

EIGENVECTORS FOR NONLINEAR MAPS

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(Submitted by A. Granas)

Dedicated to the memory of Juliusz Schauder

1. Introduction

The purpose of this paper is to present in the unified setting of index theory several results on eigenvectors and positive eigenvalues for nonlinear single-valued and multi-valued maps. These results, besides being interesting in their own right in the context of nonlinear analysis, have also applications to problems in linear algebra and population dynamics (see, for example, [1], [18]). Many researchers have made significant contributions to this area, starting with the celebrated theorem of Krein-Rutman [10]. Conjectures have been formulated and later solved. Open questions still exist. It is our intention to offer an overview and to bring the reader up to date with recent developments. We shall incorporate in the paper the case of single-valued as well as multi-valued maps. The functions we are considering belong to the class of α -contractions (k -set-contractions) or condensing maps, with respect to some measure of noncompactness. We make reference to the one proposed by Kuratowski [11], but the reader will see that the stated results hold also for other measures. The methods of proof are sometimes different between the single-valued and the multi-valued case. Moreover, some questions regarding multi-valued maps remain unsolved. For these reasons the theory regarding noncompact single-valued maps is presented separately from the theory regarding noncompact multi-valued maps. Every multi-valued map considered in this paper is assumed to have an acyclic decomposition. “Acyclic”

is intended in the Čech homology, although different homology theories could be used.

NOTATIONS AND DEFINITIONS will be introduced when needed and once stated will remain unchanged throughout the paper. We list here the ones which are used more frequently.

Spaces, cones, wedges and subsets.

- E always stands for an infinite dimensional Banach space, although many of the results we present can be rephrased in finite dimensional spaces.
- A *cone* $K \subset E$ is a closed subset of E with the following properties:
 - (i) $\mathbf{x}, \mathbf{y} \in K$ and $a, b \in [0, +\infty)$ implies $a\mathbf{x} + b\mathbf{y} \in K$;
 - (ii) $\mathbf{x} \in K, \mathbf{x} \neq \mathbf{0}$ implies $-\mathbf{x} \notin K$.
- A *wedge* $W \subset E$ is a closed subset of E which satisfies (i). A wedge is called *special* if there is a constant $\lambda > 0$ and a vector $\mathbf{x}_0 \in W$ such that $\|\mathbf{x} + \mathbf{x}_0\| \geq \lambda\|\mathbf{x}\|$ for every $\mathbf{x} \in W$. Every cone in a Banach space is a special wedge (see [12]).
- Unless otherwise stated Ω denotes an open bounded neighborhood of the origin in E or in a cone $K \subset E$. $\bar{\Omega}, \partial\Omega, \text{co}\Omega$ stand respectively for the closure, the boundary and the convex hull of Ω . When $\Omega \subset K$ its closure and boundary have to be taken with respect to K .
- $r(\Omega) = r = \sup\{\|\mathbf{x}\| : \mathbf{x} \in \partial\Omega\}$.
- Let $f : \bar{\Omega} \rightarrow E$, or $F : \bar{\Omega} \rightarrow E$ (or K) be respectively a single-valued or a multi-valued map. Then $d(f, \partial\Omega) = d = \inf\{\|f(\mathbf{x})\| : \mathbf{x} \in \partial\Omega\}$, $d(F, \partial\Omega) = d = \inf\{\|\mathbf{y}\| : \mathbf{y} \in F(\mathbf{x}), \mathbf{x} \in \partial\Omega\}$.
- $D(\mathbf{0}, r) = \{\mathbf{x} \in E : \|\mathbf{x}\| \leq r\}$; $D_{K,r} = D(\mathbf{0}, r) \cap K$.
- $B(\mathbf{0}, r) = \{\mathbf{x} \in E : \|\mathbf{x}\| < r\}$; $B_{K,r} = B(\mathbf{0}, r) \cap K$.
- $S(\mathbf{0}, r) = \{\mathbf{x} \in E : \|\mathbf{x}\| = r\}$; $S_{K,r} = S(\mathbf{0}, r) \cap K$.

Measure of noncompactness. Given a bounded subset A of a Banach space E we define, following Kuratowski [11],

$$\alpha(A) = \inf\{\varepsilon > 0 : A \text{ admits a finite covering with subsets of diameter not exceeding } \varepsilon\}.$$

The properties of α needed for our results are:

1. $\alpha(A) = 0$ if and only if A is totally bounded;
2. $\alpha(\text{co}(A)) = \alpha(A)$, where $\text{co}(A)$ denotes the convex closure of A (see [2]);
3. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;
4. $\alpha(A + B) \leq \alpha(A) + \alpha(B)$;

- 5. $\alpha(tA) = t\alpha(A)$ for every $t \geq 0$;
- 6. $\alpha(D(\mathbf{0}, 1)) = 2$ (see [8]).

A single-valued or a multi-valued function ϕ defined on the closure of a subset X of a Banach space E and with values in E is said to be an α -contraction with constant $k < 1$ (or a k -set-contraction) if $\alpha(\phi(A)) \leq k\alpha(A)$ for all bounded sets $A \subset X$. The attribute *condensing* is used for those maps F such that $\alpha(\phi(A)) < \alpha(A)$ for all $A \subset X$, A bounded and $\alpha(A) > 0$.

The index. Let X be a subset of E . A multi-valued map $F : X \multimap E$ is said to be *upper semicontinuous* if $F(\mathbf{x})$ is compact for every $\mathbf{x} \in X$, and $F^{-1}(W) = \{\mathbf{x} \in X : F(\mathbf{x}) \subset W\}$ is open in X for every W open in E . If X is the closure of an open, bounded set Ω and $\text{Fix}(F) = \{\mathbf{x} \in \bar{\Omega} : \mathbf{x} \in F(\mathbf{x})\}$ is compact in Ω , then F is said to be *admissible*. The map F will be called *acyclic* if for all $\mathbf{x} \in X$ the set $F(\mathbf{x})$ is acyclic in the Čech homology. In particular, this is true whenever $F(\mathbf{x})$ is convex for all $\mathbf{x} \in X$. Let $\{X_0, X_1, \dots, X_{n+1}\}$ be open, bounded subsets of E with $X_0 = \Omega$, $X_{n+1} = E$ and let $\Phi = \{F_0, F_1, \dots, F_n\}$ be a family of upper semicontinuous, acyclic-valued maps such that $F_i : \bar{X}_i \multimap \bar{X}_{i+1}$. Φ is an *acyclic decomposition* of F if $F = F_n \circ F_{n-1} \circ \dots \circ F_0$. For an admissible map F which is an α -contraction or is condensing and which has an acyclic decomposition Φ , one can define a fixed point index (see [6] and [20]). Although the index depends on the decomposition, it has nevertheless the following fundamental properties.

PROPERTY 1 (Decomposition). $\text{ind}(E, \Phi, \Omega) = \text{ind}(E, \Gamma, \Omega)$, where

$$\Gamma = \{F_0, \dots, F_{i-2}, F_i \circ F_{i-1}, F_{i+1}, \dots, F_n\},$$

provided that the map $F_i \circ F_{i-1}$ is still acyclic.

PROPERTY 2 (Additivity). Assume that $\Omega_1 \cup \Omega_2 \subset \Omega$ and

$$\text{Fix}(F) = (\text{Fix}(F) \cap (\Omega_1 \setminus \Omega_2)) \cup (\text{Fix}(F) \cap (\Omega_2 \setminus \Omega_1)).$$

Then $\text{ind}(E, \Phi, \Omega) = \text{ind}(E, \Phi, \Omega_1) + \text{ind}(E, \Phi, \Omega_2)$.

PROPERTY 3 (Solvability). $\text{ind}(E, \Phi, \Omega) \neq 0$ implies $\text{Fix}(F) \neq \emptyset$.

PROPERTY 4 (Normalization). $\text{ind}(E, \mathbf{x}_0, \Omega) = 1$ if $\mathbf{x}_0 \in \Omega$ (and $\text{ind}(E, \mathbf{x}_0, \Omega) = 0$ if $\mathbf{x}_0 \notin \bar{\Omega}$) where the constant map $F(\mathbf{x}) = \mathbf{x}_0$ is also denoted by \mathbf{x}_0 .

PROPERTY 5 (Homotopy). Let $H : \bar{\Omega} \times [0, 1] \multimap E$ be upper semicontinuous and condensing. Define $\text{FIX}(H) = \{\mathbf{x} \in \bar{\Omega} : \mathbf{x} \in H(\mathbf{x}, t) \text{ for some } t \in [0, 1]\}$. Assume that $\text{FIX}(H)$ is a compact subset of Ω and H has an acyclic decomposition

$$\Omega(\cdot, t) = \{H_0(\cdot, t), H_1, \dots, H_n\}.$$

Then $\text{ind}(E, \Phi(\cdot, t), \Omega)$ is defined and it is constant.

PROPERTY 6 (mod p). Assume that F and F^p have no fixed points on $\partial\Omega$. Moreover, assume that if for some $1 \leq i < p$ we have $\mathbf{y} \in F^i(\mathbf{x})$, $\mathbf{x} \in F^{p-i}(\mathbf{y})$ and $\mathbf{x} \in \Omega$, then $\mathbf{y} \in \Omega$. In this case $\text{ind}(E, \Phi, \Omega) \equiv \text{ind}(E, \Phi^p, \Omega) \pmod{p}$ (see also [21]).

The above definition and properties are obviously valid for single-valued maps and they can be extended to the class of α -contractions or condensing maps defined in an open, bounded neighborhood Ω of the origin in a cone K or in a wedge W of E .

The organization of the paper is as follows. In the first section we study the case of compact single-valued and multi-valued maps defined on $\overline{\Omega} \subset E$ or $\overline{\Omega} \subset K$. In Section 2 we extend our analysis to the class of α -contractions and condensing single-valued maps defined on $\overline{\Omega}$. Section 3 contains the results regarding multi-valued α -contractions or condensing maps defined on $\overline{\Omega}$. In Section 4 we go back to single-valued maps defined on a cone K , and which have the additional properties of being positively homogeneous of degree one and order preserving. Finally, in Section 5, the analysis of Section 4 is extended to a class of multi-valued maps defined on K .

1. The compact case

Let $f : \overline{\Omega} \rightarrow E$ be continuous and compact. The following result is well known.

THEOREM 1.1 (see [3]). Assume that (i) $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. Then $\text{ind}(E, f, \Omega) = 0$.

PROOF. Let $m > 0$ be such that $B(\mathbf{0}, m) \subset \Omega$ and let $\pi : E \rightarrow E$ be a continuous (for the existence of π see [3]) retraction such that $\pi(\mathbf{x}) = \mathbf{x}$ for every \mathbf{x} with $\|\mathbf{x}\| \geq m$ and $\|\pi(\mathbf{x})\| = m$ for every $\mathbf{x} \in B(\mathbf{0}, m)$. The map $g(\mathbf{x}) = \pi(f(\mathbf{x}))$ is continuous, compact, and it coincides with f on $\partial\Omega$. Moreover, $g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \overline{\Omega}$. Define $h(\mathbf{x}) = (2r/m)g(\mathbf{x})$.

The map h is compact, homotopic to g by a homotopy without fixed points on $\partial\Omega$ and has the property $h(\overline{\Omega}) \cap \overline{\Omega} = \emptyset$. Thus

$$0 = \text{ind}(E, h, \Omega) = \text{ind}(E, g, \Omega) = \text{ind}(E, f, \Omega).$$

□

REMARK 1.1. Condition (i) can be replaced by other assumptions. For example either one of the two conditions

- $f(\partial\Omega) \cap \text{co}\overline{\Omega} = \emptyset$, and
- $f(\partial\Omega) \cap \text{co}\Omega = \emptyset$ with $\mathbf{x} \neq f(\mathbf{x})$ for every $\mathbf{x} \in \partial\Omega$,

could replace (i). Notice that the first implies (i) and the second is implied by (i). We list two additional conditions which may be easier to verify than the ones mentioned so far:

- (ii) $r < d$;
- (iii) $0 < d$ and $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

COROLLARY 1.1. *Let f be as in Theorem 1. Then there are $\lambda > 1$ and $\mathbf{x} \in \partial\Omega$ such that $f(\mathbf{x}) = \lambda\mathbf{x}$.*

REMARK 1.2. In case (ii) we obtain $\lambda \geq d/r$.

REMARK 1.3. The condition $d > 0$ in (iii) cannot be omitted, as the following example shows.

EXAMPLE 1.1. In the Hilbert space l^2 of square summable sequences of real numbers define the compact linear operator $L\mathbf{x} = L(x_1, x_2, \dots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \dots)$. Then $L\mathbf{x} \neq t\mathbf{x}$ for all $\mathbf{x} \in B(\mathbf{0}, 1)$ and $t \in (0, 1]$. Thus $\text{ind}(l^2, L, B(\mathbf{0}, 1)) = 1$. Notice that $\inf\{\|L\mathbf{x}\| : \|\mathbf{x}\| = 1\} = 0$.

The following result extends Theorem 1.1 to multi-valued maps. It is proved with the same strategy used to establish Theorem 1.1.

THEOREM 1.2. *Let $F : \bar{\Omega} \multimap E$ be upper semicontinuous, compact, admissible and with an acyclic decomposition Φ . Assume that (j) $F(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. Then $\text{ind}(E, \Phi, \Omega) = 0$.*

REMARK 1.4. As in the single-valued case condition (j) can be replaced by other assumptions, which may be easier to verify. We list the following two.

- (jj) $r < d$;
- (jjj) $0 < d$ and $t\mathbf{x} \notin F(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

COROLLARY 1.2. *Let F be as in Theorem 1.2. Then there are $\lambda > 1$ and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.*

Again, we get $\lambda \geq d/r$ when (jj) is assumed.

COROLLARY 1.3 (see [5]). *Let $F : S(\mathbf{0}, 1) \multimap S(\mathbf{0}, 1)$ be upper semicontinuous and compact. Assume that F has an acyclic decomposition Φ . Then F has a fixed point.*

PROOF. Extend F to $D(\mathbf{0}, 1)$ by setting $F(t\mathbf{x}) = tF(\mathbf{x})$ for $t \in (0, 1]$, $\mathbf{x} \in S(\mathbf{0}, 1)$, and $F(\mathbf{0}) = \mathbf{0}$. Let $q > 1$, define $G(\mathbf{x}) = qF(\mathbf{x})$ and let Φ_1 be the acyclic decomposition of G corresponding to Φ . Then $\text{ind}(E, \Phi_1, B(\mathbf{0}, 1)) = 0$. By Corollary 1.2 there exist $\lambda > 1$ and \mathbf{x} with $\|\mathbf{x}\| = 1$ such that $\lambda\mathbf{x} \in G(\mathbf{x})$, i.e. $\mathbf{x} \in F(\mathbf{x})$. □

All previous results are valid in the case when Ω is an open, bounded neighborhood of the origin in a cone $K \subset E$. The only requirement is that the single-valued or the multi-valued functions map $\overline{\Omega}$ into K . For example, we have the following version of Theorem 1.2.

THEOREM 1.3. *Let $K \subset E$ be a cone, Ω be an open, bounded neighborhood of the origin in K , and $F : \overline{\Omega} \rightarrow K$ be upper semicontinuous, compact, admissible and with an acyclic decomposition Φ . Assume that one of the following conditions is satisfied:*

$$(j) \quad r < d;$$

$$(jj) \quad 0 < d \text{ and } tx \notin F(\mathbf{x}) \text{ for all } \mathbf{x} \in \partial\Omega \text{ and } t \in (0, 1].$$

Then $\text{ind}(K, \Phi, \Omega) = 0$.

COROLLARY 1.4. *Let F be as in Theorem 1.3. Then there are $\lambda > 1$ ($\lambda \geq d/r$) and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.*

COROLLARY 1.5. *Let $K \subset E$ be a cone and $F : S_{K,1} \rightarrow S_{K,1}$ be upper semicontinuous, compact and with an acyclic decomposition Φ . Then F has a fixed point in $S_{K,1}$.*

2. The noncompact case: single-valued maps

In this section we present results on eigenvalues and eigenvectors of functions which are not compact, but belong to the class of α -contractions or condensing maps with respect to some measure of noncompactness. To be specific we make reference to the α -measure of Kuratowski [11], but the conclusions are independent of this choice, as long as the appropriate conditions are met. Besides being of interest on their own, the results of this section provide an affirmative answer to a conjecture of Massabò-Stuart [15] (see Theorem 2.3 below). A partial answer to the conjecture was given by Nussbaum [17], in the case when Ω is star-shaped with respect to the origin. Here this restriction is removed and the conjecture is established in its full generality. Our first result extends Theorem 1.1 to α -contractions. The proof, however, is quite different since the map πf cannot be used in this case.

THEOREM 2.1. *Let $f : \overline{\Omega} \rightarrow E$ be continuous and α -contractive with constant $p < 1$. Assume that (i) $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. Then $\text{ind}(E, f, \Omega) = 0$.*

PROOF. Let $q > 1$ be such that $qp < 1$, and define $g(\mathbf{x}) = qf(\mathbf{x})$. Then there exists $\varepsilon > 0$ such that $\|\mathbf{x} - \mathbf{y}\| \geq 2\varepsilon$ for every $\mathbf{x} \in \text{co}\overline{\Omega}$ and $\mathbf{y} \in g(\partial\Omega)$. Select m so large that $(qp)^m \alpha(\Omega) < \varepsilon$ and define $V_i = g^{1-i}(\Omega)$, $i = 1, \dots, m$. By induction we obtain

$$V_m \subset V_{m-1} \subset \dots \subset V_1 = \Omega.$$

If for some index i the set V_i is empty, then obviously $\text{ind}(E, g, \Omega) = 0$, and by homotopy $\text{ind}(E, f, \Omega) = 0$. Thus we can assume, without loss of generality, that $V_i \neq \emptyset$ for all indices i . Moreover, by elementary topology, we have $h(\partial h^{-1}U) \subset \partial U$ provided that h is continuous. Thus, with $h = g$ and $U = g^{2-i}(\Omega)$ we derive

$$g(\partial V_i) = g(\partial g^{1-i}(\Omega)) \subset \partial g^{2-i}(\Omega) = \partial V_{i-1}.$$

By applying g^{i-1} to both sides we arrive at $g^i(\partial V_i) \subset g^{i-1}(\partial V_{i-1})$ for all $i = 1, \dots, m$. Hence

$$g^m(\partial V_m) \subset g^{m-1}(\partial V_{m-1}) \subset \dots \subset g(\partial \Omega).$$

Since $\alpha g^m(V_m) \leq (qp)^m \alpha(\Omega) < \varepsilon$, we can cover $g^m(V_m)$ with finitely many closed, convex sets K_1, \dots, K_s such that $\text{diam}(K_i) < \varepsilon$ for all $i = 1, \dots, s$. Define $K = \bigcup_i K_i$ and let $\pi : K \rightarrow K$ be continuous, compact and such that $\pi(K_i) \subset K_i$ (for the existence of such a map π see [7] and [19]). Since $\|\mathbf{x} - \mathbf{y}\| \geq 2\varepsilon$ for all $\mathbf{x} \in \text{co } \bar{V}_m$ and $\mathbf{y} \in g(\partial V_m)$ we deduce that πg^m and g^m are homotopic on V_m . By Theorem 1.1, $\text{ind}(E, \pi g^m, V_m) = 0$, which implies $\text{ind}(E, g^m, V_m) = 0$ as well. We can assume that m is a prime number larger than $|\text{ind}(E, g, \Omega)|$. From the mod p property we obtain

$$\text{ind}(E, g, \Omega) \equiv \text{ind}(E, g^m, V_m) \pmod{p},$$

which implies $\text{ind}(E, g, \Omega) = nm$ for some integer n . Since $|\text{ind}(E, g, \Omega)| < m$ the only possibility is $n = 0$. The maps f and g are obviously homotopic without fixed points on $\partial \Omega$. Thus $\text{ind}(E, f, \Omega) = 0$. □

In the compact case we have seen that condition (i) can be replaced by either one of the following (see Remark 1.1):

- (ii) $r < d$;
- (iii) $0 < d$ and $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial \Omega$ and $t \in (0, 1]$.

In the noncompact case (iii) needs some adjustments, as the following example shows.

EXAMPLE 2.1. In the Hilbert space l^2 of square summable sequences of real numbers define the linear operator $L\mathbf{x} = L(x_1, x_2, \dots) = \frac{1}{2}(0, x_1, x_2, \dots)$. Then L is an α -contraction with constant $p = \frac{1}{2}$. Condition (iii) is obviously satisfied, but $\text{ind}(E, L, B(0, 1)) = 1$, since L is homotopic to the constant map which sends everything into $\mathbf{0}$.

We now present how (iii) can be modified in the noncompact case. As mentioned in the introduction we define $d = \inf\{\|f(\mathbf{x})\| : \mathbf{x} \in \partial \Omega\}$, $r = \sup\{\|\mathbf{x}\| : \mathbf{x} \in \partial \Omega\}$. Let p be the α -contraction constant of f . Replace condition (iii) by

$$(iii)' \quad rp < d \text{ and } f(\mathbf{x}) \neq t\mathbf{x} \text{ for all } \mathbf{x} \in \partial \Omega \text{ and } t \in (0, 1].$$

Notice that (iii)' reduces to (iii) in the compact case since $p = 0$. To see why (iii)' works consider the map $g : \bar{\Omega} \rightarrow E$ defined by $g(\mathbf{x}) = f(\mathbf{x})/(p + \varepsilon)$, where $\varepsilon > 0$ is selected so that $p + \varepsilon < 1$ and $r(p + \varepsilon) < d$. We deduce that g is an α -contraction with constant $p/(p + \varepsilon)$ and $d' = \inf\{\|g(\mathbf{x})\| : \mathbf{x} \in \partial\Omega\} = d/(p + \varepsilon) > r$. It follows that $g(\partial\Omega) \cap \text{co}\bar{\Omega} = \emptyset$. The condition $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$ implies that f and g are homotopic via the admissible homotopy $H : \bar{\Omega} \times [0, 1] \rightarrow E$ defined by $H(\mathbf{x}, s) = (1 - s)f(\mathbf{x}) + sg(\mathbf{x})$. Thus $\text{ind}(E, f, \Omega) = \text{ind}(E, g, \Omega)$ and g satisfies the assumptions of Theorem 2.1. In the case when $r < d$ we can choose $\varepsilon > 0$ such that $p + \varepsilon < 1$. Defining g as before yields the same conclusion. Summarizing, we have the following result.

THEOREM 2.2. *Let $f : \bar{\Omega} \rightarrow E$ be continuous and α -contractive with constant $p < 1$. Assume that one of the following conditions is satisfied:*

- (ii) $r < d$;
- (iii) $rp < d$ and $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

Then $\text{ind}(E, f, \Omega) = 0$.

Theorem 2.2 remains valid in the case when Ω is an open, bounded neighborhood of the origin in a cone $K \subset E$. The only requirement is that f maps $\bar{\Omega}$ into K . Moreover, we should keep in mind that $\bar{\Omega}, \partial\Omega$ are to be taken relative to the cone.

THEOREM 2.3. *Let $K \subset E$ be a cone, Ω be an open, bounded neighborhood of the origin in K , and $f : \bar{\Omega} \rightarrow K$ be continuous and α -contractive with constant $p < 1$. Assume that one of the following conditions is satisfied:*

- (ii) $r < d$;
- (iii) $rp < d$ and $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

Then $\text{ind}(K, f, \Omega) = 0$.

The above result provides a positive answer to a conjecture of I. Massabò and C. Stuart [15], who established Theorem 2.3 under the additional assumption: *there exists a constant $c \geq 1$ such that $\|\mathbf{x} + \mathbf{y}\| \geq c\|\mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in K$ (i.e. the cone K is normal) and $rp < cd$* . They conjectured that c was not needed in the last inequality. Their conjecture was correct, as shown by Theorem 2.3. Notice that as a consequence of Theorems 2.1, 2.2 and 2.3 we deduce that f has an eigenvector $\mathbf{x}_0 \in \partial\Omega$ corresponding to an eigenvalue $\lambda > 1$. We present this result in the form of a Corollary in E . A similar result holds in K .

COROLLARY 2.1. *Let Ω be an open, bounded neighborhood of the origin in E , and $f : \bar{\Omega} \rightarrow E$ be continuous and α -contractive with constant $p < 1$. Assume that one of the following conditions is satisfied:*

- (i) $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$;
- (ii) $r < d$;
- (iii) $rp < d$ and $f(\mathbf{x}) \neq t\mathbf{x}$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

Then there exist $\lambda > 1$ ($\lambda \geq d/r$ in case (ii)) and $\mathbf{x}_0 \in \partial\Omega$ such that $\lambda\mathbf{x} = f(\mathbf{x})$.

In the case when Ω is star shaped with respect to the origin Theorem 2.1 and its consequences can be established in a more direct manner, without using the mod p theorem (see, for example, [17]).

We now turn our attention to the case when f is condensing. The proof of Theorem 2.1 needs some adjustments since we cannot define the auxiliary map g . Given an integer $m \geq 1$, define $V_m = f^{1-m}(\Omega)$ and notice that

$$\alpha(f^m(\partial V_m)) \leq \alpha(f^m(\overline{V}_m)) \leq \alpha(f^m(\overline{\Omega})).$$

As we have seen in the proof of Theorem 2.1, we have the inclusions

$$f^m(\partial V_m) \subset f^{m-1}(\partial V_{m-1}) \subset \dots \subset f(\partial\Omega).$$

It is known that for every bounded set A , $\alpha(f^n(A)) \rightarrow 0$ as $n \rightarrow \infty$ (see [16]). Therefore, by a theorem of Kuratowski [11], the set

$$K_\infty = \bigcap_m \overline{f^m(\partial V_m)}, \quad K_\infty \subset \overline{f(\partial\Omega)},$$

is nonempty and compact.

Assume that $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. We obtain $f(\mathbf{x}) \notin \overline{\Omega}$ for every $\mathbf{x} \in \partial\Omega$. That is, $\mathbf{x} \notin f^{-1}(\overline{\Omega})$. This implies $\mathbf{x} \notin \overline{V}_2$ and $\overline{V}_2 \subset \Omega$. Since

$$K_\infty \subset \overline{f(\partial\Omega)} \subset E \setminus (\text{co}\Omega \cup \partial\Omega)$$

is compact we obtain $K_\infty \cap \overline{V}_2 = \emptyset$ and $\text{dis}(K_\infty, \overline{V}_2) = 3d > 0$. Consequently,

$$d(K_\infty, \overline{V}_m) \geq 3d \quad \text{for all } m \geq 2.$$

Define $M = 1 + \sup\{\|\mathbf{x}\| : \mathbf{x} \in K_\infty\}$ and choose $q \in (1, 1 + d/M)$. Then $qK_\infty \subset E \setminus \overline{\text{co}\Omega}$, and there exists $\varepsilon, \varepsilon \in (0, \min\{1/3, d/2\})$, such that

$$qN_{2\varepsilon}(K_\infty) \subset E \setminus \overline{\text{co}\Omega}, \quad \text{where } N_h(K_\infty) = \{\mathbf{x} \in E : \text{dis}(\mathbf{x}, K_\infty) \leq h\}.$$

Select n so large that $\overline{f^m(\partial V_m)} \subset N_\varepsilon(K_\infty)$ and $\alpha(f^m(\overline{V}_m)) < \varepsilon$ for all $m \geq n$. Let $m \geq n$ be given. We can cover the set $f^m(\overline{V}_m)$ with a finite family of closed convex sets $C_i, i = 1, \dots, k$, of diameter not exceeding ε and find a compact map

$$\pi_m : \bigcup_i C_i = C \rightarrow C \quad \text{such that } \pi_m(C_i) \subset C_i \text{ and } \|f^m(\mathbf{x}) - \pi_m f^m(\mathbf{x})\| < \varepsilon$$

for all $\mathbf{x} \in \overline{V}_m$. Since $q\pi_m f^m(\partial V_m) \subset qN_{2\varepsilon}(K_\infty)$, the compact map $q\pi_m f^m$ satisfies $q\pi_m f^m(\partial V_m) \cap \overline{\text{co}\Omega} = \emptyset$. Thus $q\pi_m f^m(\partial V_m) \cap \overline{\text{co}V}_m = \emptyset$, and we can

apply Theorem 1.1 to obtain $\text{ind}(E, q\pi_m f^m, V_m) = 0$. The maps $q\pi_m f^m$ and $\pi_m f^m$ are homotopic, since

$$k(t, \mathbf{x}) = [1 + t(q - 1)]\pi_m f^m(\mathbf{x}) \in N_{2\varepsilon+d}(K_\infty) \subset N_{2d}(K_\infty) \subset E \setminus \bar{V}_2 \subset E \setminus \bar{V}_m$$

for every $\mathbf{x} \in \partial V_m$ and $t \in [0, 1]$. Hence

$$0 = \text{ind}(E, q\pi_m f^m, V_m) = \text{ind}(E, \pi_m f^m, V_m).$$

We want to show that

$$0 = \text{ind}(E, \pi_m f^m, V_m) = \text{ind}(E, f^m, V_m).$$

Define $h(t, \mathbf{x}) = (1 - t)f^m(\mathbf{x}) + t\pi_m f^m(\mathbf{x})$. If $\mathbf{x} \in \partial V_m$ we have

$$h(t, \mathbf{x}) \in N_{2\varepsilon}(K_\infty) \subset N_d(K_\infty) \subset E \setminus \bar{V}_2 \subset E \setminus \bar{V}_m.$$

This shows that h is an admissible homotopy and gives the desired result. At this point, with the same strategy used in Theorem 2.1, we obtain $\text{ind}(E, f, \Omega) = 0$. In summary:

THEOREM 2.4. *Let $f : \bar{\Omega} \rightarrow E$ be continuous and condensing. Assume that $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. Then $\text{ind}(E, f, \Omega) = 0$.*

THEOREM 2.5. *Let $\Omega \subset K$ and $f : \bar{\Omega} \rightarrow K$ be continuous and condensing. Assume $r < d$. Then $\text{ind}(K, f, \Omega) = 0$.*

COROLLARY 2.2. *Let $f : \bar{\Omega} \rightarrow E$ be continuous and condensing. Assume that $f(\partial\Omega) \cap (\text{co}\Omega \cup \partial\Omega) = \emptyset$. Then there exist $\lambda > 1$ and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} = f(\mathbf{x})$.*

COROLLARY 2.3. *Let $\Omega \subset K$ and $f : \bar{\Omega} \rightarrow K$ be continuous and condensing. Assume that $r < d$. Then there are $\lambda \geq d/r$ and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} = f(\mathbf{x})$.*

3. The noncompact case: multi-valued maps

For multi-valued, upper semicontinuous α -contractions or condensing maps the techniques of the previous section, with particular reference to the proof of Theorem 2.1, cannot be used, since we do not have enough control on the fixed points of F^n belonging to ∂V_n where $V_n = F^{1-n}(\Omega)$ and $F^{-1}(A) = \{\mathbf{x} : F(\mathbf{x}) \subset A\}$. Additional assumptions that could allow us to apply the same method to the multi-valued case include the following two, which are obviously satisfied in the single-valued case:

- (i) if $\mathbf{x} \in F(\mathbf{x})$ then $F^n(\mathbf{x}) \subset \Omega$ for all $n \geq 1$;
- (ii) if $F^n(\mathbf{x}) \cap \partial\Omega \neq \emptyset$, then $F^{n+1}(\mathbf{x}) \cap \bar{\Omega} = \emptyset$.

Here $F^{n+1}(\mathbf{x}) = F(F^n(\mathbf{x}) \cap \bar{\Omega})$. The above conditions look rather restrictive and not easy to verify. Therefore we are going to use a different approach. In this section, unless stated otherwise, Ω is assumed to be an open, *convex* and bounded neighborhood of $\mathbf{0}$.

THEOREM 3.1. *Let $F : \bar{\Omega} \rightarrow E$ be upper semicontinuous and condensing. Assume that F has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$ and $\overline{F(\partial\Omega)} \cap \bar{\Omega} = \emptyset$. Then $\text{ind}(E, \Phi, \Omega) = 0$.*

PROOF. Let $\pi : E \rightarrow \bar{\Omega}$ be the retraction $\pi(\mathbf{x}) = t(\mathbf{x})\mathbf{x}$, where $t(\mathbf{x}) = 1$ if $\mathbf{x} \in \bar{\Omega}$ and it is the largest number such that $t(\mathbf{x})\mathbf{x} \in \bar{\Omega}$ if $\mathbf{x} \in \bar{\Omega}$. Extend F to the upper semicontinuous, condensing map $G : E \rightarrow E$ defined by $G(\mathbf{x}) = F(\pi(\mathbf{x}))$ and let $\Phi_1 = \{\pi, F_0, \dots, F_n\}$. Notice that $G(E \setminus \Omega) \subset \overline{F(\partial\Omega)} \subset N_h(\mathbf{0}) \setminus \bar{\Omega}$ for some $h > 0$ such that $F(\bar{\Omega}) \cup \bar{\Omega} \subset N_h(\mathbf{0})$. By normality and due to the fact that $N_h(\mathbf{0}) \setminus \bar{\Omega}$ is acyclic, we obtain

$$\text{ind}(E, \Phi_1, N_h(\mathbf{0}) \setminus \bar{\Omega}) = 1.$$

From $\text{ind}(E, \Phi_1, N_h(\mathbf{0})) = 1$, using the Additivity Property of the index and the fact that G has no fixed points on $\partial\Omega$, we derive

$$\text{ind}(E, \Phi_1, N_h(\mathbf{0}) \setminus \bar{\Omega}) + \text{ind}(E, \Phi_1, \Omega) = 1.$$

This implies $\text{ind}(E, \Phi_1, \Omega) = 0$.

The restriction of G to Ω coincides with F . Therefore $\text{ind}(E, \Phi, \Omega) = 0$. \square

In the case when F is an α -contraction with constant $p < 1$ the above result can be strengthened somewhat. More precisely, we have

THEOREM 3.2. *Let $F : \bar{\Omega} \rightarrow E$ be an upper semicontinuous α -contraction with constant $p < 1$. Assume that F has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$ and is fixed point free on $\partial\Omega$. Then $F(\partial\Omega) \cap \Omega = \emptyset$ implies $\text{ind}(E, \Phi, \Omega) = 0$.*

PROOF. Let $q > 1$ be such that $qp < 1$. Define $G(\mathbf{x}) = qF(\mathbf{x})$, and let Φ_1 be the acyclic decomposition of G corresponding to Φ . Since $\overline{G(\partial\Omega)} \cap \bar{\Omega} = \emptyset$, Theorem 3.1 implies $\text{ind}(E, \Phi_1, \Omega) = 0$. Since F and G are homotopic the result follows. \square

COROLLARY 3.1. *Let $F : \bar{\Omega} \rightarrow E$ be an upper semicontinuous α -contraction with constant $p < 1$. Assume that F has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$ and one of the following conditions holds:*

- (i) $r < d$;
- (ii) $rp < d$ and $t\mathbf{x} \notin F(\mathbf{x})$ for all $t \in (0, 1]$ and $\mathbf{x} \in \partial\Omega$.

Then $\text{ind}(E, \Phi, \Omega) = 0$.

PROOF. Choose $\varepsilon > 0$ so that $p + \varepsilon < 1$ and $r(p + \varepsilon) < d$. Define $G(\mathbf{x}) = \frac{1}{p+\varepsilon}F(\mathbf{x})$. It is easily seen that G and F are homotopic and

$$\inf\{\|y\| : y \in G(\mathbf{x}), \mathbf{x} \in \partial\Omega\} = \frac{d}{p + \varepsilon} > r.$$

Hence G satisfies the assumptions of Theorem 3.2. □

COROLLARY 3.2. *Let $F : \bar{\Omega} \rightarrow E$ be upper semicontinuous and condensing. Assume that F has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$ and $\overline{F(\partial\Omega)} \cap \bar{\Omega} = \emptyset$. Then there exist $\lambda > 1$ and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.*

COROLLARY 3.3. *Let $F : \bar{\Omega} \rightarrow E$ be an upper semicontinuous α -contraction with constant $p < 1$. Assume that F has an acyclic decomposition and one of the following conditions holds:*

- (i) $r < d$;
- (ii) $pr < d$ and $t\mathbf{x} \notin F(\mathbf{x})$ for all $t \in (0, 1]$ and $\mathbf{x} \in \partial\Omega$.

Then there exist $\lambda > 1$ and $\mathbf{x} \in \partial\Omega$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.

We now examine some cases in which Ω is not convex, but is contained in a wedge W with the property that there exists a vector $\mathbf{x}_0 \in W$ such that $\|\mathbf{x} + \lambda\mathbf{x}_0\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in W$. Notice that, without loss of generality, we may assume $\|\mathbf{x}_0\| = 1$. Recall that W is called a *special wedge*.

THEOREM 3.3. *Let $\Omega \subset W$ and $F : \bar{\Omega} \rightarrow W$ be upper semicontinuous. Assume that F admits an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$, is an α -contraction with constant $p < 1$, and satisfies one of the following conditions:*

- (i) $r < d$;
- (ii) $rp < d$ and $t\mathbf{x} \notin F(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$ and $t \in (0, 1]$.

Then $\text{ind}(W, \Phi, \Omega) = 0$.

PROOF. Let $\varepsilon > 0$ be such that $p + \varepsilon < 1$ and $r(p + \varepsilon) < d$. Define $G(\mathbf{x}) = \frac{1}{p+\varepsilon}F(\mathbf{x})$, and let Φ_1 be the acyclic decomposition of G corresponding to Φ . G is an α -contraction with constant $p/(p + \varepsilon)$. Moreover, for every $\mathbf{x} \in \partial\Omega$ we obtain $G(\mathbf{x}) \subset W \setminus D(\mathbf{0}, r)$ and $\text{ind}(W, \Phi, \Omega) = \text{ind}(W, \Phi_1, \Omega)$. Let U and V be two open subsets of W such that

$$\mathbf{0} \in V \subset \bar{V} \subset U \subset \bar{U} \subset \Omega$$

and $\bar{\Omega} \setminus V \subset G^{-1}(W \setminus D(\mathbf{0}, r))$. There are two continuous functions $u, v : E \rightarrow \mathbb{R}$ such that

$$\begin{aligned} v(\mathbf{x}) &= 0 && \text{if } \mathbf{x} \in \bar{V}; \\ v(\mathbf{x}) &= 1 && \text{if } \mathbf{x} \in W \setminus U; \\ u(\mathbf{x}) &= 1 && \text{if } \mathbf{x} \in \bar{U}; \\ u(\mathbf{x}) &= 0 && \text{if } \mathbf{x} \in W \setminus \Omega. \end{aligned}$$

Let $M = \sup\{\|y\| : y \in G(\mathbf{x}), \mathbf{x} \in \overline{\Omega}\}$ and choose $b > M + r + 1$. Define

$$H(\mathbf{x}) = \begin{cases} G(\mathbf{x}) + bv(\mathbf{x})\mathbf{x}_0 & \text{if } \mathbf{x} \in \overline{U}, \\ u(\mathbf{x})G(\mathbf{x}) + b\mathbf{x}_0 & \text{if } \mathbf{x} \in \overline{\Omega} \setminus U, \\ b\mathbf{x}_0 & \text{if } \mathbf{x} \in W \setminus \Omega. \end{cases}$$

The map H is upper semicontinuous, α -contractive with constant $p/(p+e)$ and admits an acyclic decomposition Φ_2 . Moreover, its restriction to \overline{V} coincides with G . Let us show that $H(W \cap (D(\mathbf{0}, r) \setminus V)) \subset W \setminus D(\mathbf{0}, r)$. We examine separately the two cases (i) $\mathbf{x} \in \overline{U} \setminus V$ and (ii) $\mathbf{x} \in D(\mathbf{0}, r) \setminus \overline{U}$. In case (i) let $\mathbf{y} \in H(\mathbf{x})$, i.e. $\mathbf{y} = \mathbf{z} + bv(\mathbf{x})\mathbf{x}_0$ for some $\mathbf{z} \in G(\mathbf{x})$. We have

$$\|\mathbf{z} + bv(\mathbf{x})\mathbf{x}_0\| \geq \|\mathbf{z}\| > r$$

since $G(\overline{\Omega} \setminus V) \subset W \setminus D(\mathbf{0}, r)$. In case (ii), $\mathbf{y} = u(\mathbf{x})\mathbf{z} + b\mathbf{x}_0$ and

$$\|u(\mathbf{x})\mathbf{z} + b\mathbf{x}_0\| \geq b - \|\mathbf{z}\| \geq b - M > r + 1.$$

In particular, $\mathbf{w} \in H(\mathbf{x})$ for some $\mathbf{x} \in W$ with $\|\mathbf{x}\| = r$ implies $\|\mathbf{w}\| > r + 1$. Thus, by Corollary 3.1, $\text{ind}(W, \Phi_2, B(\mathbf{0}, r) \cap W) = 0$. Since H has no fixed points on $(B(\mathbf{0}, r) \cap W) \setminus \overline{V}$ we obtain, from the Additivity Property of the index, $\text{ind}(W, \Phi_2, V) = 0$. On the open set V we have $H = G = F$. Thus $\text{ind}(W, \Phi, V) = 0$. The map F does not have any fixed points on $\overline{\Omega} \setminus V$, and this implies, again by the Additivity Property,

$$\text{ind}(W, \Phi, \Omega) = 0.$$

□

COROLLARY 3.4. *Let $\Omega \subset W$ and $F : \overline{\Omega} \rightarrow W$ be upper semicontinuous and condensing. Assume that F has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$ and $d > r$. Then $\text{ind}(W, \Phi, \Omega) = 0$.*

PROOF. The proof is patterned after the one of Theorem 3.3. The map G is not needed since $F(\partial\Omega) \subset W \setminus D(\mathbf{0}, r)$. The map H is defined using F in place of G . □

COROLLARY 3.5. *Let $\Omega \subset W$ and $F : \overline{\Omega} \rightarrow W$ be an upper semicontinuous α -contraction with constant $p < 1$, which has an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$. Assume that one of the following conditions holds:*

- (ii) $r < d$;
- (iii) $rp < d$ and $t\mathbf{x} \notin F(\mathbf{x})$ for all $t \in (0, 1]$ and $\mathbf{x} \in \partial\Omega$.

Then there exist $\mathbf{x} \in \partial\Omega$ and $\lambda > 1$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.

PROOF. F cannot be homotopic to the constant map $G(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \overline{\Omega}$. □

COROLLARY 3.6. *Let $\Omega \subset W$ and $F : \bar{\Omega} \rightarrow W$ be upper semicontinuous, condensing, and with an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$. Assume $r < d$. Then there exist $\mathbf{x} \in \partial\Omega$ and $\lambda \geq d/r$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.*

COROLLARY 3.7. *Let $S = \{\mathbf{x} \in E : \|\mathbf{x}\| = 1\}$, and $F : S \rightarrow S$ be an upper semicontinuous α -contraction with constant $p < 1$, which admits an acyclic decomposition $\Phi = \{F_0, \dots, F_n\}$. Then F has a fixed point.*

PROOF. Let $q > 1$ be such that $qp < 1$. Define $G_q(\mathbf{x}) = qF(\mathbf{x})$ and let Φ_1 be the acyclic decomposition of G corresponding to Φ . Then G_q is an α -contraction with constant qp . Extend G_q and F to the full disk and denote the extensions with the same symbols (see Corollary 1.3). It can be shown (see for example [12]) that F and G_q are still α -contractions. If F is fixed point free on S then the index of Φ_1 on $B(\mathbf{0}, 1)$ is defined and, according to our previous results, we have $\text{ind}(E, \Phi_1, B(\mathbf{0}, 1)) = 0$. The maps F and G_q are obviously homotopic and we obtain $\text{ind}(E, \Phi, B(\mathbf{0}, 1)) = 0$. This, however, is impossible since F is homotopic to the constant map $K(\mathbf{x}) = \mathbf{0}$. □

4. Order preserving single-valued maps

Let K be a cone in E . Denote by \leq the partial ordering induced by K , i.e. $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in K$. A function $f : K \rightarrow K$ is said to be *order preserving* if $f(\mathbf{x}) \leq f(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$, and *positively homogeneous of degree one* if $f(t\mathbf{x}) = tf(\mathbf{x})$ for all $t \geq 0$ and $\mathbf{x} \in K$. The following notations and definitions will be used throughout the section.

- Given any map $f : K \rightarrow K$ which is positively homogeneous of degree 1 and such that $f(S_{K,1})$ is bounded, the *quasinorm*, $|f|$, of f is defined by setting $|f| = \sup\{\|f(\mathbf{x})\| : \mathbf{x} \in S_{K,1}\}$, and the *spectral radius*, $r(f)$, of f as

$$r(f) = \limsup_{n \rightarrow \infty} |f^n|^{1/n}.$$

Since $|f^n| \leq |f|^n$ we see that $r(f) \leq |f|$.

- For the same family of maps considered above, and using the Kuratowski measure of noncompactness, we define $\alpha(f) = \inf\{p : \alpha(f(A)) \leq p\alpha(A), \text{ for all } A \subset S_{K,1}\}$, and

$$\omega(f) = \limsup_{n \rightarrow \infty} |\alpha(f^n)|^{1/n}.$$

One can easily verify that $\omega(f) \leq \alpha(f)$.

- Given $f : K \rightarrow K$ and $\rho > 0$, we set $f_\rho(\mathbf{x}) = (1/\rho)f(\mathbf{x})$.

LEMMA 4.1 (see [14] and [17]). *Let $f : K \rightarrow K$ be positively homogeneous of degree 1 and such that $\omega(f) < 1$. Assume that there exists $\mathbf{u} \in S_{K,1}$ such that*

the sequence $\{f^n(\mathbf{u}) = \mathbf{u}_n\}$ is unbounded. Then there exists a subsequence $\{\mathbf{w}_m\}$ of $\{\mathbf{u}_n\}$ such that:

- (i) $\{\|\mathbf{w}_m\|\}$ is strictly increasing and divergent;
- (ii) $\|\mathbf{w}_m\|$ is strictly larger than the norm of every vector which comes before \mathbf{w}_m in the original sequence;
- (iii) the sequence $\{\mathbf{w}_m/\|\mathbf{w}_m\|\}$ has a convergent subsequence.

PROOF. (i) and (ii). Let $\mathbf{w}_1 = \mathbf{u}_1$. Choose as \mathbf{w}_2 the first vector of $\{\mathbf{u}_n\}$ with norm strictly larger than $\|\mathbf{u}_1\|$. Then choose as \mathbf{w}_3 the first vector of $\{\mathbf{u}_n\}$ with norm strictly larger than $\|\mathbf{w}_2\|$, etc.

(iii) Let $q \geq 1$ be such that $\alpha(f^q) < 1$. Consider the subsequence of $\{\mathbf{u}_n\}$ made of those vectors which precede the vectors of the subsequence $\{\mathbf{w}_m\}$ by exactly q positions. Denote this subsequence by $\{\mathbf{z}_{mq}\}$ and let

$$A = \{\mathbf{v}_m = \mathbf{w}_m/\|\mathbf{w}_m\|\}, \quad B_q = \{\mathbf{z}_{mq}/\|\mathbf{w}_m\|\}.$$

We have $B_q \subset D_{K,1}$, $f^q(B_q) \subset A$ and $A \setminus f^q(B_q)$ finite. Consequently,

$$\alpha(A) = \alpha(f^q(B_q)) \leq \alpha(f^q)\alpha(B_q) \leq 2\alpha(f^q).$$

Replacing q with q^2 and repeating the same procedure we obtain $\alpha(A) \leq 2[\alpha(f^q)]^2$, and, in general, $\alpha(A) \leq 2[\alpha(f^q)]^k$ for every integer $k \geq 2$. Since $\alpha(f^q) < 1$ we conclude that A is totally bounded and it contains a convergent subsequence. □

We are now ready to establish the following important result on the index of order preserving maps.

THEOREM 4.1. *Let $f : K \rightarrow K$ be continuous, positive homogeneous of degree one and order preserving. Assume that*

- (i) $\alpha(f) < r(f) < \infty$;
- (ii) *there exist $\mathbf{u} \in S_{K,1}$ and $\delta \in (\alpha(f), r(f))$ such that*

$$\limsup_{n \rightarrow \infty} \frac{\|f^n(\mathbf{u})\|}{\delta^n} > 0.$$

Then either there exists $\mathbf{x} \in S_{K,1}$ such that $f(\mathbf{x}) = \delta\mathbf{x}$ or $\text{ind}(K, f_\delta, B_{K,1}) = 0$. In this last case there exist $\mathbf{x} \in S_{K,1}$ and $\lambda \in (\delta, r(f))$ such that $f(\mathbf{x}) = \lambda\mathbf{x}$.

PROOF. Assume that $f(\mathbf{x}) \neq \delta\mathbf{x}$ for all $\mathbf{x} \in S_{K,1}$. Then there exists $\varepsilon > 0$, $\alpha(f) < \delta - \varepsilon$, such that $f(\mathbf{x}) \neq \rho\mathbf{x}$ for all $\rho \in [\delta - \varepsilon, \delta + \varepsilon]$. In fact, assume that $f(\mathbf{x}_n) = \rho_n\mathbf{x}_n$ for some sequence $\rho_n \rightarrow \delta$ and $\{\mathbf{x}_n\} \subset S_{K,1}$. From $\delta\mathbf{x}_n = f(\mathbf{x}_n) - (\rho_n - \delta)\mathbf{x}_n$ and with $A = \{\mathbf{x}_n\}$, we derive

$$\delta\alpha(A) \leq \alpha(f)\alpha(A) + \tau\alpha(A)$$

for all $\tau > 0$. Hence $\delta\alpha(A) \leq \alpha(f)\alpha(A)$. Since $\delta > \alpha(f)$ we obtain $\alpha(A) = 0$, and $\{\mathbf{x}_n\}$ must have a convergent subsequence. It is easily seen that the limit \mathbf{x}_0 of this subsequence satisfies the equality $\delta\mathbf{x}_0 = f(\mathbf{x}_0)$, contradicting the assumption $\delta\mathbf{x} \neq f(\mathbf{x})$ for all $\mathbf{x} \in S_{K,1}$. Select $\rho \in [\delta - \varepsilon, \delta)$ and consider the map f_ρ . Obviously $\alpha(f_\rho) < 1$ and the sequence

$$\|(f_\rho)^n(\mathbf{u})\| = \frac{\delta^n}{\rho^n} \cdot \frac{\|f^n(\mathbf{u})\|}{\delta^n}$$

is unbounded. Pick a positive integer k such that $\|f_\rho(\mathbf{x}) + k\mathbf{u}\| \geq 2$ for all $\mathbf{x} \in D_{K,1}$. From the Solvability Property of the index we obtain $\text{ind}(K, g, B_{K,1}) = 0$, where $g(\mathbf{x}) = f_\rho(\mathbf{x}) + k\mathbf{u}$. Consider the homotopy $H(\mathbf{x}, t) = f_\rho(\mathbf{x}) + tk\mathbf{u}$. We know that $H(\cdot, 0)$ and $H(\cdot, 1)$ are fixed point free on $S_{K,1}$. Assume $\mathbf{x} = f_\rho(\mathbf{x}) + tk\mathbf{u}$ for some $\mathbf{x} \in S_{K,1}$ and $t \in (0, 1)$. It follows that $\mathbf{x} > f_\rho(\mathbf{x})$ and $\mathbf{x} > tk\mathbf{u}$. By an induction argument we derive

$$\mathbf{x} > tk(f_\rho)^n(\mathbf{u}).$$

Let $\{\mathbf{w}_m\}$ be the subsequence of $\{(f_\rho)^n(\mathbf{u})\}$ with the properties insured by Lemma 4.1 and let \mathbf{w} , $\|\mathbf{w}\| = 1$, $\mathbf{w} \in K$, be the limit of a convergent subsequence of $\{\mathbf{w}_m/\|\mathbf{w}_m\|\}$. From $\mathbf{x} > tk(f_\rho)^n(\mathbf{u})$ we derive

$$\frac{\mathbf{x}}{\|\mathbf{w}_m\|} > tk \frac{\mathbf{w}_m}{\|\mathbf{w}_m\|}.$$

Consequently, $\mathbf{0} > \mathbf{w}$. Since this is impossible we deduce that H is fixed point free on $S_{K,1}$ for all $t \in [0, 1]$ and

$$\text{ind}(K, f_\rho, B_{K,1}) = \text{ind}(K, g, B_{K,1}) = 0.$$

The homotopy $H(\mathbf{x}, t) = tf_\rho(\mathbf{x})$ cannot be fixed point free on $S_{K,1}$ since for $t = 0$ we have $\text{ind}(K, H(\cdot, 0), B(\mathbf{0}, 1)) = 1$. Thus there exist $\mathbf{x} \in S_{K,1}$ and $t \in (0, 1)$ such that $\mathbf{x} = tf_\rho(\mathbf{x})$, i.e. $f(\mathbf{x}) = \lambda\mathbf{x}$, $\lambda \in (\delta, r(f)]$. \square

In [17] (Theorem 2.1) Nussbaum established a result which follows from Theorem 4.1, since he replaces (ii) with the stronger condition that $\|(f)^n(\mathbf{u})\|$ is unbounded.

REMARK 4.1. Notice that when $\delta = r(f)$ there is always $\mathbf{x} \in S_{K,1}$ such that $f(\mathbf{x}) = r(f)\mathbf{x}$, since otherwise $\text{ind}(K, f_{r(f)}, B_{K,1}) = 1$, a contradiction.

COROLLARY 4.1. *Let $f : K \rightarrow K$ be positively homogeneous of degree one and order preserving. Assume that*

- (i) $\alpha(f) < r(f) < \infty$;
- (ii) *there exist $\mathbf{u} \in S_{K,1}$, $\delta \in (\alpha(f), r(f)]$ and N such that $f^N(\mathbf{u}) > \delta^N \mathbf{u}$.*

Then either there exists $\mathbf{x} \in S_{K,1}$ such that $f(\mathbf{x}) = \delta\mathbf{x}$ or $\text{ind}(K, f_\delta, B_{K,1}) = 0$. In this last case there exist $\mathbf{x} \in S_{K,1}$ and $\lambda \in (\delta, r(f)]$ such that $f(\mathbf{x}) = \lambda\mathbf{x}$.

PROOF. Assume that $\delta\mathbf{x} \neq f(\mathbf{x})$ for all $\mathbf{x} \in S_{K,1}$ and let $\varepsilon > 0$, $\rho \in [\delta - \varepsilon, \delta)$ be selected as in the proof of Theorem 4.1. We want to show that the sequence $\{\|(f_\rho)^n(\mathbf{u})\|\}$ is unbounded. We have

$$(f_\rho)^N(\mathbf{u}) = \frac{1}{\rho^N} f^N(\mathbf{u}) > \frac{\delta^N}{\rho^N} \mathbf{u}.$$

By induction we obtain

$$\frac{\rho^{Nk}}{\delta^{Nk}} (f_\rho)^{Nk}(\mathbf{u}) > \mathbf{u}.$$

Since $\rho^{Nk}/\delta^{Nk} \rightarrow 0$ as $k \rightarrow \infty$ the sequence $\{(f_\rho)^{Nk}(\mathbf{u})\}$ must be unbounded. From this point on we can follow the same reasoning used in the proof of Theorem 4.1 to obtain the desired conclusion. □

REMARK 4.2. When $\delta = r(f)$ we obtain $f(\mathbf{x}) = r(f)\mathbf{x}$ for some $\mathbf{x} \in S_{K,1}$.

It is known (see for example [17]) that in the case when $f : K \rightarrow K$ is a linear map L , the condition $\omega(L) < r(L)$ implies the existence of an eigenvector \mathbf{x} corresponding to the eigenvalue $r(L)$. This result is basically a consequence of two important facts. The first is the property that every bounded linear operator L such that $r(L) > 1$ has an unbounded sequence of iterates $\{L^n(\mathbf{u})\}$, $\mathbf{u} \in S_{K,1}$. Since for every $\rho \in (\omega(L), r(L))$ we have $r(L_\rho) > 1$, the linear operator L_ρ has an unbounded sequence of iterates starting at some point of the unit sphere. This fact provides one of the two key ingredients of Theorem 4.1. The second important fact is that $\omega(L_\rho) < 1$ for every $\rho \in (\omega(L), r(L))$ and this insures that some iterate of L_ρ is an α -contraction. Since L_ρ is differentiable we can use for L_ρ the properties of the index. This is the second fundamental ingredient of Theorem 4.1. We announce below a generalization of this result to the case when the cone K is normal, i.e. there exists a constant $\gamma > 0$ such that $\|\mathbf{x} + \mathbf{y}\| \geq \gamma\|\mathbf{x}\|$ for all $\mathbf{x}, \mathbf{y} \in K$. The proof of the theorem is quite long and will appear in a forthcoming paper.

THEOREM 4.2. Let $K \subset E$ be a normal cone with nonempty interior and $f : E \rightarrow E$ be such that $f(K) \subset K$. Assume that

1. f is positively homogeneous of degree 1 and order preserving;
2. $\omega(f) < r(f) < \infty$;
3. f is of class C^1 in $E \setminus \{0\}$.

Then there exists $\mathbf{x} \in S_{K,1}$ such that $r(f)\mathbf{x} = f(\mathbf{x})$.

EXAMPLE 4.1. Let $E = C[0, 1]$ and $f(\mathbf{x})(t) = \int_0^t \sqrt{x^2(s) + x^2(1-s)} ds$. Then f is positively homogeneous of degree 1 and maps the normal cone K of positive functions into itself. Moreover, f is order preserving and of class C^1 in

$E \setminus \{0\}$. Finding an eigenvector for f corresponding to a positive eigenvalue λ is equivalent to finding a global solution (i.e. $t \in [0, 1]$) to the problem

$$\begin{cases} x'(t) = \lambda \sqrt{x^2(t) + x^2(1-t)}, \\ x(0) = 0. \end{cases}$$

Using the fact that $x'(t) = x'(1-t)$ and after a lengthy but straightforward computation we deduce that the above problem has a solution if and only if $\lambda = \sqrt{2} \ln(1 + \sqrt{2})$. The solution \mathbf{x} can be written in the form

$$x(t) = a[1 + \sinh[2 \ln(1 + \sqrt{2})t + \ln(-1 + \sqrt{2})]].$$

where $a = x(1/2)$. With $a = 1/2$ we obtain $\|\mathbf{x}\| = 1$. Theorem 4.2 implies that the spectral radius of our operator f is $r(f) = 1/\sqrt{2} \ln(1 + \sqrt{2})$.

We close this section with the following remark.

REMARK 4.3. Let $f : K \rightarrow K$ be order preserving and positively homogeneous of degree 1. Assume that $\max\{\alpha(f), r(f)\} < 1$. Then $\text{ind}(K, f, B_{K,1}) = 1$ since f is homotopic to the constant map $g(\mathbf{x}) = 0$.

When $\alpha(f) < 1 < r(f)$ and f is a linear operator L , the condition $\mathbf{x} \neq L(\mathbf{x})$ for all \mathbf{x} in $S_{K,1}$ implies that $\text{ind}(K, L, B_{K,1}) = 0$. In fact, it can be shown that there exists \mathbf{u} in $S_{K,1}$ such that the sequence $\{L^n(\mathbf{u})\}$ is unbounded. Hence, according to Theorem 4.1 we obtain $\text{ind}(K, L, B_{K,1}) = 0$. A further refinement of this result is possible by replacing $\alpha(L)$ with $\omega(L)$. In the nonlinear case the situation is different, since $r(f) > 1$ does not imply the existence of a vector \mathbf{u} in $S_{K,1}$ such that $\{f^n(\mathbf{u})\}$ is unbounded. If such a vector exists, and $\mathbf{x} \neq f(\mathbf{x})$ for all $\mathbf{x} \in S_{K,1}$, then $\text{ind}(K, f, B_{K,1}) = 0$, again as a consequence of Theorem 4.1. We are unable to provide an example of a map $f : K \rightarrow K$ which is order preserving, positively homogeneous of degree 1, with $\alpha(f) < 1 < r(f)$, $\mathbf{x} \neq f(\mathbf{x})$ for all \mathbf{x} in $S_{K,1}$ and such that $\text{ind}(K, f, B_{K,1}) \neq 0$.

5. Order preserving multi-valued maps

A multi-valued map $F : K \multimap K$ is *order preserving* if $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{z} \leq \mathbf{w}$ for every pair of points \mathbf{z}, \mathbf{w} with $\mathbf{z} \in F(\mathbf{x})$ and $\mathbf{w} \in F(\mathbf{y})$. F is *positively homogeneous of degree one* if $F(t\mathbf{x}) = tF(\mathbf{x})$ for all $\mathbf{x} \in K$ and $t \geq 0$. In this section, unless otherwise stated, all multi-valued maps are assumed to be order preserving and positively homogeneous of degree one.

- Given a bounded set $A \subset K$ we define $M(A) = \sup\{\|\mathbf{x}\| : \mathbf{x} \in A\}$. The *quasinorm* of a map $F : K \multimap K$ such that $F(S_{K,1})$ is bounded is $|F| = M(F(S_{K,1}))$. For every $\mathbf{x} \in K$ and $\mathbf{y} \in F(\mathbf{x})$ we have $\|\mathbf{y}\| \leq |F| \|\mathbf{x}\|$, and $|F|$ is the smallest possible constant for which such an inequality is true.

- Given $\mathbf{x} \in K$ and $\mathbf{z} \in F(F(\mathbf{x}))$ there exists $\mathbf{y} \in F(\mathbf{x})$ such that $\mathbf{z} \in F(\mathbf{y})$. Therefore $\|\mathbf{z}\| \leq |F|\|\mathbf{y}\| \leq |F|^2\|\mathbf{x}\|$. It follows that $|F^2| \leq |F|^2$, and in general $|F^n| \leq |F|^n$. The *spectral radius* of F is defined as

$$r(F) = \limsup_{n \rightarrow \infty} |F^n|^{1/n}.$$

We have $r(F) \leq |F|$.

- Using the Kuratowski measure of noncompactness, we define

$$\alpha(F) = \inf\{p : \alpha(F(A)) \leq p\alpha(A) \text{ for all } A \subset S_{K,1}\}, \text{ and}$$

$$\omega(F) = \limsup_{n \rightarrow \infty} |\alpha(F^n)|^{1/n}.$$

One can easily verify that $\omega(F) \leq \alpha(F)$.

- Given $\rho > 0$ and $F : K \rightarrow K$ we set $F_\rho(\mathbf{x}) = (1/\rho)F(\mathbf{x})$.

LEMMA 5.1. *Let $F : K \rightarrow K$. Assume that there exist $\mathbf{u} \in S_{K,1}$ and $\delta \leq r(F)$ such that*

$$\limsup_{n \rightarrow \infty} \frac{M(F^n(\mathbf{u}))}{\delta^n} > 0.$$

Then for every $\rho < \delta$ the sequence $\{M((F_\rho)^n(\mathbf{u}))\}$ is unbounded.

PROOF. $M((F_\rho)^n(\mathbf{u})) = M(F^n(\mathbf{u}))/\rho^n = (\delta^n/\rho^n)(M(F^n(\mathbf{u}))/\delta^n)$. Since $\limsup_{n \rightarrow \infty} M(F^n(\mathbf{u}))/\delta^n > 0$ and $\rho < \delta$ the result follows. \square

LEMMA 5.2. *Assume that $\omega(F) < 1$ and there exist $\mathbf{u} \in S_{K,1}$ such that the sequence $\{M(F^n(\mathbf{u}))\}$ is unbounded. Then there exists a sequence $\{\mathbf{y}_{n_p} \in F^{n_p}(\mathbf{u})\}$ such that $\{\mathbf{v}_{n_p} = \mathbf{y}_{n_p}/\|\mathbf{y}_{n_p}\|\}$ is convergent.*

PROOF. Let $\mathbf{w}_1 \in F(\mathbf{u})$ be such that $\|\mathbf{w}_1\| = M(F(\mathbf{u}))$. Let m_2 be the first index such that $M(F^{n_2}(\mathbf{u})) > \|\mathbf{w}_1\|$ and pick $\mathbf{w}_2 \in F^{n_2}(\mathbf{u})$ such that $\|\mathbf{w}_2\| = M(F^{n_2}(\mathbf{u}))$. Proceeding in this way we construct a sequence $\{\mathbf{w}_m \in F^{n_m}(\mathbf{u})\}$ such that $\{\|\mathbf{w}_m\|\}$ is strictly increasing, and the norm of \mathbf{w}_m is strictly larger than the norm of every vector in $F^i(\mathbf{u})$ for every index i which comes before n_m . Let $A = \{\mathbf{w}_m/\|\mathbf{w}_m\|\}$.

Choose $q \geq 1$ such that $\alpha(F^q) < 1$ and construct a set B_q as follows. Pick an index m of the sequence A and look at the corresponding index n_m such that $\mathbf{w}_m \in F^{n_m}(\mathbf{u})$. Assume that $n_m - q \geq 1$ and select a vector $\mathbf{z} \in F^{n_m - q}(\mathbf{u})$ such that $\mathbf{w}_m \in F^q(\mathbf{z})$. Define $\mathbf{z}_{m-q} = \mathbf{z}/\|\mathbf{w}_m\|$ and set $B_q = \{\mathbf{z}_{m-q}\}$. We have $B_q \subset D_{K,1}$ and $A \setminus F^q(B_q)$ is finite. Consequently,

$$\alpha(A) \leq \alpha(F^q(B_q)) \leq \alpha(F^q)\alpha(B_q) \leq 2\alpha(F^q).$$

By replacing q with q^k we obtain $\alpha(A) \leq 2[\alpha(F^q)]^k$. Hence A is totally bounded and has a convergent subsequence. \square

We are now ready to prove for multi-valued maps a result analogous to Theorem 4.1. Given a multi-valued map $F : K \rightarrow K$ which has an acyclic

decomposition $\Phi = \{F_0, \dots, F_n\}$ and given $\rho > 0$ we denote by Φ_ρ the acyclic decomposition of F_ρ induced by Φ .

THEOREM 5.1. *Let $F : K \rightarrow K$ be an upper semicontinuous map which can be expressed as a composition of acyclic-valued maps $\Phi = \{F_0, \dots, F_n\}$. Assume that*

- (i) $\alpha(F) < r(F) < \infty$;
- (ii) *there are $\delta \in (\alpha(F), r(F))$ and $\mathbf{u} \in S_{K,1}$ such that*

$$\limsup_{n \rightarrow \infty} \frac{M(F^n(\mathbf{u}))}{\delta^n} > 0.$$

Then either there exists $\mathbf{x} \in S_{K,1}$ such that $\delta\mathbf{x} \in F(\mathbf{x})$ or $\text{ind}(K, \Phi_\delta, B_{K,1}) = 0$. In this last case there are $\mathbf{y} \in S_{K,1}$ and $\lambda \in (\delta, r(F))$ such that $\lambda\mathbf{y} \in F(\mathbf{y})$.

PROOF. Assume that $\delta\mathbf{x} \notin F(\mathbf{x})$ for all $\mathbf{x} \in S_{K,1}$. Then there is $\varepsilon > 0$, $\alpha(F) < \delta - \varepsilon$, such that F_ρ is fixed point free on $S_{K,1}$ for all $\rho \in [\delta - \varepsilon, \delta)$. In fact, assume that there is a sequence $\{\rho_n\}, \rho_n \rightarrow \delta$, and $\mathbf{x}_n \in S_{K,1}$ such that $\rho_n\mathbf{x}_n \in F(\mathbf{x}_n)$. Write $\delta\mathbf{x}_n + (\rho_n - \delta)\mathbf{x}_n \in F(\mathbf{x}_n)$ or $\delta\mathbf{x}_n \in F(\mathbf{x}_n) - (\rho_n - \delta)\mathbf{x}_n$. Let $A = \{\mathbf{x}_n\}$. Then for every $\gamma > 0$ we have

$$\delta\alpha(A) \leq \alpha(F(A)) + \gamma\alpha(A) \leq \alpha(F)\alpha(A) + \gamma\alpha(A).$$

It follows that $\delta\alpha(A) \leq \alpha(F)\alpha(A)$ and since $\delta > \alpha(F)$ we obtain $\alpha(A) = 0$. Hence $\{\mathbf{x}_n\}$ has a convergent subsequence $\mathbf{x}_{n_k} \rightarrow \mathbf{x}_0, \|\mathbf{x}_0\| = 1$. Clearly

$$\rho_{n_k}\mathbf{x}_{n_k} \rightarrow \delta\mathbf{x}_0 \quad \text{and} \quad \rho_{n_k}\mathbf{x}_{n_k} \in F(\mathbf{x}_{n_k}).$$

Let $r > 0$ be given. We have $\mathbf{x}_0 \in F^{-1}(D(F(\mathbf{x}_0), r))$ where $D(F(\mathbf{x}_0), r) = \{\mathbf{y} : \|\mathbf{y} - \mathbf{z}\| \leq r \text{ for some } \mathbf{z} \in F(\mathbf{x}_0)\}$. It follows that for n_k large enough $\mathbf{x}_{n_k} \in F^{-1}(D(F(\mathbf{x}_0), r))$ and consequently $F(\mathbf{x}_{n_k}) \subset D(F(\mathbf{x}_0), r)$. This implies

$$\rho_{n_k}\mathbf{x}_{n_k} \in D(F(\mathbf{x}_0), r) \quad \text{and} \quad \delta\mathbf{x}_0 \in D(F(\mathbf{x}_0), r).$$

Since this is true for every $r > 0$ and $F(\mathbf{x}_0)$ is closed we obtain $\delta\mathbf{x}_0 \in F(\mathbf{x}_0)$. This contradiction establishes the stated property. Choose $\rho \in (\delta - \varepsilon, \delta)$. By Lemma 5.1 the sequence $\{M((F_\rho)^n(\mathbf{u}))\}$ is unbounded. Let k be so large that $k - M(F_\rho(D_{K,1})) > 1$. Define $G(\mathbf{x}) = F_\rho(\mathbf{x}) + k\mathbf{u}$ and let Φ_1 be the acyclic decomposition of G corresponding to Φ . Then $\text{ind}(K, \Phi_1, B_{K,1}) = 0$. We want to show that G and F_ρ are homotopic. Let $H(\mathbf{x}, t) = F_\rho(\mathbf{x}) + tk\mathbf{u}$ and assume $\mathbf{x} \in H(\mathbf{x}, t)$ for some $\mathbf{x} \in S_{K,1}$ and $t \in (0, 1)$. There exists $\mathbf{z} \in F_\rho(\mathbf{x})$ such that $\mathbf{x} = \mathbf{z} + tk\mathbf{u}$. Hence $\mathbf{x} > \mathbf{z}$ and $\mathbf{x} > tk\mathbf{u}$. Since F_ρ is order preserving and positively homogeneous of degree one, we obtain $\mathbf{x} > \mathbf{z} > tk\mathbf{w}_1$ for every $\mathbf{w}_1 \in (F_\rho)(\mathbf{u})$. By induction it follows that $\mathbf{x} > tk\mathbf{w}_n$ for every $\mathbf{w}_n \in (F_\rho)^n(\mathbf{u})$ and for every $n \geq 1$. Using the convergent subsequence of Lemma 5.2 we derive $\mathbf{x}/\|\mathbf{w}_m\| > tk\mathbf{w}_m/\|\mathbf{w}_m\|$. If \mathbf{v} is the limit of the subsequence we get $\|\mathbf{v}\| = 1$

and $\mathbf{v} < \mathbf{0}$. This is impossible, since $\mathbf{v} \in K$. Therefore with Φ_ρ denoting the acyclic decomposition of F_ρ induced by Φ we obtain $\text{ind}(K, \Phi_\rho, B_{K,1}) = 0$. We know that F_ρ is not homotopic to the constant map which sends everything into $\mathbf{0}$. Thus there must be $t < 1$ and $\mathbf{x} \in S_{K,1}$ such that $\mathbf{x} \in tF_\rho(\mathbf{x})$, i.e. $\lambda\mathbf{x} \in F(\mathbf{x})$ with $\lambda \in (\delta, r(F))$. \square

REMARK 5.1. In the case when $\delta = r(F)$ we conclude that F has an eigenvector corresponding to the eigenvalue $r(F)$.

COROLLARY 5.1. Let $F : K \rightarrow K$ be an upper semicontinuous map which can be expressed as a composition of acyclic-valued maps $\Phi = \{F_0, \dots, F_n\}$, is positively homogeneous of degree one and order preserving. Assume that

- (i) $\alpha(F) < r(F) < \infty$;
- (ii) there exist $\mathbf{u} \in S_{K,1}$, $\delta \in (\alpha(F), r(F)]$ and N such that $\mathbf{z} > \delta^N \mathbf{u}$ for some $\mathbf{z} \in F^N(\mathbf{u})$.

Then either there exists $\mathbf{x} \in S_{K,1}$ such that $\delta\mathbf{x} \in F(\mathbf{x})$ or $\text{ind}(K, \Phi_\delta, B_{K,1}) = 0$. In this last case there exist $\mathbf{x} \in S_{K,1}$ and $\lambda \in (\delta, r(F)]$ such that $\lambda\mathbf{x} \in F(\mathbf{x})$.

PROOF. Follow the same reasoning used in the proof of Corollary 4.1. \square

REMARK 5.2. When $\delta = r(F)$ we obtain $r(F)\mathbf{x} \in F(\mathbf{x})$ for some $\mathbf{x} \in S_{K,1}$.

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