Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 3, 1994, 189–196

APPROXIMATE TAYLOR POLYNOMIALS AND DIFFERENTIATION OF FUNCTIONS

Fon-Che Liu — Wei-Shyan Tai

Dedicated to Jean Leray

1. Introduction and preliminaries

Let D be a Lebesgue measurable set in $\mathbb{R}^n$ and k a positive integer. A real measurable function u defined on D is said to have the *Lusin property of order* k if for any $\varepsilon > 0$ there is a C^k-function g defined on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq g(x)\}| < \varepsilon$, where $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^n$. For a C^k-function g, the polynomial

$$p_g^k(x;y) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha g(x)(y-x)^\alpha$$

is called the *k-Taylor polynomial* of g at x. Polynomials of this form are sometimes referred to as polynomials centered at x. We refer to [8, p. 2] for the standard notations concerning multi-indices. If u has the Lusin property of order k on D, then it is clear that for almost every x of D there is a C^k-function g such that the set $\{z \in D : u(z) = g(z)\}$ contains x and has density one at x. Thus the following condition holds at almost every point x of D:

$$\text{ap} \lim_{y \to x} \frac{|u(y) - p_g^k(x;y)|}{|y-x|} = 0, \tag{1}$$

1991 *Mathematics Subject Classification.* Primary 26B05, 26B35.

The first author was supported in part by National Science Council-Taipei.

and hence so does the condition

$$\text{ap}\limsup_{y\to x}\frac{|u(y)-p_g^{k-1}(x;y)|}{|y-x|^k}<+\infty. \tag{2}$$

We recall that $\text{ap}\lim_{y\to x} u(y) = l$ means that the set $\{y \in D : |u(y) - l| \leq \varepsilon\}$ has density one at x for any $\varepsilon > 0$ and that $\text{ap}\limsup_{y\to x} u(y)$ is the infimum of all those $\lambda \in \mathbb{R}$ such that the set $\{y \in D : u(y) > \lambda\}$ has density zero at x.

Now some definitions are in order. A function u defined on D is said to have an *approximate* $(k-1)$*-Taylor polynomial* at x if there is a polynomial $p(x; y)$ centered at x and of degree at most $k - 1$ such that

$$\text{ap}\limsup_{y\to x}\frac{|u(y)-p(x;y)|}{|y-x|^k}<+\infty; \tag{3}$$

while u will be said to be *approximately differentiable of order* k at x if there is a polynomial $p(x; y)$ centered at x and of degree at most k such that

$$\text{ap}\lim_{y\to x}\frac{|u(y)-p(x;y)|}{|y-x|^k}=0. \tag{4}$$

If (4) is replaced by

$$\lim_{y\to x}\frac{|u(y)-p(x;y)|}{|y-x|^k}=0, \tag{5}$$

then u is said to be *differentiable of order* k at x. From (1) and (2), if u has the Lusin property of order k on D, then it is approximately differentiable of order k and has an approximate $(k-1)$-Taylor polynomial at almost every point of D. If u is approximately differentiable (differentiable) of order 1 at x, it will be simply said to be *approximately differentiable* (*differentiable*) at x.

It is our purpose in this note to relate the properties of functions defined above. Our main result is the following theorem:

THEOREM 1. *For a measurable function u defined on D the following statements are equivalent:*

(I) *u has the Lusin property of order k on D.*
(II) *u has an approximate $(k-1)$-Taylor polynomial at almost every point of D.*
(III) *u is approximately differentiable of order k at almost every point of D.*

The proof of Theorem 1 will be given in the next section. As a consequence of Theorem 1 we now establish the following theorem:

THEOREM 2. *In order for u to be differentiable of order k almost everywhere on D it is necessary and sufficient that for almost every point x of D there is a polynomial $p(x;y)$ centered at x and of degree at most $k-1$ such that*

$$\limsup_{y\to x} \frac{|u(y)-p(x;y)|}{|y-x|^k} < +\infty. \tag{6}$$

PROOF. That the extence of a polynomial $p(x;y)$ centered at x and of degree at most $k-1$ such that (6) holds for almost every $x \in D$ is necessary for u to be differentiable of order k almost everywhere in D is obvious. We show that it is also sufficient. Under this condition, since (6) implies (3), statement (II) of Theorem 1 holds; hence by Theorem 1, u has the Lusin property of order k on D. Thus given $\varepsilon > 0$, there is a C^k-function g on $\mathbb{R}^n$ such that if E is the set on which $u = g$, then $|D \setminus E| < \varepsilon$. We then choose a closed subset F of E with $|D \setminus F| < \varepsilon$ and with each of its points being a point of density of E. Then (6) holds for $x \in F$ with $p(x;y)$ replaced by $p_g^{k-1}(x;y)$. For each positive integer j let

$$C_j = \{x \in F : |u(y) - p_g^{k-1}(x;y)| \le j|y-x|^k \ \text{ for all } \ y \in D \cap B(x;1/j)\},$$

where $B(x;1/j) = \{y \in \mathbb{R}^n : |y-x| < 1/j\}$. It is clear that each C_j is a closed set and hence is measurable. Since (6) (with $p(x;y)$ replaced by $p_g^{k-1}(x;y)$) holds at every point x of F, $\bigcup_{j=1}^{+\infty} C_j = F$; as $\{C_j\}$ is an increasing sequence of sets, $|F \setminus C_j| \to 0$ as $j \to +\infty$, we can choose j_0 such $|D \setminus C_{j_0}| < \varepsilon$. Thus we can replace F by C_{j_0} and still call it F and assume that for $x \in F$ and $y \in B(x;1/j_0)$ we have

$$\limsup_{y\to x} \frac{|u(y)-p_g^{k-1}(x;y)|}{|y-x|^k} \le j_0. \tag{7}$$

Let $x \in F$ be a point of density of F. We now show that u is differentiable of order k at x. For $y \in D$ let $d(y)$ be the distance from y to F and let $z \in F$ be such that $d(y) = |y-z|$. Since x is a point of density of F,

$$d(y) = |y-z| = o(|y-x|) \qquad \text{as } \ y \to x. \tag{8}$$

For $y \in D \cap B(x;1/j_0)$, by writing $u(y) - p_g^k(x;y) = u(y) - p_g^k(z;y) + p_g^k(z;y) - p_g^k(x;y)$ and using (7) (note that $z \in F$ and $|y-z| \le |y-x| < 1/j_0$), we have

$$|u(y) - p_g^k(x;y)| \le j_0|y-z|^k + |p_g^k(z;y) - p_g^k(x;y)|. \tag{9}$$

By writing the polynomial $p_g^k(z;y)$ as a polynomial centered at x we have

$$|p_g^k(z;y) - p_g^k(x;y)| = \left| \sum_{|\alpha|\le k} \frac{1}{\alpha!}\{p_{g_\alpha}^{k-|\alpha|}(z;x) - g_\alpha(x)\}(y-x)^\alpha \right| = o(|y-x|^k), \tag{10}$$

where we write g_α for $D^\alpha g$. It then follows from (8), (9) and (10) that u is differentiable of order k at x.

Since $\varepsilon > 0$ is arbitrary and almost every point of F is a point of density of F, the proof is complete.

Theorem 2 is a generalization of a well-known result of Rademacher [3] and Stepanoff [6] to differentiability of higher order. Theorem 2 is also more general in that we do not assume D to be open.

In the remaining part of this section, we make some preparations for the proof of Theorem 1 will be considered. We remark first that if u has an approximate $(k-1)$-Taylor polynomial at $x \in D$ and if x is a point of density of D, then the polynomial $p(x;y)$ in (3) is uniquely determined. Indeed, if $q(x;y)$ is another polynomial centered at x for which (3) holds with $p(x;y)$ replaced by $q(x;y)$, then

$$\operatorname{ap}\limsup_{y\to x} \frac{|p(x;y)-q(x;y)|}{|y-x|^k} < +\infty,$$

where $y \to x$ through D. This is impossible unless $p(x;y) - q(x;y)$ is a zero polynomial, because $p(x;y) - q(x;y)$ is a polynomial of degree strictly less then k and x is a point of density of D. Hence for such x we can express the unique polynomial $p(x;y)$ for which (3) holds by

$$p(x;y) = \sum_{|\alpha|\le k-1} \frac{1}{\alpha!} u_\alpha(x)(y-x)^\alpha. \tag{11}$$

Thus if u has an approximate $(k-1)$-Taylor polynomial at almost every point of D, there are functions u_α, $|\alpha| \le k-1$, each of which is defined almost everywhere on D such that if $p(x;y)$ is defined by (11) then (3) holds for almost every x in D. It is not obvious that the functions u_α are measurable, but we will prove that in the next section.

We now quote a lemma from [1] for our later use in the proof of Theorem 1. This lemma is due to De Giorgi.

LEMMA 1. *Let E be a measurable subset of the ball $B(x;r)$ such that $|E| \ge Ar^n$ for some constant $A > 0$. Then for each positive integer k there is a positive constant C depending only on n, k and A such that*

$$|D^\alpha p(x)| \le \frac{C}{r^{n+|\alpha|}} \int_{B(x;r)} |p(y)|\,dy$$

for all polynomials p of degree at most k.

2. The proof of Theorem 1

We now prove Theorem 1. That (I) implies (III) and (III) implies (II) is obvious. It remains to show that (II) implies (I). We prove first that (I) holds under the assumption that (II) holds and all u_α, $|\alpha| \le k-1$, are measurable; then we complete the proof by showing that the u_α are indeed measurable if (II) holds.

Now suppose that (II) holds and all u_α, $|\alpha| \le k-1$, are measurable. Let

$$\rho = \frac{|B(x;|y-x|) \cap B(y;|y-x|)|}{|y-x|^n}, \qquad x, y \in \mathbb{R}^n, x \neq y;$$

then ρ is independent of x and y. For a positive integer j, $x \in D$, and $r > 0$ let

$$W_j(x;r) = B(x;r) \setminus \{y \in D : |u(y) - p(x;y)| \le j|y-x|^k\},$$

where $p(x;y)$ is defined by (11). Each $W_j(x;r)$ is a measurable set. Consider the set

$$T = \{(x,y) \in D \times D : |x-y| < r, |u(y) - p(x;y)| > j|x-y|^k\}.$$

Since all u_α are measurable T is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. It is clear that $W_j(x;r) = \{y \in D : (x,y) \in T\}$, hence it follows from the Fubini Theorem that $|W_j(x;r)|$ is a measurable function of x. For a positive integer j we now let

$$B_j = \{x \in D : |W_j(x;r)| \le \rho r^n/4,\ r \le 1/j\} \cap \{x \in D : |u_\alpha(x)| \le j,\ |\alpha| < k\};$$

since $W_j(x;r)$ is monotone in r, r only has to run through a dense sequence in the interval $(0, 1/j]$ in the definition of B_j, hence B_j is measurable. We see readily that B_j is increasing and $|D \setminus \bigcup_j B_j| = 0$.

Consider two different points x, y of B_j with $|x-y| \le 1/j$ and for $r = |x-y|$ let

$$S(x,y;r,j) = [B(x;r) \cap B(y;r)] \setminus [W_j(x;r) \cup W_j(y;r)].$$

Then

$$|S(x,y;r,j)| \ge |B(x,r) \cap B(y;r)| - |W_j(x;r)| - |W_j(y;r)| \ge \rho r^n/2.$$

For $z \in S(x,y;r,j)$ we have for $q(z) = p(y;z) - p(x;z)$ the estimate

$$|q(z)| \le |p(x;z) - u(z)| + |u(z) - p(y;z)| \le j(|z-x|^k + |y-z|^k) \le 2jr^k;$$

we now apply Lemma 1 with $E = S(x,y;r,j)$ to obtain

$$|D^\alpha q(y)| = |u_\alpha(y) - D^\alpha p(x;y)| \le \frac{C}{r^{n+|\alpha|}} \int_{S(x,y;r,j)} |q(z)|\, dz \le 2j\omega_n C r^{k-|\alpha|},$$

where ω_n is the volume of unit balls in $\mathbb{R}^n$ and C is the constant in Lemma 1 which depends only on n, k, and ρ and hence only on n and k. Given $\varepsilon > 0$, choose j_0 so that $|D \setminus B_{j_0}| \le \varepsilon/2$ and then choose a closed subset F of B_{j_0} such that $|D \setminus F| < \varepsilon$. For x, y in F, the last estimate assures that

$$|u_\alpha(y) - D^\alpha p(x;y)| \le M r^{k-|\alpha|}, \tag{12}$$

where M is a constant depending only on n, k, and j_0. From (12) and the fact that $|u_\alpha(x)| \le j_0$ for $x \in F$, we may apply the Whitney Extension Theorem (see [4, p. 177]) to find a C^k-function g on $\mathbb{R}^n$ such that $g(x) = u(x)$ for $x \in F$. This shows that (I) holds. We have thus shown that if (II) holds and all functions u_α, $|\alpha| \le k-1$, are measurable then (I) holds.

It remains to show that if (II) holds then the u_α are measurable. We do this by induction on k. When $k = 1$, $p(x;y) = u(x)$ for almost every x in D, i.e. u is u_α with $|\alpha| = 0$; but u is measurable. Suppose now that for a positive integer k, the functions u_α, $|\alpha| \le k-1$, are measurable when (II) holds. Let now u have an approximate k-Taylor polynomial at almost every point of D. We show that the functions u_α, $|\alpha| \le k$, are measurable. Since u obviously has an approximate $k-1$-Taylor polynomial at almost every point of D, by our inductive assumption the functions u_α, $|\alpha| \le k-1$, are measurable. Hence from what we have shown in the first part of the proof it follows that u Lusin property of order k on D, i.e. for any given $\varepsilon > 0$ there is a C^k-function g on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \ne g(x)\}| < \varepsilon$. Let A be the set of all points $x \in D$ with the property that the set $\{y \in D : u(y) = g(y)\}$ has density one at x, $u(x) = g(x)$ and u has an approximate k-Taylor polynomial at x. For $x \in A$ let $p(x;y)$ be the polynomial centered at x defined by

$$p(x;y) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} u_\alpha(x)(y-x)^\alpha.$$

Then

$$\operatorname{ap}\limsup_{y \to x} \frac{|u(y) - p(x;y)|}{|y-x|^{k+1}} < +\infty,$$

and hence

$$\operatorname{ap}\limsup_{y \to x} \frac{|g(y) - p(x;y)|}{|y-x|^{k+1}} < +\infty,$$

from which we infer that

$$\operatorname{ap}\lim_{y \to x} \frac{|g(y) - p(x;y)|}{|y-x|^{k}} = 0. \tag{13}$$

It follows from (13) that $p(x;y)$ is the k-Taylor polynomial of g at x, i.e. $u_\alpha(x) = D^\alpha g(x)$ for $x \in A$ and $|\alpha| \le k$. From the defining properties of the set A we infer then that the functions u_α, $|\alpha| \le k$, are approximately continuous at each point of A. Since $|D \setminus A| < \varepsilon$ and $\varepsilon > 0$ is arbitrary, the functions u_α, $|\alpha| \le k$, are approximately continuous almost everywhere on D and hence are measurable. The proof of Theorem 1 is thus complete.

3. Miscellaneous remarks

We first give some remarks concerning Theorem 1. When $k = 1$ the equivalence of statements (I) and (III) in Theorem 1 is due to Whitney [7], while the equivalence of statement (II) to both (I) and (III) is new even in this case. In [2], equivalence of (I) and (III) is established under the additional assumption in (III) that each u_α is measurable and is approximately differentiable of order

$k - |\alpha|$ almost everywhere. Results of the form of statement (I) are called by Stein and Zygmund [5] splittings of functions; it is interesting at this point to mention the following result in [4, VIII(4.2.3)]:

THEOREM 3. *Suppose that u is defined in a neighborhood of D and is locally integrable there and suppose that u has derivative in the harmonic sense everywhere on D. If the following condition holds everywhere on D:*

$$\limsup_{|y|\to 0} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < +\infty, \tag{14}$$

then u is differentiable almost everywhere on D.

The assumption that u is locally integrable in a neighborhood of D is required for derivative in the harmonic sense (see [4, p. 246]) and the assumption of existence of derivative in the harmonic sense is to assure the splitting of u on D, i.e. to assure that u has the Lusin property of order 1 on D [4, p. 248]. Since a function which is approximately differentiable almost everywhere on D has the Lusin property of order 1 on D, the proof of Theorem 3 in [4] establishes the theorem below:

THEOREM 4. *If a measurable function u defined on D is approximately differentiable almost everywhere on D and if* (14) *holds at almost every $x \in D$, then u is differentiable almost everywhere on D.*

Condition (14) is a symmetric form of condition (6) in Theorem 2 when $k = 1$, but if we replace (6) by (14), Theorem 2 (with $k = 1$) fails. Actually, in [4, p. 148] there is exhibited a nowhere differentiable function for which (14) holds everywhere. By Theorem 4 this function is not approximately differentiable on a set of positive measure. Hence condition (14) is much weaker than condition (6).

REFERENCES

[1] S. CAMPANATO, *Proprietá di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 137–160.

[2] F. C. LIU, *On a theorem of Whitney*, Bull. Inst. Math. Acad. Sinica **1** (1973), 63–70.

[3] H. RADEMACHER, *Ueber partielle und totale Differenzierbarkeit I*, Math. Ann. **79** (1919), 340–359.

[4] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[5] E. M. STEIN AND A. ZYGMUND, *On the differentiability of functions*, Studia Math. **23** (1964), 247–283.

[6] W. STEPANOFF, *Sur les conditions de l'existence de la différentielle totale*, Rec. Math. Soc. Moscou **32** (1925), 511–526.

[7] H. WHITNEY, *On totally differentiable and smooth functions*, Pacific J. Math. **1** (1951), 143–159.

[8] W. P. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag, 1989.

Manuscript received December 15, 1993

FON-CHE LIU
Institute of Mathematics
Academia Sinica, Taipei, P. R. of CHINA

E-mail address: maliufc@ccvax.sinica.edu.tw

WEI-SHYAN TAI
Department of Mathematics
Stanford University, Palo Alto, USA