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IMAGE RECOVERY BY CONVEX COMBINATIONS OF SUNNY NONEXPANSIVE RETRACTIONS

Shinya Kitahara — Wataru Takahashi

(*Submitted by Ky Fan*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

Let H be a Hilbert space, let $C_1, C_2, \dots, C_r$ be nonempty closed convex subsets of H and let I be the identity operator on H. Then the problem of image recovery in a Hilbert space setting may be stated as follows: The original (unknown) image z is known a priori to belong to the intersection C_0 of r well-defined sets $C_1, C_2, \dots, C_r$ in a Hilbert space; given only the metric projections P_i of H onto C_i $(i = 1, 2, \dots, r)$, recover z by an iterative scheme.

Recently, using the weak convergence theorem by Opial [7], Crombez [4] proved the following: Let $T = \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with $T_i = I + \lambda_i(P_i - I)$ for all $i, 0 < \lambda_i < 2, \alpha_i > 0$ for $i = 0, 1, 2, \dots, r$, $\sum_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \bigcap_{i=1}^{r} C_i$ is nonempty. Then starting from an arbitrary element x of H, the sequence $\{T^n x\}$ converges weakly to an element of C_0.

Let C be a nonempty closed convex subset of H. Then a mapping P on H is the metric projection onto C if and only if, for every $x \in H$ and $y \in C$, $(x - Px, Px - y) \geq 0$. So, the metric projection of H onto C is a sunny nonexpansive retraction, that is,

$$P(Px + t(x - Px)) = Px \qquad \text{for every } x \in C \text{ and } t \geq 0$$

and

$$\|Px - Py\| \leq \|x - y\| \qquad \text{for every } x, y \in H.$$

In this paper, we deal with the problem of image recovery in a Banach space setting. We first prove (Theorem 4) that an operator given by a convex combination of sunny nonexpansive retractions in a uniformly convex Banach space is nonexpansive and asymptotically regular. Further, we prove (Theorem 5) that the set of fixed points of the operator is equal to the intersection of the ranges of sunny nonexpansive retractions. Using Theorems 4 and 5, we prove the weak convergence theorem (Theorem 6) for the operator. Finally (in Section 4), using Theorem 6, we consider the problem of finding a common fixed point for a finite commuting family of nonexpansive mappings in a Banach space.

2. Preliminaries

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then a mapping T of C into E is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let T be a mapping of C into E. Then we denote by $F(T)$ the set of fixed points of T and by $R(T)$ the range of T. A mapping T of C into E is said to be *asymptotically regular* if, for every $x \in C$, $T^n x - T^{n+1} x$ converges to 0. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. If a mapping P of C into C is a retraction, then $Pz = z$ for every $z \in R(P)$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D.

Let E be a Banach space. Then, for every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be *uniformly convex* if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. E is also said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and

$x \neq y$. A uniformly convex Banach space is strictly convex. We can also prove the following lemma.

LEMMA 1. *Let E be a uniformly convex Banach space. Then for given $\varepsilon > 0$, $R \geq d > 0$ and $\lambda \in [0,1]$, the inequalities $\|x\| \leq d$, $\|y\| \leq d$, $\|x - y\| \geq \varepsilon$ imply $\delta(\varepsilon/R) > 0$ and $\|\lambda x + (1-\lambda)y\| \leq d\{1 - 2\lambda(1-\lambda)\delta(\varepsilon/R)\}$.*

A closed convex subset C of a Banach space E is said to have *normal structure* if for each closed bounded convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. We also know the following.

THEOREM 1 [5]. *Let E be a reflexive Banach space and let C be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into C. Then $F(T)$ is nonempty.*

Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals f on E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be *Gateaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all x and y in U. It is said to be *Fréchet differentiable* if for each x in U, this limit is attained uniformly for y in U. It is also said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit is attained uniformly for x, y in U. The following theorem is proved by Bruck [3]; see also [6].

THEOREM 2 [3]. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into C. Then the set*

$$\bigcap_{m=1}^{\infty} \overline{\text{co}}\{T^n x : n \geq m\} \cap F(T)$$

consists of at most one point.

Using Theorem 2, we can prove the following; see [3] or [6, 11].

THEOREM 3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let T be an* *asymptotically regular nonexpansive mapping of C into C with $F(T) \neq \emptyset$. Then, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.*

PROOF. Let $x \in C$ and take $y \in F(T)$. If B is the closed ball of center y and radius $\|x-y\|$, then $K = C \cap B$ is bounded and invariant under T. So, by [2], $I-T$ is demiclosed on K, i.e., if $\{x_n\}$ converges weakly to z and $\{x_n - Tx_n\}$ converges strongly to 0, then $(I-T)z = 0$. Since $\{T^n x\}$ is bounded and E is uniformly convex, there exists a subsequence $\{T^{n_i} x\}$ converging weakly to an element z of K. Since T is asymptotically regular, $\{(I-T)T^{n_i}x\}$ converges strongly to 0. By demiclosedness of T, we have $z = Tz$. Since $\{z\} = \bigcap_{m=1}^{\infty} \overline{\text{co}}\{T^n x : n \geq m\} \cap F(T)$ from Theorem 2, $\{T^n x\}$ converges weakly to z.

3. Weak convergence theorem

Throughout this paper, we denote by I the identity operator. Our weak convergence theorem (Theorem 6) will be based on the following general theorem. The proof is due to an idea of Crombez [4].

THEOREM 4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let S be an operator on C given by $S = \beta_0 I + \sum_{i=1}^{r} \beta_i S_i$, $0 < \beta_i < 1$ for $i = 0, 1, \ldots, r$, $\sum_{i=0}^{r} \beta_i = 1$, such that each S_i is nonexpansive on C and $\bigcap_{i=1}^{r} F(S_i)$ is nonempty. Then S is asymptotically regular.*

PROOF. Let $x \in C$ and take $u \in F(S)$. We may assume $x \neq u$. Putting $x_n = S^n x$ for every $n = 0, 1, 2, 3, \ldots$, the sequence $\{\|x_n - u\|\}$ is nonincreasing since S is nonexpansive. From

$$\begin{aligned} x_{n+1} - u &= Sx_n - u \\ &= \beta_0(x_n - u) + (1-\beta_0)\sum_{i=1}^{r} \frac{\beta_i}{1-\beta_0}(S_i x_n - u), \end{aligned}$$

or

$$x_{n+1} - u = \beta_0(x_n - u) + (1-\beta_0) z_n$$

with

$$z_n = \sum_{i=1}^{r} \frac{\beta_i}{1-\beta_0}(S_i x_n - u),$$

we have

$$\|z_n\| = \left\| \sum_{i=1}^{r} \frac{\beta_i}{1-\beta_0}(S_i x_n - u) \right\| \le \sum_{i=1}^{r} \frac{\beta_i}{1-\beta_0} \|S_i x_n - u\|$$
$$\le \sum_{i=1}^{r} \frac{\beta_i}{1-\beta_0} \|x_n - u\| = \|x_n - u\|$$

and hence $\|z_n\| \le \|x_n - u\|$.

We now show that the sequence $\{x_n - u - z_n\}$ has a subsequence converging to 0. If not, there exist $\varepsilon > 0$ and a positive integer n_0 such that $\|x_n - u - z_n\| \ge \varepsilon$ for all $n \ge n_0$. Then applying Lemma 1, we obtain

$$\begin{aligned}\|x_{n+1} - u\| &= \|\beta_0(x_n - u) + (1-\beta_0)z_n\| \\ &\le \|x_n - u\| \left\{ 1 - 2\beta_0(1-\beta_0)\delta\left(\frac{\varepsilon}{\|x-u\|} \right) \right\} \\ &\vdots \\ &\le \|x_{n_0} - u\| \left\{ 1 - 2\beta_0(1-\beta_0)\delta\left(\frac{\varepsilon}{\|x-u\|} \right) \right\}^{n+1-n_0}.\end{aligned}$$

Since $0 < \beta_0 < 1$, we have $\lim_{n\to\infty} \|x_n - u\| = 0 = \lim_{n\to\infty} \|z_n\|$. So, $\{x_n - u - z_n\}$ converges to 0. This is a contradiction. Thus, some subsequence $\{x_{n_i} - u - z_{n_i}\}$ converges to 0. Since $\{\|x_n - x_{n+1}\|\}$ is nonincreasing and

$$\begin{aligned}x_n - x_{n+1} &= x_n - \{u + \beta_0(x_n - u) + (1-\beta_0)z_n\} \\ &= (1-\beta_0)(x_n - u - z_n),\end{aligned}$$

we see that $x_n - x_{n+1}$ converges to 0. It now follows from $x_n - x_{n+1} = S^n x - S^{n+1} x$ that S is asymptotically regular.

The following lemma is crucial in the proof of Theorem 5.

LEMMA 2. *Let E be a strictly convex Banach space and let C be a nonempty closed convex subset of E. Let P be a sunny nonexpansive retraction of C onto D with $D \subset C$ and let $x \in C$. If $\|Px - y\| = \|x - y\|$ for some $y \in D$, then $Px = x$.*

PROOF. We may assume $x \ne y$. Since P is sunny and $\frac{1}{2}x + \frac{1}{2}Px \in C$,

$$P\left(\frac{1}{2}x + \frac{1}{2}Px\right) = P\left(Px + \frac{1}{2}(x - Px)\right) = Px.$$

So, we have

$$\begin{aligned}\|Px - y\| &= \left\|P\left(\frac{1}{2}x + \frac{1}{2}Px\right) - Py\right\| \\ &\leq \left\|\frac{1}{2}x + \frac{1}{2}Px - y\right\| \\ &= \left\|\frac{1}{2}(x - y) + \frac{1}{2}(Px - y)\right\| \\ &\leq \frac{1}{2}\|x - y\| + \frac{1}{2}\|Px - y\| \\ &\leq \|x - y\|,\end{aligned}$$

and hence $\|Px-y\| = \|\frac{1}{2}(x-y)+\frac{1}{2}(Px-y)\| = \|x-y\|$. On the other hand, assume $Px \neq x$. Then $Px - y \neq x - y$. Since E is strictly convex and $\|Px - y\| = \|x - y\|$, we have $\|\frac{1}{2}(Px - y) + \frac{1}{2}(x - y)\| < \|x - y\|$. This is a contradiction. Therefore, $Px = x$.

THEOREM 5. *Let E be a strictly convex Banach space and let C be a nonempty closed convex subset of E. Let $C_1, C_2, \dots, C_r$ be sunny nonexpansive retracts of C such that $\bigcap_{i=1}^r C_i$ is nonempty. Let T be an operator on C given by $T = \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, 2, \dots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1-\lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto C_i. Then $F(T) = \bigcap_{i=1}^r C_i$.*

PROOF. It is obvious that $\bigcap_{i=1}^r C_i = \bigcap_{i=1}^r F(T_i)$. So, it is sufficient to show $F(T) \subset \bigcap_{i=1}^r C_i$. Let $x \in F(T)$. Then, for any $y \in \bigcap_{i=1}^r C_i$, we have

$$\begin{aligned}\|x - y\| &= \|Tx - Ty\| \\ &= \left\|\sum_{i=1}^r \alpha_i T_i x - \sum_{i=1}^r \alpha_i T_i y\right\| = \left\|\sum_{i=1}^r \alpha_i (T_i x - T_i y)\right\| \\ &\leq \sum_{i=1}^r \alpha_i \|T_i x - T_i y\| \leq \sum_{i=1}^r \alpha_i \|x - y\| = \|x - y\|\end{aligned}$$

and hence $\|T_i x - T_i y\| = \|x - y\|$, $i = 1, 2, \dots, r$. So, for each i,

$$\begin{aligned}\|x - y\| = \|T_i x - T_i y\| &= \|(1 - \lambda_i)(x - y) + \lambda_i(P_i x - P_i y)\| \\ &\leq (1 - \lambda_i)\|x - y\| + \lambda_i\|P_i x - P_i y\| \\ &\leq (1 - \lambda_i)\|x - y\| + \lambda_i\|x - y\| = \|x - y\|\end{aligned}$$

and hence $\|x - y\| = \|P_i x - P_i y\| = \|P_i x - y\|$. So, it follows from Lemma 2 that $P_i x = x$. This implies $x \in C_i$. Therefore, $x \in \bigcap_{i=1}^r C_i$.

Now we prove a weak convergence theorem for nonexpansive mappings given by convex combinations of retractions.

THEOREM 6. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $C_1, C_2, \ldots, C_r$ be sunny nonexpansive retracts of C such that $\bigcap_{i=1}^r C_i$ is nonempty. Let T be an operator on C given by $T = \sum_{i=1}^r \alpha_i T_i$, $0 < \alpha_i < 1$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $T_i = (1 - \lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto C_i. Then $F(T) = \bigcap_{i=1}^r C_i$ and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.*

PROOF. Since E is uniformly convex, E is strictly convex. So, we have $F(T) = \bigcap_{i=1}^r F(T_i) = \bigcap_{i=1}^r C_i$ by Theorem 5. Let $T_i = (1-\lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$ for each $i = 1, 2, \ldots, r$, where P_i is a sunny nonexpansive retraction of C onto C_i, and let $T = \sum_{i=1}^r \alpha_i T_i$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^r \alpha_i = 1$. Choose β_1 such that $\alpha_1\lambda_1 < \beta_1 < \alpha_1$ and set $\mu_1 = \alpha_1\lambda_1/\beta_1$. Then $0 < \beta_1 < 1$, $0 < \mu_1 < 1$ and

$$\begin{aligned}
\alpha_1 T_1 &= \alpha_1\{(1-\lambda_1)I + \lambda_1 P_1\} \\
&= \alpha_1 I - \alpha_1\lambda_1 I + \alpha_1\lambda_1 P_1 \\
&= (\alpha_1 - \beta_1)I + \beta_1 I - \mu_1\beta_1 I + \mu_1\beta_1 P_1 \\
&= (\alpha_1 - \beta_1)I + \beta_1\{(1-\mu_1)I + \mu_1 P_1\}.
\end{aligned}$$

So putting $T_1' = (1-\mu_1)I + \mu_1 P_1$, T_1' is nonexpansive and $T = \alpha_1 T_1 + \sum_{i=2}^r \alpha_i T_i = (\alpha_1 - \beta_1)I + \beta_1 T_1' + \sum_{i=2}^r \alpha_i T_i$. By Theorem 4, T is asymptotically regular. So, it follows from Theorem 3 that for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.

4. Additional results

In this section, we first prove a fixed point theorem using Kirk's fixed point theorem (Theorem 1). Before proving it, we give definitions and notations. Let μ be a mean on the positive integers $\mathbb{N}$, i.e., a continuous linear functional on ℓ_∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \le \mu(a) \le \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \ldots) \in \ell_\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(a)$.

THEOREM 7. *Let E be a reflexive Banach space and let C be a nonempty closed convex subset of E which has normal structure. Let $C_1, C_2, \dots, C_r$ be bounded sunny nonexpansive retracts of C. Let T be an operator on C given by*

$$T = \sum_{i=1}^{r} \alpha_i T_i, \qquad 0 < \alpha_i < 1, \quad i = 1, 2, \dots, r, \quad \sum_{i=1}^{r} \alpha_i = 1,$$

such that for each i, $T_i = (1 - \lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto C_i. Then $F(T)$ is nonempty.

PROOF. Let $x \in C$ and consider a closed ball $B_R[x]$ of center x and radius R containing all the sets $C_1, C_2, \dots, C_r$. Then, $\sup\{\|x - T^n x\| : n = 1, 2, 3, \dots\} \leq R$ and hence $\{T^n x\}$ is bounded. Let μ be a Banach limit and define a real valued function g on C by

$$g(y) = \mu_n \|T^n x - y\|, \qquad y \in C.$$

Then the set M given by

$$M = \{z \in C : \mu_n \|T^n x - z\| = \min_{y \in C} \mu_n \|T^n x - y\|\}$$

is nonempty, bounded, closed and convex. In fact, the function g on C is continuous and convex. Further, if $\|y_n\| \to \infty$, then $g(y_n) \to \infty$. By [1], there exists an element z of C such that $g(z) = \min\{g(y) : y \in C\}$. Also, since $\{T^n x\}$ is bounded, M is bounded; see [9]. The set M is also invariant under T. In fact, if $z \in M$, then

$$\begin{aligned} \mu_n \|T^n x - Tz\| &= \mu_n \|T^{n+1} x - Tz\| \\ &\leq \mu_n \|T^n x - z\| = \min_{y \in C} \mu_n \|T^n x - y\| \end{aligned}$$

and hence $Tz \in M$. So, by Theorem 1 (Kirk's fixed point theorem), there exists an element x_0 of M such that $Tx_0 = x_0$.

Using Reich's theorem [8], we obtain the following lemma related to the existence of sunny nonexpansive retractions.

LEMMA 3. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let T be a nonexpansive mapping of C into C with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

PROOF. Put $A = I - T$. Then A is an accretive operator satisfying $C = D(A) \subset \bigcap_{r>0} R(I + rA)$ and $A^{-1}0 = F(T)$. So, by [8, 10], $\lim_{t \to \infty} J_t x$ exists for every $x \in C$, where $J_t = (I + tA)^{-1}$ for each positive number t. Putting $Px = \lim_{t\to\infty} J_t x$ for every $x \in C$, P is a sunny nonexpansive retraction of C onto $F(T)$; see [8, 10].

Using Lemma 3, we prove the following.

THEOREM 8. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $\{S_1, S_2, \ldots, S_r\}$ be a commuting family of nonexpansive mappings of C into C with $F(S_i) \neq \emptyset$ for all $i = 1, 2, \ldots, r$. Let T be an operator on C given by*

$$T = \sum_{i=1}^{r} \alpha_i T_i, \qquad 0 < \alpha_i < 1, \quad i = 1, 2, \ldots, r, \quad \sum_{i=1}^{r} \alpha_i = 1,$$

such that for each i, $T_i = (1 - \lambda_i)I + \lambda_i P_i$, $0 < \lambda_i < 1$, where P_i is a sunny nonexpansive retraction of C onto $F(S_i)$. Then $F(T) = \bigcap_{i=1}^{r} F(S_i)$ is nonempty and further, for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.

PROOF. Since E is strictly convex, $F(S_1)$ is nonempty, closed and convex. Further, $F(S_1)$ is invariant under S_2. In fact, if $x \in F(S_1)$, then $S_1 S_2 x = S_2 S_1 x = S_2 x$ and hence $S_2 x \in F(S_1)$. Since $F(S_2)$ is nonempty, for each $x \in F(S_1)$, $\{S_2^n x\}$ is bounded and $\{S_2^n x\} \subset F(S_1)$. So, $F(S_1) \cap F(S_2) \neq \emptyset$. Similarly, we can show that $\bigcap_{i=1}^{r} F(S_i) \neq \emptyset$. By Theorem 5, it is obvious that $F(T) = \bigcap_{i=1}^{r} F(S_i)$. It follows from Theorem 6 that for each $x \in C$, $\{T^n x\}$ converges weakly to an element of $F(T)$.

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SHINYA KITAHARA
Department of Information Science
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo 152, JAPAN

WATARU TAKAHASHI
Department of Information Science
Tokyo Institute of Technology
Oh-okayama, Meguro-ku
Tokyo 152, JAPAN