Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 2, 1993, 277–291

A GENERALIZATION OF FAN-BROWDER'S FIXED POINT THEOREM AND ITS APPLICATIONS

Ezio Marchi[1] — Juan-Enrique Martínez-Legaz[2]

(*Submitted by M. Lassonde*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

The famous Knaster-Kuratowski-Mazurkiewicz theorem is a fundamental result of nonlinear analysis. More than twenty years ago, Peleg, in a paper on the existence of equilibrium points in many-person games [14], gave an interesting generalization of this theorem, concerning closed subsets of a product of simplexes. But it seems that this result of Peleg has not been exploited in nonlinear analysis. The purpose of this paper is to obtain, from Peleg's theorem, some generalizations of fundamental results in this field.

[1] The first author thanks the "Dirección General de Investigación Científica y Técnica (DGICYT)" of Spain for the financial support of his stay in the University of Barcelona. He acknowledges the warm hospitality of the Department of Applied Mathematics and Analysis during his stay.

[2] Financial support from the "Dirección General de Investigación Científica y Técnica (DGICYT)", under project PB92-0615, is gratefully acknowledged.

In Section 2, we give an extension to topological spaces of Peleg's theorem which is parallel to the Horvath's extension [8] of Knaster-Kuratowski-Mazurkiewicz's theorem.

The main result in Section 3 is a generalization of Fan-Browder's fixed point theorem, involving multi-valued mappings from the product of a finite family of H-spaces (in the sense of Bardaro and Ceppitelli [1]) into each of its factors, obtained from the basic lemma of the preceding section.

Section 4 is devoted to Ky Fan type inequalities, which, analogously to the classical case, are shown to be equivalent to their corresponding Fan-Browder type theorems of Section 3.

Finally, in Section 5, we present a generalization of Ky Fan's intersection theorem for sets with convex sections.

We shall denote by "*co*" the usual convex hull operator in a vector space. Given a multi-valued mapping between two sets, $T : A \to 2^B$, we write $T^{-1}(y) = \{x \in A \mid y \in T(x)\}$ $(y \in B)$. If x is an element of a cartesian product $X_1 \times \ldots \times X_m$, its k-th component will be denoted by x_k, whereas $x^{\widehat{k}}$ will represent the element in $X_{\widehat{k}} = X_1 \times \ldots \times X_{k-1} \times X_{k+1} \times \ldots \times X_m$ obtained from x by deleting its k-th component; in this way, $(x^{\widehat{k}}, x^k)$ may be regarded as identical to x and, more generally, an element of the form $(x^{\widehat{k}}, y^k)$, with $y^k \in X_k$, will be interpreted as the point of X obtained from x by replacing its k-th component by y^k.

2. The fundamental lemmas

Our starting point is the following generalization, due to Peleg [13], of the famous Knaster-Kuratowski-Mazurkiewicz theorem:

LEMMA 1 [14]. *For $k = 1, \ldots, m$ let N_k be a nonempty finite set and*

$$S_k = \left\{\alpha^k : N_k \to \mathbb{R} \,\middle|\, \alpha^k(i) \geq 0 \text{ for all } i \in N_k,\ \sum_{i \in N_k} \alpha^k(i) = 1\right\}.$$

If C_i^k, $i \in N_k$, $k = 1, \ldots, m$, are closed subsets of $S = S_1 \times \ldots \times S_m$ such that for each $Q \subset N_k$, $k = 1, \ldots, m$,

$$\bigcup_{j \in Q} C_j^k \supset \{(\alpha^1, \ldots, \alpha^m) \in S \mid \alpha^k(i) = 0 \text{ for all } i \in N_k \setminus Q\},$$

then

$$\bigcap_{k=1}^{m} \bigcap_{i \in N_k} C_i^k \neq \emptyset.$$

When $m = 1$, the preceding lemma reduces to the classical Knaster-Kuratowski-Mazurkiewicz theorem. In the same way as Horvath obtained a generalization of the latter to topological spaces, with a family of nonempty contractible subsets playing the role of the faces of a simplex, we shall state below (Lemma 3) the corresponding more general version of Lemma 1. We shall use the following lemma, which was proved by Horvath (see the proof of Theorem 1 in [8]):

LEMMA 2 [8]. *Let X be a topological space, N a nonempty finite set and, for each $A \subset N$,*

$$S_A = \left\{ \alpha : N \to \mathbb{R} \;\middle|\; \alpha(i) \geq 0 \text{ for all } i \in N, \right.$$
$$\left. \sum_{i \in N} \alpha(i) = 1,\; \alpha(i) = 0 \text{ for all } i \in N \setminus A \right\}.$$

If $\{\Gamma_A\}_{\emptyset \neq A \subset N}$ is a family of nonempty contractible subsets of X, indexed by the nonempty subsets of N, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then there is a continuous function $f : S_N \to X$ such that $f(S_A) \subset \Gamma_A$ for each nonempty subset $A \subset N$. □

LEMMA 3. *For $k = 1, \ldots, m$, let X_k be a topological space, N_k a nonempty finite set and $\{\Gamma^k_A\}_{\emptyset \neq A \subset N_k}$ a family of nonempty contractible subsets of X_k such that $A \subset B$ implies $\Gamma^k_A \subset \Gamma^k_B$. If C^k_i, $i \in N_k$, $k = 1, \ldots, m$, are closed subsets of $X = X_1 \times \ldots \times X_m$ such that for each $(A_1, \ldots, A_m) \in (2^{N_1} \setminus \{\emptyset\}) \times \ldots \times (2^{N_m} \setminus \{\emptyset\})$,*

$$\Gamma^1_{A_1} \times \ldots \times \Gamma^m_{A_m} \subset \bigcap_{k=1}^{m} \bigcup_{i \in A_k} C^k_i,$$

then

$$\bigcap_{k=1}^{m} \bigcap_{i \in N_k} C^k_i \neq \emptyset.$$

PROOF. By Lemma 1, for each $k = 1, \ldots, m$ there is a continuous function $f_k : S_{N_k} \to X_k$ such that $f_k(S_{A_k}) \subset \Gamma^k_{A_k}$ for every nonempty subset $A_k \subset N_k$, with S_{A_k} denoting the set

$$S_{A_k} = \left\{ \alpha : N_k \to \mathbb{R} \;\middle|\; \alpha(i) \geq 0 \text{ for all } i \in N_k,\; \sum_{i \in N_k} \alpha(i) = 0 \text{ for all } i \in N \setminus A_k \right\}.$$

Let $f : S_{N_1} \times \ldots \times S_{N_m} \to X_1 \times \ldots \times X_m$ be the mapping defined by

$$f(\alpha^1, \ldots, \alpha^m) = (f_1(\alpha^1), \ldots, f_m(\alpha^m)).$$

Since f is continuous, the sets $f^{-1}(C_i^k)$ are closed for all $i \in N_k$, $k = 1, \ldots, m$. For any $(A_1, \ldots, A_m) \in (2^{N_1} \setminus \{\emptyset\}) \times \ldots \times (2^{N_m} \setminus \{\emptyset\})$, we have

$$\begin{aligned} S_{A_1} \times \ldots \times S_{A_m} &\subset f^{-1}(f(S_{A_1} \times \ldots \times S_{A_m})) \\ &= f^{-1}(f_1(S_{A_1}) \times \ldots \times f_m(S_{A_m})) \\ &\subset f^{-1}(\Gamma^1_{A_1} \times \ldots \times \Gamma^m_{A_m}) \subset f^{-1}\Big(\bigcap_{k=1}^m \bigcup_{i \in A_k} C_i^k \Big) \\ &= \bigcap_{i=1}^m \bigcup_{i \in A_k} f^{-1}(C_i^k) \end{aligned}$$

whence, by Lemma 1,

$$f^{-1}\Big(\bigcap_{k=1}^m \bigcap_{i \in A_k} C_i^k \Big) = \bigcap_{k=1}^m \bigcap_{i \in A_k} f^{-1}(C_i^k) \neq \emptyset.$$

Therefore, $\bigcap_{k=1}^m \bigcap_{i \in N_k} C_i^k \neq \emptyset$. □

The preceding lemma states that a sufficient condition for the intersection of the closed sets C_i^k to be nonempty is the existence of families $\{\Gamma_A^k\}_{\emptyset \neq A \subset N_k}$ satisfying certain properties. In fact, this is also a necessary condition, since, when $(\overline{x}_1, \ldots, \overline{x}_m) \in \bigcap_{k=1}^m \bigcap_{i \in N_k} C_i^k$, one can define $\Gamma^k_{A_k} = \{\overline{x}_k\}$ for $A_k \in 2^{N_k} \setminus \{\emptyset\}$, $k = 1, \ldots, m$.

LEMMA 4. *The statement of Lemma 3 remains valid if the N_k's are arbitrary nonempty (possibly infinite) sets and the families $\{\Gamma_A^k\}$ are indexed by the nonempty finite subsets of N_k, under the additional assumption that there exists $k \in \{1, \ldots, m\}$ and $i \in N_k$ such that C_i^k is compact.*

PROOF. By Lemma 3, the family $\{C_i^k\}_{k=1,\ldots,m, i \in N_k}$ has the finite intersection property whence, as some C_i^k is compact, it has a nonempty intersection. □

As a particular case of Lemma 4, we obtain the following infinite dimensional version of Lemma 1; for $m = 1$, it reduces to Ky Fan's generalization [5] of the Knaster-Kuratowski-Mazurkiewicz theorem:

COROLLARY 5. *For $k = 1, \ldots, m$ let X_k be an arbitrary set in a Hausdorff topological vector space Y_k and let $F_k : X_k \to 2^Y$ be a mapping taking closed values* *in the product space* $Y = Y_1 \times \ldots \times Y_m$ *such that the following conditions are satisfied:*

(i) *If, for each* $k = 1, \ldots, m$, N_k *is a nonempty finite subset of* X_k, *then*

$$co\, N_1 \times \ldots \times co\, N_m \subset \bigcap_{k=1}^{m} \bigcup_{x^k \in N_k} F_k(x^k).$$

(ii) *There exists* $k \in \{1, \ldots, m\}$ *and* $x^k \in X_k$ *such that* $F_k(x^k)$ *is compact.*

Then

$$\bigcap_{k=1}^{m} \bigcap_{x^k \in X_k} F(x^k) \neq \emptyset.$$

A more general version of Corollary 5 can be easily obtained by adapting the proof of Lemma 1 in [3], replacing the assumption that the mappings F_k are closed-valued and that one of them has a compact value by the following weaker hypotheses:

(i) For each $k = 1, \ldots, m$ and every $x^k \in X_k$, the intersection of $F_k(x^k)$ with any product of finite dimensional subspaces is closed.

(ii) If, for each $k = 1, \ldots, m, D_k$ is a finite dimensional subspace of Y_k, then

$$\overline{\bigcap_{k=1}^{m} \bigcap_{x^k \in X_k \cap D_k} F_k(x^k)} \cap (D_1 \times \ldots \times D_m) = \bigcap_{k=1}^{m} \bigcap_{x^k \in X_k \cap D_k} F_k(x^k) \cap (D_1 \times \ldots \times D_m).$$

(iii) There exists $k \in \{1, \ldots, m\}$ and $x^k \in X_k$ such that $\overline{F_k(x^k)}$ is compact.

Since we shall not use this stronger version of Corollary 5, we omit the proof.

Corollary 5 also follows from Theorem 1 in Lassonde and Schenkel [10], which deals with the case where each Y_k is a convex space (that is, a convex set in a vector space, supplied with any topology that induces the Euclidean topology on the convex hulls of its finite subsets).

All results we shall present in the subsequent sections are based on Lemma 4. The abstract setting in which this lemma finds its applicability is that of H-spaces, as defined by Bardaro and Ceppitelli [1]:

DEFINITIONS. An *H-space* is a pair $(X, \{\Gamma_A\})$ consisting of a topological space X and a family $\{\Gamma_A\}$ of nonempty finite contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. A set $D \subset X$ is called *H-convex* if, for every nonempty finite $A \subset D$, $\Gamma_A \subset D$.

We observe that, for any H-space $(X, \{\Gamma_A\})$, the pair $(X, \mathcal{C})$, where $\mathcal{C}$ is the family of all H-convex subsets of X, is an aligned space in the sense of Jamison-Waldner [9], i.e., the following properties hold:

1. $\emptyset$ and X are H-convex.
2. An arbitrary intersection of H-convex sets is H-convex.
3. The union of any family of H-convex sets totally ordered by inclusion is H-convex.

In particular, by 1. and 2., for any set $C \subset X$ there exists a smallest H-convex set containing C (namely, the intersection of all H-convex sets containing it), which will be called the *H-convex hull* of C; we shall denote it by $co_H C$. The notion of alignment is important in the study of abstract convexity [9].

The main example of H-space corresponds to the case when X is a convex subset of a Hausdorff topological vector space and, for every nonempty finite $A \subset X, \Gamma_A$ is the convex hull of A. Then the notion of H-convexity reduces to the usual convexity. All results we shall give in the following sections are new, even for this particular case of H-spaces. Although their main interest lies in their interpretation in this more restricted setting, we prefer to present them in the abstract framework of H-spaces, since neither the statements nor the proofs are more complicated at this level of generality.

3. A generalization of Fan-Browder's fixed point theorem

Throughout this section, we shall denote by $(X_k, \{\Gamma_A^k\})$, $k = 1, \ldots, m$, an H-space for which X_k is nonempty and compact and by X the product $X_1 \times \ldots \times X_m$. Based on Lemma 4, we shall first establish a generalization of Fan-Browder's fixed point theorem:

THEOREM 6. *For $k = 1, \ldots, m$ let $T_k : X \to 2^{X_k}$ be such that, for each $x^k \in X_k$, $T_k^{-1}(x^k)$ is open in X. If for each $x \in X$ there exists $k = k(x) \in \{1, \ldots, m\}$ for which $T_k(x) \neq \emptyset$, then there exists $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^m) \in X$ and $k \in \{1, \ldots, m\}$ such that $\overline{x}^k \in co_H T_k(\overline{x})$.*

PROOF. For each k, let $F_k : X_k \to 2^X$ be the mapping defined by $F_k(x^k) = X \setminus T_k^{-1}(x^k)$. Clearly, F_k is compact-valued. We have

$$\bigcap_{k=1}^{m} \bigcap_{x^k \in X_k} F_k(x^k) = X \setminus \bigcup_{k=1}^{m} \bigcup_{x^k \in X_k} T^{-1}(x^k) = \emptyset,$$

since any $x \in X$ has $T_k(x) \neq \emptyset$ for some k. Therefore, by Lemma 4, there exist nonempty finite subsets $N_k \subset X_k$, $k = 1, \ldots, m$, such that

$$\Gamma^1_{N_1} \times \ldots \times \Gamma^m_{N_m} \not\subset \bigcap_{k=1}^{m} \bigcup_{x^k \in N_k} F_k(x^k).$$

Take

$$\overline{x} = (\overline{x}^1, \ldots, \overline{x}^m) \in \Gamma^1_{N_1} \times \cdots \times \Gamma^m_{N_m} \setminus \bigcap_{k=1}^{m} \bigcup_{x^k \in N_k} F_k(x^k).$$

For some $k \in \{1, \ldots, m\}$ we have $\overline{x} \notin F_k(x^k)$ for all $x^k \in N_k$ or, equivalently, $N_k \subset T_k(\overline{x})$. Hence, we obtain

$$\overline{x}^k \in \Gamma^k_{N_k} \subset co_H T_k(\overline{x}).$$

□

Fan-Browder's fixed point theorem [5, 4] corresponds to the particular case $m = 1$ in the preceding theorem, when X_1 is a compact convex subset of a Hausdorff topological vector space and Γ^1_A is the convex hull of A. From Theorem 6 one easily obtains the following fixed point result, which is also a generalization of Fan-Browder's.

COROLLARY 7. *Let $T : X \to 2^X$ be such that, for each $x \in X$, the sets $\{y^k \in X_k \mid (x^{\widehat{k}}, y^k) \in T(x)\}$, $k = 1, \ldots, m$, are H-convex (or empty) and at least one of them is nonempty and for each $k = 1, \ldots, m$ and $y^k \in X_k$ the set $\{x \in X \mid (x^{\widehat{k}}, y^k) \in T(x)\}$ is open in X. Then T has a fixed point.*

PROOF. The mappings $T_k : X \to 2^{X_k}$ defined by $T_k(x) = \{y^k \in X_k \mid (x^{\widehat{k}}, y^k) \in T(x)\}$ satisfy the hypotheses of Theorem 6, whence there exist $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^m) \in X$ and $k \in \{1, \ldots, m\}$ such that $\overline{x}^k \in co_H T_k(\overline{x}) = T_k(\overline{x})$. But this means that $(\overline{x}^{\widehat{k}}, \overline{x}^k) \in T(\overline{x})$, i.e., that $\overline{x}$ is a fixed point of T. □

In the case when the X_k's are compact convex subsets of Hausdorff topological vector spaces and Γ^k_A denotes the convex hull of A, the mapping T satisfies the assumptions of Corollary 7 if, for each $x \in X$, the image $T(x)$ is multiconvex (in the sense of [11], i.e., its sections in each component are convex; see also [2]) and contains some point differing from x in at most one component and, for each $k \in \{1, \ldots, m\}$ and each $y^k \in X_k$, the set $\{(x, y^{\widehat{k}}) \in X \times X_{\widehat{k}} \mid (y^{\widehat{k}}, y^k) \in T(x)\}$ is open in X.

From Theorem 6, we get the following generalization of Lemma 4 in [5]:

COROLLARY 8. *Let $A_k \subset X_k \times X$, $k = 1, \dots, m$, be such that the following conditions are satisfied:*

(i) *For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid (y^k, x) \in A_k\}$ is closed in X.*

(ii) *For any $x = (x^1, \dots, x^m) \in X$ and each $k = 1, \dots, m$, $(x^k, x) \in A_k$.*

(iii) *For any $x \in X$, the sets $\{y^k \in X_k \mid (y^k, x) \notin A_k\}$, $k = 1, \dots, m$, are H-convex (or empty).*

Then there exists $\overline{x} \in X$ such that $X_k \times \{\overline{x}\} \subset A_k$ for each $k = 1, \dots, m$.

PROOF. For each k define $T_k : X \to 2^{X_k}$ by $T_k(x) = \{y^k \in X_k \mid (y^k, x) \notin A_k\}$. By assumptions (iii) and (i), the images $T_k(x)$, $x \in X$, are H-convex (or empty) and the inverse images $T_k^{-1}(y^k)$, $y^k \in X_k$, are open in X. On the other hand, by (ii), we have $x^k \notin T_k(x)$ for every $x = (x^1, \dots, x^m) \in X$ and each $k \in \{1, \dots, m\}$. Therefore, by Theorem 6, there exists $\overline{x} \in X$ such that $T_k(\overline{x}) = \emptyset$ for each k. But the emptiness of $T_k(\overline{x})$ is clearly equivalent to the inclusion $X_k \times \{\overline{x}\} \subset A_k$. □

Assumption (i) in the preceding corollary holds when the sets A_k are closed in $X_k \times X$.

COROLLARY 9. *Let $A \subset X \times X$ be such that the following conditions are satisfied:*

(i) *For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid (x^{\widehat{k}}, y^k, x) \in A\}$ is closed in X.*

(ii) *For any $x \in X$, $(x, x) \in A$.*

(iii) *For any $x \in X$, the sets $\{y^k \in X_k \mid (x^{\widehat{k}}, y^k, x) \notin A\}$, $k = 1, \dots, m$, are H-convex (or empty).*

Then there exists $\overline{x} \in X$ such that

$$\left(\bigcup_{k=1}^{m} (\{\overline{x}^{\widehat{k}}\} \times X_k) \right) \times \{\overline{x}\} \subset A.$$

PROOF. For each k, let $A_k = \{(y^k, x) \in X_k \times X \mid (x^{\widehat{k}}, y^k, x) \in A\}$. These sets satisfy the hypotheses of Corollary 8, whence there exists $\overline{x} \in X$ such that $X_k \times \{\overline{x}\} \subset A_k$ for each k. But this is equivalent to $(\bigcup_{k=1}^{m} (\{\overline{x}^{\widehat{k}}\} \times X_k)) \times \{\overline{x}\} \subset A$. □

Alternatively, Corollary 9 could have been proved from Corollary 7, using the mapping $T : X \to 2^X$ defined by $T(x) = \{y \in X \mid (y, x) \notin A\}$.

Assumption (i) in Corollary 9 is satisfied when A is closed, while condition (iii) holds when the sets $\{y \in X \mid (y,x) \notin A\}$, $x \in X$, are multiconvex (in the case of the X_k's being compact convex subsets of Hausdorff topological vector spaces and the families of contractible subsets being the convex polytopes).

We observe that the set $\bigcup_{k=1}^{m}(\{\overline{x}^{\widehat{k}}\} \times X_k)$ appearing in the conclusion of Corollary 9 consists of those points in X which differ from $\overline{x}$ in at most one component.

4. Ky Fan type inequalities

Throughout this section, as in the preceding one, $(X_k, \{\Gamma_A^k\})$, $k = 1, \dots, m$, will denote an H-space, with X_k being nonempty and compact; X will represent the product $X_1 \times \dots \times X_m$. A function $g : X_k \to \mathbb{R}$ will be called *H-quasiconcave* if the sets $g^{-1}([\lambda, +\infty))$ are H-convex, and *H-quasiconvex* if $-g$ is H-quasiconcave. Our first result generalizes Lemma 1.3 in [12], which is in turn a generalization of the famous Ky Fan minimax inequality [7]:

THEOREM 10. *Let $f_k : X \times X_k \to \mathbb{R}$, $G_k : X \to 2^{X_k}$, $k = 1, \dots, m$, be such that, for every $x^k \in X_k$ and $x \in X$,*

(i) *$f_k(\cdot, x^k)$ is lower semicontinuous,*
(ii) *$f_k(x, \cdot)$ is H-quasiconcave,*
(iii) *$G_k^{-1}(x^k)$ is open in X, and*
(iv) *$G_k(x)$ is H-convex.*

Then there exists $\overline{x} \in X$ such that

$$\sup_{y^k \in G_k(\overline{x})} f_k(\overline{x}, y^k) \leq \sup_{x^k \in G_k(x)} f_k(x, x^k)$$

for each $k = 1, \dots, m$.

PROOF. For each k, let $\mu_k = \sup_{x^k \in G_k(x)} f_k(x, x^k)$ and $A_k = \{(y^k, x) \in X_k \times X \mid f_k(x, y^k) \leq \mu_k \text{ or } y^k \notin G_k(x)\}$. The sets A_k satisfy the conditions of Corollary 8, whence there exists $\overline{x} \in X$ such that $X_k \times \{\overline{x}\} \subset A_k$ for each k. But it is easy to check that the inclusion $X_k \times \{\overline{x}\} \subset A_k$ is equivalent to the inequality

$$\sup_{y^k \in G_k(\overline{x})} f_k(\overline{x}, y^k) \leq \sup_{x^k \in G_k(x)} f_k(x, x^k).$$

□

The assumptions of the preceding theorem can be replaced by the following weaker conditions:

(i) For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid f_k(x, y^k) \leq \mu_k\} \cup (X \setminus G_k^{-1}(y^k))$ (with μ_k as in the proof of Theorem 10) is closed in X.

(ii) For any $x \in X$, the sets $\{y^k \in G_k(x) \mid f_k(x, y^k) > \mu_k\}$, $k = 1, \dots, m$, are H-convex (or empty).

Indeed, these conditions are equivalent to (i) and (iii) of Corollary 8, respectively, for the sets A_k defined in the preceding proof.

Taking $G_k(x) \equiv X_k$, $k = 1, \dots, m$, in Theorem 10, one obtains a version of Ky Fan's minimax inequality with several functions; the classical inequality corresponds to the case when $m = 1$, X_1 is a compact convex subset of a Hausdorff topological vector space and Γ_A^1 is the convex hull of A.

We derived Theorem 10 from Corollary 8 and the latter from Theorem 6. In the converse direction, Theorem 6 can be easily deduced from Theorem 10 applied to the functions $f_k : X \times X_k \to \mathbb{R}$ defined by $f_k(x, y^k) = 1$ if $y^k \in T_k(x)$, 0 otherwise, and to the mappings $G_k : X \to 2^{X_k}$ given by $G_k(x) \equiv X_k$.

COROLLARY 11. *Let $f : X \times X \to \mathbb{R}$ and $G : X \to 2^X$ be such that, for each $k = 1, \dots, m$,*

(i) the function $x \in X \to f(x, x^{\widehat{k}}, y^k) \in \mathbb{R}$ is lower semicontinuous for any $y^k \in X_k$,

(ii) $f(x, x^{\widehat{k}}, \cdot)$ is H-quasiconcave for any $x \in X$,

(iii) $\{x \in X \mid (x^{\widehat{k}}, y^k) \in G(x)\}$ is open in X for any $y^k \in X_k$, and

(iv) $\{y^k \in X_k \mid (x^{\widehat{k}}, y^k) \in G(x)\}$ is H-convex for any $x \in X$.

Then there exists $\overline{x} \in X$ such that

$$\sup_{1 \leq k \leq m} \sup_{(\overline{x}^{\widehat{k}}, y^k) \in G(\overline{x})} f(\overline{x}, \overline{x}^{\widehat{k}}, y^k) \leq \sup_{x \in G(x)} f(x, x).$$

PROOF. Apply Theorem 10 to the functions $f_k : X \times X_k \to \mathbb{R}$ defined by $f_k(x, y^k) = f(x, x^{\widehat{k}}, y^k)$ and the mappings $G_k : X \to 2^{X_k}$ given by $G_k(x) = \{y^k \in X_k \mid (x^{\widehat{k}}, y^k) \in G(x)\}$, observing that, for each k and $x = (x^1, \dots, x^m) \in X$, one has $f_k(x, x^k) = f(x, x)$ and that the conditions $x^k \in G_k(x)$ are equivalent to $x \in G(x)$. $\square$

According to the observation we made after Theorem 10, the assumptions on f and G in the preceding corollary can be replaced by the following weaker conditions:

(i) For each $k = 1, \dots, m$ and $y^k \in X_k$, the set $\{x \in X \mid f(x, x^{\widehat{k}}, y^k) \leq \sup_{x \in G(x)} f(x,x)$ or $(x^{\widehat{k}}, y^k) \notin G(x)\}$ is closed in X.

(ii) For any $x \in X$, the sets

$$\{y^k \in X_k \mid (x^{\widehat{k}}, y^k) \in G(x),\ f_k(x, y^k) > \sup_{x \in G(x)} f(x,x)\},$$

$k = 1, \dots, m$, are H-convex (or empty).

Taking $G(x) \equiv X$ in Corollary 11, for the case when the X_k's are compact convex subsets of Hausdorff topological vector spaces and Γ_A^k denotes the convex hull of A, one obtains a version of Ky Fan's minimax inequality valid for the case when the usual quasiconcavity assumption on the functions $f(x, \cdot)$ is relaxed to multi-quasiconcavity (i.e., quasiconcavity in each component [2]), which can be expressed in the following way:

$$\sup_{y \sim \overline{x}} f(\overline{x}, y) \leq \sup_{x \in X} f(x,x),$$

the relation $y \sim \overline{x}$ meaning that y differs from $\overline{x}$ in at most one component. Again when $m = 1$, this coincides with the usual Ky Fan inequality.

An alternative proof of Corollary 11 can be obtained from Corollary 9 applied to $A = \{(y,x) \in X \times X \mid f(x,y) \leq \sup_{x \in G(x)} f(x,x) \text{ or } y \notin G(x)\}$. We have already observed that Corollary 9 is an immediate consequence of Corollary 7. Conversely, Corollary 7 can be easily derived from Corollary 11, by applying it to the function $f : X \times X \to \mathbb{R}$ defined by $f(x,y) = 1$ if $y \in T(x)$, 0 if $y \notin T(x)$ and to the mapping $G : X \to 2^X$ given by $G(x) \equiv X$.

COROLLARY 12. *Let $g_k : X \times X_k \to \mathbb{R}$, $k = 1, \dots, m$, be continuous and such that $g_k(x, \cdot)$ is H-quasiconvex for every $x \in X$. Then there exists $\overline{x} = (\overline{x}^1, \dots, \overline{x}^m) \in X$ such that $g_k(\overline{x}, \overline{x}^k) = \min_{y^k \in X_k} g_k(\overline{x}, y^k)$ for each $k = 1, \dots, m$.*

PROOF. This follows from Theorem 10 applied to the functions $f_k : X \times X_k \to \mathbb{R}$ defined by $f_k(x, y^k) = g_k(x, x^k) - g_k(x, y^k)$ and the mappings $G_k : X \to 2^{X_k}$ given by $G_k(x) \equiv X_k$, observing that $\sup_{y^k \in X_k} f_k(\overline{x}, y^k) = g_k(\overline{x}, \overline{x}^k) - \inf_{y^k \in X_k} g_k(x, y^k)$ and $f_k(x, x^k) = 0$ for any $x = (x^1, \dots, x^m) \in X$. □

When one takes $m = 1$, X_1 a compact convex subset of a Hausdorff topological vector space and Γ_A^1 as the convex hull of A in Corollary 12, it reduces to Corollary

1 in [7]. On the other hand, Nash's Theorem on the existence of equilibrium points in n-person games [13] is also an immediate consequence of Corollary 12. Indeed, if the X_k's are compact convex subsets of Hausdorff topological vector spaces and $f_k : X \to \mathbb{R}$, $k = 1, \dots, m$, are continuous multiquasiconcave functions, then applying Corollary 12 to the functions $g_k : X \times X_k \to \mathbb{R}$ defined by $g_k(x, y^k) = -f(x^{\widehat{k}}, y^k)$ one gets the existence of $\overline{x} = (\overline{x}^1, \dots, \overline{x}^m) \in X$ such that $f_k(\overline{x}) = \max_{y^k \in X_k} f_k(\overline{x}^{\widehat{k}}, y^k)$ for each $k = 1, \dots, m$.

COROLLARY 13. *Let $g : X \times X \to \mathbb{R}$ be continuous and such that, for each $k = 1, \dots, m$, the function $g(x, x^k, \cdot)$ is H-quasiconvex for any $x \in X$. Then there exists $\overline{x} \in X$ such that $g(\overline{x}, \overline{x}) = \min_{y^k \in X_k} g(\overline{x}, \overline{x}^{\widehat{k}}, y^k)$ for each $k = 1, \dots, m$.*

PROOF. For each k define $g_k : X \times X_k \to \mathbb{R}$ by $g_k(x, y^k) = g(x, x^{\widehat{k}}, y^k)$. These functions satisfy the assumptions of Corollary 12, whence there exists $\overline{x} = (\overline{x}^1, \dots, \overline{x}^m) \in X$ such that

$$g(\overline{x}, \overline{x}) = g_k(\overline{x}, \overline{x}^k) = \min_{y^k \in X_k} g_k(\overline{x}, y^k) = \min_{y^k \in X_k} g(\overline{x}, \overline{x}^{\widehat{k}}, y^k)$$

for each k. □

The preceding corollary could have been deduced alternatively from Corollary 11, applied to $f : X \times X \to \mathbb{R}$ defined by $f(x, y) = g(x, x) - g(x, y)$ and $G : X \to 2^X$ given by $G(x) \equiv X$.

The conclusion in Corollary 13 can be expressed as $g(\overline{x}, \overline{x}) = \min_{y \sim \overline{x}} g(\overline{x}, y)$, $y \sim \overline{x}$ denoting that y differs from $\overline{x}$ in at most one component. This is weaker than the conclusion of Corollary 1 in [7], where $y \in X$ appears instead of $y \sim \overline{x}$. However, the quasiconvexity assumption on the functions $g(x, \cdot)$ in the latter result has been relaxed to a weak form of multiquasiconvexity in Corollary 1.3 (leaving aside the more abstract framework to which it belongs).

5. A geometric theorem on sets with convex sections

In this section, $(X_{k,i}, \{\Gamma_A^{k,i}\})$, $i = 1, \dots, n_k$ $(n_k \geq 2)$, $k = 1, \dots, m$, denote H-spaces for which $X_{k,i}$ is nonempty and compact, $X_k = X_{k,1} \times \dots \times X_{k,n_k}$, $k = 1, \dots, m$, and $X = X_1 \times \dots \times X_m$. Our first result generalizes Theorem 7 in [7] (which corresponds to the particular case when $m = 1$, each $X_{1,i}$ is a compact convex subset of a Hausdorff topological vector space and $\Gamma_A^{1,i}$ is the convex hull of A).

THEOREM 14. *Let* $f_{k,i} : X \to \mathbb{R}$, $i = 1, \ldots, n_k$, $k = 1, \ldots, m$, *be such that*

(i) *the function* $X_{\widehat{k}} \times X_{k,\widehat{i}} \ni (x^{\widehat{k}}, x^{k,\widehat{i}}) \to f_{k,i}(x^{\widehat{k}}, x^{k,\widehat{i}}, x^{k,i}) \in \mathbb{R}$ *is lower semicontinuous for any* $x^{k,i} \in X_{k,i}$, *and*

(ii) *the function* $X_{k,i} \ni x^{k,i} \to f_{k,i}(x^{\widehat{k}}, x^{k,\widehat{i}}, x^{k,i}) \in \mathbb{R}$ *is H-quasiconcave for any* $(x^{\widehat{k}}, x^{k,\widehat{i}}) \in X_{\widehat{k}} \times X_{k,\widehat{i}}$.

Let $t_{k,i} \in \mathbb{R}$, $i = 1, \ldots, n_k$, $k = 1, \ldots, m$. *If for each*

$$x = ((x^{1,1}, \ldots, x^{1,n_1}), \ldots, (x^{m,1}, \ldots, x^{m,n_m})) \in X,$$

there exists $k = k(x) \in \{1, \ldots, m\}$ *such that for every* $i = 1, \ldots, n_k$, *there is a point* $\widetilde{y}^{k,i} \in X_{k,i}$ *for which* $f_{k,i}(x^{\widehat{k}}, x^{k,\widehat{i}}, \widetilde{y}^{k,i}) > t_{k,i}$, *then there exist* $\overline{k} \in \{1, \ldots, m\}$ *and* $\overline{x} \in X$ *such that* $f_{\overline{k},i}(\overline{x}) > t_{\overline{k},i}$, $i = 1, \ldots, n_{\overline{k}}$.

PROOF. For each k and $i \in \{1, \ldots, n_k\}$, define $f_k : X \times X_k \to \mathbb{R}$ by

$$f_k(x, y^k) = \min_{1 \le i \le n_k} \{f_{k,i}(x^{\widehat{k}}, x^{k,\widehat{i}}, y^{k,i}) - t_{k,i}\}$$

for $x = ((x^{1,1}, \ldots, x^{1,n_1}), \ldots, (x^{m,1}, \ldots, x^{m,n_m})) \in X$ and $y^k = (y^{k,1}, \ldots, y^{k,n_k}) \in X_k$. These functions satisfy the assumptions of Theorem 10, whence there exists $\overline{x} \in X$ such that

$$\sup_{y^k \in X_k} f_k(\overline{x}, y^k) \le \sup_{x \in X} f_k(x, x^k),$$

$k = 1, \ldots, m$. On the other hand, for $\overline{k} = k(\overline{x})$ we have

$$\sup_{y^{\overline{k}} \in X_{\overline{k}}} f_{\overline{k}}(\overline{x}, y^{\overline{k}}) \ge f_{\overline{k}}(\overline{x}, \widetilde{y}^{\overline{k}}) > 0,$$

with $\widetilde{y}^{\overline{k}} = (\widetilde{y}^{\overline{k},1}, \ldots, \widetilde{y}^{\overline{k},n_{\overline{k}}})$ as in the statement. Combining these results, we get the existence of $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^m) \in X$ such that $f_k(\overline{x}, \overline{x}^k) > 0$. But, according to the definition of the f_k's, this is equivalent to the set of inequalities $f_{\overline{k},i}(\overline{x}) > t_{\overline{k},i}$, $i = 1, \ldots, n_{\overline{k}}$. □

As a consequence of the preceding theorem, we next obtain a generalization of an intersection theorem for sets with convex sections, due to Ky Fan [5]:

THEOREM 15. *Let* $E_{k,i} \subset X$, $i = 1, \ldots, m$, *be such that, for any* $x^{k,i} \in X_{k,i}$, *the section* $E_{k,i}(x^{k,i}) = \{(x^{\widehat{k}}, x^{k,\widehat{i}}) \in X_{\widehat{k}} \times X_{k,\widehat{i}} \mid (x^{\widehat{k}}, x^{k,\widehat{i}}, x^{k,i}) \in E_{k,i}\}$ *is open in* $X_{\widehat{k}} \times X_{k,\widehat{i}}$ *and, for any* $(x^{\widehat{k}}, x^{k,\widehat{i}}) \in X_{\widehat{k}} \times X_{k,\widehat{i}}$, *the section* $E_{k,i}^{-1}(x^{\widehat{k}}, x^{k,\widehat{i}}) = \{x^{k,i} \in X_{k,i} \mid (x^{\widehat{k}}, x^{k,\widehat{i}}, x^{k,i}) \in E_{k,i}\}$ *is H-convex (or empty). If for each* $x =$

$((x^{1,1},\dots,x^{1,n_1}),\dots,(x^{m,1},\dots,x^{m,n_m}))\in X$, *there exists* $k=k(x)\in\{1,\dots,m\}$ *such that* $E^{-1}_{k,i}(x^{\widehat{k}},x^{k,\widehat{i}})\neq\emptyset$, $i=1,\dots,n_k$, *then*

$$\bigcup_{k=1}^{m}\bigcap_{i=1}^{n_k}E_{k,i}\neq\emptyset.$$

PROOF. For each k and $i\in\{1,\dots,n_k\}$ let $f_{k,i}:X\to\mathbb{R}$ be the characteristic function of $E_{k,i}$, i.e., $f_{k,i}(x)=1$ if $x\in E_{k,i}$, 0 if $x\notin E_{k,i}$. Letting $t_{k,i}=0$, the assumptions of Theorem 14 are satisfied, whence there exists $\overline{k}\in\{1,\dots,m\}$ and $\overline{x}\in X$ such that $f_{\overline{k},i}(\overline{x})>0$, $i=1,\dots,n_{\overline{k}}$. But, in view of the definition of the $f_{k,i}$'s, the inequality $f_{\overline{k},i}(\overline{x})>0$ is equivalent to $\overline{x}\in E_{\overline{k},i}$. Therefore, we have

$$\overline{x}\in\bigcap_{i=1}^{n_{\overline{k}}}E_{\overline{k},i}\subset\bigcup_{k=1}^{m}\bigcap_{i=1}^{n_k}E_{k,i}.$$

□

The above mentioned theorem of Ky Fan corresponds to the particular case of the preceding theorem when $m=1$, each $X_{1,i}$ is a Hausdorff topological vector space and $\Gamma_A^{1,i}$ is the convex hull of A. In this particular setting, Ky Fan [6, 7] observed that Theorems 14 and 15 are equivalent. This is also true in our more general framework, since Theorem 14 can be easily derived from Theorem 15 by considering the sets $E_{k,i}=\{x\in X\mid f_{k,i}(x)>t_{k,i}\}$.

REFERENCES

[1] C. BARDARO AND R. CEPPITELLI, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities*, J. Math. Anal. Appl. **132** (1988), 484–490.

[2] K. BEN NAHIA, *Autour de la biconvexité en optimisation*, thesis, Université Paul Sabatier de Toulouse, 1986.

[3] H. BRÉZIS, L. NIRENBERG AND G. STAMPACCHIA, *A remark on Ky Fan's minimax principle*, Boll. Un. Mat. Ital. **(4) 6** (1972), 293–300.

[4] F. E. BROWDER, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[5] K. FAN, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[6] ———, *Sur un théorème de minimax*, C. R. Acad. Sci. Paris **259** (1964), 3925–3928.

[7] ———, *A minimax inequality and applications*, Inequalities III (O. Shisha, ed.), Academic Press, New York, 1972, pp. 103–113.

[8] C. D. HORVATH, *Some results on multivalued mappings and inequalities without convexity*, Nonlinear and Convex Analysis (B.-L. Lin and S. Simons, eds.), Marcel Dekker, New York, 1988, pp. 99–106.

[9] R. E. JAMISON-WALDNER, *A perspective on abstract convexity: classifying alignments by varieties*, Convexity and Related Combinatorial Geometry (D. C. Kay and M. Breen, eds.), Marcel Dekker, New York, 1982, pp. 113–151.

[10] M. LASSONDE AND C. SCHENKEL, *KKM principle, fixed points and Nash equilibria*, J. Math. Anal. Appl. **164** (1992), 542–548.

[11] E. MARCHI, *Some topics on equilibria*, Trans. Amer. Math. Soc. **220** (1976), 87–102.

[12] E. MARCHI AND J. E. MARTÍNEZ-LEGAZ, *Some results on approximate continuous selections, fixed points and minimax inequalities*, Fixed Point Theory and Applications (M. Théra and J. B. Baillon, eds.), Longman Scientific and Technical, Harlow, 1991, pp. 327–342.

[13] J. F. NASH, *Non-cooperative games*, Ann. of Math. **(2) 54** (1951), 286–295.

[14] B. PELEG, *Equilibrium points for open acyclic relations*, Canad. J. Math. **19** (1967), 366–369.

Manuscript received July 1, 1993

EZIO MARCHI
Instituto de Matemática Aplicada
Universidad Nacional de San Luis
5700 San Luis, ARGENTINA

JUAN-ENRIQUE MARTÍNEZ-LEGAZ
Dep. de Matemática Aplicada y Análisis
Universidad de Barcelona, 08071 Barcelona
and
Dep. de Economía y de Historia Económica
Universidad Autónoma de Barcelona
08193 Bellaterra, SPAIN