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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SEMILINEAR EQUATIONS WITH SUBCRITICAL NONLINEARITY

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(*Submitted by H. Brézis*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$, with $C^{2,\sigma}$-boundary $\partial\Omega$ $(\sigma > 0)$. Consider the following problem:

$$P(\beta) \qquad \begin{cases} -\Delta u = \beta f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f satisfies the following assumption:

(A) f is a positive, C^1-function on $\mathbb{R}$.

By the implicit function theorem, there exist an interval $I = (0, \widetilde{\beta})$ and a neighbourhood V of 0 in the space $C^{2,\sigma}(\overline{\Omega})$ such that, for any $\beta \in I$, there exists a unique solution $\widehat{u}(\cdot, \beta) \in V$ of the problem $P(\beta)$. Furthermore, for $\beta \in I$, $-\Delta - \beta f'(\widehat{u}(\cdot, \beta))$ has a positive first eigenvalue. (For the proof of the above fact, see Appendix.)

Suppose, in addition, that

(B) There are constants $C > 0$ and $1 < p < \widetilde{n}$ such that $f(u) < C(1 + u^p)$ for $u > 0$, where $\widetilde{n} = (n+2)/(n-2)$ if $n > 2$; $\widetilde{n} = +\infty$ if $n = 2$, and

(C) There exist constants $\mu > 2$ and $r > 0$ such that $\mu F(u) < uf(u)$ for $u > r$, where $F(u) = \int_0^u f(t)\,dt$.

Then, by the theorem of Ambrosetti-Rabinowitz, the problem $P(\beta)$ has at least one solution $\widetilde{u}(\cdot,\beta)$, different from $\widehat{u}(\cdot,\beta) \in V$, for any $\beta \in I$ (see [2], Remark 2.13). We call $\widetilde{u}(\cdot,\beta)$ a large solution of $P(\beta)$ if $\widetilde{u}(\cdot,\beta)$ is a solution of $P(\beta)$ different from the solution $\widehat{u}(\cdot,\beta) \in V$. Here we note that, under condition (B), if $u(\cdot,\beta)$ is a weak solution of $P(\beta)$, then $u(\cdot,\beta)$ must be a classical solution of $P(\beta)$, by "bootstrap" method.

In the first part of this paper, we study the asymptotic behaviour of large solutions $u(\cdot,\beta)$ as $\beta \to 0$. We impose the following additional conditions:

(D) $F(u) \geq uf(u)/(p+1)$ for $u > 0$;

(E) If $n \geq 3$, then either Ω is convex, or $f(u) \cdot u^{-\widetilde{n}}$ is decreasing on $(0,\infty)$.

For instance, if $f(u) = (1+u)^p$ or $1 + u^r + u^p$ $(1 < r < p < \widetilde{n})$, then it is not necessary that Ω is convex.

It is known that there exists a unique large solution $u(\cdot,\beta)$ of $P(\beta)$ if $f(u) = (1+u)^p$, $1 < p < \widetilde{n}$, and $\Omega = B(0,1)$ (see [5]). For the general case, we choose, for each $\beta \in I$, an arbitrary large solution $u(\cdot,\beta)$ of $P(\beta)$ and consider the class $\{u(\cdot,\beta) \mid \beta \in I\}$.

The first result is the following:

THEOREM 1. *Under the assumptions* (A), (B), (C), (D) *and* (E), *for any compact set* $K \subset \Omega$, *we have*

$$\lim_{\beta \to 0} \min_{x \in K} u(x,\beta) = +\infty$$

where $\{u(\cdot,\beta) \mid \beta \in I\}$ *is an arbitrary class of large solutions of* $P(\beta)$.

In Section 2, we split the proof of Theorem 1 into four steps:

(a) $\lim_{\beta \to 0} \|u(\cdot,\beta)\|_{L^\infty(\Omega)} = +\infty$,

(b) $\lim_{\beta \to 0} \|u(\cdot,\beta)\|_{H^1(\Omega)} = +\infty$,

(c) $\lim_{\beta \to 0} \|\beta f(u(\cdot,\beta))\|_{L^1(\Omega)} = +\infty$,

(d) $\lim_{\beta \to 0} \min_{x \in K} u(\cdot,\beta) = +\infty$, for any compact set $K \Subset \Omega$.

Having obtained Theorem 1, one may ask whether these large solutions $u(\cdot,\beta)$ have large interior oscillation. To study this problem, we assume the following condition holds:

(B′) there are positive constants c_1, c_2 and $p \in (1,\widetilde{n})$ such that, for $u > 0$, $c_1 u^p \leq f(u) \leq c_2(1 + u^p)$.

Then one has the following

THEOREM 2. *Suppose that the conditions* (A), (B′), (C), (D) *and* (E) *hold. Let* $\{u(\cdot,\beta)\mid\beta\in(1,\widetilde{\beta})\}$ *be an arbitrary class of large solutions of* $P(\beta)$ *and* $B\equiv B(\widehat{x},R)$ $\Subset\Omega$ *be an open ball. Then, for any sequence* $\{\beta_i\}$ *with* $\lim_{i\to\infty}\beta_i=0$, *either*

$$\liminf_{i\to\infty}\frac{\|u(\cdot,\beta_i)\|_{L^\infty(B)}}{\|u(\cdot,\beta_i)\|_{L^\infty(\Omega)}}=0$$

or $\lim_{i\to\infty}\mathrm{Osc}(u(\cdot,\beta_i),B)=+\infty$, *where*

$$\mathrm{Osc}(u(\cdot,\beta),B)\equiv\sup_{x\in B}u(x,\beta)-\inf_{x\in B}u(x,\beta).$$

For instance, let $\Omega=B(0,1)$ and $f(u)=1+u^p$; then for $0<r<1$, $\lim_{i\to\infty}\mathrm{Osc}(u(\cdot,\beta_i),B(0,r))=+\infty$, where $u(\cdot,\beta_i)$ is a large solution of $P(\beta_i)$ and $\lim_{i\to\infty}\beta_i=0$.

REMARK. In Theorem 2, one may replace B by any smooth subdomain $D\subseteq\overline{D}\subset\Omega$, and the result still holds.

For the case where $f(0)=0$, for instance $f(u)=u^p$ or u^r+u^p $(1<r<p<\widetilde{n})$, one can prove, in the same way, the following:

THEOREM 3. *Suppose that* f *is a nonnegative* C^1*-function on* $\mathbb{R}$ *with* $f(u)=o(u)$ *as* $u\to 0$ *and that the conditions* (B), (C), (D) *and* (E) *hold. For* $\beta>0$, *let* $u(\cdot,\beta)$ *be an arbitrary nontrivial solution of* $P(\beta)$. *Then, for any compact set* $K\Subset\Omega$,

$$\lim_{\beta\to 0}\min_{x\in K}u(x,\beta)=+\infty.$$

Suppose, in addition, that the condition (B′) *holds. Then, for any sequence* $\{\beta_i\}$ *with* $\lim_{i\to\infty}\beta_i=0$, *either*

$$\liminf_{i\to\infty}\frac{\|u(\cdot,\beta_i)\|_{L^\infty(B)}}{\|u(\cdot,\beta_i)\|_{L^\infty(\Omega)}}=0$$

or $\lim_{i\to\infty}\mathrm{Osc}(u(\cdot,\beta_i),B)=+\infty$, *where* $u(\cdot,\beta_i)$ *is a nontrivial solution of* $P(\beta_i)$ *and* $B\equiv B(x,R)\subset\overline{B}\subset\Omega$.

REMARK. For the case where $f(u)=e^u$, the result is very delicate (see [7]).

In the second part, we study the asymptotic behaviour of solutions $u(\cdot,\beta)$ of $P(\beta)$ as $\beta\to\widetilde{\beta}$. We suppose that

(A′) f is a convex positive C^1-function on $\mathbb{R}$.

(F) There exist constants $a > 0$ and $c \geq 0$ such that $\lim_{u\to\infty} f(u)/u = a$ and $f(u) \geq au + c$ for $u > 0$.

(For instance, $f(u) = \sqrt{u^2+1}$ or $u + e^{-u}$.)

The first result of the second part is the following:

THEOREM 4. *Under the assumptions* (A′) *and* (F), *we have:*

(1) *$P(\beta)$ has a solution for $\beta \in (0, \widetilde{\beta})$, where $\widetilde{\beta} = \lambda_1(-\Delta)/a$ and $\lambda_1(-\Delta)$ is the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$. $P(\beta)$ has no solutions for $\beta \geq \widetilde{\beta}$.*

(2) *Uniqueness of solution of $P(\beta)$ for $\beta \in I = (0, \widetilde{\beta})$.*

(3) *For any compact set $K \subset \Omega$, $\lim_{\beta\to\widetilde{\beta}} \min_{x\in K} u(x,\beta) = +\infty$.*

(4) *$\{u(\cdot,\beta)/\|u(\cdot,\beta)\|_{L^2(\Omega)}\}$ converges strongly to φ_1 in $H^1(\Omega)$ as $\beta \to \widetilde{\beta}$, where $\varphi_1(x)$ is the first eigenfunction of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$.*

REMARK. (i) In Theorem 4, it is not necessary that Ω is convex.

(ii) For the case where $\lim_{u\to\infty} |f(u) - (au + c)| = 0$ for some $c < 0$, one can find some results in [6].

Next, for the case where f is concave on $(0,\infty)$, for instance, $f(u) = 2 + u - e^{-u}$, we have the following:

THEOREM 5. *Let f be a positive concave C^1-function on $\mathbb{R}^+$ with $\lim_{u\to\infty} f'(u) = a > 0$. Then:*

(1) *$P(\beta)$ has a unique solution $u(\cdot,\beta)$ for $\beta \in (0, \widetilde{\beta})$, where $\widetilde{\beta} = \lambda_1(-\Delta)/a$. For $\beta \geq \widetilde{\beta}$, $P(\beta)$ has no solution.*

(2) *For any compact set $K \subset \Omega$, $\lim_{\beta\to\widetilde{\beta}} \min_{x\in K} u(x,\beta) = +\infty$.*

(3) *$\{u(\cdot,\beta)/\|u(\cdot,\beta)\|_{L^2(\Omega)}\}$ converges strongly to φ_1 in $H^1(\Omega)$ as $\beta \to \widetilde{\beta}$, where φ_1 is the first eigenfunction of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$.*

Finally, for general "almost linear" functions, for instance $f = 2 + u + \sin u$, we have the following:

THEOREM 6. *Let f be a positive C^2-function on $\mathbb{R}$. Suppose that there are positive constants a, b, c and d such that, for $u \geq 0$,*

$$au + c \leq f(u) \leq au + b \quad \textit{and} \quad |f'(u)| \leq d.$$

Then:

(1) *For $\beta < \widetilde{\beta} = \lambda_1(-\Delta)/a$, $P(\beta)$ has at least one solution $u(\cdot,\beta)$. For $\beta \geq \widetilde{\beta}$, $P(\beta)$ has no solution. There exists an interval $(0,\theta)$ such that for $\beta \in$*

$(0,\theta)$, $P(\beta)$ has a unique solution. If $f(t)/t$ is strictly decreasing on $\mathbb{R}^+$, then $\theta = \widetilde{\beta}$.

(2) *For any compact set $K \subset \Omega$, $\lim_{\beta\to\widetilde{\beta}} \min_{x\in K} u(x,\beta) = +\infty$.*

(3) *$\left\{u(\cdot,\beta)/\|u(\cdot,\beta)\|_{L^2(\Omega)}\right\}$ converges strongly to φ_1 in $H^1(\Omega)$ as $\beta \to \widetilde{\beta}$, where φ_1 is the first eigenfunction of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$.*

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2. Proof of Theorem 1

2.1. First we claim that $\lim_{\beta\to 0} \|u(\cdot,\beta)\|_{L^\infty(\Omega)} = +\infty$

Suppose, by contradiction, that there were a constant $C > 0$ and a sequence $\{\beta_i\}$ such that

$$\lim_{i\to\infty} \beta_i = 0 \quad \text{and} \quad \sup_i \|u(\cdot,\beta_i)\|_{L^\infty(\Omega)} \le C.$$

Since $f(u)$ is a C^1-function on $\mathbb{R}$, for $q \in (1,\infty)$ we have $\|f(u(\cdot,\beta_i))\|_{L^q(\Omega)} \le C$, and hence, by the L^p-estimate for solutions of P.D.E.,

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{W^{2,q}(\Omega)} = \lim_{i\to\infty} \beta_i \|f(u(\cdot,\beta_i))\|_{L^q(\Omega)} = 0.$$

By the Sobolev imbedding theorem, we see that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{C^{1,\sigma}(\overline{\Omega})} = 0.$$

This implies again, as above, that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{C^{2,\sigma}(\overline{\Omega})} = 0.$$

Using the fact that $\widehat{u}(\cdot,\beta)$ is the unique solution of $P(\beta)$ that belongs to the neighbourhood V of 0 in $C^{2,\sigma}(\overline{\Omega})$, we conclude that $u(\cdot,\beta) = \widehat{u}(\cdot,\beta)$ for large n. This contradicts the definition of a large solution $u(\cdot,\beta)$ of $P(\beta)$.

2.2. Next we prove that $\lim_{\beta\to 0} \|u(\cdot,\beta)\|_{H^1(\Omega)} = +\infty$.

Assume first that $n = 2$. It is known that, for $t \ge 1$,

$$\|u\|_{L^t(\Omega)} \le C(t)\|Du\|_{L^2(\Omega)}.$$

Suppose, by contradiction, that $\sup_i \|Du(\cdot,\beta)\|_{L^2(\Omega)}$ is bounded by C for some sequence $\{\beta_i\}$ with $\lim_{i\to\infty}\beta_i = 0$ Then $\|u(\cdot,\beta_i)\|_{L^q(\Omega)} < C(q)$ for $q > 1$. From the L^p-estimate, we deduce that for any $q > 1$,

$$\|u(\cdot,\beta_i)\|_{W^{2,q}(\Omega)} \le C\|\Delta u(\cdot,\beta_i)\|_{L^q(\Omega)} = C\beta_i\|f(u(\cdot,\beta_i))\|_{L^q(\Omega)}.$$

By the condition (B), we see that for $q > 1$,

$$\|u(\cdot,\beta_i)\|_{W^{2,q}(\Omega)} \le C\beta_i(1 + \|u(\cdot,\beta_i)\|^p_{L^q(\Omega)}) \to 0 \qquad \text{as } \beta_i \to 0.$$

By the Sobolev imbedding theorem and the L^p-estimate for solutions of P.D.E., we have

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{C^{1,\sigma}(\overline{\Omega})} = 0,$$

which implies again, as above, that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{C^{2,\sigma}(\overline{\Omega})} = 0,$$

a contradiction.

Assume that $n > 2$. Using (B) and the Sobolev inequality, we have

$$\begin{aligned} 0 &< \int_\Omega |\nabla u(x,\beta)|^2\,dx = \beta\int_\Omega f(u(x,\beta))\cdot u(x,\beta)\,dx \\ &\le c\beta\left\{\int_\Omega u(x,\beta)\,dx + \int_\Omega u(x,\beta)^{p+1}\,dx\right\} \\ &\le c\beta\left\{\left(\int_\Omega |\nabla u(x,\beta)|^2\,dx\right)^{1/2} + \left(\int_\Omega |\nabla u(x,\beta)|^2\,dx\right)^{(p+1)/2}\right\}. \end{aligned}$$

This implies that, for all $\beta \in I$,

$$1 \le c\beta\left\{\left(\int_\Omega |\nabla u(x,\beta)|^2\,dx\right)^{-1/2} + \left(\int_\Omega |\nabla u(x,\beta)|^2\,dx\right)^{(p-1)/2}\right\}.$$

Hence, for any sequence $\{\beta_i\}$ with $\lim_{i\to\infty}\beta_i = 0$, either

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{H^1(\Omega)} = 0 \qquad \text{or} \qquad = \infty.$$

Suppose, by contradiction, that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{H^1(\Omega)} = 0$$

for some such sequence. Then, by the Sobolev inequality,

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{L^{2n/(n-2)}(\Omega)} = 0.$$

Now, using (B) and the Hölder inequality, we see that

$$\text{(R)}\qquad \begin{aligned} &\int_\Omega -\Delta u(x,\beta_i)\cdot u(x,\beta_i)^{\frac{2n}{n-2}}\,dx = \beta_i \int_\Omega f(u(x,\beta_i))\cdot u(x,\beta_i)^{\frac{2n}{n-2}}\,dx \\ &\le c\beta_i \left\{\int_\Omega u(x,\beta_i)^{\frac{2n}{n-2}}\,dx + \int_\Omega u(x,\beta_i)^{p+\frac{2n}{n-2}}\,dx\right\} \\ &\le c\beta_i \left\{\int_\Omega u(x,\beta_i)^{\frac{2n}{n-2}}\,dx + \left[\int_\Omega u(x,\beta_i)^{(p-1)\cdot\frac{n}{2}}\,dx\right]^{\frac{2}{n}}\right. \\ &\quad\left.\times\left[\int_\Omega u(x,\beta_i)^{(\frac{2n}{n-2}+1)\frac{n}{n-2}}\,dx\right]^{\frac{n-2}{n}}\right\}. \end{aligned}$$

On the other hand, using the Sobolev inequality, we also see that

$$\text{(L)}\qquad \begin{aligned} &\int_\Omega -\Delta u(x,\beta_i)\cdot u(x,\beta_i)^{\frac{2n}{n-2}}\,dx \\ &= \int_\Omega \left(\frac{2n}{n-2}\right)|\nabla u(x,\beta_i)|^2 u(x,\beta_i)^{\frac{2n}{n-2}-1}\,dx \\ &= C(n)\int_\Omega \left|\nabla\left(u(x,\beta_i)^{\{\frac{1}{2}(\frac{2n}{n-2}-1)+1\}}\right)\right|^2 dx \\ &\ge C^1(\Omega,n)\left(\int_\Omega |u(x,\beta_i)|^{\{\frac{1}{2}(\frac{2n}{n-2}-1)+1\}\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \end{aligned}$$

Set $\delta(0)\equiv\frac{2n}{n-2}$ and $\delta(1)\equiv(\delta(0)+1)\frac{n}{n-2}$, note that $\delta(0)>2$, $\delta(1)>3$ and

$$\delta(1) = \left[\frac{1}{2}(\delta(0)-1)+1\right]\frac{2n}{n-2}.$$

Combining (R) and (L), we obtain

$$(*)\quad 1 \le C(n,\Omega)\beta_i\left\{\frac{\int_\Omega u(x,\beta_i)^{\delta(0)}\,dx}{\left[\int_\Omega u(x,\beta_i)^{\delta(1)}\,dx\right]^{(n-2)/n}} + \left[\int_\Omega u(x,\beta_i)^{(p-1)n/2}\,dx\right]^{2/n}\right\}.$$

From $(p-1)n/2<\delta(0)$, it follows that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{L^{(p-1)n/2}(\Omega)} = 0.$$

Then from $(*)$, we have

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{L^{\delta(1)}(\Omega)} = 0.$$

Repeat this process n times to obtain for any $k \geq 0$,

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{L^{\delta(k)}(\Omega)} = 0$$

where $\delta(k+1) \equiv (\delta(k)+1)\frac{n}{n-2}$. Note that $\delta(k+1) = [\frac{1}{2}(\delta(k)-1)+1]\frac{2n}{n-2}$.

We also note that $\delta(k) > k+2$ for any $k \geq 0$. In particular, $\delta(K) > np/2$, if $K > (n^2-2n+8)/(2n-4)$. By the Sobolev inequality and the L^p-estimate for solutions of P.D.E., we see that

$$\begin{aligned}
\|u(\cdot,\beta_i)\|_{C^0(\overline{\Omega})} &\leq C(\Omega)\|u(\cdot,\beta_i)\|_{W^{2,\delta(K)/p}(\Omega)} \\
&\leq C\|\beta_i f(u(\cdot,\beta_i))\|_{L^{\delta(K)/p}(\Omega)} \\
&\leq C\beta_i\|1+u(\cdot,\beta_i)^p\|_{L^{\delta(K)/p}(\Omega)} \\
&\leq C\beta_i\{1+\|u(\cdot,\beta_i)\|_{L^{\delta(K)}(\Omega)}\} \to 0 \qquad \text{as } i\to\infty.
\end{aligned}$$

This contradicts the fact that

$$\lim_{i\to\infty} \|u(\cdot,\beta_i)\|_{L^\infty(\Omega)} = +\infty,$$

and therefore completes the proof of our claim.

2.3. Now we prove that $\lim_{\beta\to 0}\|\beta f(u(\cdot,\beta))\|_{L^1(\Omega)} = +\infty$.

Using the Pokhozhaev identity and the condition (D), we see that

$$\begin{aligned}
&\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}(x,\beta)\right)^2(\vec{n}_x\cdot\vec{x})\,dS_x \\
&\quad = n\int_\Omega \beta F(u(x,\beta))\,dx - \frac{n-2}{2}\int_\Omega |\nabla u(x,\beta)|^2\,dx \\
&\quad \geq \frac{n}{p+1}\int_\Omega \beta f(u(x,\beta))u(x,\beta)\,dx - \frac{n-2}{2}\int_\Omega |\nabla u(x,\beta)|^2\,dx \\
&\quad = \left(\frac{n}{p+1}-\frac{n-2}{2}\right)\int_\Omega |\nabla u(x,\beta)|^2\,dx > 0
\end{aligned}$$

where $\vec{n}$ is the unit outward normal vector at $x\in\partial\Omega$. From $\lim_{\beta\to 0}\|u(\cdot,\beta)\|_{H^1(\Omega)} = +\infty$, we deduce that

$$\lim_{\beta\to 0}\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}(x,\beta)\right)^2\cdot(\vec{n}_x\cdot\vec{x})\,dS_x = +\infty.$$

To show that $\lim_{\beta\to 0}\int_\Omega \beta f(u(x,\beta))\,dx = +\infty$, we need the following two lemmas.

LEMMA 1 ([3], [4]). *Let u be a solution of the problem*

$$\begin{cases} -\Delta u = \beta f(u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose that the condition (E) *holds. Then there exist positive constants ε, r and c, independent of u, such that, for all $x \in \Omega_\varepsilon = \{x \in \Omega \mid \mathrm{dist}(x, \Omega) < \varepsilon\}$, there exists a measurable set I_x such that*

(1) *measure $(I_x) \geq r$;*
(2) *$I_x \subseteq \Omega - \Omega_{\varepsilon/2}$;*
(3) *$u(x) \leq Cu(y)$, for all $y \in I_x$.*

LEMMA 2 ([1]). *Let u be a solution of the problem*

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $W \subset \Omega$ be a neighbourhood of $\partial\Omega$. Then there exists a constant $C > 0$ such that, for all $q < n/(n-1)$ and $\alpha \in (0,1)$,

$$\|u\|_{W^{1,q}(\Omega)} + \|\nabla u\|_{C^{0,\alpha}(\partial\Omega)} \leq C(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(W)}).$$

Now we show that $\lim_{\beta\to 0} \int_\Omega \beta f(u(x,\beta))\,dx = +\infty$. Suppose, by contradiction, that there exist a constant $C_0 > 0$ and a sequence $\{\beta_i\}$ with $\lim_{i\to\infty} \beta_i = 0$ such that

$$\sup_i \int_\Omega \beta_i f(u(x,\beta_i))\,dx \leq C_0,$$

that is,

$$\sup_i \left| \int_{\partial\Omega} \frac{\partial u}{\partial n}(x,\beta_i)\,dS_x \right| \leq C_0.$$

Let φ_1 be the first eigenfunction of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_1(x) = \lambda_1\varphi_1(x),\ \varphi_1(x) > 0 & \text{in } \Omega, \\ \varphi_1(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

We see that, for all i,

$$\begin{aligned} J_{\beta_i} &\equiv \lambda_1 \int_\Omega u(x,\beta_i)\varphi_1(x)\,dx = \int_\Omega -\Delta u(x,\beta_i)\varphi_1(x)\,dx \\ &= \int_\Omega \beta_i f(u(x,\beta_i))\varphi_1(x)\,dx \leq \max_{x\in\Omega} \varphi_1(x) \cdot \int_\Omega \beta_i f(u(x,\beta_i))\,dx \\ &\leq C_0 \max_{x\in\Omega} \varphi_1(x) \equiv C_1. \end{aligned}$$

This implies that, for all compact sets $K \subset \Omega$.

$$\sup_i \int_K u(x,\beta_i)\,dx \le \frac{C_1}{\lambda_1 \cdot C(K)} \equiv C_1(K)$$

where $C(K) = \min_{x\in K} \varphi_1(x)$. By Lemma 1, for all i and for all $x \in \Omega_{\varepsilon/2}$, we have

$$u(x,\beta_i) \le \frac{c}{r}\int_{I_x} u(y,\beta_i)dy \le \frac{c}{r}\int_{\overline{\Omega}_0} u(y,\beta_i)dy \le \frac{c}{r} C_1(\overline{\Omega}_0) \equiv C_2,$$

where $\overline{\Omega}_0 \equiv \bigcup_{x\in\Omega_{\varepsilon/2}} I_x$. (Note that $\overline{\Omega}_0 \Subset \Omega$.) This implies that

$$\sup_i \|\beta_i f(u(\cdot,\beta_i))\|_{L^\infty(\Omega_{\varepsilon/2})} \le C_3.$$

Then, by Lemma 2,

$$\sup_i \|\nabla u(\cdot,\beta_i)\|_{C^0(\partial\Omega)} \le C_0 + C_3,$$

which implies that

$$\sup_i \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}(x,\beta_i)\right)^2 (\vec{n}_x \cdot \vec{x})\,dS_x \le C_4,$$

a contradiction.

2.4. Finally, we prove that, for any compact set $K \subset \Omega$,

$$\lim_{\beta\to 0}\min_{x\in K} u(x,\beta) = +\infty.$$

Set $J_\beta = \lambda_1 \int_\Omega u(x,\beta)\varphi_1(x)\,dx$. It is clear that $J_\beta = \beta\int_\Omega f(u(x,\beta))\varphi_1(x)\,dx$.

We claim that $\lim_{\beta\to 0} J_\beta = +\infty$. Otherwise there would be a constant C and a sequence $\{\beta_i\}$ with $\lim_{i\to\infty}\beta_i = 0$ such that

$$\sup_i \int \lambda_1 u(x,\beta_i)\varphi_1(x)\,dx \le C.$$

As in the proof in 2.3, this implies again that

$$\sup_i \|\beta_i f(u(\cdot,\beta_i))\|_{L^\infty(\Omega_{\varepsilon/2})} \le C$$

and thus we see that, for all i,

$$\begin{aligned}\int_\Omega \beta_i f(u(x,\beta_i))\,dx &= \int_{\Omega_{\varepsilon/2}} \beta_i f(u(x,\beta_i))\,dx + \int_{\Omega-\Omega_{\varepsilon/2}} \beta_i f(u(x,\beta_i))\,dx\\ &\le \beta_i \|f(u(\cdot,\beta_i))\|_{L^\infty(\Omega_{\varepsilon/2})}\cdot|\Omega_{\varepsilon/2}|\\ &\quad + \frac{1}{\min\limits_{x\in\Omega-\Omega_{\varepsilon/2}}\varphi_1(x)}\int_{\Omega-\Omega_{\varepsilon/2}} \beta_i f(u(x,\beta_i))\varphi_1(x)\,dx \le C.\end{aligned}$$

This means that

$$\sup_i \|\beta_i f(u(\cdot,\beta_i))\|_{L^1(\Omega)} \le C,$$

a contradiction. Therefore we have $\lim_{\beta\to 0} J_\beta = +\infty$.

Now let $G(x,y)$ be the Green function of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$. By Hopf's lemma, for all $x \in \Omega$ there exists a constant $r(x)$ such that $G(x,y) \ge r(x)\varphi_1(y)$ for $y \in \Omega$. Thus, for any compact set $K \subset \Omega$, there exists a constant $r(K) > 0$ such that $G(x,y) \ge r(K)\varphi_1(y)$ for $x \in K$, and $y \in \Omega$. Finally, for any given compact set $K \subset \Omega$, we choose, for any $\beta \in I$, $x(\beta) \in K$ such that

$$u(x(\beta),\beta) = \min_{x\in K} u(x,\beta).$$

Then

$$\begin{aligned} u(x(\beta),\beta) &= \int_\Omega G(x(\beta),y)\beta f(u(y,\beta))dy \\ &\ge r(K)\int_\Omega \beta f(u(y,\beta))\varphi_1(y)dy \\ &= r(K)J(\beta) \to +\infty \qquad \text{as } \beta \to 0, \end{aligned}$$

that is,

$$\lim_{\beta\to 0}\min_{x\in K} u(x,\beta) = +\infty$$

for any compact set $K \Subset \Omega$.

3. Proof of Theorem 2

Suppose that

$$\liminf_{i\to\infty} \frac{\|u(\cdot,\beta_i)\|_{L^\infty(B)}}{\|u(\cdot,\beta_i)\|_{L^\infty(\Omega)}} = c_0 > 0.$$

We shall prove that

$$\lim_{i\to\infty} \operatorname{Osc}(u(\cdot,\beta_i),B) = +\infty.$$

In Section 2, we have proved that

$$\lim_{\beta\to 0} \|u(\cdot,\beta)\|_{H^1(\Omega)} = +\infty.$$

From (B′), it follows that

$$\lim_{\beta\to 0} \int_\Omega \beta u(x,\beta)^{p+1}\,dx = +\infty.$$

By (B′) one sees that

$$\begin{aligned} c_1\beta \int_\Omega u^{p+1}\,dx &\le \int_\Omega |\nabla u|^2\,dx \\ &= \int_\Omega \beta f(u)u\,dx \le \beta c_2 \int_\Omega (u+u^{p+1})\,dx \\ &\le \beta c_2 \left\{ \left(\int_\Omega u^{p+1}\,dx\right)^{1/(p+1)} + \int_\Omega u^{p+1}\,dx \right\}. \end{aligned}$$

Let

$$v(x,\beta) \equiv \frac{u(x,\beta)}{[\beta \int_\Omega u^{p+1}\,dx]^{\frac{1}{2}}}.$$

Then $c_1 \le \|v(\cdot,\beta)\|_{H^1(\Omega)} \le 2c_2$, for β small.

Suppose, by contradiction, that there were a constant $M > 0$ and a subsequence of $\{\beta_i\}$, still denoted by $\{\beta_i\}$, such that, for all i, $\mathrm{Osc}(u(\cdot,\beta_i),B) \le M$. Since $\|v(\cdot,\beta)\|_{H^1(\Omega)} \le 2c_2$, there exists a subsequence of $\{v(\cdot,\beta_i)\}$, still denoted by $\{v(\cdot,\beta_i)\}$, and a function $\widetilde{v} \in H^1(\Omega)$ such that

$$\begin{cases} \nabla v(\cdot,\beta_i) \to \nabla\widetilde{v} & \text{weakly in } L^2(\Omega), \\ v(\cdot,\beta_i) \to \widetilde{v} & \text{strongly in } L^2(\Omega) \text{ and almost everywhere,} \end{cases}$$

as $i \to \infty$. We claim that $\widetilde{v} \not\equiv$ constant a.e. in B. Suppose, by contradiction, that $\widetilde{v} \equiv$ constant a.e. in B. Then $\nabla\widetilde{v} = 0$ in B. Let φ be the solution of the problem

$$\begin{cases} -\Delta\varphi = \lambda_1\varphi, \ \ \varphi > 0 & \text{in } B \equiv B(\widehat{x},R), \\ \varphi = 0 & \text{on } \partial B. \end{cases}$$

By Green's theorem, we have

$$-\int_B \varphi(x)\cdot\Delta v(x,\beta_i)\,dx = \int_B \nabla\varphi(x)\cdot\nabla v(x,\beta_i)\,dx \to \int_B \nabla\varphi(x)\cdot 0\,dx = 0 \quad \text{as } i \to \infty.$$

On the other hand, we see that, for all i,

$$-\int_B \varphi(x)\cdot\Delta v(x,\beta_i)\,dx \ge \frac{c_1\beta_i \int_B \varphi(x)u(x,\beta_i)^p\,dx}{[\beta_i \int_\Omega u(x,\beta_i)^{p+1}\,dx]^{1/2}}.$$

Since $\mathrm{Osc}(u(\cdot,\beta_i),B) \le M$ for all i and $\lim_{i\to\infty} \min_{x\in\overline{B}} u(x,\beta_i) = +\infty$, we have, for i large,

$$\frac{c_0}{2} \max_{x\in\overline{\Omega}} u(x,\beta_i) \le \max_{x\in\overline{B}} u(x,\beta_i) \le 2\min_{x\in\overline{B}} u(x,\beta_i).$$

From the maximum principle, it also follows that

$$\|u(\cdot,\beta_i)\|_{L^\infty(\Omega)} \le c\|\beta f(u)\|_{L^\infty(\Omega)} \le c\beta(1 + \|u\|^p_{L^\infty(\Omega)}),$$

which implies that there exists a constant $c_3 > 0$ such that, for $\beta \in (0, \widetilde{\beta})$,

$$1 \le c_3 \beta_i \|u(\cdot, \beta_i)\|^{p-1}_{L^\infty(\Omega)}.$$

Hence we see that, for all i sufficiently large,

$$\begin{aligned}
\left(\beta_i \int_\Omega u^{p+1}\,dx\right)^{1/2} &\le \beta_i^{1/2} \|u\|^{(p+1)/2}_{L^\infty(\Omega)} \cdot |\Omega| \\
&\le \left(c_3 \beta_i \|u\|^{p-1}_{L^\infty(\Omega)}\right)^{1/2} \cdot \beta_i^{1/2} \|u\|^{(p+1)/2}_{L^\infty(\Omega)} \cdot |\Omega| \\
&\le c_4 \beta_i \|u\|^{p}_{L^\infty(\Omega)} \cdot |\Omega| \\
&\le c_4 \beta_i \left[\frac{2}{c_0} \min_{x \in \overline{B}} u(x, \beta_i)\right]^p \cdot |\Omega| \\
&\le c_4 \beta_i \left[\frac{2}{c_0} \min_{x \in \overline{B}(\widehat{x}, R/2)} u(x, \beta_i)\right]^p \cdot |\Omega| \\
&\le c_6 \frac{1}{\min_{x \in \overline{B}(\widehat{x}, R/2)} \varphi(x)} \int_{B(\widehat{x}, R/2)} \beta_i u(x, \beta_i)^p \varphi(x)\,dx \\
&\le c_7 \int_{B(\widehat{x}, R)} \beta_i u(x, \beta_i)^p \varphi(x)\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
0 < \frac{c_1}{c_7} &\le \frac{c_1 \int_B \beta_i u(x, \beta_i)^p \varphi(x)\,dx}{\left(\beta_i \int_\Omega u(x, \beta_i)^{p+1}\,dx\right)^{1/2}} \\
&\le -\int_B \varphi(x) \Delta v(x, \beta_i)\,dx \to 0 \qquad \text{as } i \to \infty,
\end{aligned}$$

a contradiction. Thus $\widetilde{v} \not\equiv$ constant a.e. in B. Since

$$\int_\Omega \beta_i u^{p+1}\,dx \to \infty$$

and

$$\frac{u(x, \beta_i)}{\left[\int \beta_i u^{p+1}\,dx\right]^{1/2}} \to \widetilde{v}(x) \qquad \text{a.e. in } B, \text{ as } i \to \infty,$$

we have $\operatorname{Osc}(u(\cdot, \beta_i), B) \to +\infty$ as $i \to \infty$, a contradiction. This completes the proof of Theorem 2.

4. Proof of Theorem 3

By the theorem of Ambrosetti-Rabinowitz, we see that, for $\beta > 0$, $P(\beta)$ has at least one positive solution $u(\cdot, \beta)$. Since $f(0) = 0$ we see that, for $\beta > 0$, $u = 0$ is a solution of $P(\beta)$. By the implicit function theorem, there exist an interval $I = (0, \theta)$ and a neighbourhood V of 0 in $C^{2,\sigma}(\overline{\Omega})$ such that for $\beta \in I$, $P(\beta)$ has a unique solution $v(\cdot, \beta)$ that belongs to V. Hence $v(\cdot, \beta) = 0$. This implies that there exists a constant $C > 0$ such that if $u(\cdot, \beta)$ is a nontrivial solution of $P(\beta)$ for $\beta \in I$, then $\|u(\cdot, \beta)\|_{C^{2,\sigma}(\overline{\Omega})} > C$. Then we can prove, as in 2.1, that $\lim_{\beta \to 0} \|u(\cdot, \beta)\|_{L^\infty(\Omega)} = +\infty$. Following the proof of Theorems 1 and 2, one can easily complete the proof of Theorem 3.

5. Proof of Theorem 4

We split the proof of Theorem 4 into four subsections.

5.1. First we prove that $\widetilde{\beta} = \lambda_1(-\Delta)/a$, where $\lambda_1(-\Delta)$ is the first eigenvalue of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$. Furthermore, we show that the problem $P(\widetilde{\beta})$ has no solution.

Consider the problem

$$Q(\beta) \qquad \begin{cases} -\Delta u = \beta(au + b), u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where a and b are two arbitrary positive constants. It is known that, for $\beta \in (0, \lambda_1(-\Delta)/a), Q(\beta)$ has a unique solution, and that for $\beta = \lambda_1(-\Delta)/a, Q(\beta)$ has no solution. Using the assumptions (A′) and (F), one can find a constant $b(> c)$ such that, for any $u > 0, au + c \leq f(u) \leq au + b$. Now we use the supersolution-subsolution method to prove $\widetilde{\beta} \geq \lambda_1(-\Delta)/a$.

For $\beta \in (0, \lambda_1(-\Delta)/a), Q(\beta)$ has a unique solution, which is obviously a supersolution of $P(\beta)$. On the other hand, 0 is a subsolution of $P(\beta)$. Therefore $P(\beta)$ has a solution $u(\cdot, \beta)$. Since $P(\beta)$ does not have a solution for $\beta > \widetilde{\beta}$ (see Appendix), we have $\widetilde{\beta} \geq \lambda_1(-\Delta)/a$. We claim that $P(\lambda_1(-\Delta)/a)$ has no solution, and this implies easily that $\widetilde{\beta} = \lambda_1(-\Delta)/a$. Suppose, by contradiction, that w is a solution of $P(\lambda_1(-\Delta)/a)$. If $c > 0$, we see that w is a positive supersolution of $Q(\lambda_1(-\Delta)/a)$:

$$Q(\lambda_1(-\Delta)/a) \qquad \begin{cases} -\Delta u = \lambda_1(-\Delta)u + c\frac{\lambda_1(-\Delta)}{a}, u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since 0 is a subsolution of $Q(\lambda_1(-\Delta)/a)$, we conclude that $Q(\lambda_1(-\Delta)/a)$ has a positive solution. This is absurd by the Fredholm alternative. If $c = 0$, that is, $f(t) \geq at$ for $t > 0$ we calculate as follows:

$$\begin{aligned}\int_\Omega \lambda_1 w(x)\varphi_1(x)\,dx &= \int_\Omega -\Delta\varphi_1(x)\cdot w(x)\,dx \\ &= \int_\Omega -\Delta w(x)\cdot\varphi_1(x)\,dx = \int_\Omega \frac{\lambda_1}{a} f(w(x))\varphi_1(x)\,dx \\ &\geq \int_\Omega \frac{\lambda_1}{a}\cdot aw(x)\varphi_1(x)\,dx = \int_\Omega \lambda_1 w(x)\varphi_1(x)\,dx\end{aligned}$$

where $\lambda_1 = \lambda_1(-\Delta)$.

This means that

$$\int_\Omega \frac{\lambda_1}{a}[aw(x) - f(w(x))]\varphi_1(x)\,dx = 0.$$

It follows that $f(t) = at$ for $t \in (0, \max_{x\in\Omega} w(x))$, which contradicts the assumption $f(0) > 0$.

5.2. In this subsection, we prove the uniqueness of solution of $P(\beta)$ for $\beta \in (0,\widetilde{\beta})$.

Suppose, by contradiction, that there exists a solution $v(\cdot,\beta)$ of $P(\beta)$ different from the minimal solution $u(\cdot,\beta) \in V$. Then $\lambda(\beta) \equiv \lambda_1(-\Delta - \beta f'(v(\cdot,\beta))) \leq 0$ (see Appendix). Let θ be the solution of the problem

$$\begin{cases} -\Delta\theta - \beta f'(v(\cdot,\beta))\theta = \lambda(\beta)\theta & \text{in } \Omega, \\ \theta > 0 & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\beta < \widetilde{\beta} \equiv \lambda_1(-\Delta)/a$ and $f'(x) \leq a$ for $x > 0$, we obtain

$$\beta f'(v(\cdot,\beta)) + \lambda(\beta) \leq \beta a + \lambda(\beta) \leq \beta a < \lambda_1(-\Delta).$$

This implies that

$$\begin{cases} \Delta\theta + (\beta a)\theta \geq 0 & \text{in } \Omega, \\ \theta = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, $\theta(x) \leq 0$ for $x \in \Omega$, a contradiction.

5.3. Now we prove that, for any compact set $K \subset \Omega$,

$$\lim_{\beta\to\widetilde{\beta}} \min_{x\in K} u(x,\beta) = +\infty$$

which implies that

$$\lim_{i\to\infty}\int_\Omega \beta_i \frac{f(u(x,\beta_i))}{\|u(\cdot,\beta_i)\|_{L^2(\Omega)}} W(x)\,dx = \widetilde{\beta} a \int_\Omega \widetilde{v}(x)W(x)\,dx$$
$$= \lambda_1(-\Delta)\int_\Omega \widetilde{v}(x)W(x)\,dx \qquad \text{for any } W\in C_0^\infty(\Omega).$$

Therefore we obtain

$$\begin{cases} \int_\Omega \nabla\widetilde{v}(x)\nabla W(x)\,dx = \lambda_1(-\Delta)\int_\Omega \widetilde{v}(x)W(x)\,dx \ \ \forall W\in C_0^\infty(\Omega), \\ \widetilde{v}\ge 0 \ \text{ a.e. in } \ \Omega, \ \widetilde{v}=0 \ \text{ on } \ \partial\Omega, \\ \|\widetilde{v}\|_{L^2(\Omega)} = 1, \end{cases}$$

which implies that $\widetilde{v}$ is the first eigenfunction of $-\Delta$ with zero Dirichlet condition on $\partial\Omega$. Furthermore

$$\int_\Omega \frac{\beta[au(x,\beta)+c]}{\|u(\cdot,\beta)\|_{L^2(\Omega)}}\widetilde{v}(x,\beta)\,dx \le \int_\Omega \frac{\beta f(u(x,\beta))}{\|u(\cdot,\beta)\|_{L^2(\Omega)}}\widetilde{v}(x,\beta)\,dx$$
$$\le \int_\Omega \frac{\beta[au(x)+c]}{\|u(\cdot,\beta)\|_{L^2(\Omega)}}\widetilde{v}(x,\beta)\,dx,$$

which implies that

$$\text{(H)} \qquad \lim_{i\to\infty}\int_\Omega |\nabla v(x,\beta_i)|^2\,dx = \lim_{i\to\infty}\int_\Omega \frac{\beta_i f(u(x,\beta_i))}{\|u(\cdot,\beta_i)\|_{L^2(\Omega)}} v(x,\beta_i)\,dx$$
$$= \lambda_1(-\Delta)\int_\Omega \widetilde{v}(x)^2\,dx = \lambda_1(-\Delta)\int_\Omega \varphi_1(x)^2\,dx$$
$$= \int_\Omega |\nabla\varphi_1(x)|^2\,dx.$$

By (H) and the fact that $\nabla v(\cdot,\beta_i) \to \nabla\varphi_1(\cdot)$ weakly in $L^2(\Omega)$ as $i\to\infty$, we conclude that $\nabla v(\cdot,\beta_i)\to\nabla\varphi_1(\cdot)$ strongly in $L^2(\Omega)$ as $i\to\infty$.

6. Proofs of Theorems 5 and 6

6.1. Proof of Theorem 5. Using the supersolution-subsolution method as in 5.1, we see that, for $\beta < \widetilde{\beta} = \lambda_1(-\Delta)/a$, $P(\beta)$ has a minimal solution $u(\cdot,\beta)$, and that for $\beta \ge \widetilde{\beta}$, $P(\beta)$ does not have a solution. To prove the uniqueness of solution of $P(\beta)$ for $\beta < \widetilde{\beta}$, we first note that, under our assumptions, $f(t)/t$ is strictly decreasing on $(0,\infty)$. Let $\widehat{u}(\cdot,\beta)$ be the minimal solution of $P(\beta)$ and $u'(\cdot,\beta)$ be another solution of $P(\beta)$. Then

$$\begin{cases} -\Delta\widehat{u} = \beta f(\widehat{u}), \\ -\Delta u' = \beta f(u'), \end{cases}$$

which implies

$$\int_\Omega \widehat{u}u' \left\{ \frac{f(u')}{u'} - \frac{f(\widehat{u})}{\widehat{u}} \right\} dx = 0.$$

Since $u' \geq \widehat{u} > 0$ (see Appendix) and $f(t)/t$ is strictly decreasing on $(0, \infty)$, we obtain $\widehat{u} = u'$, a contradiction.

For $\beta \in (0, \widetilde{\beta})$, let $v(\cdot, \beta)$ be the solution of the problem

$$\mathrm{Q}_\beta \qquad \begin{cases} -\Delta u = \beta(au + f(0)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 4 we see that, for any compact set $K \Subset \Omega$,

$$\lim_{\beta \to \widetilde{\beta}} \min_{x \in K} v(x, \beta) = +\infty.$$

From the supersolution-subsolution method, it is clear that $u(\cdot, \beta) \geq v(\cdot, \beta)$. Thus we have proved (2). Finally, as in the proof of 5.4, one easily can obtain the conclusion (3).

6.2. Proof of Theorem 6. As usual, by the supersolution-subsolution method, we know that, for $\beta < \widetilde{\beta}$, $P(\beta)$ has a minimal solution $u(\cdot, \beta)$; and for $\beta \geq \widetilde{\beta}$, $P(\beta)$ has no solution. By the implicit function theorem, there exist an interval $I = (0, \theta)$ and a neighbourhood V of 0 in $C^{2,\sigma}(\overline{\Omega})$ such that, for $\beta \in I$, $P(\beta)$ has a unique solution $w(\cdot, \beta)$ that belongs to V. Let $v(\cdot, \beta)$ be a solution of $P(\beta)$. By our assumption,

$$\begin{aligned} 0 &< \int_\Omega |\nabla v(x, \beta)|^2 \, dx \\ &\leq \beta a \int_\Omega v(x, \beta)^2 \, dx + \beta b \int_\Omega v(x, \beta) \, dx \\ &\leq \frac{\beta a}{\lambda_1(-\Delta)} \int_\Omega |\nabla v(x, \beta)|^2 \, dx + \frac{c\beta b}{\lambda_1(-\Delta)^{1/2}} \left(\int_\Omega |\nabla v(x, \beta)|^2 \, dx \right)^{1/2}, \end{aligned}$$

which implies, for $\beta < \frac{\lambda_1(-\Delta)}{2a}$,

$$\int_\Omega |\nabla v(x, \beta)|^2 \, dx \leq c\beta.$$

Thus

$$\lim_{\beta \to 0} \|v(\cdot, \beta)\|_{H^1(\Omega)} = 0,$$

which implies, by the L^p-estimate for solutions of P.D.E., that

$$\lim_{\beta \to 0} \|v(\cdot, \beta)\|_{C^{2,\sigma}(\overline{\Omega})} = 0.$$

Therefore $v(\cdot,\beta)$ must be in V for small β, and completes the proof of uniqueness of solution of $P(\beta)$ for small β. If $f(t)/t$ is strictly decreasing on $\mathbb{R}^+$, we prove, as usual, that $P(\beta)$ has a unique solution for $\beta \in (0,\widetilde{\beta})$. Finally, one can easily prove (2) and (3) as usual.

7. Appendix

Here we recall some well-known facts about the problem

$$P(\beta) \qquad \begin{cases} -\Delta u = \beta f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see [2], [5]).

THEOREM. *Let f be a convex positive C^2-function on $\mathbb{R}$ with $\lim_{x\to\infty} f'(x) > 0$. Then:*

(1) *there exist a maximal interval $I = (0,\widetilde{\beta})$, a neighbourhood V of 0 in $C^{2,\sigma}(\overline{\Omega})$, and a unique C^1-mapping A from $I\cup\{0\}$ into V such that $A(\beta) = u(\cdot,\beta)$ is the unique solution of $P(\beta)$ belonging to V, and $-\Delta-\beta f'(u(\cdot,\beta))$ is a bijective mapping from $C^{2,\sigma}(\overline{\Omega})$ to $C^{0,\sigma}(\overline{\Omega})$;*
(2) *$\lambda_1(-\Delta - \beta f'(u(\cdot,\beta))) > 0$ for $\beta \in I$;*
(3) *$u(\cdot,\beta) > 0$, and $\frac{\partial u}{\partial \beta}(\cdot,\beta) > 0$;*
(4) *$\widetilde{\beta} \le \lambda_1(-\Delta)/a$, where $a \equiv \inf_{t>0} f(t)/t$;*
(5) *$P(\beta)$ has no solution for $\beta > \widetilde{\beta}$;*
(6) *$u(\cdot,\beta)$ is the unique solution such that $\lambda_1(-\Delta - \beta f'(u(\cdot,\beta))) > 0$.*

Furthermore, if $v(\cdot,\beta)$ is another solution of $P(\beta)$, then $v(x,\beta) \ge u(x,\beta)$ for $x \in \Omega$. (Hence we call $u(\cdot,\beta)$ the minimal solution of $P(\beta)$.)

PROOF. (1) Let $X = \{u \in C^{2,\sigma}(\overline{\Omega}) \mid u = 0 \text{ on } \Omega\}$ and $Y = C^{0,\sigma}(\overline{\Omega})$. Define

$$F : X \times \mathbb{R} \to Y, \qquad F(u,\beta) \equiv -\Delta u - \beta f(u).$$

Since $F_u(0,0)v = -\Delta v$ for v in X, we have the conclusion (1) by the implicit function theorem.

(2) Because $\beta \to u(\cdot,\beta)$ is a C^1-mapping on I,

$$\begin{aligned} \lambda_1(\beta) &\equiv \lambda_1(-\Delta - \beta f'(u(\cdot,\beta))) \\ &\equiv \inf_{\substack{\|v\|_{L^2}=1 \\ v\in H_0^1(\Omega)}} \left\{ \int_\Omega \left(|\nabla v(x,\beta)|^2 - \beta f'(u(\cdot,\beta))v^2(x)\right) dx \right\} \end{aligned}$$

is a continuous function on I with $\lambda_1(0) = \lambda_1(-\Delta) > 0$. On the other hand, we note that for $\beta \in I, \lambda_1(\beta) \neq 0$. Otherwise, there would be a $w \neq 0$ in X such that

$$-\Delta w - \beta f'(u(\cdot,\beta))w(x) = 0,$$

which contradicts the fact that $-\Delta - \beta f'(u(\cdot,\beta))$ is bijective. Therefore we conclude that $\lambda_1(\beta) > 0$ for $\beta \in I$.

(3) Since $u(\cdot,\beta)$ is a solution of $P(\beta)$ and $\beta \to u(\cdot,\beta)$ is a C^1-mapping on I, we have

$$(**) \qquad \begin{cases} -\Delta\left(\frac{\partial u}{\partial \beta}\right) - \beta f'(u(\cdot,\beta))\frac{\partial u}{\partial \beta} = f(u(x,\beta)) & \text{in } \Omega, \\ \frac{\partial u}{\partial \beta} = 0 & \text{on } \partial\Omega. \end{cases}$$

By (2) and the fact that $f(x) > 0$ on $\mathbb{R}$, we see that $\partial u/\partial\beta \geq 0$ for $\beta \in I$, $x \in \Omega$. If $\frac{\partial u}{\partial \beta}(\bar{x},\beta) = 0$ at some point $\bar{x} \in \Omega$, then

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial \beta}(\bar{x},\beta)\right) = 0,$$

$i = 1, \ldots, n$, and the matrix

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial u}{\partial \beta}\right)(\bar{x},\beta)\right)$$

is positive semi-definite. On the other hand, from $(**)$,

$$-\Delta\frac{\partial u}{\partial \beta}(\bar{x},\beta) = f(u(\bar{x},\beta)) > 0,$$

a contradiction. Thus $\frac{\partial u}{\partial \beta}(x,\beta) > 0$ on Ω. From $u(\cdot,0) = 0$, it follows that $u(\cdot,\beta) > 0$ for $\beta \in I$.

(4) We note that, under our assumptions, for any $\varepsilon > 0$ there exists $b(\varepsilon) > 0$ such that $f(x) > (a-\varepsilon)x + b(\varepsilon)$ for $x > 0$. Using the supersolution-subsolution method as in 4.1, we find that $\widetilde{\beta} \leq \lambda_1(-\Delta)/(a-\varepsilon)$ for $\varepsilon > 0$, which implies the conclusion (4).

(5) Suppose, by contradiction, that there exist $\widehat{\beta} > \widetilde{\beta}$ and $\widehat{v} \in C^{2,\sigma}(\overline{\Omega})$ such that $\widehat{v}$ is a solution of $P(\widehat{\beta})$. Since f is convex, we have, for $\beta \in I$,

$$\begin{aligned}(\mathrm{I}) \qquad & -\Delta(\widehat{v} - u(\cdot,\beta)) - \beta f'(u(\cdot,\beta))(\widehat{v} - u(\cdot,\beta)) \\ &= \widehat{\beta} f(\widehat{v}) - \beta f(u(\cdot,\beta)) - \beta f'(u(\cdot,\beta))(\widehat{v} - u(\cdot,\beta)) \\ &\geq \beta\{f(\widehat{v}) - f(u(\cdot,\beta)) - f'(u(\cdot,\beta))(\widehat{v} - u(\cdot,\beta))\} \\ &\geq 0,\end{aligned}$$

which implies, by (2), that $\widehat{v}(x) \geq u(x,\beta)$ for $x \in \Omega$ and for $\beta \in I$. By (3), L^p-estimate of solution of P.D.E. and the Ascoli-Arzelà theorem, we infer that $u(\cdot,\beta)$ converges to a C^2-function $\widetilde{u}(\cdot)$ as $\beta \to \widetilde{\beta}$, and that $\widetilde{u}(\cdot)$ is a solution of $P(\widetilde{\beta})$. Since $I = (0,\widetilde{\beta})$ is the maximal interval where the property (1) holds, we have $\lambda_1(-\Delta - \widetilde{\beta}f'(u(\cdot,\widetilde{\beta}))) = 0$. That is, there exists a W in $H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta W - \widetilde{\beta}f'(\widetilde{u}(\cdot,\widetilde{\beta}))W = 0 & \text{in } \Omega, \\ W > 0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases}$$

From (I), it also follows that $-\Delta(\widehat{u} - \widetilde{u}) - \widetilde{\beta}f'(\widetilde{u})(\widehat{u} - \widetilde{u}) \geq 0$. Then we have

$$\begin{aligned} 0 &\leq \int (-\Delta(\widehat{v} - \widetilde{u})W - \widetilde{\beta}f'(\widetilde{u})(\widehat{v} - \widetilde{u})W)\,dx \\ &= \int ((\widehat{v} - \widetilde{u})(-\Delta W) - \widetilde{\beta}f'(\widetilde{u})W \cdot (\widehat{v} - \widetilde{u}))\,dx = 0, \end{aligned}$$

which implies that

$$-\Delta(\widehat{v} - \widetilde{u}) - \widetilde{\beta}f'(\widetilde{u})(\widehat{v} - \widetilde{u}) = 0.$$

This again yields

$$-\Delta(\widehat{v} - \widetilde{u}) = \widetilde{\beta}f'(\widetilde{u})(\widehat{v} - \widetilde{u}) \leq \widetilde{\beta}[f(\widehat{v}) - f(\widetilde{u})] \leq \widehat{\beta}f(\widehat{v}) - \widetilde{\beta}f(\widetilde{u}) = -\Delta(\widehat{v} - \widetilde{u}).$$

Hence $\widehat{\beta}f(\widehat{u}(x)) = \widetilde{\beta}f(\widetilde{u}(x))$ for $x \in \Omega$. Letting $x \to \partial\Omega$, we obtain $\widehat{\beta} = \widetilde{\beta}$, a contradiction.

(6) Suppose that, for $\beta \in I$, $v(\cdot,\beta)$ is a solution of $P(\beta)$ different from $u(\cdot,\beta) \equiv A(\beta)$. Then, since f is convex, we see that

$$-\Delta(v - u) - \beta f'(u(\beta))(v - u) = \beta[f(v) - f(u) - f'(u)(v - u)] \geq 0.$$

From (2), it follows that $v(\cdot,\beta) \geq u(\cdot,\beta)$. At the same time, if $\lambda_1(\Delta - \beta f'(v(\cdot,\beta)) > 0$, we prove in the same way that $u(\cdot,\beta) \geq v(\cdot,\beta)$, a contradiction.

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