Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 2, 1993, 225–231

ON THE LERAY-SCHAUDER ALTERNATIVE

ANDRZEJ GRANAS

Dedicated to the memory of Juliusz Schauder

1. Introduction

In 1933, Jean Leray and Juliusz Schauder discovered [9] that the problem of solvability of an equation $x = Tx$, for a completely continuous operator T in a Banach space, reduces to finding a priori bounds on all possible solutions for the family of equations $x = \lambda Tx$, where $\lambda \in (0,1)$. Since then, this fact, known as the Leray-Schauder Alternative, and its various extensions and modifications, have played a basic role in various applications to nonlinear problems.

In this note, we elucidate and complement the above result. We introduce a class of nonlinear operators of the *Leray-Schauder type* and discuss its properties both in the fixed point and the coincidence setting. By elementary means and using only some known fixed point results, we show that many of the currently used nonlinear operators are of the Leray-Schauder type.

We begin with some notation and terminology. By *space* we shall understand a metric space and by a *map* a set-valued transformation.

Given a map $T : X \to Y$ between spaces, the sets Tx are the *values* of T and the set

$$\Gamma_T = \{(x,y) \in X \times Y : \ y \in Tx\}$$

is the *graph* of T. Two maps $S, T : X \to Y$ are said to have a *coincidence* provided $\Gamma_S \cap \Gamma_T \neq \emptyset$; if $T : A \to X$, where $A \subset X$, then x is a *fixed point* for T, provided $x \in Tx$.

By an *operator* we shall understand an upper semicontinuous map with non-empty compact values. An operator is said to be *compact* provided its range is relatively compact. An operator is *completely continuous* if it is compact on bounded sets. Given a class $\mathbf{M}$ of operators we let

$$\mathbf{M}(X,Y) = \{S : X \to Y : \ S \in \mathbf{M}\}, \qquad \mathbf{M}(X) = \mathbf{M}(X,X),$$

and define the class $\mathbf{M}_c$ by letting

$$\mathbf{M}_c(X,Y) = \{S = S_1 \circ S_2 \circ \cdots \circ S_k \text{ with } S_i \in \mathbf{M}, \ k = 1,2,\dots\}.$$

We now introduce four classes of operators as follows:

(1) $T \in \mathbf{K}(X,Y)$ if the values of T are convex (the *Kakutani operators*);
(2) $T \in \mathbf{A}(X,Y)$ if the values of T are R_δ-sets[1] (the *Aronszajn operators* [7]);
(3) $T \in \mathbf{E}(X,Y)$ if the values of T are acyclic (the *Eilenberg-Montgomery operators* [4]);
(4) $T \in \mathbf{N}(X,Y)$ if the values of T consist of one or m acyclic components with m fixed (the *O'Neill operators* [11]).

These classes and the classes of their composites are displayed in the following diagram:

$$\text{(D)} \qquad \begin{array}{ccccccc} \mathbf{K} & \subset & \mathbf{A} & \subset & \mathbf{E} & \subset & \mathbf{N} \\ \cap & & \cap & & \cap & & \cap \\ \mathbf{K}_c & \subset & \mathbf{A}_c & \subset & \mathbf{E}_c & \subset & \mathbf{N}_c \end{array}$$

In what follows, for a normed linear space E and a positive number ρ, we let

$$K_\rho = \{x \in E : \|x\| \le \rho\} \qquad \text{and} \qquad S_\rho = \{x \in E : \|x\| = \rho\}.$$

Given a bounded subset $A \subset E$ we let $\|A\| = \sup\{\|a\| : a \in A\}$. If $T : E \to F$ is an operator between normed linear spaces E and F, then we let $T_\rho = T \mid K_\rho : K_\rho \to F$. By $r : E \to K_\rho$ we denote the standard retraction of E onto K_ρ given by

$$r(y) = \begin{cases} y & \text{for } \|y\| \le \rho, \\ \rho \dfrac{y}{\|y\|} & \text{for } \|y\| > \rho. \end{cases}$$

The author thanks M. Lassonde for several useful comments.

2. The Leray-Schauder operators in normed linear spaces

Let E be a normed linear space and $T : E \to E$ be an operator.

[1] We recall that a compact space X is *acyclic* if all its reduced Čech homology groups over the rationals are trivial. A compact space X is *R_δ-set* provided it is the intersection of a descending sequence of compact AR's.

DEFINITION 2.1. We shall say that T is of the *Leray-Schauder type* provided for any ball K_ρ in E, either

(a) there exists $x \in K_\rho$ such that $x \in Tx$, or
(b) there exist $y \in S_\rho$ and $\lambda \in (0,1)$ such that $y \in \lambda Ty$.

We begin by a result which collects several important examples:

THEOREM 2.2. *Let $T : E \to E$ be an operator such that*

(a) *T is completely continuous,*
(b) *T belongs to one of the classes appearing in the diagram* (D).

Then T is of the Leray-Schauder type.

PROOF. Assume, for example, that $T : E \to E$ is completely continuous and is of the Eilenberg-Montgomery type. Let us fix arbitrarily a ball K_ρ in E and consider the composite map $E \xrightarrow{r} K_\rho \xrightarrow{T_\rho} E$. Since the operator $T_\rho r$ is compact and is of the Eilenberg-Montgomery type, we may apply the extended Eilenberg-Montgomery fixed point theorem given in [6], and infer that the operator $T_\rho r$ has a fixed point, i.e. $x \in Trx$ for some $x \in E$. From this, it easily follows that $r(x) = y$ is a fixed point for the composite map $rT_\rho : K_\rho \to K_\rho$, i.e.,

$$y \in rTy. \tag{2}$$

We now examine two possible cases:

(A) $\|Ty\| \leq \rho$, and
(B) $\|Ty\| > \rho$.

In case (A), $rTy = Ty$ and therefore $y \in Ty$, i.e. property (a) holds.

In case (B), there exists $z \in Ty$ such that

$$\|z\| > \rho \quad \text{and} \quad y = rz. \tag{3}$$

In view of (3) we get

$$y = \rho \frac{z}{\|z\|} \in S_\rho \quad \text{and} \quad z = \frac{\|z\|}{\rho} y \in Ty.$$

This gives $y \in \lambda Ty$ with $\lambda = \rho/\|z\| < 1$. Thus property (b) holds. The proof is complete. □

The proof for T in some other class of the diagram (D) is strictly analogous, except that another appropriate (for the class in question) fixed point theorem is used. For example, for compositions of operators in the classes **E**, **A**, and **N**, we use fixed point results given in [4], [7], and [3] respectively.

Some general properties of the Leray-Schauder operators are given in the next two results:

THEOREM 2.3. *Let $T : E \to E$ be an operator of the Leray-Schauder type; let ρ be a positive number and assume that for all $x \in S_\rho$, one of the following conditions is satisfied:*

$$\begin{array}{lll} \text{(i)} & \|Tx\| \le \|x\| & \text{(E. Rothe);} \\ \text{(ii)} & \|Tx\| \le \|x - Tx\|; & \\ \text{(iii)} & \|Tx\| \le \left(\|x - Tx\|^2 + \|x\|^2\right)^{1/2} & \text{(M. Altman);} \\ \text{(iv)} & \|Tx\| \le \max\{\|x\|, \|x - Tx\|\}. & \end{array}$$

Then the operator T has at least one fixed point in K_ρ.

PROOF. The routine verification that property (b) in Definition 2.1 cannot occur, is left to the reader. □

THEOREM 2.4 (The Leray-Schauder Alternative). *Let $T : E \to E$ be an operator of the Leray-Schauder type and let*

$$\mathcal{E}_T = \{x \in E : x \in \lambda Tx \quad \textit{for some } 0 < \lambda < 1\}.$$

Then either

(a) *the set $\mathcal{E}_T$ is unbounded, or*
(b) *the operator T has at least one fixed point.*

PROOF. Assume $\mathcal{E}_T$ is bounded and let K_ρ be a ball containing $\mathcal{E}_T$ in its interior. Since no $x \in S_\rho$ can satisfy the second property in Definition 2.1, the operator T has a fixed point and the proof is complete. □

3. The Leray-Schauder operators for coincidences

In this section E and F denote two Banach spaces and $L : E \to F$ stands for a fixed surjective linear bounded operator.

DEFINITION 3.1. We shall say that T is of the *Leray-Schauder type with respect to L*, if for any ball K_ρ in E, either:

(a) there exists $x \in K_\rho$ such that $Lx \in Tx$, or
(b) there exist $y \in S_\rho$ and $\lambda \in (0,1)$ such that $Ly \in \lambda Ty$.

We now give a result in which several important examples are collected:

THEOREM 3.2. *Let $T : E \to F$ be an operator such that*

(a) *T is completely continuous,*
(b) *T belongs to one of the classes appearing in the diagram* (D).

Then T is of the Leray-Schauder type with respect to L.

PROOF. Assume, for example, that T is of the Eilenberg-Montgomery type. Observe that the set-valued map L^{-1} from F to E satisfies the hypotheses of the well-known theorem of E. Michael [10]; consequently, L^{-1} admits a continuous single-valued selector $s : F \to E$ satisfying $Ls = \text{id}$. Consider now the operator $sT_\rho : K_\rho \to E$ and observe that sT_ρ is compact and of Eilenberg-Montgomery type. By Theorem 2.2 (applied to the class $\mathbf{E}_c$) we have: either

(i) $x \in sTx$ for some $x \in K_\rho$, or
(ii) $y \in \lambda sTy$ for some $y \in S_\rho$.

Applying L and using the fact that $Ls = \text{id}$, we conclude that, either $Lx \in Tx$ for some $x \in K_\rho$, or $Ly \in Ty$ for some $y \in S_\rho$. The proof is complete. □

THEOREM 3.3. *Let $T : E \to F$ be an operator of the Leray-Schauder type with respect to L; let ρ be a positive number and assume that for all $x \in S_\rho$, one of the following conditions is satisfied:*

(i) $\|Tx\| \le \|Lx\|$ (*E. Rothe*);
(ii) $\|Tx\| \le \|Lx - Tx\|$;
(iii) $\|Tx\|^2 \le \left(\|Lx - Tx\|^2 + \|Lx\|^2\right)^{1/2}$ (*M. Altman*);
(iv) $\|Tx\| \le \max\{\|Lx\|, \|Lx - Tx\|\}$.

Then the operators T and L have at least one point of coincidence in K_ρ.

THEOREM 3.4 (The Leray-Schauder Alternative). *Let $T : E \to F$ be an operator of the Leray-Schauder type with respect to L and let*

$$\mathcal{E}_T = \{x \in E : Lx \in Tx \text{ for some } 0 < \lambda < 1\}.$$

Then either

(a) *the set $\mathcal{E}_T$ is unbounded, or*
(b) *the operators L and T have at least one point of coincidence.*

The proofs of the last two results are strictly analogous to those of Theorems 2.3 and 2.4 and are omitted.

REMARKS.

(i) The first elementary proofs (not using the degree theory) of the Leray-Schauder Alternative in the classical setting of single-valued completely continuous operators in Banach spaces were given in [12] and [8].
(ii) The proof of Theorem 2.2 is a modification of an argument given in [2], where it is proved that a nonexpansive map in a Hilbert space is of the Leray-Schauder type. A similar argument was used earlier in the case of

single-valued completely continuous operators in normed spaces in the book [5].

(iii) Theorem 2.2 (as well as arguments in the proof) can be established also in many other situations; for example:
   (a) for the class of maps studied in [1] in normed linear spaces;
   (b) for completely continuous operators in some metric linear spaces that are not locally convex; for example, in L_p spaces with $0 < p < 1$;
   (c) for completely continuous operators appearing in the diagram (D) in the context of cones in normed linear spaces.

(iv) Arguments used in this note can also be adapted to get a simple proof of an analog of Theorem 2.2 in locally convex spaces.

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Manuscript received October 18, 1993

ANDRZEJ GRANAS
Faculty of Mathematics and Informatics
Nicolas Copernicus University
Chopina 12/18, 87-100 Toruń, POLAND
– and –
Université de Montréal
C.P. 6128, Montréal (Québec) H3C 3J7, CANADA