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PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MULTIPLE STATE-DEPENDENT TIME LAGS

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Dedicated to Jean Leray

0. Introduction

Suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies a "negative feedback condition": $h(\xi_0, \xi_1) < 0$ if $\xi_0 \geq 0$ and $\xi_1 > 0$ and $h(\xi_0, \xi_1) > 0$ if $\xi_0 \leq 0$ and $\xi_1 < 0$. Let $r : \mathbb{R} \to [0, M]$ be a locally Lipschitz map with $r(0) = \tau_0 > 0$. Consider the differential-delay equation

$$x'(t) = h(x(t), x(t - r(x(t)))). \tag{0.1}$$

Under further natural hypotheses it has been proved in [12] and [13] that equation (0.1) has a "slowly oscillating periodic solution" or "SOP solution" (see Definition 3.1 in Section 3 below).

More generally, suppose that $h : \mathbb{R}^{m+1} \to \mathbb{R}$ satisfies a negative feedback condition analogous to that above. For $1 \leq j \leq m$, let $r_j : \mathbb{R} \to [0, M]$ be a locally Lipschitz map. Assume (and this is crucial) that $r_j(0) = \tau_0$ for $1 \leq j \leq m$.

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Consider the equation

$$x'(t) = h(x(t), x(t - r_1(x(t))), x(t - r_2(x(t))), \dots, x(t - r_m(x(t)))). \tag{0.2}$$

One might hope that, under further natural assumptions, equation (0.2) has a slowly oscillating periodic solution; and indeed we shall prove this as a very special case of our later results. However, the proof in [12, 13] that there exists an SOP solution of equation (0.1) (the case $m = 1$) does not extend to the case $m > 1$; and we shall need a different approach. Ironically, the proof in [12, 13] supplanted an unpublished, earlier and less aesthetic argument by one of the authors (R.D.N.); but the earlier approach can be extended to cover equation (0.2), and it is that argument which will be refined and generalized here.

There are many variants and generalizations of equation (0.2) which also possess SOP solutions. For example, suppose that $\rho_j : \mathbb{R}^2 \to [0, M]$, for $1 \le j \le m$, is a locally Lipschitz map and that σ_j is a given real number with $0 < \sigma_j \le M$ for $1 \le j \le m$. Assume that $\rho_j(0, \xi_1) = \tau_0$ for all $\xi_1 \in \mathbb{R}$ and for $1 \le j \le m$.

Consider the equation

$$\begin{aligned} x'(t) &= h(x(t), x(t - \hat{\rho}_1), x(t - \hat{\rho}_2), \dots, x(t - \hat{\rho}_m)), \\ \hat{\rho}_j &:= \rho_j(x(t), x(t - \sigma_j)). \end{aligned} \tag{0.3}$$

Our later results (see Corollary 4.1 and Remark 4.2 in Section 4) imply that under further natural assumptions, equation (0.3) has an SOP solution.

It is obviously desirable to treat equations (0.2) and (0.3), and other examples described in Section 4 in a unified way. In order to do this, suppose that $M > 0$ and let $X_M := C([-M, 0])$ denote the Banach space of continuous, real-valued functions in the usual norm. Assume that $f : X_M \to \mathbb{R}$ is a continuous map and that f satisfies a negative feedback condition: if $\varphi \in X_M$ and $\varphi(s) > 0$ for $-M \le s \le 0$, then $f(\varphi) < 0$; and if $\varphi(s) < 0$ for $-M \le s \le 0$, then $f(\varphi) > 0$. Assume that for all $\varphi \in X_M$ with $\varphi(0) = 0$,

$$f(\varphi) = g(\varphi(-\tau_0)). \tag{0.4}$$

Here, $\tau_0 > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz map with $ug(u) < 0$ for all $u \ne 0$. Under further natural assumptions on f we shall prove (see Theorem 3.1 and Remark 3.4 in Section 3) that the equation

$$x'(t) = f(x_t) \tag{0.5}$$

has an SOP solution. (As usual [8], if $x : \mathbb{R} \to \mathbb{R}$, then $x_t \in X_M$ is defined by $x_t(s) = x(t + s)$ for $-M \le s \le 0$.) Our results about equation (0.5) will imply as very special cases theorems about equations (0.1), (0.2) and (0.3).

In practice, part of the difficulty and interest of the proof lies in finding the "appropriate, natural assumptions" on f. Even after one finds the appropriate framework and sees the basic outline of the proof, a variety of technical difficulties must be overcome. Thus it may be useful to give an overview of the proof and of various obstacles.

Even for simple examples like equations (0.1) and (0.2), the map $f : X_M \to \mathbb{R}$ in equation (0.5) will not, in general, be locally Lipschitzian. Thus questions about uniqueness and continuous dependence on initial data for the initial value problem corresponding to equation (0.5) are not standard. In Section 1 (see Definition 1.1) we introduce the concept of being "almost locally Lipschitzian" and show that our functions f, while usually never locally Lipschitzian, are usually almost locally Lipschitzian. Using this idea we are able to give satisfactory uniqueness and continuous dependence results for the initial value problem.

If $\varphi \in X_M$, we define, as usual, $\mathrm{lip}(\varphi)$ by

$$\mathrm{lip}(\varphi) = \sup\{|\varphi(s) - \varphi(t)||s - t|^{-1} : s, t \in [-M, 0] \text{ and } s \neq t\}.$$

If A, B, M and R are positive reals and τ_0 is as in equation (0.4), we define closed, bounded convex sets $G^+(-B, A, M, \tau_0)$ and $G^+(-B, A, M, \tau_0, R)$ by

$$G^+(-B, A, M, \tau_0) = \{\varphi \in X_M : -B \leq \varphi(s) \leq A \text{ for all } s \in [-M, 0], \\ \text{with } \varphi(s) \geq 0 \text{ for } -\tau_0 \leq s \leq 0, \text{ and } \varphi(0) = 0\},$$

$$G^+(-B, A, M, \tau_0, R) = \{\varphi \in X_M : \varphi \in G^+(-B, A, M, \tau_0) \text{ and } \mathrm{lip}(\varphi) \leq R\}.$$

It is also convenient to define $U^+(-B, A, M, \tau_0)$ (respectively, $U^+(-B, A, M, \tau_0, R)$) to be the set of functions $\varphi \in G^+(-B, A, M, \tau_0)$ (respectively, $G^+(-B, A, M, \tau_0, R)$) such that $\varphi(s) > 0$ for some $s \in [-\tau_0, 0)$. The set $U^+(-B, A, M, \tau_0)$ (respectively, $U^+(-B, A, M, \tau_0, R)$) is a relatively open subset of $G^+(-B, A, M, \tau_0)$ (respectively, $G^+(-B, A, M, \tau_0, R)$).

If $\varphi \in G^+(-B, A, M, \tau_0, R)$ and $x(t; \varphi) = x(t)$ denotes the solution of the initial value problem

$$x'(t) = f(x_t) \qquad \text{for } t \geq 0,$$
$$x|[-M, 0] = \varphi,$$

it is important for our results to have conditions which ensure that for all $t \geq 0$,

$$-B \leq x(t; \varphi) \leq A \quad \text{and} \quad |x'(t; \varphi)| \leq R. \tag{0.6}$$

Some results along these lines are given in Theorems 1.3, 1.4 and 1.5. See, also, Remark 3.4.

If f satisfies a negative feedback condition and equations (0.4) and (0.6) are satisfied for all $\varphi \in G^+(-B, A, M, \tau_0, R)$, it is shown in Section 2 that (for $\varphi \in U^+(-B, A, M, \tau_0, R)$) the zeros of $x(t; \varphi)$ are separated by a distance greater than τ_0. More precisely, for $\varphi \in U^+(-B, A, M, \tau_0, R)$, define $\sigma_0(\varphi)$, and $\zeta_1(\varphi)$ and $\zeta_2(\varphi)$, by

$$\begin{aligned} \sigma_0(\varphi) &= \sup\{t \geq 0 \,:\, x(s; \varphi) = 0 \text{ for all } s \in [0, t]\}, \\ \zeta_1(\varphi) &= \inf\{t \geq \tau_0 \,:\, x(t; \varphi) = 0\}, \\ \zeta_2(\varphi) &= \inf\{t > \zeta_1(\varphi) \,:\, x(t; \varphi) = 0\}. \end{aligned}$$

Then it follows that $\sigma_0(\varphi) < \tau_0$, with $\zeta_1(\varphi) > \sigma_0(\varphi) + \tau_0$ and $x(t; \varphi) < 0$ for $\sigma_0 < t < \zeta_1$, and $\zeta_2(\varphi) > \zeta_1(\varphi) + \tau_0$. It may happen that $\zeta_1(\varphi) = \infty$ or $\zeta_2(\varphi) = \infty$. Using these facts we define a map

$$\Gamma_0 : U^+(-B, A, M, \tau_0, R) \to G^+(-B, A, M, \tau_0, R)$$

by setting

$$\Gamma_0(\varphi) = x_{\zeta_2},$$

where x denotes the function $x(t; \varphi)$ and $\zeta_2 = \zeta_2(\varphi)$. If $\zeta_1(\varphi) = \infty$ or $\zeta_2(\varphi) = \infty$ we define $\Gamma_0(\varphi) = 0$. It is not hard to prove that fixed points of Γ_0 in $U^+ := U^+(-B, A, M, \tau_0, R)$ correspond to SOP solutions of equation (0.5). If $G^+ := G^+(-B, A, M, \tau_0, R)$ the strategy of the proof is to prove that Γ_0 is continuous and that $i_{G^+}(\Gamma_0, U^+)$, the fixed point index of $\Gamma_0 : U^+ \to G^+$ is defined and nonzero.

At this point, a difficulty arises: the map Γ_0 is continuous on U^+, but it is not clear (see Remark 2.1) that it can be extended continuously to G^+. In the case of equation (0.1), one can circumvent this difficulty. In [12, 13] it is proved that for $\varphi \in U^+$ and $\zeta_1(\varphi) \leq t \leq \zeta_2(\varphi)$ one has $t - r(x(t)) \geq 0$, and this observation is used in [12, 13] to define a variant of Γ_0 which can be extended continuously to G^+. This technique is no longer applicable for equation (0.2) when $m > 1$. Fortunately, as we shall show below, it is possible to define and evaluate $i_{G^+}(\Gamma_0, U^+)$ even if Γ_0 cannot be extended continuously to $\overline{U^+}$.

The basic result of this paper is contained in Theorem 3.1 and Remark 3.4. However, even to state Theorem 3.1 precisely, it is necessary to discuss the "linearization of f" at 0. In fact, for the examples of interest, the map f is rarely Fréchet differentiable at 0. In Definition 3.2 we define a weakening of Fréchet differentiability at 0, namely "almost Fréchet differentiability at 0" and "the almost Fréchet derivative at 0." It turns out that our maps f are usually almost Fréchet differentiable at 0, and this is sufficient.

The central point in Theorem 3.1 is the evaluation of $i_{G^+}(\Gamma_0, U^+)$. The basic idea of the proof in Section 3 is to consider a homotopy of equations parametrized by α, in the range $0 \le \alpha \le 2$, where $\alpha = 0$ corresponds to the original equation and $\alpha = 2$ to a linear equation

$$y'(t) = -\beta y(t) - \gamma y(t - \tau_0).$$

For each α one defines a map $\Gamma_\alpha : U^+ \to X_M$ something like Γ_0. For an appropriate retraction ρ one has $\rho\Gamma_\alpha : U^+ \to G^+$, and the key idea is to use the homotopy property of the fixed point index to prove that $i_{G^+}(\rho\Gamma_\alpha, U^+)$ is constant for $0 \le \alpha \le 2$. For $1 \le \alpha \le 2$ the map Γ_α extends continuously to G^+, and this fact, together with the additivity property of the fixed point index, simplifies the problem of computing $i_{G^+}(\rho\Gamma_\alpha, U^+)$ for $1 \le \alpha \le 2$. An analytical part of the proof (see Lemmas 3.4 and 3.4A) involves obtaining an upper bound on the minimal period of any SOP solution of one of the parametrized equations, $0 \le \alpha \le 2$. For $\alpha = 2$, the map Γ_2 corresponds to a linear equation, and the computation of $i_{G^+}(\rho\Gamma_2, U^+)$ is obtained from Lemma 3.2.

The proof of Theorem 3.1 requires use of all the basic facts about the fixed point index: the additivity, homotopy, commutativity and normalization properties. However, it is worth noting that, if we use Remark 3.2 in Section 3, our proof totally avoids the apparatus of asymptotic fixed point theory.

Theorem 3.1 and Remark 3.4 imply all of our existence results for SOP solutions. In Section 4 we verify that all the hypotheses of Theorem 3.1 are satisfied for our equations (0.1), (0.2) and (0.3). We also remark on some generalizations by R.D.N. of work of Kuang and Smith [9].

This paper is dedicated to Jean Leray in recognition of his seminal work in fixed point theory.

1. Existence, uniqueness and boundedness for the initial value problem

Throughout this paper M will denote a fixed positive constant, and we shall denote by X_M or simply X the Banach space $C([-M, 0]; \mathbb{R})$ of continuous, real-valued functions $\varphi : [-M, 0] \to \mathbb{R}$, with norm $\|\varphi\| = \max_{-M \le t \le 0} |\varphi(t)|$. If $\varphi \in X_M$, we define $\text{lip}(\varphi)$ by

$$\text{lip}(\varphi) = \sup\{|\varphi(s) - \varphi(t)||s - t|^{-1} : s, t \in [-M, 0] \text{ and } s \ne t\}.$$

As usual, if $x : [t_0 - M, t_1] \to \mathbb{R}$ is a continuous function and $t_1 \ge t_0$, then for $t_0 \le t \le t_1$ we shall define $x_t \in X_M$ by $x_t(s) = x(t + s)$ for $-M \le s \le 0$.

Our first theorem is a standard result; it follows easily from the Schauder fixed point theorem and is proved in Chapter 2 of [8].

THEOREM 1.1. *Suppose that $f : X_M \to \mathbb{R}$ is a continuous map and that $\varphi_0 \in X_M$ and $t_0 \in \mathbb{R}$. Then there exists $\delta > 0$ and a continuous function $x : [t_0 - M, t_0 + \delta] \to \mathbb{R}$, continuously differentiable on $[t_0, t_0 + \delta]$, such that*

$$x'(t) = f(x_t) \qquad \text{for } t_0 \le t \le t_0 + \delta, \tag{1.1}$$
$$x_{t_0} = \varphi_0.$$

If there exist constants C_1 and C_2 such that for all $\varphi \in X_M$,

$$|f(\varphi)| \le C_1 \|\varphi\| + C_2, \tag{1.2}$$

then x in (1.1) can be defined on $[t_0 - M, \infty)$.

We need hypotheses on f which ensure that the solution of (1.1) is unique. If f is locally Lipschitz, uniqueness holds in (1.1); but for the examples of interest to us, for example the equation (0.1), the corresponding map f is, in general, not locally Lipschitz. The reader can verify that f is not locally Lipschitz for simple examples like $f(\varphi) = \varphi(-r(\varphi(0)))$, where $r(0) = 1$ and $r'(0) \neq 0$. Thus we need a variant of the local Lipschitz condition.

DEFINITION 1.1. Let $g : D \subset X_M \to \mathbb{R}$ be a continuous map, and for each $\varphi_0 \in X_M$, and quantities $\delta > 0$ and $R > 0$, define $V(\varphi_0; \delta, R)$ by

$$V(\varphi_0; \delta, R) = \{\varphi \in X_M : \|\varphi - \varphi_0\| \le \delta \text{ and } \operatorname{lip}(\varphi) \le R\}.$$

We shall say that g is *almost locally Lipschitzian* if, for each $\varphi_0 \in D$ and $R > 0$, there exists $\delta = \delta(\varphi_0, R) > 0$ and $k = k(\varphi_0, R, \delta) \ge 0$ such that for all $\varphi, \psi \in V(\varphi_0; \delta, R) \cap D$,

$$\|g(\varphi) - g(\psi)\| \le k\|\varphi - \psi\|.$$

The map g in Definition 1.1 is defined on $D \subset X_M$. Suppose, however, that there exists a continuous retraction $\rho : X_M \to D$ of X_M onto D, which is locally Lipschitzian on X_M, and that for every $\varphi_0 \in X_M$, and $\delta > 0$ and $R > 0$, it is true that

$$\sup\{\operatorname{lip}(\rho(\varphi)) : \varphi \in V(\varphi_0; \delta, R)\} < \infty.$$

Then if $g : D \subset X_M \to \mathbb{R}$ is almost locally Lipschitzian and $G : X_M \to \mathbb{R}$ is defined by $G(\varphi) = g(\rho(\varphi))$, it follows that G is an extension of g and G is almost locally Lipschitzian. We leave the proof to the reader.

We shall apply the above observation in the following simple situation. Suppose that A and B are real numbers with $-B < A$ and define $K(-B, A, M) \equiv D \subset X_M$ by

$$D = \{\varphi \in X_M : -B \le \varphi(t) \le A \text{ for } -M \le t \le 0\}.$$

Define a retraction $r : \mathbb{R} \to [-B, A]$ by

$$r(u) = \begin{cases} A & \text{for } u \ge A, \\ u & \text{for } -B \le u \le A, \\ -B & \text{for } u \le -B. \end{cases} \tag{1.3}$$

It is easy to check that r is Lipschitz with $\mathrm{lip}(r) = 1$. Define a retraction $\rho : X_M \to D$ by

$$(\rho(\varphi))(t) = r(\varphi(t)). \tag{1.4}$$

It is easy to check that ρ is a continuous retraction of X_M onto D and that for all $\varphi, \psi \in X_M$,

$$\mathrm{lip}(\rho(\varphi)) \le \mathrm{lip}(\varphi), \qquad \text{and} \qquad \|\rho(\varphi) - \rho(\psi)\| \le \|\varphi - \psi\|.$$

It follows that if $g : D \to \mathbb{R}$ is almost locally Lipschitzian and $G = g \circ \rho$, then $G : X_M \to \mathbb{R}$ is almost locally Lipschitzian.

LEMMA 1.1. *Let $D \subset X_M$ be a closed, bounded set and let $g : D \to \mathbb{R}$ be almost locally Lipschitzian. If $R > 0$ and $D_R = \{\varphi \in D : \mathrm{lip}(\varphi) \le R\}$, then there exists a constant $k = k(R, D)$ with*

$$\|g(\varphi) - g(\psi)\| \le k\|\varphi - \psi\| \qquad \text{for all } \varphi, \psi \in D_R.$$

PROOF. The Ascoli-Arzelà theorem implies that D_R is a compact subset of X_M. Using Definition 1.1, we see that for each $\varphi \in D_R$ there exists $\delta = \delta(\varphi, R) > 0$ such that (in the notation of Definition 1.1) the restriction of g to $V(\varphi; \delta, R) \cap D_R$ is a Lipschitz map with Lipschitz constant $k(\varphi, R, \delta) \ge 0$. Compactness of D_R implies that there exist (φ_i, δ_i) for $1 \le i \le m$, with

$$D_R = \bigcup_{i=1}^{m} V(\varphi_i, \delta_i/2, R) \cap D_R. \tag{1.5}$$

Let $\delta_0 = \min\{\delta_i : 1 \le i \le m\}$ and $k_0 = \max\{k(\varphi_i, \delta_i, R) : 1 \le i \le m\}$. If $\varphi, \psi \in D_R$ and $\varphi, \psi \in V(\varphi_i, \delta_i, R)$ for some i, we obtain

$$\|g(\varphi) - g(\psi)\| \le k_0\|\varphi - \psi\|.$$

If $\varphi, \psi \in D_R$ and there does not exist i with $\varphi, \psi \in V(\varphi_i, \delta_i, R)$, select (by using (1.5)) a j with $\varphi \in V(\varphi_j, \delta_j/2, R)$, so we must have $\psi \notin V(\varphi_j, \delta_j, R)$ and

$\|\varphi-\psi\| \geq \delta_j/2 \geq \delta_0/2$. Since g is continuous on the compact set D_R, there exists a constant C with $\|g(\varphi) - g(\psi)\| \leq C$ for all $\varphi, \psi \in D_R$, so if $\|\varphi - \psi\| \geq \delta_0/2$, we have

$$\|g(\varphi) - g(\psi)\| \leq (2C\delta_0^{-1})\|\varphi - \psi\|.$$

Thus we have proved that $g|D_R$ is Lipschitz with Lipschitz constant $k = \max\{k_0, 2C\delta_0^{-1}\}$. □

Under the assumption that f is almost locally Lipschitzian we can prove uniqueness of solutions of (1.1) for Lipschitz initial conditions.

THEOREM 1.2. *Suppose that $f : X_M \to \mathbb{R}$ is a continuous map which is almost locally Lipschitzian and that $\varphi_0 \in X_M$ and $t_0 \in \mathbb{R}$ and* $\mathrm{lip}(\varphi_0) < \infty$. *If $\delta > 0$ and $y : [t_0 - M, t_0 + \delta] \to \mathbb{R}$ and $z : [t_0 - M, t_0 + \delta] \to \mathbb{R}$ both satisfy equation* (1.1), *then $y(t) = z(t)$ for $t_0 - M \leq t \leq t_0 + \delta$.*

PROOF. We define

$$t_1 = \sup\{t \geq t_0 : y(s) = z(s) \text{ for } t_0 \leq s \leq t\},$$

and we recall that $y(s) = z(s) = \varphi_0(s)$ for $t_0 - M \leq s \leq t_0$. If $t_1 < t_0 + \delta$, we must obtain a contradiction. It is an easy exercise to show that the maps $t \to y_t$ and $t \to z_t$ for $t_0 \leq t \leq t_0 + \delta$ are continuous, so $\{y_t : t_0 \leq t \leq t_0 + \delta\}$ and $\{z_t : t_0 \leq t \leq t_0 + \delta\}$ are the continuous images of a compact interval and hence compact. Since f is continuous, there exists $C > 0$ such that $|f(y_t)| \leq C$ and $|f(z_t)| \leq C$ for $t_0 \leq t \leq t_0 + \delta$, and we can also assume that $\mathrm{lip}(\varphi_0) \leq C$.

By definition of almost locally Lipschitzian, there exists $\delta_1 > 0$ such that (writing $\varphi_1 = y_{t_1} = z_{t_1}$ and using the notation of Definition 1.1) the restriction of f to $V(\varphi_1, \delta_1, C)$ is Lipschitz with Lipschitz constant k. Note that by our construction we have $\mathrm{lip}(y_t) \leq C$ and $\mathrm{lip}(z_t) \leq C$ for $t_0 \leq t \leq t_0 + \delta$. Furthermore, if $t_1 \leq t \leq t_1 + \delta_2$ and $C\delta_2 \leq \delta_1$, we see that $\|y_t - \varphi_1\| \leq C\delta_2 \leq \delta_1$, and $\|z_t - \varphi_1\| \leq \delta_1$. It follows that for $t_1 \leq t \leq t_1 + \delta_2$, where $0 < \delta_2 \leq \delta_1 C^{-1}$, we have

$$\begin{aligned}(1.6)\qquad |y(t) - z(t)| &\leq \int_{t_1}^{t} |f(y_s) - f(z_s)|\, ds \leq k \int_{t_1}^{t} \|y_s - z_s\|\, ds \\ &= k \int_{t_1}^{t} \sup_{t_1 \leq r \leq s} |y(r) - z(r)|\, ds \\ &\leq k\delta_2 \Big(\sup_{t_1 \leq r \leq t_1 + \delta_2} |y(r) - z(r)| \Big).\end{aligned}$$

If we take the supremum of (1.6) over t with $t_1 \leq t \leq t_1 + \delta_2$, we obtain

$$(1.7)\qquad \sup_{t_1 \leq t \leq t_1 + \delta_2} |y(t) - z(t)| \leq k\delta_2 \Big(\sup_{t_1 \leq r \leq t_1 + \delta_2} |y(r) - z(r)| \Big).$$

If $\delta_2 > 0$ is small enough that $k\delta_2 < 1$, then (1.7) implies that $y(t) = z(t)$ for $t_1 \le t \le t_1 + \delta_2$, which contradicts the definition of t_1 and completes the proof.□

If, under the hypotheses of Theorem 1.2, we also know that inequality (1.2) is satisfied, we can conclude from Theorems 1.1 and 1.2 that there exists a unique continuous function $x|[t_0 - M, \infty)$, continuously differentiable on $[t_0, \infty)$, such that $x_{t_0} = \varphi_0$ and $x'(t) = f(x_t)$ for $t \ge t_0$. Note again that we need that φ_0 be Lipschitzian.

In order to use Theorem 1.2 we need some examples of maps f which are almost locally Lipschitzian.

PROPOSITION 1.1. *Suppose that $r_i : X_M \to [0, M]$ is continuous and almost locally Lipschitzian for $1 \le i \le m$. Let $g : \mathbb{R}^{m+1} \to \mathbb{R}$ be locally Lipschitzian. If $f : X_M \to \mathbb{R}$ is defined by*

$$f(\varphi) = g(\varphi(0), \varphi(-r_1(\varphi)), \varphi(-r_2(\varphi)), \dots, \varphi(-r_m(\varphi))),$$

then f is continuous and almost locally Lipschitzian.

PROOF. To show that f is continuous, it suffices to prove that the map $\varphi \to \varphi(-r_i(\varphi))$ is continuous for $1 \le i \le m$. To see this, select $\varphi_0 \in X_M$ and $\varepsilon > 0$. If $\|\varphi - \varphi_0\| < \delta$ we obtain

$$\begin{aligned} |\varphi(-r_i(\varphi)) - \varphi_0(-r_i(\varphi_0))| \le& |\varphi(-r_i(\varphi)) - \varphi_0(-r_i(\varphi))| \\ &+ |\varphi_0(-r_i(\varphi)) - \varphi_0(-r_i(\varphi_0))| \\ \le& \delta + |\varphi_0(-r_i(\varphi)) - \varphi_0(-r_i(\varphi_0))|. \end{aligned} \tag{1.8}$$

Because φ_0 is continuous, there exists $\delta_1 > 0$ with $|\varphi_0(s) - \varphi_0(t)| < \varepsilon/2$ for all $s, t \in [-M, 0]$ with $|s - t| < \delta_1$. Because r_i is continuous, we can select $\delta > 0$ so that $|r_i(\varphi) - r_i(\varphi_0)| < \delta_1$ for $\|\varphi - \varphi_0\| \le \delta$ and so that $\delta < \varepsilon/2$. For this choice of δ, we obtain from (1.8) that

$$|\varphi(-r_i(\varphi)) - \varphi_0(-r_i(\varphi_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and we conclude that f is continuous.

To prove that f is almost locally Lipschitzian, select $\varphi_0 \in X_M$ and $R > 0$. Because r_i, for $1 \le i \le m$, is almost locally Lipschitzian, there exist $\delta > 0$ and $C_i > 0$ such that for all $\varphi, \psi \in V(\varphi_0; \delta, R)$ we have

$$|r_i(\varphi) - r_i(\psi)| \le C_i \|\varphi - \psi\|.$$

(Here $V(\varphi_0; \delta, R)$ is defined in Definition 1.1.) Because g is locally Lipschitzian, the restriction of g to the set

$$\{y \in \mathbb{R}^{m+1} : |y_i| \le \|\varphi_0\| + \delta \text{ for } 1 \le i \le m+1\}$$

is Lipschitz. Thus there exists a constant C so that if $\|\varphi-\varphi_0\| \le \delta$ and $\|\psi-\varphi_0\| \le \delta$ we have

$$|f(\varphi) - f(\psi)| \le C\Big(|\varphi(0) - \psi(0)| + \sum_{i=1}^{m} |\varphi(-r_i(\varphi)) - \psi(-r_i(\psi))|\Big).$$

If $\varphi, \psi \in V(\varphi_0; \delta, R)$ we obtain also that

$$\begin{aligned}|\varphi(-r_i(\varphi)) - \psi(-r_i(\psi))| &\le |\varphi(-r_i(\varphi)) - \psi(-r_i(\varphi))| + |\psi(-r_i(\varphi)) - \psi(-r_i(\psi))| \\ &\le \|\varphi - \psi\| + R|r_i(\varphi) - r_i(\psi)| \\ &\le (C_i R + 1)\|\varphi - \psi\|.\end{aligned}$$

Combining these estimates we see that

$$|f(\varphi) - f(\psi)| \le C\Big(m + 1 + R\sum_{i=1}^{m} C_i\Big)\|\varphi - \psi\|,$$

so f is almost locally Lipschitz. □

COROLLARY 1.1. *Suppose that $r_i : \mathbb{R} \to [0, M]$ is locally Lipschitzian for $1 \le i \le m$ and that $g : \mathbb{R}^{m+1} \to \mathbb{R}$ is locally Lipschitzian. If $f : X_M \to \mathbb{R}$ is defined by*

$$f(\varphi) = g(\varphi(0), \varphi(-\rho_1), \varphi(-\rho_2), \ldots, \varphi(-\rho_m)),$$

where $\rho_i = r_i(\varphi(0))$, then f is continuous and almost locally Lipschitzian.

PROOF. This follows immediately from Proposition 1.1. □

We shall need results which imply that solutions of equation (1.1) remain bounded. To describe our results we need some notation and definitions. If $\varphi, \psi \in X_M$ we shall write $\phi \le \psi$ if $\varphi(t) \le \psi(t)$ for all $t \in [-M, 0]$, and $\varphi < \psi$ if $\varphi(t) < \psi(t)$ for all $t \in [-M, 0]$. If $C \in \mathbb{R}$ we shall write $\varphi \le C$ if $\varphi(t) \le C$ for all $t \in [-M, 0]$ and $\varphi < C$ if $\varphi(t) < C$ for all $t \in [-M, 0]$. If $-B$ and A are real numbers with $-B < A$, and $R > 0$, we define

$$\begin{aligned} K(-B, A, M) &= \{\varphi \in X_M : -B \le \varphi \le A\}, \\ K(-B, A, M, R) &= \{\varphi \in K(-B, A, M) : \operatorname{lip}(\varphi) \le R\}. \end{aligned} \tag{1.9}$$

If $-B < 0 < A$, we also write

$$\begin{aligned} K_0(-B, A, M) &= \{\varphi \in K(-B, A, M) : \varphi(0) = 0\}, \\ K_0(-B, A, M, R) &= \{\varphi \in K(-B, A, M, R) : \varphi(0) = 0\}. \end{aligned}$$

DEFINITION 1.2. If $f : D \subset X_M \to \mathbb{R}$ is a map, we say that f *satisfies a negative feedback condition* if

(a) $f(\varphi) \le 0$ for all $\varphi \in D$ with $\varphi \ge 0$, and $f(\varphi) < 0$ for all $\varphi \in D$ with $\varphi > 0$; and

(b) $f(\varphi) \ge 0$ for all $\varphi \in D$ with $\varphi \le 0$, and $f(\varphi) > 0$ for all $\varphi \in D$ with $\varphi < 0$.

The negative feedback condition of Definition 1.2 will play a crucial role in later results.

THEOREM 1.3. *Let $-B$ and A be real numbers with $-B < A$ and suppose that $f : K(-B, A, M) \to \mathbb{R}$ is continuous, almost locally Lipschitzian and bounded on $K(-B, A, M) \equiv K$. Assume that*

$$f(\varphi) \le 0 \text{ for all } \varphi \in K \text{ with } \varphi(0) = A, \text{ and}$$
$$f(\varphi) \ge 0 \text{ for all } \varphi \in K \text{ with } \varphi(0) = -B.$$

Then for every Lipschitz $\varphi \in K$ and every $t_0 \in \mathbb{R}$ there exists a unique continuous function $x : [t_0 - M, \infty) \to [-B, A]$, differentiable on $[t_0, \infty)$, and satisfying

$$x'(t) = f(x_t) \qquad \text{for } t \ge t_0,$$
$$x_{t_0} = \varphi.$$

PROOF. Let $\rho : X_M \to K$ be the retraction defined by equations (1.3) and (1.4). As noted before, $\varphi \to f(\rho(\varphi))$ is continuous and almost locally Lipschitz on X_M. Furthermore, because f is bounded on K, the composition $f \circ \rho$ is bounded on X_M. Theorem 1.2 implies that if $\varphi \in K$ is Lipschitz, then the equation

$$x'(t) = f(\rho(x_t)) \qquad \text{for } t \ge t_0,$$
$$x_{t_0} = \varphi,$$

has a unique solution $x : [t_0 - M, \infty) \to \mathbb{R}$. To complete the proof, it suffices to show that $-B \le x(t) \le A$ for all $t \ge t_0$. If $\psi \in X_M$ and $\psi(0) \ge A$, notice that $(\rho(\psi))(0) = A$, so $f(\rho(\psi)) \le 0$. Similarly, we see that if $\psi(0) \le -B$, then $f(\rho(\psi)) \ge 0$. If $x(t_1) > A$ for some $t_1 > t_0$, then we define $\tau = \sup\{t \in [t_0, t_1] : x(t) = A\}$, so $x(\tau) = A$ and $x(t) > A$ for $\tau < t \le t_1$. The mean value theorem implies that $x'(t) > 0$ for some t with $\tau < t < t_1$, which contradicts the fact that $(\rho(x_t))(0) = A$ and

$$x'(t) = f(\rho(x_t)) \le 0.$$

This shows that $x(t) \leq A$ for all $t \geq t_0$, and a similar argument yields $x(t) \geq -B$ for all $t \geq t_0$. □

Theorem 1.3 immediately yields the following boundedness result.

COROLLARY 1.2. *Let hypotheses and notation be as Proposition* 1.1. *Assume that there exist real numbers $-B$ and A, with $-B < A$, such that for all $\zeta \in \mathbb{R}^m$ with $-B \leq \zeta_i \leq A$ for $1 \leq i \leq m$,*

$$g(A, \zeta) \leq 0 \quad \textit{and} \quad g(-B, \zeta) \geq 0.$$

If $\varphi_0 \in K(-B, A, M)$ and $\operatorname{lip}(\varphi_0) < \infty$, *the solution of the equation*

$$\begin{aligned} &x'(t) = g(x(t), x(t - r_1(x_t)), x(t - r_2(x_t)), \dots, x(t - r_m(x_t))) \qquad \textit{for } t \geq t_0, \\ &x_{t_0} = \varphi_0, \end{aligned}$$

is defined for all $t \geq t_0$ and satisfies $-B \leq x(t) \leq A$ for all $t \geq t_0$.

Theorem 1.3 also gives information about examples like

$$x'(t) = -\lambda x(t) - \lambda k x(t - r), \qquad \text{with } r = 1 + cx(t), \tag{1.10}$$

where $\lambda > 0$, and $k > 0$ and $c > 0$. Such equations are discussed in [12] and [13].

COROLLARY 1.3. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitzian map with $g(\zeta_0, \zeta_1) \leq 0$ for all ζ_0, ζ_1 satisfying $\zeta_0 \geq 0$ and $\zeta_1 \geq 0$, and $g(\eta, \eta) \geq 0$ for all $\eta \leq 0$. Suppose that A and B are positive reals with*

$$\sup_{-B \leq \zeta_1 \leq 0} g(A, \zeta_1) \leq 0. \tag{1.11}$$

Let $r : K(-B, A, M) \to [0, M]$ be a continuous, almost locally Lipschitzian map such that $r(\varphi) = 0$ for all $\varphi \in K(-B, A, M)$ with $\varphi(0) = -B$. If $\varphi_0 \in K(-B, A, M)$ and $\operatorname{lip}(\varphi_0) < \infty$, *the solution of the equation*

$$\begin{aligned} &x'(t) = g(x(t), x(t - r(x_t))) \qquad \textit{for } t \geq t_0, \\ &x_{t_0} = \varphi_0, \end{aligned}$$

is defined for all $t \geq t_0$ and satisfies $-B \leq x(t) \leq A$ for all $t \geq t_0$.

PROOF. We apply Theorem 1.3 with $f(\varphi) = g(\varphi(0), \varphi(-r(\varphi)))$. If $\varphi \in K(-B, A, M)$ and $\varphi(0) = -B$, we obtain $f(\varphi) = g(-B, -B) \geq 0$. If $\varphi(0) = A$ and $\zeta_1 = \varphi(-r(\varphi))$, then (1.11) implies that $g(A, \zeta_1) \leq 0$ for $-B \leq \zeta_1 \leq 0$. The assumptions on g yield $g(A, \zeta_1) \leq 0$ if $0 \leq \zeta_1 \leq A$. Thus we find $f(\varphi) \leq 0$ for all $\varphi \in K(-B, A, M)$ with $\varphi(0) = A$. Corollary 1.1 implies that f is continuous and almost locally Lipschitzian, so Corollary 1.3 follows from Theorem 1.3. □

Corollary 1.3 immediately applies to equations studied in [12], namely,

$$x'(t) = -\lambda x(t) + \lambda f(x(t-r)) \qquad \text{with } r = r(x(t)) \text{ and } \lambda > 0. \tag{1.12}$$

Assume in (1.12) that f and r are locally Lipschitz maps from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $\zeta f(\zeta) < 0$ for all $\zeta \neq 0$ and that there exists $-B < 0$ with $r(-B) = 0$. Define A by

$$A = \sup_{-B \leq \zeta \leq 0} f(\zeta)$$

and assume that $r(\zeta) \geq 0$ for $-B \leq \zeta \leq A$. If M is defined by

$$M = \sup\{r(\zeta) \,:\, -B \leq \zeta \leq A\}$$

we are now in the situation of Corollary 1.3. If $\varphi_0 \in K(-B, A, M)$ and $\text{lip}(\varphi_0) < \infty$ and $t_0 \in \mathbb{R}$, Corollary 1.3 implies that there is a continuous map $x : [t_0 - M, \infty] \to [-B, A]$ which is continuously differentiable on $[t_0, \infty)$, satisfies $x_{t_0} = \varphi_0$ and satisfies equation (1.12) for $t \geq t_0$. Equation (1.10) is a very special case with $B = c^{-1}, A = kc^{-1}$ and $M = 1 + k$.

Classical examples like Wright's equation,

$$x'(t) = -\alpha x(t-1)(1 + x(t)), \tag{1.13}$$

do not satisfy the hypotheses of Theorem 1.3. Our next two theorems are variants of Theorem 1.3 which allow us to cover such examples.

THEOREM 1.4. *Let B be a positive real and $D = \{\varphi \in X_M \,:\, \varphi \geq -B\}$ and suppose that $f : D \to \mathbb{R}$ is a continuous map which is almost locally Lipschitzian, bounded on bounded subsets of D and satisfies a negative feedback condition. Assume that $f(\varphi) \geq 0$ for all $\varphi \in D$ with $\varphi(0) = -B$ and suppose that there are positive constants C_1 and C_2 with*

$$f(\varphi) \leq C_1|\varphi(0)| + C_2 \qquad \textit{for all } \varphi \in D.$$

Define $A = (C_2/C_1)\exp(C_1 M)$. If $\varphi \in D$, with $\text{lip}(\varphi) < \infty$ and $\varphi(0) = 0$, there is a unique solution $x(t; \varphi) = x(t)$ of

$$\begin{aligned} x'(t) &= f(x_t) && \textit{for } t \geq 0, \\ x(t) &= \varphi(t) && \textit{for } -M \leq t \leq 0. \end{aligned}$$

Furthermore, $x(t)$ is defined for all $t \geq 0$ and $-B \leq x(t) < A$ for all $t \geq 0$.

PROOF. Select $A_1 \geq A$ and $A_1 \geq \|\varphi\|$. Let ρ be the usual retraction of X_M onto $K(-B, A_1, M)$ and define $\widetilde{f}(\psi) = f(\rho(\psi))$ for $\psi \in X_M$. Theorems 1.1 and

1.2 imply that if $\varphi \in X_M$ and $\mathrm{lip}(\varphi) < \infty$, the initial value problem

$$x'(t) = \widetilde{f}(x_t) \qquad \text{for } t \geq 0,$$
$$x_0 = \varphi,$$

has a unique solution $x(t;\varphi)$ defined for all $t \geq 0$. Thus, taking $\varphi \in D$, with $\varphi(0) = 0$ and $\mathrm{lip}(\varphi) < \infty$, it suffices to prove that

$$-B \leq x(t;\varphi) < A \qquad \text{for } t \geq 0.$$

If $\psi \in X_M$ and $\psi(0) = -B$, we know that $\widetilde{f}(\psi) \geq 0$, so the same argument given in Theorem 1.3 shows that $x(t;\varphi) \geq -B$ for all $t \geq 0$.

It remains to prove that $x(t) := x(t;\varphi) < A$ for $t \geq 0$. If not, let τ be the first time $t > 0$ with $x(t) = A$ and define $\tau_1 = \sup\{t \in [0,\tau) : x(t;\varphi) = 0\}$; note that τ_1 is defined because $\varphi(0) = 0$. The definition of τ implies that $x'(\tau) = f(x_\tau) \geq 0$. Since f satisfies a negative feedback condition, we conclude that $\tau - \tau_1 \leq M$. For $\tau_1 \leq t \leq \tau$ we have

$$x'(t) = f(x_t) \leq C_1 x(t) + C_2,$$

and this differential inequality gives

$$x(\tau) = A \leq (C_2/C_1)(\exp(C_1(\tau - \tau_1)) - 1) < (C_2/C_1)\exp(C_1 M),$$

which contradicts the choice of A. □

The reader can easily verify that Theorem 1.4 is directly applicable to equations like Wright's equation (1.13).

Our next theorem is another easy variant of Theorem 1.3, but it applies to examples which are not covered by Theorem 1.3 or 1.4.

THEOREM 1.5. *Let A, B and M be positive reals and let*

$$f : X_M \supset K(-B,A,M) \to \mathbb{R}$$

be a continuous, almost locally Lipschitzian map which satisfies a negative feedback condition (Definition 1.2). Assume that for all $\varphi \in K(-B,A,M)$ we have $-BM^{-1} \leq f(\varphi) \leq AM^{-1}$. If $\varphi_0 \in K_0(-B,A,M)$, so that $\varphi_0(0) = 0$, and if also $\mathrm{lip}(\varphi_0) < \infty$ and $t_0 \in R$, then there exists a unique continuous map $x : [t_0 - M, \infty) \to [-B, A]$, differentiable on $[t_0,\infty)$, with

$$\begin{aligned} x'(t) &= f(x_t) \qquad \text{for } t \geq t_0, \\ x_{t_0} &= \varphi_0. \end{aligned} \tag{1.14}$$

PROOF. By using the usual retraction $\rho : X_M \to K(-B,A,M)$, we can assume that $f : X_M \to \mathbb{R}$ is continuous, almost locally Lipschitzian, satisfies a negative feedback condition and satisfies $-BM^{-1} \le f(\varphi) \le AM^{-1}$ for all $\varphi \in X_M$. For this extended f, Theorems 1.1 and 1.2 imply that equation (1.14) has a unique solution $x(t)$ defined on $[t_0 - M, \infty)$. To complete the proof, it suffices to prove that $-B \le x(t) \le A$ for all $t \ge t_0$. We shall prove that $x(t) \le A$ for all $t \ge t_0$, the proof that $x(t) \ge -B$ for $t \ge t_0$ being analogous. Suppose, by way of contradiction, that $x(t_2) > A$ for some $t_2 > t_0$. Since we assume that $x(t_0) = 0$, we can define $t_1 = \sup\{t \in [t_0, t_2] : x(t) = 0\}$ and note that $x(t) > 0$ for $t_1 < t \le t_2$. Because $x'(t) = f(\rho(x_t)) \le AM^{-1}$ for all t, we must have $t_2 - t_1 > M$ and $x(t_1 + M) \le A$. On the other hand, the negative feedback condition on f implies that $x'(t) \le 0$ for $t_1 + M \le t \le t_2$, so $x(t_2) \le x(t_1 + M) \le A$. Thus we have obtained a contradiction, and we conclude that $x(t) \le A$ for all $t \ge t_0$. □

Like the question of uniqueness of the initial value problem, the question of continuous dependence of solutions on initial data is not completely standard in our context. For completeness we sketch the proof of a result which will suffice for our purposes.

THEOREM 1.6. *Let $-B < A$ and $M > 0$ be real numbers and suppose that $f : K(-B, A, M) \to \mathbb{R}$ and $g : K(-B, A, M) \to \mathbb{R}$ are continuous, almost locally Lipschitzian maps and that $|f(\varphi)| \le R$ and $|g(\varphi)| \le R$ for all $\varphi \in K(-B, A, M)$. Suppose that $\varphi_0, \psi_0 \in K(-B, A, M, R)$ and that $\rho : X_M \to K(-B, A, M)$ is the retraction given by equations (1.3) and (1.4). Let $x : [t_0 - M, \infty) \to \mathbb{R}$ and $y : [t_0 - M, \infty) \to \mathbb{R}$ denote the unique solutions of*

$$\begin{aligned} x'(t) &= f(\rho(x_t)) \qquad \text{for } t \ge t_0, \\ x_{t_0} &= \varphi_0, \end{aligned}$$

and

$$\begin{aligned} y'(t) &= g(\rho(y_t)) \qquad \text{for } t \ge t_0, \\ y_{t_0} &= \psi_0, \end{aligned}$$

respectively. Define $\delta \ge 0$ by

$$\delta = \sup\{|f(\varphi) - g(\varphi)| : \varphi \in K(-B, A, M, R)\}.$$

By Lemma 1.1 the restriction of f to $K(-B, A, M, R)$ is a Lipschitz map; let C denote its Lipschitz constant, that is,

$$|f(\theta_1) - f(\theta_2)| \le C\|\theta_1 - \theta_2\| \qquad \text{for all } \theta_1, \theta_2 \in K(-B, A, M, R).$$

Then for all $t \geq t_0$ we have

$$|y(t) - x(t)| \leq e^{C(t-t_0)}\|\varphi_0 - \psi_0\| + (e^{C(t-t_0)} - 1)\delta/C. \tag{1.15}$$

PROOF. Our assumptions imply that $|x'(t)| \leq R$ and $|y'(t)| \leq R$ for all $t \geq t_0$, so we have $\rho(x_t), \rho(y_t) \in K(-B, A, M, R)$ for all $t \geq t_0$. If we define $z(t) = |y(t) - x(t)|$, we obtain for $t \geq t_0$,

$$\begin{aligned} z(t) &\leq z(t_0) + \int_{t_0}^{t} |g(\rho(y_s)) - f(\rho(x_s))|\, ds \\ &\leq z(t_0) + \int_{t_0}^{t} |g(\rho(y_s)) - f(\rho(y_s))|\, ds + \int_{t_0}^{t} |f(\rho(y_s)) - f(\rho(x_s))|\, ds \\ &\leq z(t_0) + \delta(t - t_0) + C\int_{t_0}^{t} \|y_s - x_s\|\, ds. \end{aligned} \tag{1.16}$$

For $\tau \geq t_0$, define $\zeta(\tau)$ by

$$\zeta(\tau) = \max\{|z(t)| : t_0 - M \leq t \leq \tau\}.$$

Since $\|y_s - x_s\| \leq \zeta(s)$ we deduce from (1.16) that for $t_0 \leq t \leq \tau$,

$$\begin{aligned} z(t) &\leq z(t_0) + \delta(t - t_0) + C\int_{t_0}^{t} \zeta(s)\, ds \\ &\leq z(t_0) + \delta(\tau - t_0) + C\int_{t_0}^{\tau} \zeta(s)\, ds. \end{aligned}$$

For $t_0 - M \leq t \leq t_0$ we have $z(t) \leq \|\psi_0 - \varphi_0\|$, so we conclude that

$$\zeta(\tau) = \max_{t_0 - M \leq t \leq \tau} z(t) \leq \|\varphi_0 - \psi_0\| + \delta(\tau - t_0) + C\int_{t_0}^{\tau} \zeta(s)\, ds. \tag{1.17}$$

Writing

$$I(\tau) = \int_{t_0}^{\tau} \zeta(s)\, ds,$$

we see that (1.17) gives

$$I'(\tau) - CI(\tau) \leq \|\varphi_0 - \psi_0\| + \delta(\tau - t_0). \tag{1.18}$$

By using (1.18) to estimate $I(\tau)$ or by applying Gronwall's inequality to (1.17) we find

$$\zeta(t) \leq e^{C(t-t_0)}\|\varphi_0 - \psi_0\| + (e^{C(t-t_0)} - 1)\delta/C,$$

which implies (1.15). □

2. The operator of translation along trajectories

Throughout this section we assume A, B and M are positive numbers, and that $f : K(-B, A, M) \to \mathbb{R}$ is a map. As in Section 1, $\rho : X_M \to K(-B, A, M)$ and $r : \mathbb{R} \to [-B, A]$ will always be as given by equations (1.3) and (1.4). We shall suppose that f satisfies one or more of the following hypotheses. (See equation (1.9) for the definitions of $K(-B, A, M)$ and $K(-B, A, M, R)$).

H2.1. $f : K(-B, A, M) \subset X_M \to \mathbb{R}$ is continuous and almost locally Lipschitzian (see Definition 1.1). In addition f is bounded on $K(-B, A, M)$, so there is a constant L with $|f(\varphi)| \leq L$ for all $\varphi \in K(-B, A, M)$.

H2.2. $f : K(-B, A, M) \to \mathbb{R}$ satisfies a negative feedback condition (see Definition 1.2). In addition there exist $\tau_0 \in (0, M]$, and a locally Lipschitz map $g : \mathbb{R} \to \mathbb{R}$ such that $\xi g(\xi) < 0$ for all $\xi \neq 0$ and

$$f(\varphi) = g(\varphi(-\tau_0))$$

for all $\varphi \in K(-B, A, M)$ with $\varphi(0) = 0$.

We shall work with subsets $G^+(-B, A, M, \tau_0) \subset K(-B, A, M)$ and $G^+(-B, A, M, \tau_0, R) \subset K(-B, A, M, R)$ defined as follows. Let

$$G^+(-B, A, M, \tau_0) = \{\varphi \in K(-B, A, M) : \varphi(t) \geq 0 \quad \text{for } -\tau_0 \leq t \leq 0 \text{ and } \varphi(0) = 0\} \tag{2.1}$$

and

$$G^+(-B, A, M, \tau_0, R) = \{\varphi \in G^+(-B, A, M, \tau_0) : \text{lip}(\varphi) \leq R\}. \tag{2.2}$$

Analogously, we define $G^-(-B, A, M, \tau_0)$ and $G^-(-B, A, M, \tau_0, R)$, corresponding to functions φ with $\varphi(t) \leq 0$ for $-\tau_0 \leq t \leq 0$. In our further work, τ_0 will always be a fixed number with $-M \leq \tau_0 < 0$, and τ_0 will be as in H2.2 if H2.2 is assumed to hold.

We shall need a boundedness condition for solutions of equation (1.1).

H2.3. The function $f : K(-B, A, M) \to \mathbb{R}$ satisfies H2.1 and $\tau_0 \in (0, M]$ is a given number. For every $\varphi \in G^+(-B, A, M, \tau_0)$ with $\text{lip}(\varphi) < \infty$, the unique solution $x(t)$ of

$$x'(t) = f(x_t) \qquad \text{for } t \geq t_0,$$
$$x_{t_0} = \varphi,$$

satisfies $-B \le x(t) \le A$ for all $t \ge t_0$.

There is a sharpening of H2.3 which is actually useful in some applications.

H2.3A. The function $f : K(-B,A,M) \to \mathbb{R}$ satisfies H2.1 and $\tau_0 \in (0,M]$ is a given number. There exists $R > 0$ such that for every $\varphi \in G^+(-B,A,M,\tau_0,R)$, the unique solution $x(t)$ of

$$x'(t) = f(x_t) \qquad \text{for } t \ge t_0,$$
$$x_{t_0} = \varphi,$$

satisfies $-B \le x(t) \le A$ and $|x'(t)| \le R$ for all $t \ge t_0$.

Theorems 1.3, 1.4 and 1.5 give examples where H2.3 is satisfied.

If $0 < \tau_0 \le M$ and $0 \le \alpha \le 1$, we define a map $Q_\alpha : X_M \to X_M$ by

(2.3)
$$(Q_\alpha(\varphi))(t) = \begin{cases} \varphi(t) & \text{if } -(1-\alpha)M - \alpha\tau_0 \le t \le 0, \\ \varphi(-(1-\alpha)M - \alpha\tau_0) & \text{if } -M \le t \le -(1-\alpha)M - \alpha\tau_0. \end{cases}$$

We leave it to the reader to check that $Q_\alpha : X_M \to X_M$ is a bounded linear projection of norm one for each α, that $Q_\alpha(K(-B,A,M)) \subset K(-B,A,M)$ and $Q_\alpha(K(-B,A,M,R)) \subset K(-B,A,M,R)$, and that Q_0 is the identity map.

If $f : K(-B,A,M) \to \mathbb{R}$ we shall define $F_\alpha : K(-B,A,M) \to \mathbb{R}$, for $0 \le \alpha \le 1$, by

$$F_\alpha(\varphi) := F(\varphi,\alpha) := f(Q_\alpha(\varphi)). \tag{2.4}$$

LEMMA 2.1. *Assume that $f : K(-B,A,M) \to \mathbb{R}$ and that F_α and $F : K(-B,A,M) \times [0,1] \to \mathbb{R}$ are defined by equation* (2.4). *If f is continuous, then F is continuous; and if f satisfies a negative feedback condition, then F_α satisfies a negative feedback condition for $0 \le \alpha \le 1$. If f is almost locally Lipschitzian, then the maps F_α, for $0 \le \alpha \le 1$, are almost locally Lipschitzian, uniformly in α, that is, for each $R > 0$, there exists a constant k_R with*

$$|F_\alpha(\theta) - F_\alpha(\varphi)| \le k_R\|\theta - \varphi\|$$

for all $\theta, \varphi \in K(-B,A,M,R)$ and all α satisfying $0 \le \alpha \le 1$. If f satisfies the hypotheses of Theorem 1.3, *Theorem* 1.4 *or Theorem* 1.5, *then, for $0 \le \alpha \le 1$, the function F_α satisfies the hypotheses of, respectively, Theorem* 1.3, 1.4 *or* 1.5.

PROOF. Assume that f is continuous. The composition of continuous functions is continuous, so to prove that F is continuous, it suffices to prove that

$(\varphi, \alpha) \to Q_\alpha(\varphi)$ is continuous. If $\alpha_k \to \alpha$ and $\varphi_k \to \varphi$ in X_M, we obtain (since $\|Q_\alpha\| = 1$)

$$\begin{aligned}\|Q_{\alpha_k}(\varphi_k) - Q_\alpha(\varphi)\| &\le \|Q_{\alpha_k}(\varphi_k - \varphi)\| + \|(Q_{\alpha_k} - Q_\alpha)(\varphi)\| \\ &\le \|\varphi_k - \varphi\| + \sup\{|\varphi(s) - \varphi(t)| \,:\, |s-t| \le |\alpha_k - \alpha|(M - \tau_0)\}.\end{aligned}$$

This estimate implies that $Q_{\alpha_k}(\varphi_k) \to Q_\alpha(\varphi)$ in X_M.

To prove that F_α satisfies a negative feedback condition if f does, it suffices to observe that if $\varphi \ge 0$ or $\varphi > 0$ (respectively, $\varphi \le 0$ or $\varphi < 0$) then $\theta \ge 0$ or $\theta > 0$ (respectively, $\theta \le 0$ or $\theta < 0$), where $\theta = Q_\alpha(\varphi)$ and $0 \le \alpha \le 1$.

Assume next that f is almost locally Lipschitzian. Observe that if $\varphi \in K(-B, A, M)$ and $\mathrm{lip}(\varphi) < \infty$, then

$$\mathrm{lip}(Q_\alpha(\varphi)) \le \mathrm{lip}(\varphi). \tag{2.5}$$

Lemma 1.1 implies that there exists a constant $k = k_R$ such that for all $\varphi, \psi \in K(-B, A, M, R)$ we have

$$|f(\varphi) - f(\psi)| \le k\|\varphi - \psi\|. \tag{2.6}$$

Combining (2.5) and (2.6) we conclude that for all $\varphi, \psi \in K(-B, A, M, R)$,

$$\begin{aligned}|F_\alpha(\varphi) - F_\alpha(\psi)| &= |f(Q_\alpha(\varphi)) - f(Q_\alpha(\psi))| \\ &\le k\|Q_\alpha(\varphi) - Q_\alpha(\psi)\| \le k\|\varphi - \psi\|,\end{aligned}$$

which is the desired estimate.

Our definition of $Q_\alpha(\varphi)$ implies that for $0 \le \alpha \le 1$,

$$(Q_\alpha(\varphi))(0) = \varphi(0) \qquad \text{and} \qquad (Q_\alpha(\varphi))(-\tau_0) = \varphi(-\tau_0).$$

If f satisfies H2.2 and $\varphi \in K(-B, A, M)$ and $\varphi(0) = 0$, it follows that

$$F_\alpha(\varphi) = f(Q_\alpha(\varphi)) = g((Q_\alpha(\varphi))(-\tau_0)) = g(\varphi(-\tau_0)),$$

so F_α also satisfies H2.2.

If $f : K(-B, A, M) \to \mathbb{R}$ is continuous or almost locally Lipschitzian or satisfies a negative feedback condition, then we have seen that F_α is respectively continuous or almost locally Lipschitzian or satisfies a negative feedback condition. If $f(\varphi) \le 0$ (respectively, $f(\varphi) \ge 0$) whenever $\varphi \in K(-B, A, M)$ and $\varphi(0) = A$ (respectively, $\varphi(0) = -B$), then, because $(Q_\alpha\varphi)(0) = \varphi(0)$, the same condition is satisfied by F_α. Obviously, if f is bounded on $K(-B, A, M)$, then F_α is bounded on $K(-B, A, M)$. Thus, if f satisfies the hypotheses of Theorem 1.3, then F_α also satisfies the hypotheses of Theorem 1.3.

The verification that F_α satisfies the hypotheses of Theorem 1.4 if

$$f : D = \{\varphi \in X_M \,:\, \varphi \geq -B\} = K(-B, \infty, M) \to \mathbb{R}$$

satisfies the hypotheses of Theorem 1.4 is equally straightforward and is left to the reader. Similarly, the verification that F_α satisfies Theorem 1.5 if f does is also left to the reader. □

If $f : K(-B, A, M) \to \mathbb{R}$ and F_α, for $0 \leq \alpha \leq 1$, is defined by equation (2.4), we can extend F_α to a map on X_M by

$$F_\alpha(\varphi) := F_\alpha(\rho(\varphi)).$$

For notational convenience, if $\varphi \in X_M$ and $\mathrm{lip}(\varphi) < \infty$, we shall define $x(t) = x(t; \varphi, \alpha)$ to be the unique solution of

$$\begin{aligned} x'(t) &= F_\alpha(x_t) = f(Q_\alpha(\rho(x_t))) && \text{for } t \geq 0, \\ x(t) &= \varphi(t) && \text{for } -M \leq t \leq 0. \end{aligned} \tag{2.7}$$

Theorems 1.1 and 1.2 and Lemma 2.1 imply that the solution $x(t; \varphi, \alpha)$ of equation (2.7) is unique and is defined for all $t \geq 0$.

If f satisfies the hypotheses of Theorem 1.4, so f maps $D = \{\varphi \in X_M \,:\, \varphi \geq -B\}$ into $\mathbb{R}$, there is some ambiguity about how F_α should be "extended" to X_M. In the case of Theorem 1.4, let A be as defined in Theorem 1.4, let ρ be the standard retraction onto $K(-B, A, M)$ and define

$$F_\alpha(\varphi) = f(Q_\alpha(\rho(\varphi))).$$

Since $\widetilde{F}_\alpha(\varphi) = f(Q_\alpha(\varphi))$ is defined for all $\varphi \in D$, the function F_α is, in general, an extension of $\widetilde{F}_\alpha | K(-B, A, M)$ but not of $\widetilde{F}_\alpha : D \to \mathbb{R}$.

If f satisfies H2.3, then for every $\varphi \in G^+(-B, A, M, \tau_0)$ with $\mathrm{lip}(\varphi) < \infty$ we have $-B \leq x(t; \varphi, 0) \leq A$ for all $t \geq 0$. Note, however, that we do not necessarily know that $-B \leq x(t; \varphi, \alpha) \leq A$ for $0 < \alpha \leq 1$. If f satisfies the hypotheses of Theorem 1.3, 1.4 or 1.5 and $\varphi \in G^+(-B, A, M, \tau_0)$ with $\mathrm{lip}(\varphi) < \infty$, Lemma 2.1 implies that $-B \leq x(t; \varphi, \alpha) \leq A$ for $0 \leq \alpha \leq 1$.

Our next lemma provides information about the dependence of $x(t; \varphi, \alpha)$ on (t, φ, α).

Lemma 2.2. *Assume that f satisfies* H2.1 *and let $x(t; \varphi, \alpha)$ denote the solution of equation* (2.7). *Given $\varepsilon > 0, T > 0$ and $R > 0$, there exists $\delta > 0$ such that*

$$\sup_{0 \leq t \leq T} |x(t; \varphi_1, \alpha_1) - x(t; \varphi_2, \alpha_2)| < \varepsilon$$

for all $\varphi_1, \varphi_2 \in K(-B, A, M, R)$ *with* $\|\varphi_1 - \varphi_2\| < \delta$ *and all* $\alpha_1, \alpha_2 \in [0, 1]$ *with* $|\alpha_1 - \alpha_2| < \delta$.

PROOF. By virtue of Lemma 2.1, we know that F_α is almost locally Lipschitzian, uniformly in α. Thus, with the aid of Theorem 1.6, we see that it suffices to prove that, given any $\eta > 0$, there exists $\delta_1 > 0$ with

$$\text{(2.8)} \quad \sup\{|F(\varphi, \alpha) - F(\varphi, \beta)| : \varphi \in K(-B, A, M, R), \\ \text{with } |\beta - \alpha| \leq \delta_1 \text{ for } \alpha, \beta \in [0, 1]\} < \eta.$$

However, the Ascoli-Arzelà theorem implies that $K(-B, A, M, R)$ is a compact subset of X_M, and we conclude from Lemma 2.1 that $(\varphi, \alpha) \to F(\varphi, \alpha)$ is continuous on the compact set $K(-B, A, M, R) \times [0, 1]$. Inequality (2.8) now follows from the fact that a continuous map on a compact metric space is uniformly continuous. □

Our next lemma will be crucial for all our remaining work. Basically, it asserts that, if f satisfies H2.1 and H2.2 and $\varphi \in G^+(-B, A, M, \tau_0)$, with $\text{lip}(\varphi) < \infty$ and $\varphi(t) > 0$ for some $t \in [-\tau_0, 0]$, then successive zeros of $x(t; \varphi, \alpha)$ for $t > 0$ are separated by more than τ_0. The precise statement of the lemma accounts for the possibility that there exists $\sigma_0 > 0$ with $x(t; \varphi, \alpha) = 0$ for all t with $0 \leq t \leq \sigma_0$.

For convenience we introduce some further notation before stating Lemma 2.3; let

$$U^+(-B, A, M, \tau_0) = \{\varphi \in G^+(-B, A, M, \tau_0) : \varphi(t) > 0 \text{ for some } t \in [-\tau_0, 0)\}$$

and

$$\text{(2.9)} \qquad U^+(-B, A, M, \tau_0, R) = \{\varphi \in U^+(-B, A, M, \tau_0) : \text{lip}(\varphi) \leq R\}.$$

LEMMA 2.3. *Assume that* $h : K(-B, A, M) \to \mathbb{R}$ *satisfies* H2.1 *and* H2.2 *and that* $\rho : X_M \to K(-B, A, M)$ *is given by equations* (1.3) *and* (1.4). *Assume that* $\varphi \in X_M$, *with* $\text{lip}(\varphi) < \infty$ *and* $\rho(\varphi) \in U^+(-B, A, M, \tau_0)$, *and let* $x(t)$ *denote the unique solution of*

$$x'(t) = h(\rho(x_t)) \qquad \textit{for } t \geq 0,$$
$$x|[-M, 0] = \varphi.$$

Define

$$\sigma_0 = \sup\{s \geq 0 : x(t) = 0 \textit{ for all } t \in [0, s]\}.$$

Then it follows that $\sigma_0 < \tau_0$ *and* $x(t) < 0$ *for* $\sigma_0 < t \leq \sigma_0 + \tau_0$. *Define* $z_1 = \inf\{t > \sigma_0 : x(t) = 0\}$ *and* $z_1 = \infty$ *if* $x(t) < 0$ *for all* $t > \sigma_0$. *If* $z_1 < \infty$

it follows that $x'(z_1) > 0$ *and* $x(t) > 0$ *for* $z_1 < t \le z_1 + \tau_0$. *If we define* $z_2 = \inf\{t > z_1 : x(t) = 0\}$ *and we define* $z_2 = \infty$ *if* $x(t) > 0$ *for all* $t > z_1$, *it follows that* $x'(z_2) < 0$ *if* $z_2 < \infty$.

PROOF. Let g be as in H2.2, so

$$h(\psi) = g(\psi(-\tau_0))$$

for all $\psi \in K(-B, A, M)$ with $\psi(0) = 0$. It follows that

$$x'(t) = g(r(x(t - \tau_0))) = 0 \qquad \text{for } 0 \le t \le \sigma_0. \tag{2.10}$$

Because $\xi g(\xi) < 0$ for all $\xi \neq 0$, we conclude from equation (2.10) that

$$x(t - \tau_0) = \varphi(t - \tau_0) = 0 \qquad \text{for } 0 \le t \le \sigma_0. \tag{2.11}$$

If $\sigma_0 \ge \tau_0$, (2.11) implies that $\varphi(s) = 0$ for $-\tau_0 \le t \le 0$, which contradicts the assumption that $\rho(\varphi) \in U^+(-B, A, M, \tau_0)$.

To proceed further we need a simple estimate on h. Select $R \ge 1$ so that $\operatorname{lip}(\varphi) \le R$ and $|h(\psi)| \le R$ for all $\psi \in K(-B, A, M)$. Define $\psi_1 \in X_M$ by

$$\psi_1(t) = \begin{cases} \tau_0^{-1}(t + \tau_0) & \text{for } -\tau_0 \le t \le 0, \\ 0 & \text{for } -M \le t \le -\tau_0. \end{cases}$$

For $\psi \in X_M$ with $\operatorname{lip}(\psi) \le R$ we have $\rho(\psi) \in K(-B, A, M, R)$ and

$$h(\rho(\psi)) = h(\rho(\psi)) - h(\rho(\psi - \psi(0)\psi_1)) + h(\rho(\psi - \psi(0)\psi_1)). \tag{2.12}$$

If we define $\theta = \rho(\psi - \psi(0)\psi_1)$, we note that $\theta(0) = 0$, and $\theta(-\tau_0) = r(\psi(-\tau_0))$, and so

$$\begin{aligned} \operatorname{lip}(\theta) &\le \operatorname{lip}(\psi - \psi(0)\psi_1) \le \operatorname{lip}(\psi) + |\psi(0)|\operatorname{lip}(\psi_1) \\ &\le R + \tau_0^{-1} \max\{A, B\} = R_1. \end{aligned} \tag{2.13}$$

It follows from (2.13) and the fact that h is almost locally Lipschitzian that there is a constant $k_{R_1} = k$ such that for all $\psi \in X_M$ with $\operatorname{lip}(\psi) \le R$,

$$\begin{aligned} |h(\rho(\psi)) - &h(\rho(\psi - \psi(0)\psi_1))| \\ &\le k\|\rho(\psi) - \rho(\psi - \psi(0)\psi_1)\| \\ &\le k\|\psi - (\psi - \psi(0)\psi_1)\| = k|\psi(0)|. \end{aligned} \tag{2.14}$$

Also, H2.2 implies that

$$h(\rho(\psi - \psi(0)\psi_1)) = g(r(\psi(-\tau_0))). \tag{2.15}$$

Using (2.14) and (2.15) in (2.12) we see that there is a constant k (depending on R) such that

$$(2.16)\qquad -k|\psi(0)| + g(r(\psi(-\tau_0))) \le h(\rho(\psi)) \le k|\psi(0)| + g(r(\psi(\tau_0)))$$

for all $\psi \in X_M$ with $\text{lip}(\psi) \le R$. Notice that, if h satisfies H2.1 and H2.2, the constant k in equation (2.16) can be any number such that h is Lipschitz with Lipschitz constant k on $K(-B, A, M, R_1)$, with $R_1 = R + \tau_0^{-1}\max\{A, B\}$.

We now apply equation (2.16). Select R so that $\text{lip}(\varphi) \le R$ and $|h(\psi)| \le R$ for all $\psi \in K(-B, A, M)$. It then follows that $\text{lip}(x_t) \le R$ for all $t \ge 0$ and (using (2.16)) that there is a constant k with

$$(2.17)\qquad -k|x(t)| + g(r(x(t-\tau_0))) \le h(\rho(x_t)) \le k|x(t)| + g(r(x(t-\tau_0))).$$

Because $g(\xi) \le 0$ for $\xi \ge 0$, we obtain from (2.17) that

$$(2.18)\qquad h(\rho(x_t)) \le k|x(t)| \qquad \text{for } \sigma_0 \le t \le \sigma_0 + \tau_0.$$

We claim that $x(t) \le 0$ for $\sigma_0 \le t \le \sigma_0 + \tau_0$. If not, there exist t_2 and t_1 with $\sigma_0 \le t_1 < t_2 \le \sigma_0 + \tau_0$, where $x(t_1) = 0$ and $x(t) > 0$ for $t_1 < t \le t_2$. Using (2.18) we see that

$$\frac{d}{dt}(\exp(-k(t-t_1))x(t)) \le 0,$$

which implies that

$$\exp(-k(t_2-t_1))x(t_2) \le x(t_1) = 0,$$

a contradiction.

By definition of σ_0, there exists a sequence $\{t_j\}$, with $t_j > \sigma_0$ and $\lim_{j\to\infty} t_j = \sigma_0$, and with $x(t_j) \ne 0$. Because we know that $x(t) \le 0$ on $[\sigma_0, \sigma_0 + \tau_0]$, we can assume that $x(t_j) < 0$. On the interval $[t_j, \sigma_0 + \tau_0]$, (2.18) gives

$$\frac{d}{dt}(\exp(k(t-t_j))x(t)) \le 0,$$

which implies that

$$x(t) \le \exp(-k(t-t_j))x(t_j) < 0 \qquad \text{for } t_j \le t \le \sigma_0 + \tau_0.$$

Allowing t_j to approach σ_0 yields $x(t) < 0$ for $\sigma_0 < t \le \sigma_0 + \tau_0$.

If $z_1 < \infty$, with z_1 as in the statement of Lemma 2.3, we see that

$$x'(z_1) = h(\rho(x_{z_1})) = g(r(x(z_1 - \tau_0))) > 0,$$

because $x(z_1 - \tau_0) < 0$.

The remainder of the lemma follows by arguments exactly like those already given. The details are left to the reader. □

In the terminology of Lemma 2.3, we need precise information about $\|x_t\|$ when t is large and $z_1 = \infty$ or $z_2 = \infty$. In our next lemma we prove that in fact, if $z_1 = \infty$ or $z_2 = \infty$, then $\lim_{t\to\infty}\|x_t\| = 0$.

LEMMA 2.4. *Assume that $h : K(-B, A, M) \to \mathbb{R}$ satisfies* H2.1 *and* H2.2. *Let g be as in* H2.2 *and select R so that $R \geq |h(\varphi)|$ for all $\varphi \in K(-B, A, M)$. If $R_1 = R + \tau_0^{-1}\max\{A, B\}$ and if k is such that h is Lipschitzian on $K(-B, A, M, R_1)$ with Lipschitz constant k, then*

$$-k|\varphi(0)| + g(r(\varphi(-\tau_0))) \leq h(\rho(\varphi)) \leq k|\varphi(0)| + g(r(\varphi(-\tau_0))) \tag{2.19}$$

for all $\varphi \in X_M$ with $\operatorname{lip}(\varphi) \leq R$. Because g is locally Lipschitzian, there exists C with

$$|g(\xi)| \leq C|\xi| \qquad \text{for } \xi \in [-B, A].$$

If $\varphi \in X_M$, $\operatorname{lip}(\varphi) \leq R$ and $\rho(\varphi) \in U^+(-B, A, M, \tau_0)$, and if $x(t) = x(t;\varphi)$ denotes the solution of

$$x'(t) = h(\rho(x_t)) \qquad \text{for } t \geq 0,$$
$$x_0 = \varphi,$$

and $\sigma_0 = \sigma_0(\varphi)$, and $z_1 = z_1(\varphi)$ and $z_2 = z_2(\varphi)$, are defined as in Lemma 2.3, then

$$x(t) \geq -Ck^{-1}(e^{kM} - e^{k(M-\tau_0)}) \sup_{-\tau_0 \leq s \leq 0} r(\varphi(s)) \qquad \text{for } 0 \leq t \leq z_1 \tag{2.20}$$

and

$$x(t) \leq Ck^{-1}(e^{kM} - e^{k(M-\tau_0)}) \sup_{z_1-\tau_0 \leq s \leq z_1} |r(x(s))| \qquad \text{for } z_1 \leq t \leq z_2. \tag{2.21}$$

For every $\delta > 0$, there exists $T = T(\delta, R, h)$ such that if $\rho(\varphi) \in U^+(-B, A, M, \tau_0)$ for some $\varphi \in X_M$ satisfying $\operatorname{lip}(\varphi) \leq R$, and if $z_1(\varphi) > T$ (respectively, $z_2(\varphi) - z_1(\varphi) > T$), then $|x(t)| < \delta$ for $T \leq t \leq z_1$ (respectively, for $z_1 + T \leq t \leq z_2$).

PROOF. Inequality (2.19) was obtained in the proof of Lemma 2.3. For $\sigma_0 \leq t \leq \sigma_0 + \tau_0$ we have

$$\begin{aligned} x'(t) = h(\rho(x_t)) &\geq kr(x(t)) + g(r(x(t-\tau_0))) \\ &\geq kx(t) - C\sup\{r(\varphi(s)) : -\tau_0 \leq s \leq 0\}, \end{aligned}$$

where $r : \mathbb{R} \to [-B, A]$ is the retraction in equation (1.3). It follows, writing

$$\|\psi\|_{\tau_0} = \sup\{|\psi(s)| : -\tau_0 \leq s \leq 0\} \qquad \text{for } \psi \in X_M,$$

that we have

$$\frac{d}{dt}(\exp(-k(t-\sigma_0)x(t))) \geq -C\|\rho(\varphi)\|_{\tau_0}\exp(-k(t-\sigma_0)),$$

which implies that

$$0 \geq x(t) \geq -C\|\rho(\varphi)\|_{\tau_0}k^{-1}(e^{k(t-\sigma_0)}-1) \qquad \text{for } \sigma_0 \leq t \leq \sigma_0+\tau_0.$$

If $\sigma_0+\tau_0 \leq t \leq \min\{\sigma_0+M, z_1\}$, we obtain from (2.19) that

$$x'(t) \geq kr(x(t)) + g(r(x(t-\tau_0))) \geq kx(t),$$

which implies that for $\sigma_0+\tau_0 \leq t \leq \min\{\sigma_0+M, z_1\}$ we have

$$x(t) \geq e^{k(t-\sigma_0-\tau_0)}x(\sigma_0+\tau_0) \geq -C\|\rho(\varphi)\|_{\tau_0}k^{-1}(e^{k(t-\sigma_0)}-e^{k(t-\sigma_0-\tau_0)}).$$

A simple calculus exercise now implies that

$$x(t) \geq -C\|\rho(\varphi)\|_{\tau_0}k^{-1}(e^{kM}-e^{k(M-\tau_0)}) \qquad \text{for } 0 \leq t \leq \min\{\sigma_0+M, z_1\}.$$

If $\sigma_0+M < z_1$, the negative feedback condition on h implies that $x'(t) > 0$ for $\sigma_0+M < t < z_1$, so we have established (2.20) for $0 \leq t \leq z_1$.

The proof of equation (2.21) is analogous and is left to the reader.

To define $T(\delta, R, h)$, select $\delta > 0$ and define

$$S_\delta = \{\varphi \in K(-B, A, M, R) : |\varphi(t)| \geq \delta \text{ for all } t \in [-M, 0]\}.$$

The set S_δ is a closed subset of the compact set $K(-B, A, M, R)$ and hence is compact. The negative feedback condition on h implies that $h(\varphi) \neq 0$ for $\varphi \in S_\delta$; and using the compactness of S_δ, we see that there exists $c_1 > 0$ with $h(\varphi) \leq -c_1$ for all $\varphi \in S_\delta$ with $\varphi > 0$ and $h(\varphi) \geq c_1$ for all $\varphi \in S_\delta$ with $\varphi < 0$.

If $\varphi \in X_M$, with $\operatorname{lip}(\varphi) \leq R$ and $\rho(\varphi) \in U^+(-B, A, M, \tau_0)$, and if $z_1 = z_1(\varphi) > \sigma_0(\varphi) + 2M$ and $x(\sigma_0+2M) < -\delta$, define τ to be the first time $t \geq \sigma_0 + 2M$ with $x(t) = -\delta$ (*a priori*, τ may equal ∞). Since x is increasing on $[\sigma_0+M, z_1]$, we see that $\rho(x_t) \in S_\delta$ for $\sigma_0+2M \leq t \leq \tau$. It follows that

$$x'(t) = h(\rho(x_t)) \geq c_1 \qquad \text{for } \sigma_0+2M \leq t \leq \tau,$$

and because $x(\sigma_0+2M) \geq -RM$, we conclude that

$$\tau \leq \sigma_0 + 2M + (RM-\delta)c_1^{-1} < \tau_0 + 2M + (RM-\delta)c_1^{-1} \equiv T_1$$

and $|x(t)| < \delta$ for $T_1 < t \leq z_1$. A similar argument (left to the reader) shows that if

$$z_2 - z_1 > T_2 \equiv 2M + (RM-\delta)c_1^{-1},$$

then $|x(t)| < \delta$ for $z_1+T_2 < t \leq z_2$. If we define $T = T_1$, we obtain the statement of the lemma. □

With the aid of Lemmas 2.1–2.4 we can define the operator of translation along trajectories. Assume that $f : K(-B,A,M) \to \mathbb{R}$ satisfies H2.1 and H2.2 and recall that $x(t;\varphi,\alpha)$ denotes a solution of equation (2.7). Let R be such that

$$R \geq \sup\{|f(\varphi)| : \varphi \in K(-B,A,M)\}. \tag{2.22}$$

For $\varphi \in X_M$ with $\operatorname{lip}(\varphi) < \infty$ and $\rho(\varphi) \in U^+(-B,A,M,\tau_0)$, as in Lemma 2.3 we define $\sigma_0 = \sigma_0(\varphi,\alpha)$, and $z_1 = z_1(\varphi,\alpha)$ and $z_2 = z_2(\varphi,\alpha)$, as

$$\sigma_0(\varphi,\alpha) = \sup\{s \geq 0 : x(t;\varphi,\alpha) = 0 \text{ for all } t \in [0,s]\},$$

and

$$\begin{aligned} z_1(\varphi,\alpha) &= \inf\{t > \sigma_0(\varphi,\alpha) : x(t;\varphi,\alpha) = 0\}, \\ z_2(\varphi,\alpha) &= \inf\{t > z_1(\varphi,\alpha) : x(t;\varphi,\alpha) = 0\}. \end{aligned} \tag{2.23}$$

We define $z_1 = \infty$ if $x(t,\varphi,\alpha) < 0$ for all $t > \sigma_0$, and $z_2 = \infty$ if $z_1 = \infty$ or if $x(t;\varphi,\alpha) > 0$ for all $t > z_1$. Lemma 2.3 implies that $\sigma_0 < \tau_0$ and $z_1 - \sigma_0 > \tau_0$, and that $z_2 - z_1 > \tau_0$ if $z_1 < \infty$. Furthermore, $x'(z_1) > 0$ if $z_1 < \infty$ and $x'(z_2) < 0$ if $z_2 < \infty$. We define a map $\widetilde{\Gamma} : U^+(-B,A,M,\tau_0,R) \times [0,1] \to X_M$ by

$$\widetilde{\Gamma}(\varphi,\alpha) := \widetilde{\Gamma}_\alpha(\varphi) := \begin{cases} x_{z_2} & \text{if } z_2 = z_2(\varphi,\alpha) < \infty, \\ 0 & \text{if } z_2(\varphi,\alpha) = \infty. \end{cases} \tag{2.24}$$

We define $\Gamma : U^+(-B,A,M,\tau_0,R) \times [0,1] \to X_M$ by

$$\Gamma(\varphi,\alpha) := \Gamma_\alpha(\varphi) := Q_\alpha(\widetilde{\Gamma}(\varphi,\alpha)), \tag{2.25}$$

where Q_α is defined in equation (2.3) for $0 \leq \alpha \leq 1$.

THEOREM 2.1. *Assume that f satisfies* H2.1 *and* H2.2 *and that R satisfies equation* (2.22). *If Γ is defined by equation* (2.25) *and we write $U^+ = U^+(-B,A,M,\tau_0,R)$, then Γ is a continuous map of $U^+ \times [0,1]$ into X_M. If $\Gamma(\varphi,\alpha) = \varphi$ for $(\varphi,\alpha) \in U^+ \times [0,1]$ and $x(t) = x(t;\varphi,\alpha)$ denotes the solution of equation* (2.7) *for $t \geq -M$ and $z_2 = z_2(\varphi,\alpha)$ is given by equation* (2.23), *then $x(t+z_2) = x(t)$ for all $t \geq 0$. Furthermore, if $x(t)$ is defined for $t \leq 0$ by demanding that x be periodic of period z_2, it follows that* (*since $\rho Q_\alpha = Q_\alpha \rho$*)

$$x'(t) = f(\rho(Q_\alpha(x_t))) = f(Q_\alpha(\rho(x_t))) \qquad \textit{for all } t.$$

There exists $\delta > 0$ such that if $(\psi,\alpha) \in U^+ \times [0,1]$ and if $\psi(s) \leq \delta$ for $-\tau_0 \leq s \leq 0$, then $-B \leq x(t;\psi,\alpha) \leq A$ for $0 \leq t \leq z_2(\psi,\alpha)$ and $\Gamma(\psi,\alpha) \in$

$G^+(-B, A, M, \tau_0, R) = G^+$. *If f also satisfies* H2.3, *then* $\varphi \to \Gamma(\varphi, 0)$ *is a continuous map of* U^+ *into* G^+. *If f also satisfies the hypotheses of Theorem* 1.3, 1.4 *or* 1.5, *then* $\Gamma(U^+ \times [0,1]) \subset G^+$.

PROOF. To prove that Γ is continuous, it suffices to prove that $\widetilde{\Gamma}$ given by equation (2.24) is continuous. Indeed, suppose we have shown that $\widetilde{\Gamma}$ is continuous. Assume that $(\varphi_j, \alpha_j) \in U^+ \times [0,1]$ converges to (φ, α) in $X_M \times [0,1]$. If $\psi_j = \widetilde{\Gamma}(\varphi_j, \alpha_j)$ and $\psi = \widetilde{\Gamma}(\varphi, \alpha)$, we know that $\mathrm{lip}(\psi_j) \leq R$ and $\mathrm{lip}(\psi) \leq R$, and

$$\lim_{j \to \infty} \|\psi_j - \psi\| = 0.$$

It follows that

$$\begin{aligned}
\|\Gamma(\varphi_j, \alpha_j) - \Gamma(\varphi, \alpha)\| &= \|Q_{\alpha_j}(\psi_j) - Q_\alpha(\psi)\| \\
&\leq \|Q_{\alpha_j}(\psi_j - \psi)\| + \|Q_{\alpha_j}(\psi) - Q_\alpha(\psi)\| \\
&\leq \|\psi_j - \psi\| + R|\alpha_j - \alpha|,
\end{aligned}$$

which proves continuity of Γ.

To prove continuity of $\widetilde{\Gamma}$, suppose that $(\varphi_0, \alpha_0) \in U^+ \times [0,1]$ and that (φ_j, α_j), for $j \geq 1$, is a sequence in $U^+ \times [0,1]$ which converges to (φ_0, α_0). We have to show that $\widetilde{\Gamma}(\varphi_j, \alpha_j)$ converges to $\widetilde{\Gamma}(\varphi_0, \alpha_0)$. For notational convenience, we define (see equation (2.7)) $x_0(t) = x(t; \varphi_0, \alpha_0)$ and $x_j(t) = x(t; \varphi_j, \alpha_j)$ for $j \geq 1$, and $u_0 = z_1(\varphi_0, \alpha_0)$ and $v_0 = z_2(\varphi_0, \alpha_0)$, and finally $u_j = z_1(\varphi_j, \alpha_j)$ and $v_j = z_2(\varphi_j, \alpha_j)$ for $j \geq 1$ (see equation (2.23)).

We consider two cases: $v_0 < \infty$ and $v_0 = \infty$. If $v_0 < \infty$, select $\delta > 0$ with $\delta < u_0 - \tau_0$ and $\delta < \tau_0/2$ and note that Lemma 2.3 implies that $x_0(t) < 0$ for $\tau_0 \leq t \leq u_0 - \delta$, and that $x_0(t) > 0$ for $u_0 + \delta \leq t \leq v_0 - \delta$, and also $x_0(v_0 + \delta) < 0$. Lemma 2.2 implies that for all sufficiently large j we have $x_j(t) < 0$ for $\tau_0 \leq t \leq u_0 - \delta$, and $x_j(t) > 0$ for $u_0 + \delta \leq t \leq v_0 - \delta$, and also $x_j(v_0 + \delta) < 0$. We conclude with the aid of Lemma 2.3 that $|u_j - u_0| < \delta$ and $|v_j - v_0| < \delta$. We also obtain from Lemma 2.2 that for all sufficiently large j we have $|x_j(t) - x_0(t)| < \delta$ for $-M \leq t \leq v_0 + \delta$. If we recall that $\mathrm{lip}(x_j) \leq R$ and $\mathrm{lip}(x_0) \leq R$, we see that for all sufficiently large j,

$$\begin{aligned}
\|\widetilde{\Gamma}(\varphi_j, \alpha_j) - \widetilde{\Gamma}(\varphi_0, \alpha_0)\| &= \sup\{|x_j(v_j + t) - x_0(v_0 + t)| : -M \leq t \leq 0\} \\
&\leq \sup\{|x_j(v_0 + t) - x_0(v_0 + t)| : -M \leq t \leq 0\} \\
&\quad + \sup\{|x_j(v_j + t) - x_j(v_0 + t)| : -M \leq t \leq 0\} \\
&\leq \delta + R\delta.
\end{aligned}$$

Since $\delta > 0$ was arbitrary, we have proved that $\widetilde{\Gamma}$ is continuous at (φ_0, α_0) if $z_2(\varphi_0, \alpha_0) = v_0 < \infty$.

It remains to consider the case $v_0 = \infty$. Here we have two subcases: either (a) $u_0 < \infty$ and $v_0 = \infty$ or (b) $u_0 = \infty$. Given $\delta > 0$, there exists (by virtue of Lemma 2.4) $T > 0$ so that $|x(t; \varphi, \alpha)| < \delta$ for all $(\varphi, \alpha) \in U^+ \times [0, 1]$ and t such that $T \le t \le z_1(\varphi, \alpha)$ or $z_1(\varphi, \alpha) + T \le t \le z_2(\varphi, \alpha)$. If $u_0 < \infty$, the same arguments already used show that for all sufficiently large j we have $|u_j - u_0| < \delta$ and $x_j(t) > 0$ for $u_0 + \delta \le t \le u_0 + T + M + \delta$. It follows that for all sufficiently large j we have $v_j - u_j > T + M$ and

$$\|\widetilde{\Gamma}(\varphi_j, \alpha_j)\| = \sup\{|x_j(v_j + t)| : -M \le t \le 0\} < \delta.$$

(If $v_j = \infty$ then $\widetilde{\Gamma}(\varphi_j, \alpha_j) = 0$ by definition.) This shows that, in subcase (a), $\widetilde{\Gamma}(\varphi_j, \alpha_j)$ converges to $0 = \widetilde{\Gamma}(\varphi_0, \alpha_0)$.

In subcase (b), we see with the aid of Lemma 2.2 that for all sufficiently large j, we have $x_j(t) < 0$ for $\tau_0 \le t \le T + M$ and that $u_j > T + M$. (We can assume $v_j < \infty$, because $\widetilde{\Gamma}(\varphi_j, \alpha_j) = 0$ if $v_j = \infty$.) It follows from Lemma 2.4 that $|x_j(t)| < \delta$ for all sufficiently large j and for $u_j - M \le t \le u_j$, and Lemma 2.4 (see equation (2.21)) also implies that there is a constant C_1 with $|x_j(t)| < C_1\delta$ for all large j and for $u_j \le t \le v_j$. Since $\delta > 0$ is arbitrary, we conclude again that $\widetilde{\Gamma}(\varphi_j, \alpha_j)$ approaches 0 as $j \to \infty$, and this completes the proof of the continuity of $\widetilde{\Gamma}$.

Suppose next that $\varphi \in U^+$ and let ψ denote any Lipschitz function in X_M with

$$\psi(t) = \varphi(t) \qquad \text{for } -(1-\alpha)M - \alpha\tau_0 \le t \le 0.$$

Define $x(t) = x(t; \varphi, \alpha)$ and set $\widetilde{x}(t) = x(t)$ for $t \ge 0$ and $\widetilde{x}(t) = \psi(t)$ for $-M \le t \le 0$. It is easy to see that $Q_\alpha(\widetilde{x}_t) = Q_\alpha(x_t)$ for $t \ge 0$, so $\widetilde{x}_0 = \psi$ and

$$\widetilde{x}'(t) = f(\rho(Q_\alpha(\widetilde{x}_t))) \qquad \text{for } t \ge 0.$$

Using our uniqueness results for solutions of the initial value problem, we conclude that $\widetilde{x}(t) = x(t; \psi, \alpha)$ and that $x(t; \psi, \alpha) = x(t; \varphi, \alpha)$ for $t \ge 0$. If $\Gamma(\varphi, \alpha) = \varphi$ and we write $z_2 = z_2(\varphi, \alpha)$, we obtain from the definition of Γ that

$$x(t; \varphi, \alpha) = x(z_2 + t; \varphi, \alpha) \qquad \text{for } -(1-\alpha)M - \alpha\tau_0 \le t \le 0.$$

Define $\psi(t) = x(z_2 + t; \varphi, \alpha)$ for $-M \le t \le 0$, so $\psi(t) = \varphi(t)$ for $-(1-\alpha)M - \alpha\tau_0 \le t \le 0$, and define $y(t) = x(z_2 + t; \varphi, \alpha)$ for $-M \le t < \infty$. By construction we have that $y(t) = x(t; \psi, \alpha)$, so our previous remarks imply that

$$x(z_2 + t; \varphi, \alpha) = y(t) = x(t; \psi, \alpha) = x(t; \varphi, \alpha) \qquad \text{for } t \ge 0.$$

Clearly, if we extend $x(t;\varphi,\alpha)$ for all t by periodicity, the extended function $x(t;\varphi,\alpha)$ will still satisfy equation (2.7).

If we recall (see Lemma 2.1) that $F_\alpha(\varphi) \equiv f(Q_\alpha(\varphi))$ is Lipschitzian on $K(-B,A,M,R)$ with Lipschitz constant k, and that k can be chosen independent of α, in the range $0 \leq \alpha \leq 1$, and if we apply Lemma 2.4 to $h \equiv F_\alpha$, we see that equations (2.19), (2.20) and (2.21) are satisfied for $h = F_\alpha$. In particular, there is a constant C_1 such that for all $(\psi,\alpha) \in U^+ \times [0,1]$ we have

$$\sup\{|x(t;\psi,\alpha)| \,:\, 0 \leq t \leq z_2(\psi,\alpha)\} \leq C_1 \sup\{\psi(s) \,:\, -\tau_0 \leq s \leq 0\}.$$

In particular, if $\delta > 0$ is chosen so that $C_1\delta < \min\{A,B\}$, we obtain that $-B \leq x(t;\psi,\alpha) \leq A$ for $0 \leq t \leq z_2(\psi,\alpha)$ if $(\psi,\alpha) \in U^+ \times [0,1]$ and $\sup\{\psi(s) : -\tau_0 \leq s \leq 0\} \leq \delta$. We have arranged that $|x'(t;\psi,\alpha)| \leq R$ for all $t \geq 0$ and all $(\psi,\alpha) \in U^+ \times [0,1]$, so we conclude that $\Gamma(\psi,\alpha) \in G^+$ if $(\psi,\alpha) \in U^+ \times [0,1]$ and $\sup\{\psi(s) : -\tau_0 \leq s \leq 0\} \leq \delta$.

If f satisfies H2.3, then by definition we have that $-B \leq x(t;\psi,0) \leq A$ for all $t \geq 0$ and $\psi \in U^+$, so $\Gamma(\psi,0) \in G^+$. If f satisfies the hypotheses of Theorem 1.3, 1.4 or 1.5, then Lemma 2.1 implies that F_α satisfies the hypotheses of, respectively, Theorem 1.3, 1.4 or 1.5, so $-B \leq x(t;\psi,\alpha) \leq A$ for all $t \geq 0$ and $(\psi,\alpha) \in U^+ \times [0,1]$, and so $\Gamma(\psi,\alpha) \in G^+$. □

REMARK 2.1. It is unclear when, under the hypotheses of Theorem 2.1, Γ can be extended continuously to $G^+(-B,A,M,\tau_0,R) \times [0,1]$. If $\varphi_0 \in G^+$ and $\varphi_0(t) = 0$ for $-\tau_0 \leq t \leq 0$, it seems necessary (but perhaps not sufficient) for continuity of Γ at (φ_0,α_0) that $\Gamma(\varphi_0,\alpha_0) = 0$. However, if there exists a sequence $(\varphi_j,\alpha_j) \in U^+ \times [0,1]$ with

$$\lim_{j\to\infty} (\varphi_j,\alpha_j) = (\varphi_0,\alpha_0) \quad \text{and}$$
$$\lim_{j\to\infty} z_2(\varphi_j;\alpha_j) = \zeta < (1-\alpha_0)M - (1-\alpha_0)\tau_0,$$

then in general Γ will not be continuous at (φ_0,α_0). To see this, note that

$$(\Gamma(\varphi_j,\alpha_j))(-(1-\alpha_j)M - \alpha_j\tau_0) \to \varphi_0(\zeta - (1-\alpha_0)M - \alpha_0\tau_0)$$

and

$$\zeta - (1-\alpha_0)M - \alpha_0\tau_0 \equiv -\gamma_0 < -\tau_0.$$

Thus, if $\varphi_0(-\gamma_0) \neq 0$, then Γ will not be continuous at (φ_0,α_0).

On the other hand, if we recall that $z_2(\psi,\beta) > 2\tau_0$ for all $(\psi,\beta) \in U^+ \times [0,1]$ and if we use Lemma 2.4, we see that Γ will be continuous at (φ_0,α_0) if

$$2\tau_0 - (1-\alpha_0)M - \alpha_0\tau_0 \geq -\tau_0. \tag{2.26}$$

If $3\tau_0 \geq M$, equation (2.26) will be satisfied for $0 \leq \alpha \leq 1$ and Γ can be extended continuously to $G^+ \times [0,1]$. If $M > 3\tau_0$, then Γ can be extended continuously to

$$\{(\varphi,\alpha) \in G^+ \times [0,1] : \text{either } \varphi \in U^+, \text{ or } (3-\alpha)\tau_0 \geq (1-\alpha)M \text{ and } \varphi|[-\tau_0,0]=0\}.$$

In particular, if $(3-\alpha)\tau_0 \geq (1-\alpha)M$ (as will be true for α near to 1), then Γ_α can always be extended continuously to G^+. In all cases, we define $\Gamma(\varphi,\alpha)=0$ when $\varphi|[-\tau_0,0]=0$.

Generally, suppose that $f : X_M \to \mathbb{R}$ is continuous and almost locally Lipschitzian and that there exist constants C_1 and C_2 with

$$|f(\varphi)| \leq C_1 + C_2\|\varphi\|. \tag{2.27}$$

If $\varphi \in X_M$ and $\operatorname{lip}(\varphi) < \infty$, let $x(t;\varphi,\alpha)$ denote the unique solution of

$$\begin{aligned} x'(t) &= f(Q_\alpha(x_t)) \qquad \text{for } t \geq 0, \\ x|[-M,0] &= \varphi. \end{aligned} \tag{2.28}$$

Previously, we assumed that $f : K(-B,A,M) \to \mathbb{R}$ and extended f to X_M by $f(\varphi) = f(\rho(\varphi))$, so equation (2.27) was automatic. Assume that f satisfies H2.2 (except that f is defined on X_M instead of $K(-B,A,M)$). If A and B are any positive reals and $\varphi \in U^+(-B,A,M,\tau_0)$ and $\operatorname{lip}(\varphi) < \infty$, we can define $z_1(\varphi,\alpha)$ and $z_2(\varphi,\alpha)$ by equation (2.23). Furthermore, we can prove that there exists a constant R with $|x'(t;\varphi,\alpha)| \leq R$ for $0 \leq t \leq z_2(\varphi,\alpha)$ and for all $(\varphi,\alpha) \in U^+(-B,A,M,\tau_0) \times [0,1]$ with $\operatorname{lip}(\varphi) < \infty$. If we fix A, B and R and define

$$G = \{\varphi \in X_M : \operatorname{lip}(\varphi) \leq R \text{ and } \varphi(0)=0, \text{ and } \varphi(t) \geq 0 \text{ for } -\tau_0 \leq t \leq 0\}$$

and

$$W = \{\varphi \in X_M : -B < \varphi < A \text{ and } \varphi(t) > 0 \text{ for some } t \in [-\tau_0, 0]\},$$

then Γ_α (given by equation (2.25)) is a well-defined continuous map from $W \cap G$ to G and fixed points φ of Γ_α correspond to periodic solutions of equation (2.28) as described in Theorem 2.1. Since $W \cap G$ is a relatively open subset of the closed convex set G and Γ_α is locally compact, one can try to prove the existence of fixed points of Γ_α by showing that the fixed point index $i_G(\Gamma_\alpha, W \cap G)$ is defined and nonzero. This general approach to proving existence of periodic solutions has proved very powerful, and it is the method we shall use here.

3. Existence of slowly oscillating periodic solutions

Assume that $f : K(-B, A, M) \to \mathbb{R}$ satisfies hypotheses H2.1 and H2.2 and that τ_0 is as in H2.2. Generalizing the terminology of [10, 11, 12], we shall be interested in a special class of periodic solutions $x(t)$ of

$$x'(t) = f(x_t). \tag{3.1}$$

DEFINITION 3.1. A periodic solution $x(t)$ of equation (3.1) is called an *SOP solution*, or a *slowly oscillating periodic solution*, or a *P_2-solution* if there exist numbers z_0, z_1 and z_2 with $z_1 - z_0 > \tau_0$, and $z_2 - z_1 > \tau_0$, such that $x(t) < 0$ for $z_0 < t < z_1$, and $x(t) > 0$ for $z_1 < t < z_2$, and such that $x(t + z_2 - z_0) = x(t)$ for all t.

The term *slowly* refers to the fact that zeros of $x(t)$ are separated by distances greater than τ_0. In general (see [10, 15]) there may be periodic solutions of (3.1) with zeros separated by a distance less than τ_0. *A priori*, it may also happen that there exists a periodic solution $x(t)$ which is alternately negative or positive on intervals $[z_{j-1}, z_j]$, for $1 \le j \le 2m$, of length greater than τ_0 and which satisfies $x(t + z_{2m} - z_0) = x(t)$ for all t. As in [11], we shall call such solutions *P_{2m}-solutions*. Under certain circumstances one can prove that every P_{2m}-solution is necessarily a P_2-solution: see Theorem 2.6 in [12]. In general, equation (3.1) may possess P_{2m}-solutions which are not P_2-solutions, but little is rigorously known.

Our goal in this section is to establish the existence of an SOP solution of equation (3.1). Theorem 2.1 shows that this is equivalent to finding a nonzero fixed point of Γ_0 in $U^+(-B, A, M, \tau_0, R) = U^+$, where Γ_0 is defined by equation (2.25). The basic tool which we shall use to prove the existence of a fixed point of Γ_0 in U^+ is the fixed point index, which can be considered a generalization of the topological degree of a mapping. Expositions of the classical fixed point index can be found in [3] and [5]; expositions of more general forms of the fixed point index which are more suitable for applications in analysis are given in [2], [7], [17], [18], and [19]. These articles also contain further references to the literature, in particular, to seminal work of J. Leray.

For the reader's convenience we briefly summarize some facts about a special case of the fixed point index. Suppose that X is a Banach space, K is a closed, convex subset of X, and U is a relatively open subset of K (so $U = W \cap K$, where W is an open subset of X). Note that the interior of K in X may be empty. If U_1 and K_1 are topological Hausdorff spaces, a continuous map $h : U_1 \to K_1$ is called *locally compact* if, for each $x \in U_1$, there exists an open neighborhood $V_x \subset U_1$ of x such that the closure of $h(V_x)$ in K_1 is compact. Obviously, if K_1

is compact, h is locally compact. If (for U a relatively open subset of a closed, convex set K as above) $h : U \to K$ is a continuous, locally compact map and $S = \{x \in U : h(x) = x\}$ is compact (possibly empty), then there is defined an integer $i_K(h, U)$, the fixed point index of h on U. If $i_K(h, U) \neq 0$, then h has a fixed point in U. If U_1 and U_2 are relatively open subsets of K, with $h : U = U_1 \cup U_2 \to K$ continuous and locally compact, such that $h(x) \neq x$ for all $x \in U_1 \cap U_2$ and $S = \{x \in U : h(x) = x\}$ is compact (and possibly empty), then $i_K(h, U_j)$ is defined for $j = 1, 2$ and

$$i_K(h, U) = i_K(h, U_1) + i_K(h, U_2).$$

This is the so-called additivity property of the fixed point index. It follows from the additivity property that if U is a relatively open subset of a closed, convex set K in a Banach space and $h : U \to K$ is continuous and locally compact and if $S = \{x \in U : h(x) = x\}$ is compact, then for any relatively open set $U_1 \subset K$ with $S \subset U_1 \subset U$, one has $i_K(h, U) = i_K(h, U_1)$.

Suppose that $\Omega \subset K \times [0, 1]$ is open as a subset of $K \times [0, 1]$ in the relative topology, that $h : \Omega \to K$ is continuous and locally compact and that $\Sigma = \{(x, t) \in \Omega : h(x, t) = x\}$ is compact. Define $\Omega_t = \{x : (x, t) \in \Omega\}$ and $h_t : \Omega_t \to K$ by $h_t(x) = h(x, t)$. Then the homotopy property of the fixed point index asserts that

$$i_K(h_0, \Omega_0) = i_K(h_1, \Omega_1).$$

If Ω_t is empty, $i_K(h_t, \Omega_t) = 0$, by definition.

Finally, we shall need the commutativity property of the fixed point index. Suppose that U_j is a relatively open subset of a closed, convex set K_j in a Banach space X_j. Assume that $h_1 : U_1 \to K_2$ is continuous and locally compact and that $h_2 : U_2 \to K_1$ is continuous. Define

$$V_1 = h_1^{-1}(U_2) = \{x \in U_1 : h_1(x) \in U_2\},$$
$$V_2 = h_2^{-1}(U_1) = \{y \in U_2 : h_2(y) \in U_1\},$$

and assume that

$$S_1 = \{x \in V_1 : h_2(h_1(x)) = x\}$$

is compact (possibly empty). Then

$$S_2 = \{y \in V_2 : h_1(h_2(y)) = y\}$$

is compact and

$$i_{K_1}(h_2 h_1, V_1) = i_{K_2}(h_1 h_2, V_2).$$

A special case of the commutativity property is important. Suppose that U is a relatively open subset of a closed, convex set K in a Banach space X. Suppose that $h : U \to K$ is continuous, locally compact and $S = \{x \in U : h(x) = x\}$ is compact. Assume that $h(U) \subset K_1 \subset K$, where K_1 is a closed, convex subset of K. Then it follows that

$$i_K(h, U) = i_{K_1}(h, U \cap K_1).$$

Our goal is to compute the fixed point index of

$$\Gamma_0 : U^+(-B, A, M, \tau_0, R) = U^+ \to G^+(-B, A, M, \tau_0, R) = G^+.$$

In particular, we must prove that the fixed point index of $\Gamma_0 : U^+ \to G^+$ is defined. Our strategy is to reduce the problem of computing this index to the much simpler problem of computing a corresponding fixed point index for the linear equation

$$y'(t) = -\beta y(t) - \gamma y(t - \tau_0), \tag{3.2}$$

where we shall usually assume that $\beta \geq 0$ and $\gamma \geq 0$, and also $\tau_0 > 0$.

Thus we begin by recalling some results about equation (3.2). Denote by $X = X_{\tau_0}$ the Banach space of continuous functions $\varphi : [-\tau_0, 0] \to \mathbb{R}$ with the sup norm, and write

$$K = \{\varphi \in X_{\tau_0} : \varphi \geq 0 \text{ and } \varphi(0) = 0\}. \tag{3.3}$$

If $\varphi \in X$ and β and γ are any real numbers and $\tau_0 > 0$, there exists a unique solution $y(t) = y(t; \varphi, \beta, \gamma)$, defined for all $t \geq -\tau_0$, of

$$\begin{aligned} y'(t) &= -\beta y(t) - \gamma y(t - \tau_0) && \text{for } t \geq 0, \\ y(t) &= \varphi(t) && \text{for } -\tau_0 \leq t \leq 0. \end{aligned} \tag{3.4}$$

The map $(t, \varphi, \beta, \gamma) \to y(t; \varphi, \beta, \gamma)$ is continuous. These are standard results.

Next assume that $\tau_0 > 0$, and that $\beta \geq 0$ and $\gamma \geq 0$ in equation (3.4). If $\varphi \in K$ and $\gamma = 0$, or if $\varphi = 0$, then it is clear that $y(t; \varphi, \beta, \gamma) = 0$ for all $t \geq 0$. Thus we assume that $\gamma > 0$ and $\varphi \in K - \{0\}$ and define $\zeta_1 = \zeta_1(\varphi, \beta, \gamma)$ and $\zeta_2 = \zeta_2(\varphi, \beta, \gamma)$ by

$$\begin{aligned} \zeta_1 &= \inf\{t \geq \tau_0 : y(t; \varphi, \beta, \gamma) = 0\}, \\ \zeta_2 &= \inf\{t > \zeta_1 : y(t; \varphi, \beta, \gamma) = 0\}. \end{aligned} \tag{3.5}$$

As usual, we define $\zeta_1 = \infty$ if $y(t; \varphi, \beta, \gamma) < 0$ for all $t \geq \tau_0$, and $\zeta_2 = \infty$ if $\zeta_1 = \infty$ or if $y(t; \varphi, \beta, \gamma) > 0$ for all $t > \zeta_1$. An easier version of the argument in Lemma 2.3 shows that there exists $\sigma_0 \in [0, \min\{\tau_0, \zeta_1 - \tau_0\})$, with $y(t) = y(t; \varphi, \beta, \gamma) = 0$

for $0 \le t \le \sigma_0$ and $y(t) < 0$ for $\sigma_0 < t < \zeta_1$; the proof is left to the reader. Also, as in Lemma 2.3, we find that $\zeta_2 - \zeta_1 > \tau_0$, and $y'(\zeta_1) > 0$ and $y'(\zeta_2) < 0$.

We now define the operator of translation along trajectories for equation (3.4). If $\varphi \in K - \{0\}$, and $\beta \ge 0$ and $\gamma > 0$, and if $\zeta_2 = \zeta_2(\varphi,\beta,\gamma) < \infty$, we define (writing $y(t) = y(t;\varphi,\beta,\gamma)$) the map $S_{\beta,\gamma} : K - \{0\} \to K - \{0\}$ by

$$S_{\beta,\gamma}(\varphi) = y_{\zeta_2}. \tag{3.6}$$

As usual, $y_{\zeta_2} \in K$ is defined by

$$y_{\zeta_2}(s) = y(s + \zeta_2; \varphi, \beta, \gamma) \qquad \text{for } -\tau_0 \le s \le 0.$$

If $\zeta_2 = \infty$, we define $S_{\beta,\gamma}(\varphi) = 0$. Finally, if $\varphi = 0$ or if $\varphi \in K$ and $\gamma = 0$, we define $S_{\beta,\gamma}(\varphi) = 0$. Easier variants of the arguments in Section 2 (particularly equations (2.20) and (2.21) in Lemma 2.4) show that $(\varphi,\beta,\gamma) \to S_{\beta,\gamma}(\varphi)$ is continuous for $\varphi \in K$, and $\beta \ge 0$ and $\gamma \ge 0$, and that $(\varphi,\beta,\gamma) \to S_{\beta,\gamma}(\varphi)$ takes bounded sets to sets with compact closure. If $\varphi \in K - \{0\}$ and $S_{\beta,\gamma}(\varphi) = \varphi$ and $y(t) = y(t;\varphi,\beta,\gamma)$ is extended to $t \le -\tau_0$ by $y(t) = y(t + \zeta_2)$ (where $\zeta_2 = \zeta_2(\varphi,\beta,\gamma)$), then $y(t)$ is a P_2-solution of equation (3.2). The details of these arguments are left to the reader.

It is convenient to note that we obtain explicit estimates in terms of $\|\varphi\|$ for $y(t) = y(t;\varphi,\beta,\gamma)$ on $[0,\zeta_2]$. Specifically, for $0 \le t \le \tau_0$ and $\varphi \in K$, we have

$$\frac{d}{dt}(e^{\beta t} y(t)) = -\gamma y(t - \tau_0) e^{\beta t} \ge -\gamma \|\varphi\| e^{\beta t}.$$

If we interpret $(1 - e^{-\beta t})\beta^{-1} = t$ for $\beta = 0$, we obtain

$$y(t) \ge -\gamma \|\varphi\| (1 - e^{-\beta t})\beta^{-1} \qquad \text{for } 0 \le t \le \tau_0.$$

Since $y'(t) \ge 0$ for $\tau_0 \le t \le \zeta_1 = \zeta_1(\varphi,\beta,\gamma)$, we conclude that

$$y(t) \ge -\gamma \|\varphi\| (1 - e^{-\beta\tau_0})\beta^{-1} \qquad \text{for } 0 \le t \le \zeta_1.$$

If $\zeta_1 \ge 2\tau_0$ and $2\tau_0 \le t < \zeta_1$, we note that $y'(t) \ge -(\beta + \gamma) y(t)$, so

$$-\gamma \|\varphi\| \beta^{-1}(1 - e^{-\beta\tau_0}) e^{-(\beta+\gamma)(t - 2\tau_0)} \le y(t) < 0 \qquad \text{for } 2\tau_0 \le t < \zeta_1.$$

Analogous arguments imply that

$$y(t) \le \gamma \|y_{\zeta_1}\| (1 - e^{-\beta(t - \zeta_1)})\beta^{-1} \qquad \text{for } \zeta_1 \le t \le \zeta_1 + \tau_0,$$

and

$$y(t) \le \gamma \|y_{\zeta_1}\| (1 - e^{-\beta\tau_0})\beta^{-1} \qquad \text{for } \zeta_1 \le t \le \zeta_2.$$

If $\zeta_1 + 2\tau_0 \le \zeta_2$ and $\zeta_1 + 2\tau_0 \le t < \zeta_2$ we also obtain

$$0 < y(t) \le \gamma \|y_{\zeta_1}\| (1 - e^{-\beta\tau_0})\beta^{-1} e^{-(\beta+\gamma)(t - \zeta_1 - 2\tau_0)}.$$

We note that the map $\varphi \to S_{\beta,\gamma}(\varphi)$ is not linear, even though equation (3.2) is linear. However, it is easy to see that $S_{\beta,\gamma}$ is homogeneous of degree one:

$$S_{\beta,\gamma}(\lambda\varphi) = \lambda S_{\beta,\gamma}(\varphi) \qquad \text{for } \varphi \in K \text{ and } \lambda \geq 0. \tag{3.7}$$

As is well known, understanding equation (3.2) depends on understanding the location of zeros of the associated characteristic equation

$$z = -\beta - \gamma \exp(-\tau_0 z) \qquad \text{for } z \in \mathbb{C}. \tag{3.8}$$

Our next lemma lists some known results about equations (3.2) and (3.8). All of these facts, together with references to the literature, can be found in [10, pp. 119–125].

LEMMA 3.1. *Assume that $\beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$ and that $\tau_0 > 0$. If $\beta < \gamma$ define $\nu = \nu(\beta/\gamma, \tau_0)$ to be the unique solution ν with $\pi/2 \leq \nu\tau_0 < \pi$ of*

$$\cos(\nu\tau_0) = -\frac{\beta}{\gamma} =: c.$$

If $\beta < \gamma$ and

$$\gamma(1 - c^2)^{1/2} = (\gamma^2 - \beta^2)^{1/2} > \nu = \nu(\beta/\gamma, \tau_0),$$

then equation (3.8) has precisely one solution z with $\mathrm{Re}(z) > 0$ and $0 < \mathrm{Im}(z) < \pi/\tau_0$. If $\gamma \leq \beta$, or if $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} < \nu = \nu(\beta/\gamma, \tau_0)$, then equation (3.8) has no solution z with $\mathrm{Re}(z) \geq 0$. If $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} = \nu$, then $\pm i\nu$ are the only pure imaginary solutions of equation (3.8). If $y(t)$ is a periodic solution of equation (3.2) which is not identically zero and which is nonnegative on some interval of length $\ell \geq \tau_0$, then $\gamma > \beta$ and $(\gamma^2 - \beta^2)^{1/2} = \nu$, and $y(t) = a\cos(\nu t) + b\sin(\nu t)$ where $\nu = \nu(\beta/\gamma, \tau_0)$ and a and b are reals with $a^2 + b^2 > 0$.

The assumption that $\gamma \leq \beta$, or that $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} \neq \nu(\beta/\gamma, \tau_0)$ is equivalent to the assumption that equation (3.8) has no pure imaginary root $i\omega$ with $|\omega| \leq \pi/\tau_0$, supposing that $\beta \geq 0$ and $\gamma \geq 0$, with $\beta + \gamma > 0$ and $\tau_0 > 0$.

We next need to discuss the fixed point index of $S_{\beta,\gamma} : K \to K$.

LEMMA 3.2. *Assume that $\beta \geq 0$ and $\gamma \geq 0$, and that $\tau_0 > 0$. Let K be defined by (3.3), let $S_{\beta,\gamma} : K \to K$ be given by equation (3.6) and let $U \subset K$ denote a relatively open subset of K with $0 \in K$. If $\gamma \leq \beta$ or if $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} \neq \nu(\beta/\gamma, \tau_0)$, for ν as in Lemma 3.1, then 0 is the only fixed point of $S_{\beta,\gamma}$ in K, so the additivity property of the fixed point index implies that*

$$i_K(S_{\beta,\gamma}, K) = i_K(S_{\beta,\gamma}, U).$$

If $\gamma \le \beta$, or if $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} < \nu(\beta/\gamma, \tau_0)$, then

$$i_K(S_{\beta,\gamma}, U) = 1.$$

If $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} > \nu(\beta/\gamma, \tau_0)$, then

$$i_K(S_{\beta,\gamma}, U) = 0.$$

PROOF. If $\gamma = 0$, Lemma 3.2 is immediate, so we assume $\gamma > 0$. Suppose that $\gamma \le \beta$, or $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} \neq \nu(\beta/\gamma, \tau_0)$. Our previous remarks imply that if $S_{\beta,\gamma}(\varphi) = \varphi$ for some $\varphi \in K - \{0\}$, then $y(t; \varphi, \beta, \gamma)$ (defined by equation (3.4)) can be extended to a P_2-solution of equation (3.2). Lemma 3.1 implies that, under the given assumptions on β, γ and τ_0, equation (3.2) has no P_2-solution. Thus 0 is the only fixed point of $S_{\beta,\gamma}$ in K, and the first part of Lemma 3.2 follows.

Next suppose that $\gamma \le \beta$, or that $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} < \nu(\beta/\gamma, \tau_0)$. Lemma 3.1 implies that equation (3.8) has no solution z with $\mathrm{Re}(z) \ge 0$. Standard theory for linear functional differential equations [8] then implies that for every $\varphi \in X_{\tau_0}$,

$$\lim_{t\to\infty} y(t; \varphi, \beta, \gamma) = 0. \tag{3.9}$$

Let U be any relatively open neighborhood of 0 in K and consider the homotopy

$$(\lambda, \varphi) \to \lambda S_{\beta,\gamma}(\varphi) =: T_\lambda(\varphi)$$

for $0 \le \lambda \le 1$ and $\varphi \in U$. We claim that $\lambda S_{\beta,\gamma}(\varphi) = \varphi$ for $(\varphi, \lambda) \in U \times [0,1]$ if and only if $\varphi = 0$. For $\lambda = 0$ this is obvious, and we already know that $S_{\beta,\gamma}(\varphi) \neq \varphi$ for $\varphi \neq 0$. Thus we assume that $0 < \lambda < 1$. If $\varphi \in K - \{0\}$ and

$$S_{\beta,\gamma}(\varphi) = \lambda^{-1}\varphi \qquad \text{for some } \lambda \in (0,1),$$

then by using the homogeneity of $S_{\beta,\gamma}$ (equation (3.7)) we see that

$$S^m_{\beta,\gamma}(\varphi) = \lambda^{-m}\varphi,$$

where $S^m_{\beta,\gamma}$ denotes the m-th iterate of $S_{\beta,\gamma}$. However, this contradicts equation (3.9). Thus the hypotheses of the homotopy property are satisfied and

$$i_K(T_1, U) = i_K(S_{\beta,\lambda}, U) = i_K(T_0, U).$$

However, T_0 is a constant map with $T_0(\varphi) = 0 \in U$ for all $\varphi \in U$, so it is well-known that

$$i_K(T_0, U) = 1.$$

(Formally, we are using the so-called "normalization property" of the fixed point index; see [3], [4], [7], [17], [18], [19].)

It remains to consider the case that $\beta < \gamma$ and $(\gamma^2 - \beta^2)^{1/2} > \nu(\beta/\gamma, \tau_0)$. We first reduce to the case $\beta = 0$. To accomplish this we consider the homotopy

$$(\lambda, \varphi) \to S_{\lambda\beta,\gamma}(\varphi) \qquad \text{for } 0 \le \lambda \le 1 \text{ and } \varphi \in K.$$

It is easy to check that for $0 \le \lambda \le 1$,

$$(\gamma^2 - (\lambda\beta)^2)^{1/2} \ge (\gamma^2 - \beta^2)^{1/2} > \nu(\beta/\gamma, \tau_0) \ge \nu((\lambda\beta)/\gamma, \tau_0),$$

so Lemma 3.1 implies that 0 is the only fixed point of $S_{\lambda\beta,\gamma}$ for $0 \le \lambda \le 1$. Thus we obtain from the homotopy property that

$$i_K(S_{\beta,\gamma}, U) = i_K(S_{0,\gamma}, U),$$

so we can assume that $\beta = 0$ and $\gamma > \nu(0, \tau_0) = \pi/(2\tau_0)$.

We shall view γ as fixed, and for notational convenience we shall write $T = S_{0,\gamma}$. We shall write $y(t;\varphi) = y(t;\varphi,0,\gamma)$ for the solution of equation (3.4) with $\beta = 0$; and for $\zeta_1(\varphi,0,\gamma)$ and $\zeta_2(\varphi,0,\gamma)$ as in equation (3.5), we shall set $\zeta_1(\varphi) = \zeta(\varphi,0,\gamma)$ and $\zeta_2(\varphi) = \zeta_2(\varphi,0,\gamma)$. We define a cone $K_1 \subset X_{\tau_0}$ by

$$K_1 = \{\varphi \in X_{\tau_0} : \varphi(-\tau_0) = 0 \text{ and } \varphi \text{ is increasing on } [-\tau_0, 0]\}.$$

(We say φ is increasing on $[-\tau_0, 0]$ if $\varphi(s) \le \varphi(t)$ whenever $-\tau_0 \le s \le t \le 0$.) We define $T_1 : K \to K_1$ by $T_1(\varphi) = y_{\zeta_1+\tau_0}$, where $\zeta_1 = \zeta_1(\varphi)$ and $y(t) = y(t;\varphi)$. One can easily see from equation (3.2) and the fact that $\beta = 0$ that $y|[\zeta_1, \zeta_1+\tau_0]$ is increasing; one can also check that T_1 is continuous and takes bounded sets to sets with compact closure. If $\psi \in K_1 - \{0\}$, define $z_1 = z_1(\psi)$ to be the first $t > 0$ such that $y(t;\psi) = 0$, and $z_2 = z_2(\psi)$ to be the first $t > z_1(\psi)$ with $y(t;\psi) = 0$. (Because $\gamma > \pi/(2\tau_0) > 1/\tau_0$, an easy direct argument as in Lemma 2.3 on page 271 of [14] implies that $z_1(\psi) < \infty$ and $z_2(\psi) < \infty$, and that $\zeta_1(\varphi) < \infty$ and $\zeta_2(\varphi) < \infty$ for all $\varphi \in K - \{0\}$.) Define $T_2 : K_1 \to K$ by $T_2(0) = 0$ and $T_2(\psi) = y_{z_1}$ for $\psi \in K_1 - \{0\}$ and $y(t) = y(t;\psi)$. One can check that T_2 is continuous (though not necessarily locally compact) and that $T = T_2T_1$. If we write $\widetilde{T} = T_1T_2$, then $\widetilde{T}$ can be described as follows. If $\psi \in K_1 - \{0\}$ and we set $y(t) = y(t;\psi)$ and $z_2 = z_2(\psi)$, then $\widetilde{T}(\psi) = y_{z_2+\tau_0}$. Of course we have $\widetilde{T}(0) = 0$. The commutativity property of the fixed point index directly yields

$$i_K(T, K) = i_K(T_2T_1, K) = i_{K_1}(T_1T_2, K_1) = i_{K_1}(\widetilde{T}, K_1). \tag{3.10}$$

Our characterization of $\widetilde{T}$ above shows that it is the same operator of translation along trajectories considered in Section 2 of [14]. Thus we can directly apply Lemma 2.8, page 275, in [14] and conclude that 0 is an ejective fixed point of $\widetilde{T} : K_1 \to K_1$. (Ejective points of a map are defined in Section 1 of [14].) We can now directly apply Lemma 1.2, page 322, in [15] and conclude that

$$i_{K_1}(\widetilde{T}, V) = 0, \tag{3.11}$$

where V is any relatively open neighborhood of 0 in K_1. If U is any relatively open neighborhood of 0 in K, we conclude from (3.10) and (3.11) that

$$i_K(T, U) = i_K(T, K) = i_{K_1}(\widetilde{T}, K_1) = 0,$$

which completes the proof. □

REMARK 3.1. In Lemma 1.2 of [15] it is assumed that the ejective fixed point x_0 is also an extreme point of the closed convex set C in question. However, the conclusions of Lemma 1.2 remain valid if x_0 is not an extreme point, but C is infinite dimensional. The proof is similar. In Lemma 3.2, $x_0 = 0$ and $C = K_1$, with 0 an ejective point of $\widetilde{T}$ and an extreme point of the infinite dimensional, closed, convex set K_1.

REMARK 3.2. Our proof that $i_{K_1}(\widetilde{T}, K_1) = 0$ used some sophisticated ideas from asymptotic fixed point theory. One can obtain the same result by using Proposition 2, page 248, in [16], which is a conceptually simpler theorem. Let $U = \{\psi \in K_1 : \|\psi\| < 1\}$. It is a special case of Proposition 2 in [16] that $i_{K_1}(\widetilde{T}, U) = 0$ if (a) $\lambda\widetilde{T}(\psi) \neq \psi$ for all $\lambda \geq 1$ and all $\psi \in K_1$ with $\|\psi\| = 1$; and (b) we have

$$\inf\{\|\widetilde{T}(\psi)\| : \psi \in K_1, \|\psi\| = 1\} > 0.$$

The second hypothesis can be proved by elementary estimates, although it is worth noting that

$$\inf\{\|T(\varphi)\| : \varphi \in K \text{ and } \|\varphi\| = 1\} = 0.$$

For the first hypothesis, we already know from the linear theory that 0 is the only fixed point of $\widetilde{T}$. If $\lambda\widetilde{T}(\psi) = \psi$ for some $\psi \in K_1$ with $\|\psi\| = 1$ and some $\lambda > 1$, the homogeneity of $\widetilde{T}$ implies that

$$\widetilde{T}^m(\psi) = \lambda^{-m}\psi, \quad \text{hence} \quad \lim_{m\to\infty} \|\widetilde{T}^m(\psi)\| = 0.$$

On the other hand, the homogeneity of $\widetilde{T}$ and the fact that 0 is an ejective fixed point of $\widetilde{T}$ imply that for all $\psi \in K_1 - \{0\}$,

$$\limsup_{m\to\infty} \|\widetilde{T}^m(\psi)\| = \infty.$$

It follows that the hypotheses of Proposition 2 in [16] are satisfied.

Not surprisingly, Proposition 2 and Corollary 1, pages 248–249 in [16], are related to work of Grafton [6] about periodic solutions of functional differential equations. See [16] for further details.

We shall need to prove that equation (3.1) or, more generally, equation (2.7), has no P_2-solutions which are small in the L^∞ norm. To prove this it is natural to make appropriate assumptions about the Fréchet derivative at 0 of the map f in equation (3.1). The difficulty which must be addressed is that, for the examples of interest, f is not Fréchet differentiable at 0. To handle this problem we introduce a weakening of the idea of differentiability at 0.

Assume that $(Y, \|\cdot\|)$ and $(Y_1, \|\cdot\|_1)$ are Banach spaces, that Y_1 is a dense subset of Y (in the $\|\cdot\|$ norm), and that the inclusion map is continuous. If $y_0 \in Y_1$ and U is an open neighborhood of y_0 in Y, a function $f : U \to \mathbb{R}$ is called *almost Fréchet differentiable at* y_0 in case $f_1 \equiv f|Y_1$ is Fréchet differentiable at y_0 as a map from Y_1 to $\mathbb{R}$, that is, if there exists a bounded linear map $Df_1(y_0) : Y_1 \to \mathbb{R}$ with

$$|f(y) - f(y_0) - (Df_1(y_0))(y - y_0)| = o(\|y - y_0\|_1) \qquad \text{as } \|y - y_0\|_1 \to 0.$$

We shall only apply this definition when $Y = X_M = C([-M, 0])$ and $Y_1 = C^{0,1}([-M, 0])$, the Banach space of Lipschitz continuous real-valued maps. For definiteness, we explicitly state the definition in this case.

DEFINITION 3.2. Suppose that $\delta_0 > 0$ and that $f : \{\varphi \in X_M : \|\varphi\| < \delta_0\} \to \mathbb{R}$ is a map. We say that f is *almost Fréchet differentiable at* 0 if there exists a continuous linear map $L : C^{0,1}([-M, 0]) \to \mathbb{R}$ and a function $\sigma : (0, \delta_0) \to (0, \infty)$ with $\lim_{\varepsilon \to 0^+} \varepsilon^{-1}\sigma(\varepsilon) = 0$ such that

$$|f(\varphi) - f(0) - L(\varphi)| \le \sigma(\|\varphi\| + \mathrm{lip}(\varphi))$$

for all Lipschitz $\varphi \in X_M$ with $\|\varphi\| + \mathrm{lip}(\varphi) < \delta_0$.

A priori, the map L in Definition 3.2 need not extend as a continuous linear map to all of X_M. However, in our applications this will be the case and, in fact L will take a very simple form.

By replacing σ in Definition 3.2 by $\widetilde{\sigma}$, where

$$\widetilde{\sigma}(\varepsilon) = \sup\{\sigma(s) : 0 < s \le \varepsilon\},$$

we can assume that $\widetilde{\sigma}$ is increasing. We shall therefore always assume that σ is increasing.

We shall refer to L in Definition 3.2 as *the almost Fréchet derivative of f at* 0. The next lemma lists some elementary properties of the almost Fréchet derivative.

LEMMA 3.3. *Suppose that $\delta_0 > 0$ and that $f : \{\varphi \in X_M : \|\varphi\| < \delta_0\} \to \mathbb{R}$ is a map which is almost differentiable at* 0 *and has almost Fréchet derivative L at* 0. *If $\varphi \in X_M = C([-M,0])$ is Lipschitzian, we have*

$$L(\varphi) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(f(\varepsilon\varphi) - f(0)). \tag{3.12}$$

If f is almost locally Lipschitzian, the map L has a unique extension (also denoted by L) to a continuous linear map from X_M to $\mathbb{R}$. If f satisfies H2.2 *and g and τ_0 are as in* H2.2, *then there exist nonnegative real numbers β and γ with $\gamma = -g'(0)$ so that*

$$L(\varphi) = -\beta\varphi(0) - \gamma\varphi(-\tau_0)$$

for all Lipschitzian $\varphi \in X_M$.

PROOF. If $Y_M = C^{0,1}([-M,0])$, the Banach space of Lipschitzian real-valued maps $\varphi : [-M,0] \to \mathbb{R}$, our definition implies that $f|Y_M$ is Fréchet differentiable at 0, so equation (3.12) follows immediately and $L : Y_M \to \mathbb{R}$ is a continuous linear map. If f is almost locally Lipschitzian, Lemma 1.1 implies that there is a constant k such that $|f(\varphi) - f(0)| \le k\|\varphi\|$ for all Lipschitzian $\varphi \in X_M$ with $\|\varphi\| \le \delta_0$ and $\mathrm{lip}(\varphi) \le \delta_0$. (Here $\|\cdot\|$ denotes, as usual, the sup norm on X_M.) It follows from equation (3.12) that if $\varphi \in X_M$ and φ is Lipschitzian, then

$$\begin{aligned} |L(\varphi)| = |\lim_{\varepsilon \to 0}(f(\varepsilon\varphi) - f(0))| &\le \limsup_{\varepsilon \to 0} |\varepsilon|^{-1}|f(\varepsilon\varphi) - f(0)| \\ &\le \limsup_{\varepsilon \to 0} |\varepsilon|^{-1}k\|\varepsilon\varphi\| = k\|\varphi\|. \end{aligned}$$

Since Y_M is dense in $(X_M, \|\cdot\|)$, we obtain from the previous inequality that L extends uniquely to a continuous linear map from X_M to $\mathbb{R}$; and the norm of this extension is bounded by k.

If f satisfies H2.2 and $\varphi \in X_M$ is Lipschitzian with $\varphi(0) = 0$, we see that

$$L(\varphi) = \lim_{\varepsilon \to 0} \varepsilon^{-1}f(\varepsilon\varphi) = \lim_{\varepsilon \to 0} \varepsilon^{-1}g(\varepsilon\varphi(-\tau_0)).$$

This equation implies that g is differentiable at 0 and

$$L(\varphi) = g'(0)\varphi(-\tau_0).$$

The assumptions on g in H2.2 imply that $g'(0) = -\gamma \le 0$. In general, let ψ be any fixed, nonnegative Lipschitz function in X_M with $\psi(0) = 1$ and $\psi(-\tau_0) = 0$, and define $\beta = -L(\psi)$. If $\varphi \in X_M$ and $\mathrm{lip}(\varphi) < \infty$, our remarks above imply that

$$L(\varphi - \varphi(0)\psi) = -\gamma(\varphi(-\tau_0) - \varphi(0)\psi(-\tau_0)) = -\gamma\varphi(-\tau_0).$$

It follows that

$$L(\varphi) = L(\varphi - \varphi(0)\psi) + \varphi(0)L(\psi) = -\beta\varphi(0) - \gamma\varphi(-\tau_0).$$

The negative feedback condition for f implies that

$$-\beta = L(\psi) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1} f(\varepsilon\psi) \le 0,$$

as claimed. □

In order to establish our main results we also need an upper bound on the minimal period of any P_2-solution of equation (2.6).

LEMMA 3.4. *Let $\rho : X_M \to K(-B, A, M)$ be the retraction given by equations* (1.3) *and* (1.4), *and let Q_α, for $0 \le \alpha \le 1$, be the linear projection defined by equation* (2.3). *Assume that $f : K(-B, A, M) \to \mathbb{R}$ is almost Fréchet differentiable at* 0 *(Definition* 3.2) *and satisfies* H2.1 *and* H2.2. *There exists a constant N so that if x is a P_2-solution of*

$$x'(t) = F_\alpha(\rho(x_t)) := f(Q_\alpha(\rho(x_t))) \tag{3.13}$$

for some $\alpha \in [0, 1]$, then the minimal period of x is less than or equal to N.

PROOF. Suppose that x is a P_2-solution of (3.13) for some α, with $0 \le \alpha \le 1$. By a time translation we can assume that $x(0) = 0$ and $x(t) > 0$ on $[-\tau_0, 0)$. As usual, we define $z_1 = \inf\{t \ge \tau_0 : x(t) = 0\}$ and $z_2 = \inf\{t > z_1 : x(t) = 0\}$, so z_2 is the minimal period of x. Select R so that $|f(\varphi)| \le R$ for all $\varphi \in K(-B, A, M)$, define $R_1 = R + \tau_0^{-1} \max\{A, B\}$ and select k so that f is Lipschitzian with Lipschitz constant k on $K(-B, A, M, R_1)$. If $F_\alpha(\varphi) := f(Q_\alpha(\varphi))$, it is easy to see that F_α is also Lipschitz with Lipschitz constant k on $K(-B, A, M, R_1)$ and that $|F_\alpha(\varphi)| \le R$ for all $\varphi \in K(-B, A, M)$. It follows from Lemma 2.4 (see equations (2.20) and (2.21)) that there exists $C_1 \ge 1$, with C_1 independent of α, such that for any P_2-solution x of equation (3.13) with $x(0) = 0$ and $x(t) > 0$ for $-\tau_0 \le t < 0$,

$$\begin{aligned} &\sup\{|x(t)| : 0 \le t \le z_1\} \le C_1 \sup\{r(x(t)) : -\tau_0 \le t \le 0\}, \\ &\sup\{x(t) : z_1 \le t \le z_2\} \le C_1 \sup\{|r(x(t))| : z_1 - \tau_0 \le t \le 0\}. \end{aligned} \tag{3.14}$$

Furthermore, by using the last part of Lemma 2.4, we see that for each $\delta > 0$, there exists $T = T(\delta, f)$ such that if x is a P_2-solution of (3.13) for some α with $0 \le \alpha \le 1$, and $z_1 \ge T$ (respectively, $z_2 - z_1 \ge T$) then $|x(t)| \le \delta$ for $T \le t \le z_1$ (respectively, $|x(t)| \le \delta$ for $z_1 + T \le t \le z_2$). (Note that an examination of the constants in the proof of Lemma 2.4 shows that T can be chosen independently of such α.) If $z_2 \ge 2T(\delta, f) + 2\tau_0$, it follows that either $z_1 \ge T(\delta, f) + \tau_0$ or $z_2 - z_1 \ge T(\delta, f) + \tau_0$. If we assume, for definiteness, that $z_1 \ge T(\delta, f) + \tau_0$, we obtain

$$\sup\{|x(t)| : z_1 - \tau_0 \le t \le z_1\} \le \delta.$$

This implies, using equation (3.14), that

$$\sup\{x(t)\,:\,z_1 \le t \le z_2\} \le C_1\delta.$$

By the periodicity of x and the fact that $z_2 - z_1 > \tau_0$ (Lemma 2.3) we see that

$$\begin{aligned}\sup\{|x(t)|\,:\,0 \le t \le z_1\} &\le C_1 \sup\{x(t)\,:\,-\tau_0 \le t \le 0\}\\ &\le C_1 \sup\{x(t)\,:\,z_1 \le t \le z_2\} \le C_1^2\delta.\end{aligned}$$

Thus we conclude that if $z_2 \ge 2T(\delta, f) + 2\tau_0$, then

$$\sup_t |x(t)| \le C_1^2\delta.$$

Taking the contrapositive, we conclude that if x is a P_2-solution of equation (3.13) and $\sup_t|x(t)| > \delta$, then the minimal period z_2 of x satisfies

$$z_2 \le 2T(C_1^{-2}\delta, f) + 2\tau_0 =: T_*(\delta, f). \tag{3.15}$$

By using equation (3.15), we can restrict our attention to P_2-solutions x for which $\sup_t |x(t)|$ is small. In particular, we shall always assume that $-B \le x(t) \le A$ for all t, so $\rho(x_t) = x_t$ for all t.

We shall prove Lemma 3.4 by finding upper estimates for z_1 and for $z_2 - z_1$. We restrict attention to bounding z_1, since the argument for bounding $z_2 - z_1$ is the same. Note that our remarks above imply that if $z_1 \ge T(C_1^{-2}\delta, f) + \tau_0$, then $\sup_t|x(t)| \le \delta$.

We can assume that $z_1 \ge 3M + \tau_0$, or we already have an upper bound. We claim that

$$|x(t)| \ge C_1^{-2}|x(t - 2M)| \qquad \text{for } 2M \le t \le z_1 - \tau_0. \tag{3.16}$$

For suppose that (3.16) fails for some t_0 with $2M \le t_0 \le z_1 - \tau_0$. If we recall that x is increasing on $[M, z_1]$ we obtain from equation (3.14) that

$$\begin{aligned}\sup\{|x(s)|\,:z_1 - \tau_0 \le s \le z_1\}&\\ \le |x(t_0)| &< C_1^{-2}|x(t_0 - 2M)|\\ &\le C_1^{-2}C_1 \sup\{x(s)\,:\,-\tau_0 \le s \le 0\}.\end{aligned} \tag{3.17}$$

However, equation (3.14) and the periodicity of x imply that

$$\begin{aligned}\sup\{x(s)\,:\,-\tau_0 \le s \le 0\} &\le \sup\{x(s)\,:\,z_1 \le s \le z_2\}\\ &\le C_1 \sup\{|x(s)|\,:\,z_1 - \tau_0 \le s \le z_1\},\end{aligned}$$

and this contradicts equation (3.17) and proves that equation (3.16) holds. Assuming that $z_1 \ge 3M + \tau_0$, we also obtain from (3.16) that

$$|x(z_1 - \tau_0)| \ge C_1^{-2}|x(z_1 - \tau_0 - 2M)| \ge C_1^{-2}|x(t - 2M)| \tag{3.18}$$

for $z_1 - \tau_0 \le t \le z_1$.

Recall that f is almost differentiable at 0, so Lemma 3.3 implies that

$$f(\psi) = -\beta\psi(0) - \gamma\psi(-\tau_0) + R(\psi),$$
$$|R(\psi)| \le \sigma(\|\psi\| + \mathrm{lip}(\psi)),$$

where $\beta \ge 0$ and $\gamma \ge 0$, and $\|\psi\| + \mathrm{lip}(\psi) < \delta_0$, and where σ is an increasing function with $\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\sigma(\varepsilon) = 0$. To complete the proof we shall have to exploit this differentiability. For any t, equation (2.19) gives (for g as in H2.2)

$$-k|x(t)| + g(x(t-\tau_0)) \le x'(t) = f(Q_\alpha(x_t)) \le k|x(t)| + g(x(t-\tau_0)).$$

If C is a Lipschitz constant for g on $[-B, A]$, we conclude that

$$|x'(t)| \le k|x(t)| + C|x(t-\tau_0)| \tag{3.19}$$

for all t. If $3M \le \tau \le z_1 - \tau_0$ and if we recall that x is increasing on $[M, z_1]$ and use equation (3.16), we obtain that

$$\|Q_\alpha(x_\tau)\| \le \|x_\tau\| = |x(\tau - M)| \le |x(\tau - 2M)| \le C_1^2|x(\tau)|. \tag{3.20}$$

Similarly, by exploiting (3.19) we see that

$$\begin{aligned} \mathrm{lip}(Q_\alpha(x_\tau)) \le \mathrm{lip}(x_\tau) &= \sup\{|x'(t)| \,:\, \tau - M \le t \le \tau\} \\ &\le k\|x_\tau\| + C\|x_{\tau-\tau_0}\| \\ &= k|x(\tau - M)| + C|x(\tau - \tau_0 - M)| \\ &\le (k+C)|x(\tau - 2M)| \le (k+C)C_1^2|x(\tau)|. \end{aligned} \tag{3.21}$$

Thus, for $3M \le \tau \le z_1 - \tau_0$ we have

$$\|Q_\alpha(x_\tau)\| + \mathrm{lip}(Q_\alpha(x_\tau)) \le (k + C + 1)C_1^2|x(\tau)| := C_2|x(\tau)|. \tag{3.22}$$

If δ_0 is as in Definition 3.2, we can assume, by virtue of equation (3.15), that $C_2|x(t)| < \delta_0$ for all t, so we obtain from (3.22) and the almost differentiability of f at 0 that, for $3M \le t \le z_1 - \tau_0$,

$$\begin{aligned} x'(t) &= f(Q_\alpha(x_t)) = -\beta x(t) - \gamma x(t-\tau_0) + R(x_t), \\ |R(x_t)| &\le \sigma(C_2|x(t)|). \end{aligned} \tag{3.23}$$

For the range $z_1 - \tau_0 \le \tau \le z_1$ the final inequality in (3.20) is no longer valid; however, we do have

$$|x(\tau - 2M)| \le |x(z_1 - \tau_0 - 2M)| \le C_1^2|x(z_1 - \tau_0)|,$$

giving

$$\|Q_\alpha(x_\tau)\| \le C_1^2|x(z_1 - \tau_0)|.$$

The final estimate in (3.21) must be similarly modified to give

$$\mathrm{lip}(Q_\alpha(x_\tau)) \le (k+C)C_1^2|x(z_1-\tau_0)|.$$

Thus, for $z_1-\tau_0 \le t \le z_1$ we have the estimate

$$|R(x_t)| \le \sigma(C_2|x(z_1-\tau_0)|) \tag{3.24}$$

of the remainder term, in place of the estimate in (3.23) above.

We now consider two cases, depending on the size of $\beta+\gamma$.

CASE I. Assume that $\beta + C_1^2\gamma < 1/(2\tau_0)$. Select ε_0 so that

$$\varepsilon^{-1}\sigma(C_2\varepsilon) \le 1/(2\tau_0) \qquad \text{for } 0<\varepsilon\le\varepsilon_0.$$

Equation (3.15) yields an upper bound on the period of P_2-solutions x of equation (3.13) with $\sup_t|x(t)| > \varepsilon_0$. Thus to complete the proof in Case I, it suffices to assume that $\sup_t|x(t)| \le \varepsilon_0$ and derive a contradiction. Using equation (3.18) we see that for $z_1-\tau_0\le t\le z_1$ we have $|x(t)| \le |x(z_1-\tau_0)|$ and

$$|x(t-\tau_0)| \le |x(z_1-\tau_0-2M)| \le C_1^2|x(z_1-\tau_0)|,$$

and so by equation (3.24),

$$\begin{aligned}
x'(t) &= -\beta x(t) - \gamma x(t-\tau_0) + R(x_t)\\
&\le (\beta + C_1^2\gamma)|x(z_1-\tau_0)| + \sigma(C_2|x(z_1-\tau_0)|)\\
&= (\beta + C_1^2\gamma + \sigma(C_2|x(z_1-\tau_0)|)|x(z_1-\tau_0)|^{-1})|x(z_1-\tau_0)|\\
&< \tau_0^{-1}|x(z_1-\tau_0)|.
\end{aligned}$$

However, this implies that

$$\begin{aligned}
x(z_1) &= x(z_1-\tau_0) + \int_{z_1-\tau_0}^{z_1} x'(s)\,ds\\
&< x(z_1-\tau_0) + \int_{z_1-\tau_0}^{z_1} \tau_0^{-1}|x(z_1-\tau_0)|\,ds < 0,
\end{aligned}$$

which is a contradiction.

CASE II. Assume that $\beta + C_1^2\gamma \ge 1/(2\tau_0)$, so that $\beta+\gamma \ge 1/(2C_1^2\tau_0) := 2c_3$. We derive from equation (3.23) and the fact that x is increasing on $[M, z_1]$ that, for $3M \le t \le z_1 - \tau_0$,

$$x'(t) = -\beta x(t) - \gamma x(t-\tau_0) + R(x_t) \ge -(\beta+\gamma)x(t) - \sigma(C_2|x(t)|).$$

Select $\varepsilon_1 > 0$ so that

$$\varepsilon^{-1}\sigma(C_2\varepsilon) \le (\beta+\gamma)/2 \qquad \text{for } 0<\varepsilon\le\varepsilon_1.$$

If $\sup_t |x(t)| > \varepsilon_1$, equation (3.15) yields an upper bound for the period of x. Thus we assume that $\sup_t |x(t)| \le \varepsilon_1$, and we derive from the above inequalities that

$$x'(t) \ge -((\beta+\gamma)/2)x(t) \ge -C_3 x(t) \qquad \text{for } 3M \le t \le z_1 - \tau_0.$$

This differential inequality implies that

$$0 > x(t) \ge \exp(-C_3(t-3M))x(3M) \qquad \text{for } 3M \le t \le z_1 - \tau_0.$$

If we use this estimate for $t = z_1 - \tau_0$ we find that

$$\begin{aligned}\sup\{|x(s)| : z_1 - \tau_0 \le s \le z_1\} &= |x(z_1-\tau_0)| \\ &\le \exp(-C_3(z_1 - 3M - \tau_0))|x(3M)| \\ &\le \exp(-C_3(z_1 - 3M - \tau_0)) \sup\{|x(s)| : 0 \le s \le z_1\}.\end{aligned}$$

If we combine this estimate with equation (3.14), we obtain

$$\begin{aligned}\sup\{x(t) : -\tau_0 \le t \le 0\} &\le \sup\{x(t) : z_1 \le t \le z_2\} \\ &\le C_1 \sup\{|x(t)| : z_1 - \tau_0 \le t \le z_1\} \\ &\le C_1 \exp(-C_3(z_1 - 3M - \tau_0)) \sup\{|x(s)| : 0 \le s \le z_1\} \\ &\le C_1^2 \exp(-C_3(z_1 - 3M - \tau_0)) \sup\{x(t) : -\tau_0 \le t \le 0\}.\end{aligned}$$

We conclude from this that

$$C_1^2 \exp(-C_3(z_1 - 3M - \tau_0)) \ge 1,$$

which yields an explicit upper bound for z_1. □

REMARK 3.3. The proof of Lemma 3.4 provides very crude, but explicit upper bounds for the minimal period of a P_2-solution of equation (3.13). If one only wants to prove the existence of an upper bound, a shorter proof can be given. Assume to the contrary that there exists a sequence (x^n, α^n), where x^n is a P_2-solution of

$$(x^n)'(t) = F_{\alpha^n}(\rho(x_t^n))$$

and the minimal period p^n of x^n approaches ∞. By using equation (3.15) one can see that if one defines

$$\|x^n\| := \sup\{|x^n(t)| : t \in \mathbb{R}\},$$

then $\|x^n\| \to 0$. As in Lemma 3.4 one can assume that $x^n(0) = 0$ and $x^n(t) > 0$ on $[-\tau_0, 0)$. If one defines $y^n(t) = x^n(t)\|x^n\|^{-1}$, then by taking a subsequence and using an argument like that in Theorem 3.1 below, one can assume that

$\alpha^n \to \alpha$, that $y^n(t)$ converges uniformly on compact subsets of $\mathbb{R}$ to a C^1 function $y(t)$ and that

$$y'(t) = -\beta y(t) - \gamma y(t - \tau_0). \tag{3.25}$$

However, by exploiting equation (3.14) and the assumption that $p^n \to \infty$, one can obtain a contradiction from equation (3.25).

We shall also need a slight variant of Lemma 3.4. The proof is essentially the same as that of Lemma 3.4 and is omitted.

LEMMA 3.4A. *For $j = 0, 1$, assume that $h_j : K(-B, A, M) \to \mathbb{R}$ is almost Fréchet differentiable at* 0 *(Definition* 3.2*) and satisfies* H2.1 *and* H2.2, *with the number τ_0 in* H2.2 *the same for h_0 and h_1. For $0 \le \lambda \le 1$, define h_λ : $K(-B, A, M) \to \mathbb{R}$ by*

$$h_\lambda(\varphi) = (1 - \lambda)h_0(\varphi) + \lambda h_1(\varphi),$$

and let $\rho : X_M \to K(-B, A, M)$ be the retraction given by equations (1.3) *and* (1.4). *There exists a constant N so that if x is a P_2-solution of*

$$x'(t) = h_\lambda(\rho(x_t))$$

for some $\lambda \in [0, 1]$, then the minimal period of x is less than or equal to N.

We can now prove our main theorem concerning the existence of slowly oscillating periodic solutions of equation (3.1). Recall that hypotheses H2.1, H2.2 and H2.3 are given in Section 2.

THEOREM 3.1. *Let A, B and M be positive reals and suppose that $f : K(-B, A, M) \to \mathbb{R}$ satisfies* H2.1, H2.2 *and* H2.3. *Assume that f is almost Fréchet differentiable at* 0 *(Definition* 3.2*) with almost Fréchet derivative L at* 0, *so (by Lemma* 3.3*)*

$$L(\varphi) = -\beta\varphi(0) - \gamma\varphi(-\tau_0) = \lim_{\varepsilon \to 0} \varepsilon^{-1} f(\varepsilon\varphi)$$

with $\beta \ge 0$ and $\gamma \ge 0$. Select

$$R \ge \sup\{|f(\varphi)| : \varphi \in K(-B, A, M)\}$$

and let $G^+(-B, A, M, \tau_0, R) =: G^+$ be given by equation (2.2), *and $U^+(-B, A, M, \tau_0, R) =: U^+$ be given by equation* (2.9). *Let $\Gamma_0 : U^+ \to G^+$ be given by equation* (2.25) *and Theorem* 2.1. *Assume that either $\gamma \le \beta$, or that $\gamma > \beta$ and $(\gamma^2 - \beta^2)^{1/2} \ne \nu$, where $\nu = \nu(\beta/\gamma, \tau_0)$ is defined in Lemma* 3.1. *Then there exists $\delta > 0$ such that $\Gamma_0(\varphi) \ne \varphi$ for all $\varphi \in U^+$ with* $\sup\{\varphi(s) : -\tau_0 \le s \le 0\} \le \delta$, *and the fixed point index $i_{G^+}(\Gamma_0, U^+)$ is defined. If $\gamma \le \beta$, or if*

$\gamma > \beta$ and $(\gamma^2 - \beta^2)^{1/2} < \nu(\beta/\gamma, \tau_0)$, then $i_{G^+}(\Gamma_0, U^+) = 0$. If $\gamma > \beta$ and $(\gamma^2 - \beta^2)^{1/2} > \nu(\beta/\gamma, \tau_0)$, then $i_{G^+}(\Gamma_0, U^+) = 1$ and equation (3.1) has a slowly oscillating periodic solution x with $-B \le x(t) \le A$ for all t.

PROOF. Let $\Gamma : U^+ \times [0,1] \to X_M$ be defined by equation (2.25). It is easy to see from the form of F_α and from equation (2.7) that $\mathrm{lip}(\Gamma_\alpha(\varphi)) \le R$ for all $\varphi \in U^+$. It follows from Theorem 2.1 that $\varphi \to \rho(\Gamma_\alpha(\varphi))$ is a continuous map of U^+ to the compact, convex set G^+.

Our first claim is that

$$i_{G^+}(\Gamma_0, U^+) = i_{G^+}(\rho\Gamma_1, U^+). \tag{3.26}$$

Hypothesis H2.3 implies that $\Gamma_0(U^+) \subset G^+$ and $\rho\Gamma_0 = \Gamma_0$, so to prove equation (3.26) it suffices to use the homotopy property of the fixed point index and prove that

$$\{(\varphi, \alpha) \in U^+ \times [0,1] \,:\, \rho(\Gamma(\varphi, \alpha)) = \varphi\}$$

is compact. Because G^+ is compact, it suffices to prove that there exists $\delta > 0$ such that $\rho(\Gamma(\varphi, \alpha)) \ne \varphi$ for all $(\varphi, \alpha) \in U^+ \times [0,1]$ with $\sup\{\varphi(s) \,:\, -\tau_0 \le s \le 0\} \le \delta$. To prove the latter claim, we assume the contrary and obtain a contradiction. Thus, suppose that there exists a sequence $(\varphi^j, \alpha^j) \in U^+ \times [0,1]$ with $\lim_{j\to\infty}(\sup\{\varphi^j(t) \,:\, -\tau_0 \le t \le 0\}) = 0$ and $\rho(\Gamma(\varphi^j, \alpha^j)) = \varphi^j$. By taking a subsequence we can assume that $\alpha^j \to \alpha$. We define $x^j(t) = x(t; \varphi^j, \alpha^j)$ for $t \ge -\tau_0$, and set $q^j = z_1(\varphi^j, \alpha^j)$ and $p^j = z_2(\varphi^j, \alpha^j)$. Theorem 2.1 implies that by deleting the first few terms of the sequence we can assume that $-B < x^j(t) < A$ for all t and hence

$$\rho(\Gamma(\varphi^j, \alpha^j)) = \Gamma(\varphi^j, \alpha^j) = \varphi^j,$$

and so

$$\frac{d}{dt}x^j(t) = f(Q_{\alpha^j}(x_t^j))$$

for all t. Theorem 2.1 also implies that $x^j(t + p^j) = x^j(t)$ for all $t \ge -\tau_0$. Furthermore, if we extend x^j for all t by defining it to be periodic of period p^j, then x^j is a P_2-solution. In fact, by using equations (2.20) and (2.21) in Lemma 2.4 we see that there exists a constant C_2, independent of j, with

$$\|x^j\| := \sup\{|x^j(t)| \,:\, t \in \mathbb{R}\} \le C_2 \sup\{x^j(t) \,:\, -\tau_0 \le t \le 0\}, \tag{3.27}$$

and so $\|x^j\| \to 0$.

We now define $y^j(t) = x^j(t)\|x^j\|^{-1}$ and observe that $y^j(0) = 0$ and $\|y^j\| = 1$, and

$$y^j(t) = \int_0^t \|x^j\|^{-1} f(Q_{\alpha^j}(x_s^j))\, ds. \tag{3.28}$$

Because f is bounded by R on $K(-B, A, M)$ we see that $x^j_s \in K(-B, A, M, R)$ for all j and all s. Because f is almost Lipschitzian, the restriction of f to $K(-B, A, M, R)$ is Lipschitz with Lipschitz constant $k = k_R$. It follows that

$$\left|\frac{d}{dt}x^j(t)\right| = |f(Q_{\alpha^j}(x^j_t))| \leq k\|x^j_t\| \leq k\|x^j\| \tag{3.29}$$

and hence

$$\left|\frac{d}{dt}y^j(t)\right| \leq k \qquad \text{for all } t.$$

The Ascoli-Arzelà theorem implies that by taking a subsequence we can assume that y^j converges uniformly on compact intervals of $\mathbb{R}$ to a continuous function y. Because $x^j(t) \geq 0$ on $[-\tau_0, 0]$ and $x^j(0) = 0$, we see that $y(t) \geq 0$ for $-\tau_0 \leq t \leq 0$ and $y(0) = 0$. Equation (3.27) implies that $\sup\{y^j(t) : -\tau_0 \leq t \leq 0\} \geq C_2^{-1}$, so we also have that $\sup\{y(t) : -\tau_0 \leq t \leq 0\} \geq C_2^{-1}$. Lemma 3.4 implies that p^j is a bounded sequence, so by taking a further subsequence we can assume that $p^j \to p < \infty$. It is a straightforward exercise, which we leave to the reader, to prove that y is periodic of period p, and we already have shown that y is not identically zero.

Equation (3.29) implies that for all s,

$$\operatorname{lip}(Q_{\alpha^j}(x^j_s)) \leq \operatorname{lip}(x^j_s) \leq k\|x^j\|.$$

If we use the definition of almost differentiability, we see that

$$\begin{aligned} f(Q_{\alpha^j}(x^j_s)) &= -\beta x^j(s) - \gamma x^j(s - \tau_0) + \eta^j(s), \\ |\eta^j(s)| &\leq \sigma((k+1)\|x^j\|), \end{aligned} \tag{3.30}$$

where $\lim_{\varepsilon \to 0^+} \varepsilon^{-1}\sigma(\varepsilon) = 0$. If we use (3.30) in (3.28) and recall that $y^j(s)$ converges to $y(s)$ uniformly on compact intervals, we obtain

$$y(t) = -\int_0^t (\beta y(s) + \gamma y(s - \tau_0))\, ds.$$

Thus y satisfies equation (3.2), which contradicts Lemma 3.1. This completes the proof of equation (3.26).

For $\varphi \in G^+$ define $\|\varphi\|_{\tau_0} = \sup\{\varphi(s) : -\tau_0 \leq s \leq 0\}$. Our argument above shows that there exists $\delta_* > 0$ with $(\rho\Gamma_1)(\varphi) \neq \varphi$ for $\varphi \in G^+$ satisfying $0 < \|\varphi\|_{\tau_0} \leq \delta_*$. By Theorem 2.1, we can shrink δ_* and assume that $(\rho\Gamma_1)(\varphi) = \Gamma_1(\varphi)$ for $\|\varphi\|_{\tau_0} \leq \delta_*$. Remark 2.1 implies that Γ_1 can be extended continuously to G^+ by defining $\Gamma_1(\varphi) = 0$ for $\varphi \in G^+$ with $\|\varphi\|_{\tau_0} = 0$. For $0 < \delta < \delta_*$, we define V_δ and W_δ by

$$\begin{aligned} W_\delta &= \{\varphi \in G^+ : \|\varphi\|_{\tau_0} < \delta\}, \\ V_\delta &= \{\varphi \in G^+ : \|\varphi\|_{\tau_0} > \delta\}. \end{aligned}$$

If $0 < \delta_1 < \delta < \delta_*$, it follows easily from the additivity property for the fixed point index that

$$i_{G^+}(\rho\Gamma_1, U^+) = i_{G^+}(\rho\Gamma_1, V_{\delta_1}),$$
$$i_{G^+}(\rho\Gamma_1, V_{\delta_1}) + i_{G^+}(\rho\Gamma_1, W_\delta) = i_{G^+}(\rho\Gamma_1, G^+) = 1.$$

(We know that $\rho\Gamma_1$ is a continuous map of the compact, convex set G^+ into itself and hence is homotopic to a constant map of G^+ to G^+, so $i_{G^+}(\rho\Gamma_1, G^+) = 1$.) Combining this information with (3.26), we see that for $0 < \delta < \delta_*$,

$$i_{G^+}(\Gamma_0, U^+) = i_{G^+}(\rho\Gamma_1, U^+) = 1 - i_{G^+}(\Gamma_1, W_\delta). \tag{3.31}$$

Thus to complete the proof it suffices to evaluate $i_{G^+}(\Gamma_1, W_\delta)$ for δ small.

For β and γ as in the statement of our theorem, and for $\psi \in U^+$ and $1 \leq \lambda \leq 2$, we consider solutions $u(t) = u(t; \psi, \lambda)$ of

$$\begin{aligned} u'(t) &= (2-\lambda) f(Q_1(\rho(u_t))) \\ &\quad + (\lambda - 1)(-\beta u(t) - \gamma u(t - \tau_0)) \qquad \text{for } t \geq 0, \\ u|[-M, 0] &= \psi. \end{aligned} \tag{3.32}$$

By using Lemma 2.3 we see that we can define $\zeta_1(\psi, \lambda)$ and $\zeta_2(\psi, \lambda)$ by

$$\zeta_1(\psi, \lambda) = \inf\{t \geq \tau_0 \,:\, u(t; \psi, \lambda) = 0\},$$
$$\zeta_2(\psi, \lambda) = \inf\{t > \zeta_1(\psi, \lambda) \,:\, u(t; \psi, \lambda) = 0\}.$$

Writing $u(t)$ for $u(t; \psi, \lambda)$, and ζ_1 for $\zeta_1(\psi, \lambda)$, and so forth, we have by Lemma 2.3 that $u(t) \leq 0$ for $0 \leq t \leq \zeta_1$, that $u(t) < 0$ for $\zeta_1 - \tau_0 \leq t < \zeta_1$, and that $\zeta_2 - \zeta_1 > \tau_0$ and $u(t) > 0$ for $\zeta_1 < t < \zeta_2$. As usual, we allow $\zeta_1 = \infty$ or $\zeta_2 = \infty$. For $1 \leq \lambda \leq 2$ and $\psi \in U^+$ we define $\widetilde{\Gamma}_\lambda(\psi) := \widetilde{\Gamma}(\psi, \lambda)$ by

$$\widetilde{\Gamma}(\psi, \lambda) = u_{\zeta_2},$$

where u is the solution of (3.32) and $\zeta_2 := \zeta_2(\psi, \lambda)$. If $\zeta_1(\psi, \lambda) = \infty$ or $\zeta_2(\psi, \lambda) = \infty$, we define $\widetilde{\Gamma}(\psi, \lambda) = 0$. The same arguments used in Theorem 2.1 show that $\widetilde{\Gamma}$ is a continuous map of $U^+ \times [1, 2]$ to X_M. We define $\Gamma : U^+ \times [1, 2] \to X_M$ by

$$\Gamma(\psi, \lambda) = Q_1(\widetilde{\Gamma}(\psi, \lambda)),$$

and we extend Γ to $G^+ \times [1, 2]$ by defining $\Gamma(\psi, \lambda) = 0$ for $\psi \in G^+$ with $\psi|[-\tau_0, 0] = 0$. By using Lemma 2.4, we see that there exists $\delta_1 > 0$ such that $\Gamma(\psi, \lambda) \in G^+$ if $\|\psi\|_{\tau_0} \leq \delta_1$. Arguing as in Remark 2.1, we also see that Γ is continuous on $G^+ \times [1, 2]$. Also, the same argument as in Theorem 2.1 shows that if $\Gamma(\psi, \lambda) = \psi$ for $(\psi, \lambda) \in U^+ \times [1, 2]$, then $u(t + \zeta_2) = u(t)$ for all $t \geq -\tau_0$, where

$u(t) := u(t;\psi,\lambda)$ and $\zeta_2 = \zeta_2(\psi,\lambda)$. Furthermore, if this solution is extended to $\mathbb{R}$ by defining $u(t+\zeta_2) = u(t)$ for all $t \le -\tau_0$, then u is a P_2-solution of

$$u'(t) = (2-\lambda)f(Q_1(\rho(u_t))) + (\lambda-1)(-\beta u(t) - \gamma u(t-\tau_0)). \tag{3.33}$$

We now argue essentially as in the proof of equation (3.26). We claim that for all sufficiently small $\delta > 0$,

$$i_{G^+}(\Gamma_1, W_\delta) = i_{G^+}(\Gamma_2, W_\delta). \tag{3.34}$$

To prove equation (3.34), we use the homotopy property of the fixed point index and observe that it suffices to prove that there exists $\delta_2 > 0$ with $\Gamma(\psi,\lambda) \ne \psi$ for $(\psi,\lambda) \in G^+ \times [1,2]$ and $0 < \|\psi\|_{\tau_0} \le \delta_2$. If not, there exists a sequence $(\psi^j,\lambda^j) \in U^+ \times [1,2]$, with $\|\psi^j\|_{\tau_0} \to 0$ as $j \to \infty$ and $\Gamma(\psi^j,\lambda^j) = \psi^j$. We define $\zeta_2^j = \zeta_2(\psi^j,\lambda^j)$ and define $u^j(t)$ to be the P_2-solution of equation (3.33) of period ζ_2^j with $u^j(t) = u(t;\psi^j,\lambda^j)$ for $t \ge -\tau_0$. By using Lemma 2.4 we see that there exists $C_3 > 0$ with

$$\|u^j\| := \sup\{|u^j(t)| : t \in \mathbb{R}\} \le C_3 \sup\{|u^j(t)| : -\tau_0 \le t \le 0\},$$

so $\|u^j\| \to 0$ as $j \to \infty$. Lemma 3.4A implies that there is a constant N with $\zeta_2^j \le N$ for all j. If we define $v^j(t) = u^j(t)\|u^j\|^{-1}$ and use the Ascoli-Arzelà theorem as in the proof of equation (3.26), we find that by taking an appropriate subsequence we can assume that $v^j(t) \to v(t)$ for all t, with uniform convergence on compact t intervals, $\lambda^j \to \lambda$, and $v(t)$ is a P_2-solution of

$$v'(t) = -\beta v(t) - \gamma v(t-\tau_0).$$

This contradicts Lemma 3.2 and proves equation (3.34).

We are now almost in a position to use Lemma 3.2 to complete the proof of Theorem 3.1. Let K and $S_{\beta,\gamma} : K \to K$ be as in Lemma 3.2 and for $\delta > 0$ define $B_\delta = \{\varphi \in K : \|\varphi\| < \delta\}$ and $D = \{\varphi \in K : \|\varphi\| \le A\}$. For δ sufficiently small, Lemma 2.4 implies that $S_{\beta,\gamma}(B_\delta) \subset D \subset K$, so the commutativity property of the fixed point index implies that

$$i_K(S_{\beta,\gamma}, B_\delta) = i_D(S_{\beta,\gamma}, B_\delta). \tag{3.35}$$

We claim that

$$i_D(S_{\beta,\gamma}, B_\delta) = i_{G^+}(\Gamma_2, W_\delta).$$

The proof is a further application of the commutativity property. Define the extension map $j : D \to G^+$ by $j(\theta) = \psi$, where

$$\psi(s) = \begin{cases} \theta(s) & \text{for } -\tau_0 \le s \le 0, \\ \theta(-\tau_0) & \text{for } -M \le s \le -\tau_0. \end{cases}$$

Define the restriction map $\pi : G^+ \to D$ by $\pi(\theta) = \theta|[-\tau_0, 0]$. For fixed $\delta > 0$ sufficiently small, we view Γ_2 as a map from W_δ to G^+. We leave to the reader the easy verification that

$$\Gamma_2 = \Gamma_2 j \pi.$$

Because nonzero fixed points of Γ_2 must lie in U^+ and correspond to P_2-solutions of equation (3.2), we know that 0 is the only fixed point of Γ_2 in W_δ. The commutativity property implies that

$$i_{G^+}(\Gamma_2, W_\delta) = i_{G^+}((\Gamma_2 j)\pi, W_\delta) = i_D(\pi(\Gamma_2 j), B_\delta). \tag{3.36}$$

However, it is not hard to see that on B_δ,

$$\pi(\Gamma_2 j) = S_{\beta,\gamma},$$

so equations (3.35) and (3.36) yield

$$i_{G^+}(\Gamma_2, W_\delta) = i_K(S_{\beta,\gamma}, B_\delta). \tag{3.37}$$

If we now combine equations (3.31), (3.34) and (3.37) we see that for all sufficiently small $\delta > 0$,

$$i_{G^+}(\Gamma_0, U^+) = 1 - i_K(S_{\beta,\gamma}, B_\delta). \tag{3.38}$$

Equation (3.38) and Lemma 3.2 give the assertions about the fixed point index $i_{G^+}(\Gamma_0, U^+)$. In particular, if $i_{G^+}(\Gamma_0, U^+) = 1$, we know that Γ_0 has a fixed point in U^+, and Theorem 2.1 then implies that equation (3.1) has a P_2-solution x with $-B \le x(t) \le A$ for all t. □

REMARK 3.4. Modify the hypotheses of Theorem 3.1 by assuming that f satisfies H2.3A instead of H2.3 and by selecting R as in the statement of H2.3A. For this R define $\widetilde{f} : K(-B, A, M, R) \to \mathbb{R}$ by $\widetilde{f}(\varphi) = f(\varphi)$ if $|f(\varphi)| \le R$, and $\widetilde{f}(\varphi) = \pm R$ if $\pm f(\varphi) > R$. By using H2.3A we can see that if x is a P_2-solution of equation (3.1) and $x|[-M, 0] \in U^+(-B, A, M, \tau_0, R)$, then x is also a P_2-solution of $x'(t) = \widetilde{f}(x_t)$. Furthermore, the definition of Γ_0 is unchanged if one uses $\widetilde{f}$ instead of f. Thus, for purposes of proving Theorem 3.1 under these weaker hypotheses, it suffices (for this choice of R) to use $\widetilde{f}$ instead of f. An examination of the proof of Theorem 3.1 shows that the same arguments remain valid, so all conclusions of Theorem 3.1 remain true if we assume H2.3A instead of H2.3 and take R as in H2.3A. This technical observation is useful in treating some examples.

REMARK 3.5. It is of interest to treat versions of Theorem 3.1 in which one considers parametrized families of functional differential equations

$$x'(t) = f(x_t, \lambda) \qquad \text{for } a < \lambda < b, \tag{3.39}$$

and seeks global bifurcation theorems for the P_2-solutions of equation (3.39). Results of this type were given by Nussbaum in [15] and [18]. Extensions of these results and more detailed expositions can be found in Section 4 of [18],and in [19] and [20]. By using the determination of the fixed point index given in Theorem 3.1, R.D.N. has obtained global bifurcation theorems which are applicable to equations of the type treated in Theorem 3.1. Some care is necessary because Γ_0 may not extend continuously to G^+. Details will be given in another paper.

A related bifurcation theorem for equation (0.2) has been given by P.P. [22].

4. Applications

In this section we shall illustrate how Theorem 3.1 can be applied to prove existence of P_2-solutions for a wide variety of examples.

We begin with a lemma concerning almost Fréchet differentiability at 0. For $M > 0$, recall that $X_M = C([-M, 0])$ is the Banach space of real-valued continuous functions $\varphi : [-M, 0] \to \mathbb{R}$ with the sup norm $\|\varphi\|$. We shall denote elements $\zeta \in \mathbb{R}^{m+1}$ by $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_m)$.

LEMMA 4.1. *For $\delta_0 > 0$, let*

$$U = \{\zeta \in \mathbb{R}^{m+1} : |\zeta_i| < \delta_0 \text{ for } 0 \le i \le m\}$$

and suppose that $h : U \to \mathbb{R}$ is continuously differentiable. For a given $M > 0$, let $B = \{\varphi \in X_M : \|\varphi\| < \delta_0\}$ and assume that, for $0 \le i \le m$, there is a continuous, almost locally Lipschitzian map $r_i : B \to [0, M]$. (Recall Definition 1.1.) Define $f : B \to \mathbb{R}$ by

$$f(\varphi) = h(\varphi(-r_0(\varphi)), \varphi(-r_1(\varphi)), \dots, \varphi(-r_m(\varphi))).$$

Then f is continuous, almost locally Lipschitzian and almost Fréchet differentiable at 0. The almost Fréchet derivative L of f at 0 is given by

$$L(\varphi) = -\sum_{i=0}^{m} \lambda_i \varphi(-\sigma_i), \tag{4.1}$$

where $-\lambda_i = (\partial h/\partial \zeta_i)(0)$ and $\sigma_i = r_i(0)$.

PROOF. The fact that f is continuous and almost locally Lipschitzian follows from Proposition 1.1.

If we select δ_1 satisfying $0 < \delta_1 < \delta_0$, and set $V = \{\varphi \in X_M : \|\varphi\| + \mathrm{lip}(\varphi) \leq \delta_1\}$, the same argument as in Proposition 1.1 shows that there is a constant k such that $r_i|V$ is Lipschitz with Lipschitz constant k for $0 \leq i \leq m$. Because h is C^1 we know that

$$\left| f(\varphi) + \sum_{i=0}^{m} \lambda_i \varphi(-r_i(\varphi)) \right| \leq \rho(\|\varphi\|), \tag{4.2}$$

where

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \rho(\varepsilon) = 0.$$

If $0 < \delta \leq \delta_1$ and $\|\varphi\| + \mathrm{lip}(\varphi) \leq \delta$, we obtain that

$$\begin{aligned} \left| \sum_{i=0}^{m} \lambda_i (\varphi(-r_i(\varphi)) - \varphi(-r_i(0))) \right| &\leq \mathrm{lip}(\varphi) \Big(\sum_{i=0}^{m} |\lambda_i| |r_i(\varphi) - r_i(0)| \Big) \\ &\leq \mathrm{lip}(\varphi) \Big(\sum_{i=0}^{m} |\lambda_i| \Big) k \|\varphi\| \leq k \Big(\sum_{i=0}^{m} |\lambda_i| \Big) \delta^2. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) we see that for $0 < \|\varphi\| + \mathrm{lip}(\varphi) \leq \delta$,

$$\left| f(\varphi) + \sum_{i=0}^{m} \lambda_i \varphi(-\sigma_i) \right| \leq \rho(\|\varphi\|) + k \Big(\sum_{i=0}^{m} |\lambda_i| \Big) \delta^2, \tag{4.4}$$

and equation (4.4) implies that f is almost Fréchet differentiable at 0 with almost Fréchet derivative given by equation (4.1). □

We wish to apply Theorem 3.1 to equations of the form

$$x'(t) = h(x(t), x(t - r_1(x_t)), x(t - r_2(x_t)), \ldots, x(t - r_m(x_t))).$$

In order to do this, it is convenient to list various hypotheses on h and r_i, with $1 \leq i \leq m$. Recall that for A, B and M positive reals, $K(-B, A, M) = \{\varphi \in X_M : -B \leq \varphi \leq A\}$; and if $0 < \tau_0 \leq M$, then $G^+(-B, A, M, \tau_0)$ is given by equation (2.1).

H4.1. There are positive numbers A, B and M and continuous maps $r_i : K(-B, A, M) \to [0, M]$, for $1 \leq i \leq m$, such that each r_i is almost locally Lipschitzian. Furthermore, there exists $\tau_0 \in (0, M]$ such that $r_i(\varphi) = \tau_0$ for all $\varphi \in K(-B, A, M)$ with $\varphi(0) = 0$ and for $1 \leq i \leq m$.

Note that τ_0 is independent of i, for $1 \leq i \leq m$.

H4.2. There are positive numbers A and B, and a locally Lipschitzian function $h : H(-B,A) \to \mathbb{R}$, where

$$H(-B,A) := \{\zeta \in \mathbb{R}^{m+1} : -B \le \zeta_i \le A \text{ for } 0 \le i \le m\}.$$

If

$$\mathring{K}^{m+1} = \{\zeta \in \mathbb{R}^{m+1} : \zeta_i > 0 \text{ for } 0 \le i \le m\},$$

then $h(\zeta) < 0$ for all $\zeta \in H(-B,A)$ with $\zeta \in \mathring{K}^{m+1}$ and $h(\zeta) > 0$ for all $\zeta \in H(-B,A)$ with $-\zeta \in \mathring{K}^{m+1}$. If $g(u) := h(0,u,u,\dots,u)$, then $ug(u) < 0$ for all $u \in [-B,A], u \neq 0$.

H4.3. There is a positive number τ_0, an open neighborhood U of 0 in $\mathbb{R}^{m+1}$ and a continuously differentiable function $h : U \to \mathbb{R}$. If $-\lambda_i = (\partial h/\partial \zeta_i)(0)$ for $0 \le i \le m$, and if

$$\beta := \lambda_0 \quad \text{and} \quad \gamma := \sum_{i=1}^{m} \lambda_i,$$

then $0 \le \beta < \gamma$ and

$$(\gamma^2 - \beta^2)^{1/2} > \nu(\beta/\gamma, \tau_0) \tag{4.5}$$

where $\nu(\beta/\gamma,\tau_0)$ is the unique solution ν with $\pi/2 \le \nu\tau_0 < \pi$ of $\cos(\nu\tau_0) = -\beta/\gamma$.

Recall (see Lemma 3.1) that equation (4.5) is satisfied if and only if the equation

$$z = -\beta - \gamma \exp(-\tau_0 z)$$

has a complex solution z with $\mathrm{Re}(z) > 0$.

If h and r_i, for $1 \le i \le m$, satisfy H4.1 and H4.2, we define $f : K(-B,A,M) \to \mathbb{R}$ by

$$f(\varphi) = h(\varphi(0), \varphi(-r_1(\varphi)), \varphi(-r_2(\varphi)), \dots, \varphi(-r_m(\varphi))). \tag{4.6}$$

THEOREM 4.1. *Assume that h and r_i, for $1 \le i \le m$, satisfy* H4.1, H4.2 *and* H4.3 *and that f is given by equation* (4.6). *In addition assume that there exists $R > 0$ such that for every $\varphi \in G^+(-B,A,M,\tau_0,R) = G^+$ (see* (2.2)) *the equation*

$$x'(t) = f(x_t) \qquad \text{for } t \ge 0,$$
$$x|[-M,0] = \varphi,$$

has a solution $x(t) = x(t;\varphi)$ *with* $-B \le x(t) \le A$ *and* $|x'(t)| \le R$ *for all* $t \ge 0$. *If* $U^+ = U^+(-B, A, M, \tau_0, R)$ *(see* (2.9)*) and* $\Gamma_0 : U^+ \to G^+$ *is defined by equation* (2.25), *then* $i_{G^+}(\Gamma_0, U^+) = 1$, *and there exists a slowly oscillating periodic solution* y *satisfying* $y'(t) = f(y_t)$, *and* $-B \le y(t) \le A$ *and* $|y'(t)| \le R$ *for all* t.

PROOF. This follows immediately from the variant of Theorem 3.1 stated in Remark 3.4. Lemma 4.1 implies that f satisfies H2.1 and that f is almost Fréchet differentiable at 0 with almost Fréchet derivative L given by

$$L(\varphi) = -\beta\varphi(0) - \gamma\varphi(-\tau_0).$$

(Here β and γ are as in H4.3.) Hypothesis H2.2 follows directly from H4.1 and H4.2, and we directly assume that f satisfies H2.3A. □

We need assumptions which directly imply that $-B \le x(t;\varphi) \le A$ for all $t \ge 0$ and for all Lipschitzian $\varphi \in G^+(-B, A, M, \tau_0)$.

COROLLARY 4.1. *Assume that* h *and* r_i, *for* $1 \le i \le m$, *satisfy* H4.1, H4.2 *and* H4.3 *and that* f *is given by equation* (4.6). *For every* $\varphi \in K(-B, A, M)$ *with* $\varphi(0) = A$, *assume that* $f(\varphi) \le 0$; *and for every* $\varphi \in K(-B, A, M)$ *with* $\varphi(0) = -B$ *assume* $f(\varphi) \ge 0$. *Then there exists a slowly oscillating periodic solution* y *with* $y'(t) = f(y_t)$ *and* $-B \le y(t) \le A$ *for all* t.

PROOF. In the notation of Theorem 4.1, Theorem 1.3 implies that for all $\varphi \in K(-B, A, M)$ with $\text{lip}(\varphi) < \infty$, we have $-B \le x(t;\varphi) \le A$ for all $t \ge 0$. Thus Corollary 4.1 follows immediately from Theorem 4.1. □

COROLLARY 4.2. *Let* A, B, M *and* τ_0 *be positive reals with* $\tau_0 \le M$. *Assume that* h *is a function which satisfies* H4.2 *and* H4.3. *For* $1 \le i \le m$, *let* $\rho_i : [-B, A] \to [0, M]$ *be a Lipschitzian map such that* $\rho_i(0) = \tau_0$. *For* $\varphi \in K(-B, A, M)$, *define* $r_i(\varphi) = \rho_i(\varphi(0))$ *and let* f *be defined by equation* (4.6). *Assume that if* $\varphi \in K(-B, A, M)$ *and* $\varphi(0) = A$, *then* $f(\varphi) \le 0$; *and if* $\varphi \in K(-B, A, M)$ *and* $\varphi(0) = -B$, *then* $f(\varphi) \ge 0$. *Then there exists a* P_2*-solution* y *which satisfies*

$$\begin{aligned} y'(t) &= f(y_t) \\ &:= h(y(t), y(t - \rho_1(y(t))), y(t - \rho_2(y(t))), \dots, y(t - \rho_m(y(t)))), \end{aligned}$$

and $-B \le y(t) \le A$ *for all* t.

PROOF. This follows immediately from Corollary 4.1. □

REMARK 4.1. If $m = 1$, Corollary 4.2 yields all the existence results for P_2-solutions which were obtained in [12] and slightly generalized in [13]. As noted in the introduction, the arguments in [12] and [13] do not generalize to give Corollary 4.2.

REMARK 4.2. Corollary 4.1 allows far more general functions r_i than are used in Corollary 4.2. We mention two examples, both of which follow immediately from Corollary 4.1. For $1 \le i \le m$, suppose that n_i is a nonnegative integer and σ_{ij} is a real number satisfying $0 \le \sigma_{ij} \le M$ for $1 \le j \le n_i$. Assume that

$$\rho_i : [-B, A]^{n_i+1} := \prod_{j=0}^{n_i} [-B, A] \to [0, M]$$

is a Lipschitzian map and that $\rho_i(\zeta) = \tau_0$ whenever $\zeta \in \mathbb{R}^{n_i+1}$ and ζ_0, the first component of ζ, equals 0. For $\varphi \in K(-B, A, M)$ and $1 \le i \le m$, define $r_i(\varphi)$ by

$$r_i(\varphi) = \rho_i(\varphi(0), \varphi(-\sigma_{i1}), \varphi(-\sigma_{i2}), \dots, \varphi(-\sigma_{in_i})).$$

Assume that h satisfies H4.2 and H4.3 and f is defined by equation (4.6). If $\varphi \in K(-B, A, M)$ and $\varphi(0) = A$ (respectively, $\varphi(0) = -B$) assume that $f(\varphi) \le 0$ (respectively, $f(\varphi) \ge 0$). Then there exists a P_2-solution x of $x'(t) = f(x_t)$ satisfying $-B \le x(t) \le A$.

As a second example, suppose that, for $1 \le i \le m$, the function $\sigma_i : K(-B, A, M) \to [0, M]$ is almost locally Lipschitzian and that $\sigma_i(\varphi) = 0$ whenever $\varphi(0) = 0$. Let ρ_i be as in Corollary 4.2 and define $r_i(\varphi) = \rho_i(\varphi(-\sigma_i(\varphi)))$. Then r_i satisfies H4.1, and if h satisfies H4.2 and H4.3 and f (given by equation (4.6)) satisfies the condition of Corollary 4.1, one obtains a P_2-solution x of $x'(t) = f(x_t)$.

As an immediate consequence of Corollary 4.1, we also obtain the following result.

COROLLARY 4.3. *Assume that h and r_i, for $1 \le i \le m$, satisfy* H4.1, H4.2 *and* H4.3 *and that f satisfies equation* (4.6). *For every $\zeta \in [-B, A]^{m+1}$ such that ζ_0 (the first component of ζ) equals A (respectively, equals $-B$) assume that $h(\zeta) \le 0$ (respectively, $h(\zeta) \ge 0$). Then there exists a P_2-solution y with $y'(t) = f(y_t)$ and $-B \le y(t) \le A$ for all t.*

One may lose considerable information by using Corollary 4.3 instead of Corollary 4.1. The next corollaries illustrate this point.

COROLLARY 4.4. *Let A, B, M and τ_0 be positive reals with $\tau_0 \le M$. Assume that $h : [-B, A] \times [-B, A] \to \mathbb{R}$ satisfies* H4.2 *and* H4.3 *(with $m = 1$) and that*

$$\sup\{h(A, \zeta_1) : -B \le \zeta_1 \le 0\} \le 0. \tag{4.7}$$

Let $r_1 : K(-B, A, M) \to [0, M]$ be a continuous, almost locally Lipschitzian map. Assume that $r_1(\varphi) = \tau_0$ (respectively, $r_1(\varphi) = 0$) for all $\varphi \in K(-B, A, M)$ such that $\varphi(0) = 0$ (respectively, $\varphi(0) = -B$). Then there exists a slowly oscillating periodic solution y which satisfies $y'(t) = h(y(t), y(t - r_1(y_t)))$ and $-B \le y(t) \le A$ for all t.

PROOF. We know that h and r_1 satisfy H4.1, H4.2 and H4.3. If $\varphi \in K(-B, A, M)$ and $\varphi(0) = A$, equation (4.7) and the negative feedback condition on h imply that

$$f(\varphi) = h(A, \zeta_1) \le 0.$$

If $\varphi \in K(-B, A, M)$ and $\varphi(0) = -B$, it follows that

$$f(\varphi) = h(-B, -B) \ge 0,$$

and so Corollary 4.4 is a consequence of Corollary 4.1. □

REMARK 4.3. Corollary 4.4 can be easily applied to examples in which

$$h(\zeta_0, \zeta_1) = -\lambda\zeta_0 + \lambda g(\zeta_1),$$

where $\zeta_1 g(\zeta_1) < 0$ for all $\zeta_1 \ne 0$, and where $g'(0) = -k < -1$, and λ is sufficiently large. In this case $A = \sup\{g(\zeta_1) : -B \le \zeta_1 \le 0\}$. Details are left to the reader.

Our next corollary is meant as an illustrative example of Corollary 4.1.

COROLLARY 4.5. *Let c_i, for $1 \le i \le m$, be positive numbers with $c_i < c_{i+1}$ for $1 \le i < m$. Let k_i, for $1 \le i \le m$, be positive reals such that*

$$\sum_{i=1}^{m-1} k_i \le 1 \quad \textit{and} \quad \sum_{i=1}^{m} k_i > 1. \tag{4.8}$$

Let $\lambda > 0$ be such that

$$\lambda\left(\left(\sum_{i=1}^{m} k_i\right)^2 - 1\right)^{1/2} > \nu,$$

where

$$\cos\nu = -\left(\sum_{i=1}^{m} k_i\right)^{-1} \quad \textit{and} \quad \pi/2 < \nu < \pi.$$

Define

$$B = c_m^{-1} \quad and \quad A = c_m^{-1} \sum_{i=1}^{m} k_i.$$

Then there exists a P_2-solution x of the equation

$$x'(t) = -\lambda x(t) - \lambda \sum_{i=1}^{m} k_i x(t - 1 - c_i x(t)),$$

with $-B \leq x(t) \leq A$ for all t.

PROOF. Set $M = 1 + \sum_{i=1}^{m} k_i$ and $\tau_0 = 1$ and define $r_i : K(-B, A, M) \to [0, M]$ by $r_i(\varphi) = 1 + c_i\varphi(0)$, for $1 \leq i \leq m$. One can see that H4.1 is satisfied. In our case,

$$h(\zeta_0, \zeta_1, \dots, \zeta_m) = -\lambda\zeta_0 - \sum_{i=1}^{m} \lambda k_i \zeta_i,$$

and h satisfies H4.2 and H4.3. If $\varphi \in K(-B, A, M)$ and f is given by equation (4.6), then

$$f(\varphi) = -\lambda\varphi(0) - \lambda \sum_{i=1}^{m} k_i \varphi(-1 - c_i\varphi(0)).$$

If $\varphi(0) = -B = -c_m^{-1}$, then using the definition of A gives

$$f(\varphi) = \lambda\Big((k_m + 1)c_m^{-1} - \sum_{i=1}^{m-1} k_i\varphi(-1 - c_i\varphi(0))\Big)$$

$$\geq \lambda\Big((k_m + 1)c_m^{-1} - \sum_{i=1}^{m-1} k_i c_m^{-1}\Big(\sum_{i=1}^{m} k_i\Big)\Big).$$

Thus $f(\varphi) \geq 0$ for $\varphi(0) = -B$ provided

$$k_m + 1 - \Big(\sum_{i=1}^{m-1} k_i\Big)\Big(\sum_{i=1}^{m} k_i\Big) =: S \geq 0.$$

Using (4.8), we find that

$$S \geq k_m + 1 - \sum_{i=1}^{m} k_i = 1 - \sum_{i=1}^{m-1} k_i \geq 0.$$

If $\varphi(0) = A$, we find that

$$f(\varphi) \leq -\lambda A + \lambda \sum_{i=1}^{m} k_i B = 0.$$

Thus the hypotheses of Corollary 4.1 are satisfied. □

Corollary 4.1 is obtained from Theorem 4.1 with the aid of Theorem 1.3. Similarly, one can use Theorem 1.4 or Theorem 1.5 to obtain other versions of Theorem 4.1. The arguments are straightforward.

There are less obvious corollaries of Theorem 3.1. We mention without proof a result which has been obtained by one of the authors (R.D.N.) and concerns equations of the form

$$x'(t) = -\sum_{j=1}^{m} f_j(x(t - r_j(x_t))). \tag{4.9}$$

To describe the theorem, we need to list some assumptions on f_j and r_j.

H4.4. For $1 \leq j \leq m$ there are functions $f_j : \mathbb{R} \to \mathbb{R}$ which are locally differentiable and C^1 on a neighborhood of 0. For all nonzero ζ we have $\zeta f_j(\zeta) > 0$. There exists $\gamma_j < \infty$ with

$$\limsup_{|\zeta| \to \infty} \frac{f_j(\zeta)}{\zeta} = \gamma_j.$$

If $M > 0$ and $r : X_M = C([-M, 0]) \to \mathbb{R}$ is a map we shall write

$$\lim_{|\varphi(0)| \to \infty} r(\varphi) = \sigma_0$$

if, for every $\delta > 0$, there exists $C = C(\delta)$ with

$$|r(\varphi) - \sigma_0| < \delta$$

for all $\varphi \in X_M$ such that $|\varphi(0)| \geq C(\delta)$.

H4.5. For some positive M, and for $1 \leq j \leq m$, there are continuous, almost locally Lipschitzian maps $r_j : X_M \to [0, M]$. There exist numbers $\tau_0 > 0$ and $\sigma_0 \geq 0$ (independent of j) such that $r_j(\varphi) = \tau_0$ for all $\varphi \in X_M$ with $\varphi(0) = 0$ and such that

$$\lim_{|\varphi(0)| \to \infty} r_j(\varphi) = \sigma_0.$$

If $\rho_j : \mathbb{R} \to [0, M]$, for $1 \leq j \leq m$, are locally Lipschitzian maps such that

$$\rho_j(0) = \tau_0 \quad \text{and} \quad \lim_{|u| \to \infty} \rho_j(u) = \sigma_0$$

for each j, and if one defines $r_j(\varphi) = \rho_j(\varphi(0))$, then H4.5 is obviously satisfied. However, one can easily construct much more complicated examples.

H4.6. If γ_j, σ_0 and τ_0 are as given in H4.4 and H4.5, then

$$\sigma_0 \sum_{j=1}^{m} \gamma_j < 3/2 \quad \text{and} \quad \tau_0 \sum_{j=1}^{m} {f'}_j(0) > \pi/2.$$

COROLLARY 4.6. *Assume that hypotheses* H4.4, H4.5 *and* H4.6 *are satisfied. Then there exists a slowly oscillating periodic solution x of equation* (4.9).

REMARK 4.4. Corollary 4.6 generalizes a result by Yang Kuang and H. L. Smith in [9]. Smith and Kuang consider equations of the form

$$x'(t) = -f(x(t - \rho(x(t)))). \tag{4.10}$$

They assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian and C^1 near 0 and that $uf(u) > 0$ for all nonzero u. It is assumed that

$$\sup\{u^{-1}f(u) \ : \ u \in \mathbb{R} - \{0\}\} = \gamma < \infty.$$

The map $\rho : \mathbb{R} \to [0, \infty)$ is C^1 and satisfies

$$\rho(0) = \tau_0 > 0 \quad \text{and} \quad \lim_{|u| \to \infty} \rho(u) = \sigma_0.$$

Furthermore, it is assumed that

$$\lim_{|u| \to \infty} |u\rho'(u)| = 0. \tag{4.11}$$

Finally, it is supposed that $\gamma\sigma_0 < 3/2$ and $\tau_0 f'(0) > \pi/2$. Under these assumptions, Kuang and Smith prove the existence of a P_2-solution of (4.10).

Corollary 4.6 is more general in allowing multiple time lags. However, even for one time lag, Corollary 4.6 makes assumptions on the limit of $u^{-1}f(u)$ as $|u| \to \infty$, rather than on the supremum of this quantity for all $u \in \mathbb{R}$. Also, no assumption like equation (4.11) is needed.

Even for simple examples like

$$\begin{aligned} x'(t) &= -kx(t - \rho(x(t))), \\ \rho(u) &= (1 + e^{-u^2})/2, \end{aligned} \tag{4.12}$$

it is unlikely that Corollary 4.6 or the Kuang-Smith result [9] is best possible. Corollary 4.6 implies existence of P_2-solutions of equation (4.12) for $\pi/2 < k < 3$. However, numerical studies (see [9]) suggest existence of P_2-solutions for a much larger range of $k > \pi/2$.

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