Pi(H')$ is an element of $\Pi(H)$. If $H<H_0$, then $(V\times V_0)^{K'}=V^{K'}\times V_0=V^{\overline{K}}\times V_0$ if $H<K'<H_0$, $\overline{K}\cap H_0=K'$ with $V^{\overline{K}}=V^{K'}$ if K' is not an isotropy subgroup for V, while $(V\times V_0)^K=V^K$ if K is not a subgroup of H_0. Thus, if F belongs to $\Pi(H)$, $(F,x)^{K'}$ will be $(F^{\overline{K}},x)$ in the first case, or (F^K,x) if $H<K'=K\cap H_0$, or F^K if K is not a subgroup of H_0, that is, in all cases different from zero. That is, $\Sigma_{V_0}F$ belongs to $\Pi(H')$ for $H'=H$.

Hence, if $[F]_\Gamma$ belongs to $\Pi^\Gamma_{SV}(S^W)$, then $[F]_\Gamma=\sum[F_H]_\Gamma$, where F_H belongs to $\Pi(H)$, $\Sigma_{V_0}F_H$ belongs to $\Pi(H')$ with $H'=H$, Σ_{V_0} is an isomorphism if H is not a subgroup of H_0. If $H<H_0$, then F_H is given by a sum of obstruction classes, coming from $\Pi^H_{Sn}(S^{W^H})$, for $n\leq\dim V^H-\dim\Gamma/H$, as seen in Remark 8.6. In this case x will remain as a dummy variable at each stage of the fundamental cells, that is, the obstruction classes for $\Pi(H')$ will be in $\Pi^H_{Sn\times V_0}(S^{W^H\times V_0})$. In particular, for (F_H,x) the obstruction classes will be the suspension of the classes for F_H. From the suspension theorem, one will have an isomorphism if $n\leq 2\dim W^H-2$, for any $n\leq\dim V^H-\dim\Gamma/H$. In this case Σ_{V_0} will be one-to-one from $\Pi(H)$ into $\Pi(H')$ and for any element G in $\Pi(H')$, the obstruction classes will be suspensions by x, that is, $[G]_\Gamma=\sum[F_k,x]_\Gamma=[\sum F_k,x]_\Gamma$, therefore Σ_{V_0} is onto.

We have proved the first part of the following result

THEOREM 9.1. (a) *If* $\dim W^H\geq k+2-\dim\Gamma/H$ *for all isotropy subgroups* H *for* V *with* $H<H_0$, *then* Σ_{V_0} *is one-to-one.* Σ_{V_0} *is also onto if* H_0 *is an isotropy subgroup for* V.

(b) *If* H_0 *is not an isotropy subgroup for* V, *then* Σ_{V_0} *will be onto only if* $k=0$ *and* $\Gamma/H_0\simeq S^1$.

PROOF. In order to prove (b) one has to show that $\Pi(H')=0$ for all $H'=H\cap H_0$ which are not isotropy subgroups for V. Then Σ_{V_0} will be onto (from the ordinary suspension theorem) from $\Pi(K)$ onto $\Pi(K')$, with $K'=K<H_0$, provided $\dim W^K\geq k+1-\dim\Gamma/K$. If $k=0$, this condition will always be met.

Now, for $H'=H\cap H_0=\overline{H}\cap H_0$, with $V^{\overline{H}}=V^{H'}$, $\Pi(H')$ will vanish, from Theorem 3.1, provided $\dim(V\times V_0)^{H'}-\dim\Gamma/H'<\dim(W\times V_0)^{H'}$, i.e. if $\dim W^{\overline{H}}+$

$k+\dim V_0-\dim \Gamma/H' < \dim W^{\overline{H}}+\dim V_0$, or else if $k < \dim \Gamma/H'$. This inequality has to be true in particular for $H' = H_0$, where $\dim \Gamma/H_0$ is 0 or 1. Hence, $k = 0$ and $\Gamma/H_0 \simeq S^1$. Then $\dim \Gamma/H' > 0$ for any $H' = H \cap H_0$. □

REMARK 9.1. The classical equivariant suspension theorem gives the following conditions for Σ_{V_0} to be onto:

1) $\dim W^H \geq k+1$ for any isotropy subgroup H for V.

2) $\dim W^{K'} - \dim W^H \geq k+1$ for any isotropy subgroup H for V and isotropy subgroup K' for $V \times V_0$, with $K' < H_0$, $K' < H$ but H not a subgroup of H_0.

For an isomorphism one needs $k+2$ instead of $k+1$. The argument is by proving that the suspension from $\Pi^H_{S^n}(S^W)$ into $\Pi^H_{S^{n\times V_0}}(S^{W\times V_0})$ is an onto or one-to-one map. Now, this last group is the product of $\Pi(K')$, with K' an isotropy subgroup for $\mathbb{R}^{n+1} \times V_0$ and $K' < H$, hence, K' is H itself or $H \cap H_0$ if H is not a subgroup of H_0. If $H < H_0$, then this suspension is the ordinary suspension and (1) will suffice. If H is not a subgroup of H_0, then $\Pi(H \cap H_0)$ will be zero if, by Theorem 3.1, $\dim(B^{n+1} \times V_0)^{H\cap H_0} - \dim H/H\cap H_0 < \dim(I \times W \times V_0)^{H\cap H_0}$, i.e. if $n - \dim H/H \cap H_0 < \dim W^{H\cap H_0}$ for all $n \leq \dim V^H - \dim \Gamma/H$.

One obtains the condition $\dim W^{H\cap H_0} - \dim W^H \geq k+1-\dim \Gamma/H\cap H_0$, which improves (2). Since this condition has to be true also for $\overline{H} \cap H_0$ if H_0 is not an isotropy subgroup for V, it follows that $W^{\overline{H}\cap H_0} = W^{H\cap H_0} = W^{\overline{H}}$ and one recovers the condition $k < \dim \Gamma/H'$.

REMARK 9.2. Given the explicit generators for the subgroups $\Pi(H)$ if $\dim \Gamma/H = k$ or if $k = 0$ or 1, it is apparent that Σ_{V_0} is one-to-one for any H_0 provided $\dim W^\Gamma \geq k+2$ if $H_0 = \Gamma$ and $\dim W^H \geq 2$ (always true if W^H contains a complex variable) for $\Pi(H)$ if $k = \dim \Gamma/H$.

Our final result is the following

THEOREM 9.2. *If $k = \dim \Gamma/H$ respectively, if $k = 0$ or 1, then any element of $\Pi(H)$ respectively, of $\Pi^\Gamma_{SV}(S^W)$, is the Γ-degree of a map from Ω into W provided $\Omega^H \neq \emptyset$.*

PROOF. Given $f : \Omega \to W$, $\partial\Omega \to W\backslash\{0\}$, recall that $\deg_\Gamma(f;\Omega)$ is the class of $(2t-1+2\varphi(\lambda,X), \widetilde{f}(\lambda,X))$ in $\Pi^\Gamma_{SV}(S^W)$. Since $\Omega^H \neq \emptyset$ and Ω^H is open in V^H, there is an $X^0 = (\lambda^0, X^0_0, y^0_i, z^0_j)$ with y^0_i and z^0_j different from 0 in Ω^H. By changing variables we shall assume that $\lambda^0 = X^0_0 = 0$.

a) If $\dim \Gamma/H = k > 0$ and $\dim W^\Gamma \geq 1$, let $X_0 = (x_0, \widetilde{X}_0)$ be a decomposition of U^Γ. Define $x'_j = x_j/|x^0_j|$, $x'_0 = x_0/R$ where R is the radius of a large ball

containing Ω. Let

$$f(\lambda, X) = \Big(x_0' - 2\Big(\prod |x_j'|^2 - 1\Big), \widetilde{X}_0, (\lambda_1 + i(|z_2'|^2 - 1))z_1^{l_1}, \dots,$$
$$(\lambda_{k-1} + i(|z_k'|^2 - 1))z_{k-1}^{l_{k-1}}, (\lambda_k + ix_0')^d z_k^{l_k}, (P_j(x_1', \dots, x_j') - 1)x_j^{l_j},$$
$$(P_j(y_1', \dots, y_j') - 1)y_j, (Q_j(y_j') - 1)y_j, x_s^{l_s}\Big),$$

where $P_j(y_1', \dots, y_j')$ is the invariant polynomial based on the real variables for $k_j = 1$ and $Q_j(y_j') = y_j'$ is used for the case $k_j = 2$.

Since $|x_0'| \leq 1$ in Ω^H, the zeros of f on Ω^H have $x_j \neq 0$ for all j's and for $|x_j'| = 1$, as in Theorem 7.1. For z_j in $\mathbb{R}^+$, $j = 1, \dots, k$, there are $\prod k_j$ zeros, equal to γX^0, for some γ in Γ. For the map $(2t - 1 + 2\varphi(\lambda, X), f(\lambda, X))$ one may deform φ to 0 on $\partial(I \times B^H)$ and rotate $2t - 1$ and x_0' to obtain $(-x_0', F_d)$, where F_d is the generator of Theorem 7.1. Hence, one has the suspension of F_d, a map of Γ-degree equal to d.

If $\dim W^\Gamma = 0$, then $f(\lambda, X_0, 0) = 0$ and one needs that $\overline{\Omega}^\Gamma = \emptyset$ in order to define the Γ-degree of f. As before, let $(0, x_j^0)$ be a point in Ω^H with $x_j^0 \neq 0$ and define $\lambda_k' = \lambda_k / R$. Let

$$f(\lambda, X) = \Big((\lambda_1 + i(|z_2'|^2 - 1))z_1^{l_1}, \dots, (\lambda_{k-1} + i(|z_k'|^2 - 1))z_{k-1}^{l_{k-1}},$$
$$\Big(\lambda_k' + 2\sum(|x_j'|^2 - 1)^2 + i(|z_1'|^2 - 1)\Big)z_k^{l_k}, (|z_k'|P_j(x_1', \dots, x_j') - 1)x_j^{l_j},$$
$$(|z_k'|Q_j(y_j') - 1)y_j, (|z_k'|P_j(y_1', \dots, y_j') - 1)y_j, x_s^{l_s}\Big).$$

If $f(\lambda, X) = 0$ and $z_k = 0$, then $x_j' = 0$ for all j's and $(\lambda, 0)$ is in $\Omega^\Gamma = \emptyset$. Thus, $z_k \neq 0, |z_1'| = 1 = \dots = |z_k'|$ and, if $x_j' = 0$, then $\lambda_k' + 2\sum(|x_j'|^2 - 1)^2 \geq \lambda_k' + 2 > 0$. Hence, the zeros are for $|x_j'| = 1$, $X = \gamma X^0$. For the map $(2t - 1 + 2\varphi(\lambda, X), f(\lambda, X))$ one may easily check that its usual degree on the fundamental cell C is d.

b) If $k = 0$ and one has at least one complex x_j, then, as in Theorem 7.1, one replaces P_j by P_j^d in the preceding maps and, if all variables in Ω^H are real, then one uses the maps of Theorem 7.1 in order to get the result.

c) If $k = 1$ and $|\Gamma/H| < \infty$ and $\dim W^\Gamma \geq 1$, one takes the generators η_j and $\widetilde{\eta}$ of Theorems 8.1 and 8.2, replacing $2t - 1$ by x_0' and x_j by x_j', as we have done above.

If $\dim W^\Gamma = 0$, take the map

$$\Big((|z_j'|Q_i(y_i') - 1)y_i, (|z_j'|P_i(y_1', \dots, y_i') - 1)y_i, (|z_j'|P_i(x_1', \dots, x_i') - 1)x_i^{l_i},$$
$$\Big(\mu' + 2\sum(|x_i'|^2 - 1)^2 + i(|z_j'|^2 - 1)\Big)^d z_j^{l_j}, x_s^{l_s}\Big).$$

The zeros in Ω^H are for $|z_j'| = 1$, $|x_i'| = 1$ and it is easy to see that its Γ-degree is $d\eta_j$, as in Theorem 8.2.

For a map having the Γ-degree of $d\widetilde{\eta}$, consider

$$f(\lambda, X) = \left(\left(\epsilon^{-1}|x_n'|\left|\prod|x_i'|P_n' - \epsilon_n\right|P_i' - \epsilon_i\right)x_i^{l_i},\right.$$
$$\left.\left(\mu + i\left(\epsilon^2 - \left|\prod|x_i'|P_n' - \epsilon_n\right|^2|x_n'|^2\right)\left(\prod|x_i'|P_n' - \epsilon_n\right)x_n^{l_n}\right)\right),$$

where P_i' stands for $P_i(y_1', \dots, y_i')$, $Q_i(y_i')$ or $P_i(x_1', \dots x_i')$ according to the different cases. Again the zeros of f in Ω^H are for $|x_i'| = 1$, $x_i = \gamma x_i^0$, $i < n$, $|x_n - \gamma x_n^0| = \epsilon$ and one has to compute the class of $(2t - 1 + 2\varphi(\lambda, X), f(\lambda, X))$ on $\partial\mathcal{C}$. There one may deform $x_j^{l_j}$ to $x_j^{0l_j}$ and then to 1 and look at $(2t - 1 + 2\varphi, \epsilon^{-1}|P_n' - \epsilon_n|P_i' - \epsilon_i, (\mu + i(\epsilon^2 - |P_n' - \epsilon_n|^2))^d(P_n' - \epsilon_n))$. One may deform linearly the first component to $2t - 1 + 2(|P_n' - \epsilon_n|^2 - \epsilon^2)$, replace $2t - 1$ by $\tau(2t - 1)$ in this component and $i(\epsilon^2 - |P_n' - \epsilon_n|^2)$ by $\tau(\epsilon^2 - |P_n' - \epsilon_n|^2) + (1 - \tau^2)(2t - 1)/2$, arriving at $(|P_n' - \epsilon_n|^2 - \epsilon^2, \epsilon^{-1}|P_n' - \epsilon_n|P_i' - \epsilon_i, (\mu + i(2t - 1)/2)^d(P_n' - \epsilon_n))$. One may clearly replace $\epsilon^{-1}|P_n' - \epsilon_n|$ by 1 and get the map $d\widetilde{\eta}$. □

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