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NONLINEAR EIGENVALUE PROBLEMS GOVERNED BY A VARIATIONAL INEQUALITY OF VON KARMAN'S TYPE: A DEGREE THEORETIC APPROACH

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(*Submitted by A. Marino*)

Dedicated to the memory of Juliusz Schauder

1. Introduction

Let X be a Hilbert space, K a closed convex subset of X and $F : \mathbb{R} \times K \to X$ a nonlinear mapping given by $(\lambda, u) \to F(\lambda, u)$. This paper is mainly devoted to the study of a general nonlinear eigenvalue problem governed by a variational inequality V.I. (F, λ, K) defined as follows:

$$\text{V.I. } (F,\lambda,K) : \qquad \begin{cases} \text{find } \lambda^* \in \mathbb{R},\ u^* \in K \text{ such that} \\ \langle F(\lambda^*, u^*), v - u^* \rangle \geq 0 \text{ for each } v \in K. \end{cases}$$

This variational inequality arises in numerous engineering problems such as the buckling of plates [1], [2], [5], [6], [8], [9], [14], the bending of a beam [10], the double membrane problem [7], and reaction-diffusion systems [19].

[1]This work was completed while this author was visiting the University of Milano under a grant of CNR of Italy.

Our present study leads to a mathematical description of a buckling phenomenon which subsumes the case of a thin elastic plate subjected to unilateral conditions [5], [6], [8], [9], [13], [17] (case 1; see Example 2.3.1); to unilateral and transversal conditions [13] (case 2; see Example 2.3.3); and also the case of plates lying on a linear elastic body [17] (case 3; see Example 2.3.2.). Indeed, these problems fit within the framework of a variational inequality V.I. (F, λ, K), with $F(\lambda, u) := Tu - \lambda \cdot Lu + Au$, where K stands for a closed convex cone of a suitable Sobolev space, T is a nonlinear operator homogeneous of order $p = 3$, L is a linear operator and A a linear operator (cases 1 and 3) or a nonlinear operator homogeneous of order 1 (case 2). These examples drawn from elasticity theory are discussed in tandem with the mathematical theory developed for this general variational inequality. They are not meant to cover all the possible areas of applications of the theory, but rather to illustrate and motivate the paper.

In the literature, this study is generally done by using the Ly usternik-Schnirelmann theory [14], [15] and the Galerkin approximation of cones [5], [13]. In this paper, following the work by P. Quittner, we will use instead the Leray-Schauder degree theory. This will be done in order to obtain a bifurcation theory for this general variational inequality which subsumes all of previous cited cases. Eventually it should be observed that, in contrast to the other approaches, our technique can also be applied to variational inequalities not involving a gradient operator.

The paper is organized as follows:

In the first two sections we recall those aspects of the Leray-Schauder theory that we need. We will also briefly outline models which give the equilibrium of an elastic plate (i) subjected to unilateral conditions; (ii) subjected to unilateral and tranversal conditions; (iii) lying on a linear elastic body. In Section 3 a general existence theorem is proved. In Section 4 we develop a bifurcation theory for the nonlinear eigenvalue problem governed by the general variational inequality V.I. (F, λ, K). Finally, in Section 5 we discuss some aspects of a spectral theory relative to V.I. (F, λ, K).

2. Preliminaries

2.1 Topological degree. Let X and Y be two Banach spaces, K a retract of X. In what follows, the topological notions (open, closed, boundary) will refer to the relative topology on K. Let $U \subset X$ be a bounded open subset of K with closure $\overline{U}$ and boundary ∂U in K, and let $f : \overline{U} \to Y$. We recall that f is a *compact* mapping if f is continous and if for every bounded subset B of $\overline{U}$, $f(B)$ is a relatively compact subset of Y. $f : X \to Y$ is said to be *strongly continuous* if for every sequence $\{x_n : n \in \mathbb{N}\}$ which converges weakly to x ($x_n \overset{w}{\to} x$ for short), then $f(x_n)$ tends to

$f(x)$ for the norm convergence ($f(x_n) \to f(x)$ for short). In general the two classes of mappings just defined are not comparable: an example of a compact mapping which is not strongly continuous and of a strongly continuous mapping which is not compact can be found in [10]. However, if X is a reflexive Banach space then each strongly continuous operator is compact while if X is a reflexive Banach space and f a linear operator, then strong continuity and compactness are equivalent [10].

Given a compact mapping $f : \overline{U} \to X$, set $\Phi(x) := x - f(x)$ for each $x \in \overline{U}$. If $p \notin \Phi(\partial U)$, we may define the *topological Leray-Schauder* degree of Φ with respect to U and p, ($\deg(\Phi, U, p)$, for short). $\deg(\Phi, U, p)$ is an integer which can be viewed as an estimate of the number of solutions of the equation $\Phi(x) = p$, $x \in U$. Let us summarize some properties of the Leray-Schauder degree which will be used later on:

P.1. If $\deg(\Phi, U, p) \neq 0$ then there exists $x \in U$ such that $p = \Phi(x)$.

P.2. Let f_t be a homotopy of compact transformations of $\overline{U}$ such that $\Phi_t := I - f_t$ and $p \notin \Phi_t(\partial U)$. Then $\deg(\Phi_t, U, p)$ is independent of $t \in [0, 1]$. In particular, if f and g are compact mappings defined on $\overline{U}$ such that $f = g$ on ∂U and $p \notin (I - f)(\partial U)$, then

$$\deg(I - f, U, p) = \deg(I - g, U, p).$$

P.3. If $U = \bigcup_{i \in I} U_i$, with U_i open and $U_i \cap U_j = \emptyset$ for each $i \neq j$, $i, j \in I$ (I is a finite subset of $\mathbb{N}$) and $p \notin \Phi(\bigcup_{i \in I}(\partial U_i))$, then

$$\deg(\Phi, U, p) = \sum_{i \in I} \deg(\Phi, U_i, p).$$

P.4. If K is a closed subset of $\overline{U}$ and $p \notin \Phi(K)$, then

$$\deg(\Phi, U, p) = \deg(\Phi, U \backslash K, p).$$

P.5. If $p \in U$ (respectively $p \notin \overline{U}$), then $\deg(I, U, p) = 1$ (respectively $\deg(I, U, p) = 0$).

P.6. Suppose that U^* is a bounded open subset of $[0, 1] \times X$, and that $f : U^* \to X$ is compact. Let Φ_t denote the mapping $x \to x - f(t, x)$ and let $U_t = \{x \in X : (t, x) \in U^*\}$. If $p \notin \Phi_t(\partial U_t)$ for $0 \leq t \leq 1$, then $\deg(\Phi_t, U_t, p)$ is independent of t in $[0, 1]$.

For more details about the topological degree, the reader is invited to consult for instance [16].

2.2 Variational inequality, complementarity and fixed point formulation. In what follows, X will be a real Hilbert space, whose scalar product is denoted by $\langle \cdot, \cdot \rangle$, K a nonempty closed convex cone in X, $A : X \to X$ an operator defined on X, and $f \in X$ a fixed element. The problem

$$\text{V.I. } (A; f, K): \qquad \begin{cases} \text{find } u \in K \text{ such that} \\ \langle v - u, Au - f \rangle \geq 0, \text{ for each } v \in K, \end{cases}$$

is called the *variational inequality* associated with A, f and K.

If $K^* = \{y \in X : \langle y, x \rangle \geq 0 \text{ for each } x \in K\}$ denotes the dual cone of K, we may define the general *complementarity problem* C.P. $(A; f, K)$:

$$\text{C.P. } (A; f, K): \qquad \begin{cases} \text{find } u \in K \text{ such that} \\ A(u) - f \in K^* \text{ and } \langle Au - f, u \rangle = 0. \end{cases}$$

This problem has now been extensively studied since it has important applications in various areas of applied mathematics, such as for instance, optimization, mechanics, economic equilibrium and elasticity theory.

The basic relation between problems V.I. $(A; f, K)$ and C.P. $(A; f, K)$ is the following:

PROPOSITION 2.2.1. *Let X be a real Hilbert space, K a closed convex cone with vertex at the origin in X, $f \in X$ and $A : K \to X$. Then u^* is solution of* V.I. $(A; f, K)$ *if and only if u^* is a solution of* C.P. $(A; f, K)$.

Let the set-valued mapping $P_A : X \to 2^K$ be defined by

$$P_A(f) := \{u \in K : u \text{ is a solution of V.I. } (A; f, K)\}.$$

It has been shown by A. Szulkin in [23] that, if $A : K \to X$ has the following properties:

[H_1] $A : K \to X$ is continuous on finite dimensional subspaces (i.e. the restriction of A to the intersection of K with any finite dimensional subspace of X is weakly continuous);

[H_2] there exist $\alpha > 0$, $q > 1$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|u - v\|^q \quad \text{for each } u, v \in K,$$

then P_A is single-valued, bounded and continuous.

Let $A, L, T : X \to X$ be given, and let g be fixed in X. Let us now suppose that the mapping A is the sum of two operators A_1 and A_2, with A_1 satisfying Szulkin's assumptions [H_1] and [H_2].

It is by now well known that the complementarity problem admits an equivalent fixed point formulation [7]; more precisely, we have:

PROPOSITION 2.2.2. *Let U be an open bounded set in K, $\lambda \in \mathbb{R}$, and consider the following problem:*

$$\text{V.I. } (A, L, T, g, \lambda, \overline{U}) : \begin{cases} \text{find } u \in \overline{U},\ \lambda \in \mathbb{R} \text{ such that} \\ \langle Tu, v-u \rangle \geq \langle \lambda \cdot Lu - Au + g, v - u \rangle \text{ for each } v \in K. \end{cases}$$

If A_1 satisfies assumptions [H_1] *and* [H_2], *then $u \in \overline{U}$ is a solution of* V.I. $(A, L, T, g, \lambda, \overline{U})$ *if and only if u is a solution of the following fixed point problem:*

$$\text{F.P. } (A, L, T, \lambda, g, \overline{U}) : \begin{cases} \text{find } u \in \overline{U},\ \lambda \in \mathbb{R} \text{ such that} \\ u = P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u + g). \end{cases}$$

If $P_{A_1}(-Tu + \lambda \cdot Lu - A_2u + g)$ is compact and if V.I. $(A, L, T, g, \lambda, \overline{U})$ *has no solution on ∂U, then the topological degree of the mapping*

$$\Phi := I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u + g)$$

with respect to U and 0 is well defined.

In Section 3, this fixed point formulation associated with the topological degree will be the main ingredient in order to obtain existence results for the eigenvalue problem V.I. (A, L, T, g, λ, K). Section 4 is devoted to the existence of nontrivial solutions and bifurcation points for the variational inequality V.I. $(A, L, 0, \lambda, K)$ (V.I. (A, L, T, λ, K) for short).

REMARK 2.2.1. The degree-theoretic approach to variational inequalities is due to A. Szulkin and yields useful information relative to our specific problem V.I. (A, L, T, g, λ, K). Indeed, this method seems more appropriate than P. Quittner's.

Throughout the following, we will denote by $\sigma_K(A, L)$ the set of all $\lambda \in \mathbb{R}$ such that there exists a nontrivial solution to the inequality

$$\text{V.I. } (A, L, \lambda, K) : \begin{cases} \text{find } u \in K,\ \lambda \in \mathbb{R} \text{ such that} \\ \langle Au, v-u \rangle \geq \langle \lambda \cdot Lu, v - u \rangle \text{ for each } v \in K \end{cases}$$

or to the equivalent fixed point problem

$$\text{F.P. } (A, L, \lambda, U) : \begin{cases} \text{find } u \in X,\ \lambda \in \mathbb{R} \text{ such that} \\ u = P_{A_1}(\lambda \cdot Lu - A_2 u). \end{cases}$$

2.3 Motivation: examples in elasticity. As mentioned in the introduction, our study is motivated by problems arising in the engineering literature: contact problems of sheet piles, tunnels and certain foundation structures. All these problems may be formulated as problems of type V.I. (A, L, T, g, λ, K).

EXAMPLE 2.3.1. (Elastic plates subjected to unilateral conditions). Let Ω be a thin plate identified with a bounded open connected subset of $\mathbb{R}^2$ referred to a coordinate system Ox_1x_2. Assume that Ω is clamped on $\Gamma_1 \subset \partial\Omega$ and simply supported on $\Gamma_2 = \partial\Omega\backslash\Gamma_1$. $\partial\Omega$ is supposed regular (i.e. $\partial\Omega$ is a 1-dimensional manifold of class C^m, $m \geq 1$, and Ω is located on one side of $\partial\Omega$).

Let X be the subspace of the Sobolev space.

$$H^2(\Omega) := \left\{u \in L^2(\Omega) \,:\, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L_2(\Omega);\, i,j = 1,2\right\},$$

defined by

$$X := \left\{u \in H^2(\Omega) \,:\, u = 0 \text{ on } \Gamma,\, \frac{\partial u}{\partial n} = 0 \text{ a.e. on } \Gamma_1\right\}$$

and let K be the closed convex cone of X defined by:

$$K := \{u \in X \,:\, u \geq 0 \text{ a.e. in } \Omega\}.$$

In the case of an elastic plate subjected to unilateral conditions, for a fixed real parameter λ measuring the magnitude of lateral loading, the equilibrium of the plate is governed by the following variational inequality [5], [8]:

$$\text{find } u \in K \text{ such that } \langle u - \lambda \cdot Lu + Tu, v - u\rangle \geq 0, \text{ for each } v \in K,$$

where L is a linear operator describing the lateral loading in the plane of the plate, and T a nonlinear operator introduced in the Von Karman nonlinear theory of plates (see for instance [1] or [4]). The scalar product $\langle\cdot,\cdot\rangle$ is defined by a continuous bilinear form $a(u,v)$ [17], which is coercive in the case of Γ_1 nonempty and the norm defined on X by $(a(u,v))^{1/2}$ is equivalent to the usual norm on $H^2(\Omega)$.

These operators satisfy the following additional properties:

(i) L is a self-adjoint compact linear operator;
(ii) T is strongly continuous and positively homogeneous of order $p = 3$;
(iii) $\langle x, Tx\rangle > 0$ for each $x \in K\backslash\{0\}$; $T(0) = 0$.

Moreover, if the plate is subjected to a body force of density g, the equilibrium of the plate is governed by the variational inequality

$$\text{find } u \in K \text{ such that } \langle u - \lambda \cdot Lu + Tu, v - u\rangle \geq \langle g, v - u\rangle, \text{ for each } v \in K.$$

EXAMPLE 2.3.2. (Elastic plate lying on a linear elastic body). Let, moreover, Ω' be a linear elastic body identified as a bounded open connected subset of $\mathbb{R}^3$, referred to a coordinate system $Ox_1x_2x_3$, and let Γ_1' be a part of the boundary $\partial\Omega'$ supposed to be regular.

Let Z be the subspace of the Sobolev space

$$H^1(\Omega') := \left\{ z \in L^2(\Omega') : \frac{\partial z}{\partial x_i} \in L^2(\Omega'),\ i = 1, 2, 3 \right\},$$

defined by

$$Z := \{ z \in (H^1(\Omega'))^3 : z = 0 \text{ on } \Gamma_1' \},$$

and K' be the closed convex cone of $X \times Z$ (X as in Example 2.3.1) defined by

$$K' := \{ (u, z) \in X \times Z : u - z_3 \geq 0 \text{ a.e. in } \Omega \}.$$

In the case of an elastic plate subjected to unilateral conditions, for a fixed λ, the equilibrium of the plate is governed by the variational inequality [17]

find $(u, z) \in K'$ such that

$$\langle u - \lambda \cdot Lu + Tu, v - u \rangle + a'(z, x - z) \geq 0, \quad \text{for each } (v, x) \in K',$$

where L, T and $\langle \cdot , \cdot \rangle$ are as in Example 2.3.1 and $a'(u, v)$ is the bilinear coercive continuous form of the strain energy of Ω' [17].

By the Riesz theorem, there exists a continuous linear map $A : Z \to Z$ such that $a'(u, v) = (Au, v)$, where $(\cdot , \cdot)$ is the scalar product on Z.

If we write

$$\begin{aligned} T' &: X \times Z \to X \times Z, & (u, z) &\to (Tu, 0), \\ L' &: X \times Z \to X \times Z, & (u, z) &\to (Lu, 0), \\ A' &: X \times Z \to X \times Z, & (u, z) &\to (u, Az), \end{aligned}$$

and $\langle\langle \cdot , \cdot \rangle\rangle = \langle \cdot , \cdot \rangle + (\cdot , \cdot)$, the problem may then be rewritten as

find $y = (u, z) \in K'$, $\lambda \in \mathbb{R}$ such that

$$\langle\langle A'y - \lambda \cdot L'y + T'y, h - y \rangle\rangle \geq 0, \quad \text{for each } h \in K'.$$

EXAMPLE 2.3.3. (Elastic plates subjected to unilateral and transversal conditions. In this case, for a fixed λ, the equilibrium of the plate is governed by the variational inequality [13]

find $u \in K$ such that $\langle u - \lambda \cdot Lu + Tu + Ru, v - u \rangle \geq 0$, for each $v \in K$,

where L, T, $\langle \cdot , \cdot \rangle$ are as in Example 2.3.1 and R, which describes the transversal load, is positive, strongly continuous, nonlinear, and homogeneous of order 1; R is called the *contact* operator.

3. A general existence theorem

Let X be a real Hilbert space, and K a closed convex cone. To avoid repetitions, throughout this section we will refer to the following assumptions:

(1) $A : X \to X$ is such that $A = A_1 + A_2$, where
   (1.1) $A_2 : X \to X$ is strongly continuous, positively homogeneous of order 1
      (i.e. $A_2(tu) = t \cdot A_2(u)$ for each $u \in X$, $t > 0$);
   (1.2) $A_1 : X \to X$ is bounded, linear and α-coercive, i.e.
      $\langle A_1 u, u\rangle \geq \alpha\|u\|^2$, for each $u \in X$;

(2) $L : K \to X$ is strongly continuous and positively homogeneous of order 1;

(3) $T : K \to X$ is strongly continuous and positively homogeneous of order $p > 1$;

(4) $\langle Tu, u\rangle > 0$, for each $u \in K\backslash\{0\}$.

Let us denote by

$$K_r := \{x \in K \,:\, \|x\| < r\},$$

the open ball in K of radius $r > 0$.

REMARK 3.1. (i) It should be observed that Szulkin's assumptions are fulfilled for A_1 and therefore P_{A_1} is single-valued, bounded and continuous.

(ii) Assumptions (1) to (4) are fulfilled in the previous examples: in Example 2.3.1, take $A_1 = I$, $A_2 = 0$; in Example 2.3.2, take $A_1 = A'$, $A_2 = 0$, while for Example 2.3.3, take $A_1 = I$ and $A_2 = R$.

We can now state the following:

LEMMA 3.1. *Assume that hypotheses* (1)–(4) *hold. Then there exists* $r_0 > 0$ *depending on* $\lambda \in \mathbb{R}$ *and* $g \in X$ *such that, for each* $r \geq r_0$,

$$\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u + g), K_r, 0) = 1.$$

PROOF. The map $u \to -Tu + \lambda \cdot Lu - A_2 u + g$ is compact and since P_{A_1} is continuous, the map $u \to P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u + g)$ is compact.

Let U be a bounded open set in X such that $0 \notin \Phi(\partial U)$ where $\Phi : \overline{U} \to X$ is given by $x \to x - P_{A_1}(-Tx + \lambda \cdot Lx - A_2 x + g)$. Since the topological degree of Φ with respect to U and 0 is clearly well-defined, we may define the homotopy

$$H_\lambda(t, u) := P_{A_1}(-Tu + t \cdot (\lambda \cdot Lu - A_2 u + g)).$$

We claim that there exists $r_0 > 0$ such that for each $r \geq r_0$, $(I - H_\lambda(t, \cdot)(\partial K_r)) \neq 0$ for each $t \in [0, 1]$. Indeed, suppose on the contrary, we may find sequences $\{u_n; n \in$

$\mathbb{N}\}$ and $\{t_n : n \in \mathbb{N}\}$ such that $u_n \in K$, $t_n \in [0,1]$, $\lim_{n\to\infty} \|u_n\| = \infty$ and

$$\langle Tu_n + A_1 u_n, v - u_n\rangle \geq t_n\langle \lambda \cdot Lu_n - A_2 u_n + g, v - u_n\rangle, \qquad \text{for each } v \in K.$$

In particular, for $v = 0$ we obtain

$$\langle Tu_n + A_1 u_n, u_n\rangle \leq t_n\langle \lambda \cdot Lu_n - A_2 u_n + g, u_n\rangle. \tag{3.1}$$

We claim that there exists some $\tau > 0$ such that $\langle Tu_n, u_n\rangle \geq \tau\|u_n\|^{p+1}$ for all $n \in \mathbb{N}$. Otherwise, on relabelling if necessary and on setting $v_n := u_n/\|u_n\|$ we would obtain $\lim_{n\to\infty}\langle Tv_n, v_n\rangle = 0$. Since we may assume that $w\text{-}\lim_{n\to\infty} v_n = v_0$, $v_0 \in K$, by strong continuity of T we would obtain $\langle Tv_0, v_0\rangle = 0$, and therefore by Assumption (4), $v_0 = 0$.

Using (3.1) and Assumptions (1.2) and (4) we have

$$t_n\lambda\langle Lu_n, u_n\rangle \geq \langle A_1 u_n, u_n\rangle + t_n\langle A_2 u_n, u_n\rangle - t_n\langle g, u_n\rangle,$$

and therefore

$$t_n\lambda\langle Lv_n, v_n\rangle \geq \alpha + t_n\langle A_2 v_n, v_n\rangle - t_n\langle g, v_n\rangle/\|u_n\|.$$

Hence by passing to a subsequence, if necessary (this is possible since $t_n \in [0,1]$), we may assume that $\lim_{n\to\infty} t_n = t^*$ and we get

$$t^*\,\lambda\langle Lv_0, v_0\rangle \geq \alpha + t^*\langle A_2 v_0, v_0\rangle$$

and then $\alpha \leq 0$, a contradiction.

By applying again (3.1), Assumption (1.2) and the previous claim we have

$$\begin{aligned}\alpha\|u_n\|^2 + \tau\|u_n\|^{p+1} &\leq \langle Tu_n + A_1 u_n, u_n\rangle \\ &\leq |\lambda|\|Lu_n\| \cdot \|u_n\| + \|A_2 u_n\| \cdot \|u_n\| + \|g\| \cdot \|u_n\|.\end{aligned}$$

In particular, dividing the last inequality by $\|u_n\|^{p+1}$ we obtain:

$$\alpha\|u_n\|^{1-p} + \tau \leq |\lambda|\|Lu_n\|/\|u_n\|^p + \|A_2 u_n\|/\|u_n\|^p + \|g\|/\|u_n\|^p.$$

Since A_2 and L are continuous positively homogeneous of order 1, there exist $\Gamma_{A_2}, \Gamma_L > 0$ such that

$$\|A_2 x\| \leq \Gamma_{A_2}\|x\|, \qquad \text{for each } x \in K,$$

and

$$\|Lx\| \leq \Gamma_L\|x\|, \qquad \text{for each } x \in K.$$

This yields

$$\alpha\|u_n\|^{1-p} + \tau \leq |\lambda|\Gamma_L\|u_n\|^{1-p} + \Gamma_{A_2}\|u_n\|^{1-p} + \|g\| \cdot \|u_n\|^{-p},$$

and therefore by taking the limit as n tends to $+\infty$ we obtain $\tau \leq 0$, a contradiction.

Using now Property P.2 of the topological degree we have

$$\begin{aligned}\deg(\Phi, K_r, 0) &= \deg(I - H_\lambda(1,\cdot), K_r, 0)\\ &= \deg(I - H_\lambda(0,\cdot), K_r, 0)\\ &= \deg(I - P_{A_1}(-Tu), K_r, 0).\end{aligned}$$

We now define the homotopy $G_\lambda(t,u) := P_{A_1}(-t\cdot Tu)$; we claim that for each $r > 0$, $I - G_\lambda(t,\cdot)(\partial K_r) \neq 0$ for each $t \in [0,1]$.

Indeed, suppose, on the contrary, that there exist $r > 0$, $t^* \in [0,1]$ and $u^* \in K$ with $\|u^*\| = r$ such that

$$u^* = P_{A_1}(-t\cdot Tu^*),$$

or equivalently,

$$\langle A_1u^* + t^*\cdot Tu^*, v - u^*\rangle \geq 0, \qquad \text{for each } v \in K.$$

For $v = 0$, we get

$$\langle A_1u^* + t^*\cdot Tu^*, u^*\rangle \leq 0,$$

from which, by Assumption (4) and properties of A_1 we derive $\alpha\|u^*\|^2 \leq 0$. This yields $u^* = 0$, a contradiction. Thus,

$$\begin{aligned}\deg(\Phi, K_r, 0) &= \deg(I - P_{A_1}(-Tu), K_r, 0)\\ &= \deg(I - G_\lambda(1,\cdot), K_r, 0)\\ &= \deg(I - G_\lambda(0,\cdot), K_r, 0)\\ &= \deg(I - P_{A_1}(0), K_r, 0).\end{aligned}$$

Since A_1 is coercive, necessarily $P_{A_1}(0) = 0$, and therefore by virtue of property P.5 of the topological degree we obtain $\deg(\Phi, K_r, 0) = 1$ and the desired result. □

Theorem 3.1. *Assume that hypotheses* (1)–(4) *hold. Let $g \in X$ be fixed. If there exists $u_0 \in K$ such that $\langle g, u_0\rangle > 0$, then for each $\lambda \in \mathbb{R}$, there exists $u(\lambda) \in K$ such that* (i) *$u(\lambda) \neq 0$ and* (ii) *$u(\lambda) \in K$ and*

$$\langle Au(\lambda) - \lambda\cdot Lu(\lambda) + Tu(\lambda), v - u(\lambda)\rangle \geq \langle g, v - u(\lambda)\rangle, \qquad \textit{for each } v \in K.$$

Proof. The existence of $u(\lambda)$, solution of V.I. (A, L, T, g, λ, K) follows from Lemma 3.1 and property P.1 of the Leray-Schauder degree. For zero to be a solution, it is necessary that $\langle g, v\rangle \leq 0$, for each $v \in K$, and thus $u(\lambda) \neq 0$. □

Example 3.1. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary Γ. Let p^* be the critical exponent for the Sobolev imbedding $H_1 \hookrightarrow L^p$, i.e. $p^* := 2N/(N-2)$ if $2 < N$, and $+\infty$ if not. Let $\lambda \in \mathbb{R}$, $g \in L^2(\Omega)$, $2 < p < p^*$ and $K := \{u \in H_0^1(\Omega) : u \geq 0\text{a.e. in } \Omega\}$. If there exists $u_0 \in K$ such that $\int_\Omega g u_0\,dx > 0$, then by a simple application of Theorem 3.1, the variational inequality

$$u \in K : \int_\Omega \nabla u \nabla (v-u)\,dx - \lambda \int_\Omega u(v-u)\,dx + \int_\Omega u^{p-1}(v-u)\,dx \geq \int_\Omega g(v-u)\,dx \qquad \forall v \in K,$$

has a solution.

EXAMPLE 3.2. If the plates in 2.3.1–2.3.3 are subjected to a body force of density g, and if there exists $u_0 \in K$ such that $\langle g, u_0\rangle > 0$, then we may apply Theorem 3.1 to get the existence of an equilibrium of such plates.

Theorem 3.1 gets interesting when zero is not a solution for V.I. (A, L, T, g, λ, K). If $g = 0$, then $u = 0$ is a solution of V.I. (A, L, T, λ, K) and in this case, it is necessary to study the existence of a nontrivial solution and possible bifurcation from the line of trivial solutions. This case is the object of the following section.

4. Bifurcation theory for the nonlinear eigenvalue problem governed by the variational inequality V.I. (A, L, T, λ, K)

We denote by $C_0 = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$ the curve of trivial solutions of V.I. (A, L, T, λ, K). We say that λ_0 is a bifurcation point for V.I. (A, L, T, λ, K) (with respect to the curve C_0) if in every neighbourhood of $u(\lambda, 0)$ there exists a solution of V.I. (A, L, T, λ, K) which is not contained in C_0 (that is, a nontrivial solution), or equivalently if there exist sequences $\{\lambda_n : n \in \mathbb{N}\}$, $\{u_n : n \in \mathbb{N}\}$ of solutions of V.I. (A, L, T, λ, K) such that $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, and $\lim_{n\to\infty} \lambda_n = \lambda_0$.

Let us define

$$\Gamma := \mathrm{cl}\,\{(\lambda, u) \in \mathbb{R} \times K \setminus \{0\} : \ u \text{ solution of V.I. } (A, L, T, \lambda, K)\},$$

(we recall that cl A stands for the closure of A). The following lemmas turned out to be important for the sequel:

LEMMA 4.1. *Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space, K a closed convex cone in X, $F : \mathbb{R} \times X \to X$ a strongly continuous operator and $G : X \to X$ a weakly continuous operator (i.e. $x_n \to x \Rightarrow Gx_n \xrightarrow{w} Gx$) such that the map $x \to \langle x, Gx\rangle$ is weakly lower semicontinuous. If $\{u_n : n \in \mathbb{N}\} \subset K$, $\{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{R}$, and $\{\varepsilon_n : n \in \mathbb{N}\} \subset \mathbb{R}$ are sequences such that for each $n \in \mathbb{N}$,*

$$\langle Gu_n - F(\lambda_n, u_n), v - u_n\rangle \geq -\varepsilon_n \|v - u_n\|, \quad \textit{for each } v \in K, \tag{4.1}$$

and

$$w\text{-}\lim_{n\to\infty} u_n = u^*, \qquad \lim_{n\to\infty} \lambda_n = \lambda^*, \qquad \lim_{n\to\infty} \varepsilon_n = 0,$$

then

(i) $\lim_{n\to\infty}\langle Gu_n, u_n\rangle = \langle Gu^*, u^*\rangle$,
(ii) $\langle Gu^* - F(\lambda^*, u^*), v - u^*\rangle \geq 0$, *for each* $v \in K$.

Moreover, if G *is the identity mapping, then* $\lim_{n\to\infty} \|u_n - u^*\| = 0$.

PROOF. (i) Since the mapping $x \to \langle Gx, x\rangle$ is weakly lower semicontinuous we have $\langle Gu^*, u^*\rangle \leq \liminf_{n\to\infty}\langle Gu_n, u_n\rangle$. Also, it suffices to prove the reverse inequality. If we take $v = 0$ in (4.1) we obtain

$$\langle Gu_n, u_n\rangle \leq \langle F(\lambda_n, u_n), u_n\rangle + \varepsilon_n \|u_n\|;$$

and by virtue of the assumptions, $\limsup_{n\to\infty}\langle Gu_n, u_n\rangle \leq \langle F(\lambda^*, u^*), u^*\rangle$. On the other hand, (4.1) applied to $v = u_n + u^*$ gives

$$\langle Gu_n, u^*\rangle \geq \langle F(\lambda_n, u_n), u^*\rangle - \varepsilon_n \|u^*\|,$$

from which we get $\langle Gu^*, u^*\rangle \geq \langle F(\lambda^*, u^*), u^*\rangle$, and therefore

$$\langle Gu^*, u^*\rangle \geq \lim_{n\to\infty}\langle Gu_n, u_n\rangle.$$

(ii) For each $v \in K$, combine (4.1) and (i). If G is the identity, then $\lim_{n\to\infty} \|u_n\| = \|u^*\|$, and by using the fact that in every Hilbert space the norm is Kadec, we derive that $\lim_{n\to\infty} u_n = u^*$ and the proof is complete. □

LEMMA 4.2. *Let* X, $\langle \cdot, \cdot \rangle$ *be a Hilbert space,* K *a closed convex cone in* X, $F : \mathbb{R} \times X \to X$ *a strongly continuous operator and* $G : X \to X$ *an operator satisfying assumptions* [H₁] *and* [H₂]. *If* $\{u_n : n \in \mathbb{N}\} \subset K$ *and* $\{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{R}$ *are sequences such that for each* $n \in \mathbb{N}$

$$\langle Gu_n - F(\lambda_n, u_n), v - u_n\rangle \geq 0 \quad \textit{for each } v \in K,$$

and

$$w\text{-}\lim_{n\to\infty} u_n = u^*, \qquad \lim_{n\to\infty} \lambda_n = \lambda^*,$$

then

(i) $\lim_{n\to\infty} u_n = u^*$,
(ii) $\langle Gu^* - F(\lambda^*, u^*), v - u^*\rangle \geq 0$, *for each* $v \in K$.

PROOF. Using the fixed point formulation, we have

$$u_n = P_G(F(\lambda_n, u_n)),$$

and by strong continuity of the map $P_G(F(\lambda, u))$, we get our result. □

PROPOSITION 4.1. *Let X be a real Hilbert space and K a closed convex cone in X. Suppose L, $T : K \to X$ are compact mappings and $A : K \to X$ satisfies $A = A_1 + A_2$, with A_1 bounded, linear, α-coercive and A_2 compact. Let $\lambda_1 < \lambda_2$ and $r_0 > 0$ be given. Then there exists a bifurcation point $\lambda_0 \in [\lambda_1, \lambda_2]$ of* V.I. *(A, L, T, λ, K) provided the following conditions are fulfilled:*

(1) V.I. (A, L, T, λ_1, K) *and* V.I. (A, L, T, λ_2, K) *have no solution on ∂K, for each $0 < r < r_0$.*
(2) $\deg(u - P_{A_1}(-Tu + \lambda_1 \cdot Lu - A_2 u), K_r, 0) \neq \deg(u - P_{A_1}(-Tu + \lambda_2 \cdot Lu - A_2 u), K_r, 0)$ *for each $0 < r < r_0$ (i.e. r sufficiently small).*

PROOF. By compactness of A_2, L and T, and Assumption (1), the degree of the map $u \to u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u)$ with respect to K_r $(0 < r < r_0)$ and 0 is well defined for λ_1 and λ_2. If we suppose that there is no bifurcation point $\lambda_0 \in [\lambda_1, \lambda_2]$ for V.I. (A, L, T, λ, K), then this problem has no solution for each $\lambda \in [\lambda_1, \lambda_2]$ on ∂K_r for r sufficiently small.

Thus for each r sufficiently small, $r < r_0$, the homotopy

$$H(t, u) := P_{A_1}\big((-Tu + [(1-t)\lambda_1 + t\lambda_2] \cdot Lu - A_2 u), K_r, 0\big)$$

is such that $u \neq H(t, u)$ for each u on ∂K_r, and each $t \in [0,1]$. Hence by Property P.2 of the topological degree, we obtain

$$\begin{aligned}\deg\big(u - P_{A_1}(-Tu + \lambda_1 \cdot Lu - A_2 u), K_r, 0\big)& \\ &= \deg(u - H(0, u), K_r, 0) \\ &= \deg(u - H(1, u), K_r, 0) \\ &= \deg\big(u - P_{A_1}(-Tu + \lambda_2 \cdot Lu - A_2 u), K_r, 0\big),\end{aligned}$$

a contradiction to Assumption (2). □

REMARK 4.1. Proposition 4.1 is closely related to the results of [18] and [19].

REMARK 4.2. Lemma 3.1 determines the corresponding degree for r large enough and this knowledge does not seem to be useful for the localization of bifurcation points.

PROPOSITION 4.2. *Assume that hypotheses* (1)–(4) *hold. Then each bifurcation point λ_0 for* V.I. *(A, L, T, λ, K) belongs to $\sigma_K(A, L)$.*

PROOF. Let λ_0 be a bifurcation point for V.I. (A, L, T, λ, K). Then choose $\{\lambda_n : n \in \mathbb{N}\}$ and $\{u_n : n \in \mathbb{N}\}$ such that $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} \lambda_n = \lambda_0$

and

$$\langle Tu_n - \lambda_n \cdot Lu_n + Au_n, v - u_n \rangle \geq 0 \quad \text{for each} \quad v \in K.$$

According to Proposition 2.1, we also have

$$\langle Tu_n, u_n \rangle = \langle \lambda_n \cdot Lu_n - Au_n, u_n \rangle \tag{4.2}$$

and

$$\langle Tu_n, y \rangle \geq \langle \lambda_n \cdot Lu_n - Au_n, y \rangle, \quad \text{for each } y \in K. \tag{4.3}$$

If we divide (4.2) by $\|u_n\|$, we obtain

$$\langle Tu_n, u_n \rangle / \|u_n\|^2 = \lambda_n \langle L(u_n/\|u_n\|) - A(u_n/\|u_n\|), u_n/\|u_n\|) \rangle.$$

Set $v_n = u_n/\|u_n\|$; on relabelling, if necessary, we may suppose that $w\text{-}\lim_{n\to\infty} v_n = v^*$. We have

$$\langle Av_n - \lambda_n Lv_n, v - v_n \rangle \geq -\left\langle \frac{Tu_n}{\|u_n\|}, v - v_n \right\rangle, \quad \text{for each } v \in K;$$

since by Assumption (3), there exists $C > 0$ such that $\|Tu\| \leq C\|u\|^p$, for all $u \in K$, we derive

$$\langle Av_n - \lambda_n \cdot Lv_n, v - v_n \rangle \geq -C\|u_n\|^{p-1}\|v - v_n\|, \quad \text{for each } v \in K.$$

Hence, using Lemma 4.1, we get

$$\langle Av^* - \lambda_0 Lv^*, v - v^* \rangle \geq 0, \quad \text{for each} \quad v \in K.$$

We claim that $v^* \neq 0$. Indeed, we have

$$\alpha \leq \langle Av_n, v_n \rangle = \lambda_n \langle Lv_n, v_n \rangle - \langle Tv_n, v_n \rangle / \|u_n\|^{p-1} - \langle A_2 v_n, v_n \rangle;$$

computing the limit we derive

$$\alpha \leq \lambda_0 \langle Lv^*, v^* \rangle - \langle A_2 v^*, v^* \rangle.$$

By contradiction, this yields $v^* \neq 0$ and thus $\lambda_0 \in \sigma_K(A, L)$. □

REMARK 4.4. All the assumptions of Propositions 4.1–4.2 are satisfied in the examples of Section 2.

REMARK 4.5. Using the homotopy $H_\lambda(t, u) = P_A(-t \cdot Tu + \lambda \cdot Lu - A_2 u + g)$, it is also easy to prove that for each $\lambda \notin \sigma_K(A, L)$, for r small enough, the topological degree of the map $u \to P_{A_1}(Tu + \lambda \cdot Lu - A_2 u)$ with respect to K_r and 0 does not depend on T; i.e. $\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u), K_r, 0) = \deg(u - P_{A_1}(\lambda \cdot Lu - A_2 u), K_r, 0)$. See also [18].

In the sequel we will assume that

$$\{x \in K : \langle Ax, x\rangle = 0\} = \{0\}. \tag{5}$$

(This assumption is realized if for instance A_2 is positive, as is the case in the examples given below.)

The following three theorems contain interesting information for our practical problems.

Let ρ and ρ' be defined by

$$1/\rho = \sup_{x\in K\setminus\{0\}} \langle Lx, x\rangle / \langle Ax, x\rangle,$$

$$1/\rho' = \inf_{x\in K\setminus\{0\}} \langle Lx, x\rangle / \langle Ax, x\rangle$$

In the case of:

Example 2.3.1: $1/\rho = \sup_{K\setminus\{0\}} \langle Lu, u\rangle / \|u\|^2$,

Example 2.3.2: $1/\rho = \sup_{K'\setminus\{0\}} \langle Lu, u\rangle / (\|u\|^2 + a'(z, z))$,

Example 2.3.3: $1/\rho = \sup_{K\setminus\{0\}} \langle Lu, u\rangle / (\|u\|^2 + \langle Ru, u\rangle)$.

These numbers are well known to be the positive critical load of the plate (negative critical load for the corresponding ρ'). In the sequel we consider the case of positive λ. Our theory can easily be extended for negative λ.

PROPOSITION 4.3. *Assume that hypotheses* (1)–(5) *hold. Suppose* $0 < \rho < +\infty$. *Then for each* $\lambda \in (0, \rho]$, $u = 0$ *is the unique solution to* V.I. (A, L, T, λ, K).

PROOF. If u is a nontrivial solution of V.I. (A, L, T, λ, K), then by Proposition 2.1, we have

$$\langle Au, u\rangle - \lambda\langle Lu, u\rangle = -\langle Tu, u\rangle.$$

Since $\lambda \neq 0$ and $u \neq 0$, using Assumptions (4) and (5) we have

$$1/\lambda < \langle Lu, u\rangle / \langle Au, u\rangle \leq 1/\rho,$$

a contradiction. □

The following lemma completes our information about the number $\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0)$.

LEMMA 4.3. *Assume that hypotheses* (1)–(5) *hold. Suppose* $0 < \rho < +\infty$. *Then:*

(i) *for each* $\lambda \in (0, \rho]$ *and each* $r > 0$ *we have* $\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) = 1$;

(ii) *Assume* A *and* L *linear. If there exists* $\overline{u}_\rho \in \ker(A^* - \rho L^*) \cap K$ *such that*

$A^*\overline{u}_\rho \in \operatorname{int} K^*$ *(that means* ρ *is an eigenvalue for the couple of adjoints* (A^*, L^*) *and we can choose a corresponding eigenvector* $\overline{u}_\rho$ *in* K *and such that* $\langle A^*\overline{u}_\rho, v\rangle > 0$*, for each* $v \in K \setminus \{0\}$*), then for each* $\lambda \in (\rho, +\infty)$*, we have* $\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) = 0$*, for all* r *small enough.*

(iii) *Assume* A *and* L *linear, and suppose that* ρ *is an isolated eigenvalue for the couple* (A, L)*. If* $\beta(\lambda) := \dim\ker(A - \lambda L) = 1$ *for* $\beta = \rho$*, and if there exist* $u_\rho \in \ker(A - \rho L) \cap \operatorname{int} K$ *and* $\overline{u}_\rho \in \ker(A^* - \rho L^*) \cap \operatorname{int} K$ *such that* $\langle Au_\rho, \overline{u}_\rho\rangle > 0$*, then for each* $\lambda > \rho$*,* λ *close to* ρ*, we have* $\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) = 0$*, for all* r *small enough.*

PROOF. (i) Suppose that there exists $r > 0$ such that

$$\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) \neq 1.$$

For R sufficiently large, thanks to Lemma 3.1 we have

$$\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_R, 0) = 1,$$

and therefore by virtue of Property P.3 of the topological degree we get

$$\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_R \backslash K_r, 0) \neq 0.$$

As a result, we derive the existence of a nontrivial solution and a contradiction with Proposition 4.3.

(ii) We shall split the proof of (ii) into several steps.

Step I: Define the homotopy

$$H_\lambda(t, u) := P_{A_1}(-t \cdot Tu + \lambda \cdot Lu - A_2u).$$

We claim that for r small enough, and for each $t \in [0, 1]$,

$$I - H_\lambda(t, \cdot)(\partial K_r) \neq 0.$$

Indeed, suppose that, on the contrary, we may find sequences $\{u_n : n \in \mathbb{N}\}$ and $\{t_n : n \in \mathbb{N}\}$ such that $u_n \in K$, $t_n \in [0, 1]$, $\lim_{n\to\infty} \|u_n\| = 0$ and

$$\langle Au_n, v - u_n\rangle \geq \langle \lambda \cdot Lu_n, v - u_n\rangle - t_n\langle Tu_n, v - u_n\rangle, \quad \text{for each } v \in K.$$

Set $v_n := u_n/\|u_n\|$; by considering a subsequence if necessary, we may assume that $w\text{-}\lim v_n = v^*$ and $v^* \neq 0$ (if not we get $\alpha \leq 0$, a contradiction). As usual, using Proposition 2.1, we get

$$\langle Av^*, v - v^*\rangle \geq \langle \lambda \cdot Lv^*, v - v^*\rangle, \quad \text{for each } v \in K. \tag{4.4}$$

If we put $v := \overline{u}_\rho + v^*$, we obtain

$$\langle Av^*, \overline{u}_\rho\rangle \geq \lambda\langle Lv^*, \overline{u}_\rho\rangle,$$

and therefore

$$\rho\langle v^*, L^*\overline{u}_\rho\rangle = \langle v^*, A^*\overline{u}_\rho\rangle \geq \lambda\langle v^*, L^*\overline{u}_\rho\rangle.$$

Hence $\rho \geq \lambda$, a contradiction. Thus, for $\lambda > \rho$, and r small enough, we have

$$\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) = \deg(I - P_{A_1}(\lambda \cdot Lu - A_2u), K_r, 0).$$

Step II: Define the homotopy

$$G_\lambda(t, u) := P_{A_1}(\lambda \cdot Lu - A_2u + t \cdot \overline{u}_\rho).$$

We claim that for each $t \in [0, 1]$, $I - G_\lambda(t, \cdot)(\partial K_r) \neq 0$ $(r > 0)$.

Indeed, if the claim is not true, there exist $t \in [0, 1]$, $u \in K$ such that $\|u\| = r$, and

$$\langle Au - \lambda \cdot Lu - t\overline{u}_\rho, v - u\rangle \geq 0, \quad \text{for each } v \in K. \tag{4.5}$$

Put $v := u + \overline{u}_\rho$ in (4.5). Then we get

$$t\|\overline{u}_\rho\|^2 \leq \langle Au - \lambda \cdot Lu, \overline{u}_\rho\rangle = (\rho - \lambda)\langle Lu, \overline{u}_\rho\rangle = ((\rho - \lambda)/\rho)\langle u, A^*\overline{u}_\rho\rangle.$$

Hence $t\|\overline{u}_\rho\|^2 < 0$, a contradiction. Thus, for $\lambda > \rho$, and r small enough, we have

$$\deg(I - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) = \deg(I - P_{A_1}(\lambda \cdot Lu - A_2u + \overline{u}_\rho), K_r, 0).$$

The proof of the claim is complete.

Step III: To complete the proof of (ii), it is sufficient to check that

$$\deg(I - P_{A_1}(\lambda \cdot Lu - A_2u - \overline{u}_\rho), K_r, 0) = 0.$$

Indeed, assume that, on the contrary, there exists $u \in K$ such that

$$\langle Au - \lambda u - \overline{u}_\rho, v - u\rangle \geq 0, \qquad \text{for each } v \in K. \tag{4.6}$$

Necessarily $u \neq 0$, since otherwise we get $\langle \overline{u}_\rho, v\rangle \leq 0$, for each $v \in K$, and in particular $\|\overline{u}_\rho\|^2 \leq 0$, a contradiction.

Put $v := \overline{u}_\rho + u$ in (4.6). Then we get

$$\|\overline{u}_\rho\|^2 \leq ((\rho - \lambda/\rho)\langle u, A^*\overline{u}_\rho\rangle < 0,$$

a contradiction.

(iii) Under our assumptions, ρ is isolated in $\sigma_K(A, L)$ (see Theorem 5.2). Define the homotopy

$$C_\lambda(t, u) := P_{A_1}(t\overline{u}_\rho + \lambda \cdot Lu - A_2u).$$

We claim that for $\lambda > \rho$, λ close to ρ and r small enough,

$$\deg(I - P_{A_1}(\lambda \cdot Lu - A_2u), K_r, 0) = 0.$$

Indeed, for $t = 0$ the map $C_\lambda(t, u)$ is admissible, while for $0 < t \leq 1$, the equation $u = C_\lambda(t, u)$ is not solvable for λ close to ρ.

Otherwise, we can choose $\{u_n : n \in \mathbb{N}\} \subset K$ and $\{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{R}$ such that $\lambda_n > \rho$, $\lim_{n\to\infty} \lambda_n = \rho$, and

$$\langle -t\overline{u}_\rho + \lambda_n, Lu_n + Au_n, v - u_n\rangle \geq 0, \qquad \text{for each } v \in K. \tag{4.7}$$

If we put $v := \overline{u}_\rho + u_n$ in (4.7), we obtain

$$\langle Au_n, \overline{u}_\rho\rangle \leq -t\rho|\overline{u}_\rho|^2/(\lambda_n - \rho).$$

The last relation proves that $\{u_n : n \in \mathbb{N}\}$ is unbounded, so that we may define $v_n := u_n/\|u_n\|$. By considering a subsequence if necessary, we may assume that $w\text{-}\lim v_n = v^*$ and $v^* \neq 0$. It is easy to prove (see [17] or the proof of Theorem 5.1) that $v^* \notin \operatorname{Ker}(A - \rho L)$.

Thus, since $\beta(\rho) = 1$, there exists $\alpha > 0$ such that $v^* = \alpha u_\rho$. But we have $\langle Av^*, \overline{u}_\rho\rangle \leq 0$, a contradiction. □

THEOREM 4.1. *Assume that hypotheses* (1)–(5) *are fulfilled, with A and L linear. Let $\rho > 0$ be as in Proposition* 4.3. *If ρ is isolated in $\sigma_K(A, L)$, and if there exists $\overline{u}_\rho \in \ker(A^* - \rho L^*) \cap K$ such that $A^*\overline{u}_\rho \in \operatorname{int} K^*$, then $\Gamma_\rho := \Gamma \cup \{(\rho, 0)\}$ contains a subcontinuum Γ_0 such that $(\rho, 0) \in \Gamma_0$, which either* (i) *is bounded, or* (ii) $\Gamma_0 \cap \{\mathbb{R} \times \{0\}\} \neq \{(\rho, 0)\}$.

PROOF. If we suppose that each subcontinuum Γ_0 meeting $(\rho, 0)$ is bounded, then using Remark 4.5 and the fact that ρ is isolated in $\sigma_K(A, L)$, we can prove that there exists a bounded open subset D of $\mathbb{R} \times K$ such that:

- (i) $(\rho, 0) \in D$;
- (ii) $\partial D \cap \Gamma = \emptyset$ and
- (iii) D contains no trivial solutions of V.I. (A, L, T, λ, K) except those in $B((\rho, 0), \delta)$, where

$$\delta < \operatorname{dist}(\rho, \sigma_K(A, L) \setminus \{\rho\}).$$

(this is classical in bifurcation theory and the proof is similar to the one given in [22] for example).

Now let $D_\lambda := \{u \in K : (\lambda, u) \in D\}$. If μ is a large number then by (ii) and (iii), $D\mu = \emptyset$. For λ close to ρ, $\deg(I - P_{A_1}(-Tu + \lambda Lu - A_2 u), D_\lambda, 0)$ is constant. By choosing r small enough, it can be assumed that, for $0 < |\lambda - \rho| \leq \delta$, $(\lambda, 0)$ is the only solution of V.I. (A, L, T, λ, K) and for $|\lambda - \rho| \geq \delta$, $0, \lambda \leq \mu$, $\overline{K}_r \cap D_\lambda = \emptyset$.

Hence, by Property P.6 of the topological degree, we have, for λ close ρ, $\lambda \neq \rho$,

$$\begin{aligned}
&\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2 u), , 0) \\
&\qquad = \deg(u - P_{A_1}(-Tu + \mu \cdot Lu - A_2 u), \overline{K}_r \setminus D_\lambda, 0) \\
&\qquad = \deg(u - P_{A_1}(-Tu + \mu \cdot Lu - A_2 u), \emptyset, 0) = 0.
\end{aligned}$$

Thus for $\lambda < \rho$, λ close to ρ, by virtue of Lemma 4.3(i), we get

$$\begin{aligned}\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), D_\lambda, 0) \\ = \deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), K_r, 0) \\ + \deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), \overline{K}_r \setminus D_\lambda, 0) = 1,\end{aligned}$$

while for $\lambda > \rho$, λ close to ρ, by using Lemma 4.3(ii), we have

$$\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), D_\lambda, 0) = 0.$$

Since $\deg(u - P_{A_1}(-Tu + \lambda \cdot Lu - A_2u), D_\lambda, 0)$ is constant for λ close to ρ, we get a contradiction. The proof of Theorem 4.1 is complete. □

If $\rho > 0$ is isolated in $\sigma_K(A, L)$, then from Theorem 4.1 we can deduce the existence of a subcontinuum of nontrivial solutions which either meets infinity or meets $(\overline{\mu}, 0)$ where $\overline{\mu} \neq \rho$. In Section 5 we give some conditions for ρ to be isolated in $\sigma_K(A, L)$. Using part (iii) of Lemma 4.3, we derive, similarly to Theorem 4.1, also the following:

THEOREM 4.2. *Assume that hypotheses* (1)–(5) *hold, with* A *and* L *linear. Let* $\rho > 0$ *be as in Proposition* 4.3. *If* ρ *is isolated in* $\sigma_K(A, L)$, $\beta(\rho) = 1$, *and if there exist* $u_\rho \in \ker(A - \rho L) \cap \operatorname{int} K$ *and* $\overline{u}_\rho \in \ker(A^* - \rho L^*) \cap \operatorname{int} K$ *such that* $\langle Au_\rho, \overline{u}_\rho \rangle > 0$, *then* Γ_ρ *contains a subcontinuum* Γ_0 *such that* $(\rho, 0) \in \Gamma_0$, *which either* (i) *is unbounded, or* (ii) $\Gamma_0 \cap \{\{0\} \times \mathbb{R}\} \neq \{(\rho, 0)\}$.

THEOREM 4.3. *Assume the hypotheses* (1)–(5) *hold, with* A *and* L *linear. Let* $\rho > 0$, $(\rho' < 0)$ *be as in Proposition* 4.3. *If there exists* $\overline{u}_\rho \in \ker(A^* - \rho L^*) \cap K$ *such that* $A^*\overline{u}_\rho \in \operatorname{int} K^*$, *then:*

(i) *for each* $\lambda \in (0, \rho]$, V.I. (A, L, T, λ, K) *has a nontrivial solution;*
(ii) ρ *is a bifurcation point for* V.I. (A, L, T, λ, K).

PROOF. (i) Use Lemma 3.1 (with $g = 0$), Lemma 4.1, and Property P.3 of the topological degree.

(ii) We give the proof for $\rho > 0$. Let $\varepsilon > 0$ be sufficiently small. By Lemma 4.3 (i), for each $r > 0$ we have

$$\deg(u - P_{A_1}(-Tu + (\rho - \varepsilon)Lu - A_2u), K_r, 0) = 1,$$

and by Lemma 4.3 (ii), for r small enough we have

$$\deg(u - P_{A_1}(-Tu + (\rho + \varepsilon)Lu - A_2u), K_r, 0) = 0.$$

Thus by Proposition 4.1, there exists a bifurcation point in $[\rho - \varepsilon, \rho + \varepsilon]$. Since ε is arbitrary, ρ is necessarily a bifurcation point for V.I. (A, L, T, λ, K). □

5. Spectral analysis of the set $\sigma_K(A, L)$

Let $F : K \to X$ be a continuous positively homogeneous operator of order 1.

PROPOSITION 5.1. *Assume that hypotheses* (1)–(2) *hold. Then the set* $\sigma_K(A, L)$ *is closed.*

PROOF. Take a sequence $\{\lambda_n : n \in \mathbb{N}\}$ with $\lambda_n \in \sigma_K(A, L)$ and such that $\lim_{n\to\infty} \lambda_n = \lambda^*$. Since $\lambda_n \in \sigma_K(A, L)$, there exists $u_n \in K \setminus \{0\}$ such that $\|u_n\| = 1$, and

$$\langle Au_n - \lambda_n \cdot Lu_n, u_n\rangle = 0$$

and

$$\langle Au_n - \lambda_n \cdot Lu_n, \gamma\rangle \geq 0, \quad \text{for each } \gamma \in K.$$

On relabelling if necessary, we can suppose that

$$w\text{-}\lim_{n\to\infty} u_n = u^*,$$

and thus taking the limit in the above relations, we obtain, by Lemma 4.1

$$\langle Au^* - \lambda^* \cdot Lu^*, u^*\rangle = 0$$

and

$$\langle Au^* - \lambda^* \cdot Lu^*, \gamma\rangle \geq 0, \quad \text{for each } \gamma \in K.$$

Lemma 4.1, applied to $F(\lambda, u) := A_2 u - \lambda Lu$ and $Gu := A_1 u$, gives $\langle A_1 u^*, u^*\rangle = \lim_{n\to\infty}\langle Au_n, u_n\rangle \geq \alpha$ and therefore $u^* \in K \setminus \{0\}$. Hence $\lambda^* \in \sigma_K(A, L)$ and the proof is complete. □

The following proposition gives some bounds for the set $\sigma_K(A, L)$.

Let $\rho > 0$ be as in Section 4. The following result tells us that if A and L are linear and self-adjoint, ρ always belongs to $\sigma_K(A, L)$. This agrees with Theorem 4.2, since ρ is a bifurcation point for V.I. (A, L, T, λ, K).

PROPOSITION 5.2. *Let X be a real Hilbert space, and K a closed convex cone in X. Suppose that L, $A : K \to X$ are linear and self-adjoint. Then $\rho \in \sigma_K(A, L)$.*

PROOF. By the definition of ρ, we have

$$\langle Au - \rho \cdot Lu, u\rangle \geq 0, \quad \text{for each } u \in K, \tag{5.1}$$

and there exists $u^* \in K \setminus \{0\}$ such that

$$\langle Au^* - \rho \cdot Lu^*, u^*\rangle = 0. \tag{5.2}$$

Let v be arbitrary in K, $\alpha > 0$ and put $u := \alpha \cdot u^* + v$ in (5.1). We obtain, by (5.2),

$$\langle Av - \rho \cdot Lv, v\rangle + 2\alpha\langle Au^* - \rho \cdot Lu^*, v\rangle \geq 0.$$

Hence,

$$(1/\alpha)\langle Av - \rho \cdot Lv, v\rangle + 2\langle Au^* - \rho \cdot Lu^*, v\rangle \geq 0, \quad \text{for each } v \in K.$$

Taking the limit as $\alpha \to +\infty$, we obtain

$$\langle Au^* - \rho \cdot Lu^*, v\rangle \geq 0, \quad \text{for each } v \in K.$$

By virtue of (5.2) and Proposition 2.1 we get

$$\langle Au^* - \rho \cdot Lu^*, v - u^*\rangle \geq 0, \quad \text{for each } v \in K,$$

and therefore, $\rho \in \sigma_K(A, L)$. □

Suppose now that $A, L : X \to X$ are bounded and linear; we may then define the *resolvent set* $r(A, L)$ of the pair (A, L) as the set of all $\lambda \in \mathbb{R}$ such that $A - \lambda \cdot L$ has a bounded inverse, and the *spectrum* of the pair (A, L) as $\mathrm{Sp}(A, L) = \mathbb{R} \backslash r(A, L)$. We say that $\lambda \in \mathrm{Sp}(A, L)$ is an *eigenvalue* for the couple (A, L) if $\dim\ker(A - \lambda \cdot L) \geq 1$. We denote by $\sigma(A, L)$ the set of eigenvalues of the pair (A, L) [3].

Moreover, $\lambda \in \mathbb{R}$ is said to be a *simple eigenvalue* for the pair (A, L) if [3]:

(i) $\dim\ker(A - \lambda \cdot L) = 1$,
(ii) $\operatorname{codim} R(A - \lambda \cdot L) = 1$,
(iii) $\ker(A - \lambda \cdot L) \oplus R(A - \lambda \cdot L) = X$.

This concept generalizes the notion of simple eigenvalue of the classical problem $Au = \lambda \cdot u$; some properties of this concept can be found in [7]. The number $\beta(\lambda) = \dim\ker(A - \lambda \cdot L)$ is called the *geometric multiplicity* of λ relative to the pair (A, L).

LEMMA 5.1. *Let X be a Hilbert space. Then*

$$y \in K^*,\ y \neq 0,\ z \in K \text{ and } \langle y, z\rangle = 0 \Rightarrow z \in \partial K.$$

PROOF. By contradiction, suppose that $z \in \operatorname{int}(K)$. Then for each $\Phi \in X$, there exists $t > 0$ such that $z \pm t\Phi \in K$.

Hence for each $y \in K^*$, we have $\langle y, z\rangle \pm t\langle y, \Phi\rangle \geq 0$ from which, since $\langle y, z\rangle = 0$, we derive $\langle y, \Phi\rangle = 0$, for all $\Phi \in X$, a contradiction. □

By a simple review of a similar result of P. Quittner [18], we now obtain a sufficient condition for $\lambda \in \mathbb{R}$ to be isolated in $\sigma_K(A, L)$:

THEOREM 5.2. *Assume that hypotheses* (1) *and* (2) *hold, with A and L linear. Let $\lambda_0 > 0$ be an isolated eigenvalue in $\sigma(A, L)$ satisfying:*

(i) $\beta(\lambda_0) = 1$;

(ii) *there exists* $u_0 \in \ker(A - \lambda_0 \cdot L) \cap \operatorname{int} K$;

(iii) *there exists* $u_0^* \in \ker(A^* - \lambda_0 \cdot L^*) \cap \operatorname{int} K$.

Then λ_0 is isolated in $\sigma_K(A, L)$.

PROOF. Indeed, suppose that, on the contrary, there exist sequences $\{\lambda_n : n \in \mathbb{N}\}$ and $\{u_n : n \in \mathbb{N}\} \subset K$ ($u_n \neq 0$) such that

$$\lim_{n\to\infty} \lambda_n = \lambda_0, \quad \langle Au_n - \lambda_n \cdot Lu_n, v - u_n\rangle \geq 0 \quad \text{for each } v \in K. \tag{5.3}$$

By using the fact that λ_0 is isolated in $\sigma(A, L)$ we know that $Au_n \neq \lambda_n \cdot Lu_n$; hence by virtue of Proposition 2.1 and Lemma 5.1, $u_n \in \partial K$. Also, without loss of generality, we may suppose $w\text{-}\lim_{n\to\infty} u_n = z$; computing the limit in (5.3), we obtain, by application of Lemma 4.2, $\lim_{n\to\infty} u_n = z$,

$$\langle Az - \lambda_0 \cdot Lz, v - z\rangle \geq 0 \quad \text{for each } v \in K,$$

and also since ∂K is closed, $z \in \partial K$.

Now let $\Phi \in X$; since $u_0^* \in \operatorname{int} K$ there exists $\delta > 0$ such that

$$v = z + u_n^* \pm \delta \cdot \Phi \in K.$$

Therefore, we have

$$\begin{aligned} 0 &\leq \pm\delta\langle Az - \lambda_0 \cdot Lz, \Phi\rangle + \langle Az - \lambda_0 \cdot Lz, u_0^*\rangle \\ &\leq \pm\delta\langle Az - \lambda_0 \cdot Lz, \Phi\rangle + \langle z, A^*u_0^* - \lambda_0 \cdot L^*u_0^*\rangle \\ &\leq \pm\delta\langle Az - \lambda_0 \cdot Lz, \Phi\rangle. \end{aligned}$$

This yields $Az - \lambda_0 \cdot Lz \in X^\perp = \{0\}$.

Now, let $z \in \partial K$; since for each $\alpha > 0$, $\alpha \cdot u_0 \in \operatorname{int} K$, there does not exist any $t > 0$ such that $u_0 = t \cdot z$. As a result, we have $\dim \ker(Au - \lambda_0 \cdot Lu) > 1$, a contradiction with the fact that $\beta(\lambda_0) = 1$. □

REFERENCES

[1] M. S. BERGER, *Nonlinearity and Functional Analysis*, Lecture on Nonlinear Problems in Mathematical Analysis, Academic Press, 1977.

[2] G. BEZINE, A. CIMETIERE AND J. P. GELBERT, *Unilateral buckling of thin elastic plates by the boundary integral equation method*, Internat. J. Numer Methods Engrg **21** (1985), 2189–2199.

[3] S.-N. CHOW AND J. K. HALE, *Methods of Bifurcation Theory*, Springer,, New York, 1982.

[4] PH. G. CIARLET AND P. RABIER, *Les Equations de Von Karman*, Lecture Notes in Math. **826** (1980), Springer.

[5] A. CIMETIERE, *Un problème de flambement unilatéral en théorie des plaques*, Journal de Mécanique **19** (1980), 183–202.

[6] ———, *Méthode de Lyapounov–Schmidt et branche de bifurcation pour une classe d'inéquations variationnelles*, C. R. Acad. Sci. Paris,t. 300, Série I, No. 15 (1985), 565–568.

[7] R. W. COTTLE, F. GIANNESSI AND J.-L. LIONS, *Variational Inequalities and Complementarity Problems*, John Wiley & Sons,, New York, 1980.

[8] M. C. DO, *Bifurcation theory for elastic plates subjected to unilateral conditions*, J. Math. Anal. and Appl. **60** (1977), 435–448.

[9] ———, *Problème de valeurs propres pour une inéquation variationnelle sur un cône et application au flambement unilatéral d'une plaque mince*, C. R. Acad. Sc. Paris **280** (1975), 45–48.

[10] S. FUČIK, J. NEČAS, J. SOUČEK AND V. SOUČEK, *Spectral analysis of nonlinear operator*, Lecture Notes in Math. **346** (1973), Springer.

[11] D. GOELEVEN, *Thése de doctorat* (1993), Université de Zimoges.

[12] G. HELMBERG, *Introduction to Spectral Theory in Hilbert Space*, North Holland Publishing Company, 1975.

[13] G. ISAC AND M. THÉRA, *Complementarity problem and the existence of the postcritical equilibrium state of a thin elastic plate*, Journal of Optimization Theory and Applications **58**(2) (1988), 241–257.

[14] K. S. KUBRUSLY AND J. T. ODEN, *Nonlinear eigenvalue problems characterized by variational inequalities with applications to the postbuckling analysis of unilaterally–supported plates*, Nonlinear Analysis, Theory, Methods & Applications **5**(12) (1981), 1265–1284.

[15] M. KUČERA, J. NEČAS AND J. SOUČEK, *The eigenvalue problem for variational inequalities and a new version of the Lusternik–Schnirelmann theory*, in Nonlinear Analysis (L. Cesari, R. Kannan and H. F. Weinberger, eds.), 1978.

[16] N. G. LLOYD, *Degree Theory*, Cambridge University Press, 1978.

[17] P. D. PANAGIOTOPOPOULOS, *Inequality Problems in Mechanics and Applications: Convex and Nonconvex Energy Functions*, Birkhäuser, 1985.

[18] P. QUITTNER, *Spectral analysis of variational inequalities*, Comment. Math. Univ. Carolinae **27**(3) (1986), 605–629.

[19] ———, *Bifurcation points and eigenvalues of inequalities of reaction–diffusion type*, J. Reine Angew. Math. **380** (1987), 1–13.

[20] ———, *Solvability and multiplicity results for variational inequalities*, Comment. Math. Univ. Carolinae **30**(2) (1989), 281–302.

[21] ———, *On the principle of linearized stability for variational inequalities*, Math. Ann. **283** (1989), 257–270.

[22] P. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487–513.

[23] A. SZULKIN, *Positive solutions of variational inequalities: a degree–theoric approach*, Jour. of Differential Equations **57** (1985), 90–111.

[24] ———, *Existence and nonuniqueness of solutions of a noncoercive elliptic variational inequality*, Proc. of Symp. in Pure Mat. **45**, Part 2 (1986), 413–418.

[25] D. GOELEVEN, V. H. NGUYEN AND M. THÉRA, *Méthode du degré topologique et branches de bifurcation pour les inéquations variationnelles de Von Karman*, C. R. Acad. Sci. (1993), Paris, à paraitre.

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