Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 3, 1994, 27–54

THE EFFECT OF THE DOMAIN SHAPE ON THE EXISTENCE OF POSITIVE SOLUTIONS OF THE EQUATION $\Delta u + u^{2^*-1} = 0$

DONATO PASSASEO

Dedicated to Jean Leray

1. Introduction

The aim of this paper is to establish some existence and multiplicity results for positive solutions of the following Dirichlet problem:

$$P(\Omega) \qquad \begin{cases} \Delta u + u^{2^*-1} = 0, \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $2^* = 2n/(n-2)$ (2^* is the critical Sobolev exponent for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$).

It is well known that the solutions of Problem $P(\Omega)$ correspond to the non-negative functions u which are constrained critical points for the functional

$$f(u) = \int_\Omega |Du|^2 \, dx$$

constrained on the manifold

$$V(\Omega) = \left\{ u \in H_0^{1,2}(\Omega) \, : \, \int_\Omega |u|^{2^*} \, dx = 1 \right\}.$$

But, because of the presence of the critical Sobolev exponent, this functional does not satisfy the well known Palais-Smale compactness condition. Thus, the standard variational methods to obtain critical points do not apply: even the minimum of f on $V(\Omega)$ does not exist (see Proposition 2.2).

Indeed, the first contribution to the study of Problem $P(\Omega)$ was a nonexistence result due to Pokhozhaev (see [27]): if the bounded domain Ω is star-shaped, then $P(\Omega)$ has no solution.

On the other hand, it is easy to verify (as pointed out by Kazdan-Warner in [16]) that, if Ω is an annulus (i.e. $\Omega = \{x \in \mathbb{R}^n : 0 < r_1 < |x| < r_2\}$), then Problem $P(\Omega)$ has a radial solution. Thus, the existence of solutions for $P(\Omega)$ seems to be strictly related to the geometrical properties of Ω.

Many important researches have been devoted to the study of the effect of the domain shape on the existence and the multiplicity of positive solutions of problems of this type. In this direction, a first result was obtained by Coron in [11]: he showed that, if the domain Ω has a "hole" of sufficiently small size, then Problem $P(\Omega)$ has at least one positive solution. But the most remarkable result in this direction is the following theorem of Bahri and Coron (see [1]): Problem $P(\Omega)$ has at least one positive solution if the domain Ω has nontrivial topology, in a suitable sense (i.e. if some homology groups are nontrivial).

Thus, the following natural question arises, which was pointed out by Brezis in [4]: can one replace in Pokhozhaev's Theorem the assumption "Ω is star-shaped" by "Ω has trivial topology", in other words, are there domains Ω with trivial topology (for example contractible domains) such that Problem $P(\Omega)$ has a solution? Several papers have been devoted to this question (see, for instance, [12], [13], [18], [8]). In [8] some nonexistence results are obtained in bounded domains, which are contractible but not star-shaped, while in [12], [13], [18] the existence of contractible bounded domains Ω where $P(\Omega)$ has solution is proved. In particular, in [18] it is shown that, for every positive integer k, there exists a contractible bounded domain Ω_k such that $P(\Omega_k)$ has at least k distinct positive solutions.

Notice that in [13] it is proved that, under suitable assumptions, the solution obtained by Coron (see [11]) in a domain with a "small hole" persists with respect to a perturbation which changes this domain into a contractible domain. More generally, in [12] it is proved that also the solution given by Bahri and Coron (see [1]) in nontrivial domains persists with respect to similar perturbations, which allows one to obtain contractible domains Ω such that $P(\Omega)$ has a solution. On the contrary, in [18] we find, in the same contractible domains, a new type of solutions which vanish as the perturbation goes to zero, while the ones obtained in [13] and [12] converge to the solution of the limit problem, given by [11] or [1] (see also Remark 3.10 for more details).

In this paper we continue the research begun in [18] and we study in a systematic way the existence and the multiplicity of positive solutions of Problem $P(\Omega)$ depending on the shape of Ω. Some results here exposed have been announced in [18].

Let us remark that the domains considered in [18] have a radial symmetry with respect to an axis, which plays a very important rôle in the proofs given in [18]: in fact, this symmetry allows one to obtain the solutions as local minimum points in the subspace of the radial functions.

On the contrary, no symmetry assumption is required in the results stated here; the solutions are obtained not by minimization techniques, but by means of more complex topological-variational methods (similar methods were also used in [2], [3], [9], [10] for problems with subcritical growth).

The main results obtained in this paper are stated in Theorems 2.5 and 3.1.

As it is also pointed out by means of some simple examples (see Examples 2.17 and 2.18), these results show that every perturbation of a given domain $\widetilde{\Omega}$, which is obtained by removing a closed subset K of small capacity in such a way that the domain $\Omega = \widetilde{\Omega} \setminus K$ has a different type of homotopy than $\widetilde{\Omega}$, gives rise to a positive solution of Problem $P(\widetilde{\Omega} \setminus K)$; moreover, this solution converges to zero as the capacity of K goes to zero, and its existence is not related to the solvability of Problem $P(\widetilde{\Omega})$. Notice that one can also show that several independent perturbations of this type can give rise to several distinct positive solutions (see Remark 3.12). It is evident that this allows one to obtain domains Ω which have a very complex shape but a trivial topology (in particular, contractible domains), where one can prove the existence of a number of positive solutions arbitrarily large, without any symmetry assumption.

Finally, let us mention that in [25] one can find other existence and multiplicity results for Problem $P(\Omega)$ with $\Omega = \widetilde{\Omega} \setminus K$ (and $\operatorname{cap} K$ small enough), where we evaluate the number of positive solutions in terms of relative category of the domain $\widetilde{\Omega}$ with respect to $\widetilde{\Omega} \setminus K$ (see [14]).

2. Existence of positive solutions

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $2^* = 2n/(n-2)$ (the critical Sobolev exponent for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$). It is easy to verify that a function u in $H^{1,2}(\Omega)$, $u \geq 0$ in Ω and $u \not\equiv 0$, is a solution of

Problem $P(\Omega)$ if and only if

$$\int_\Omega |Du|^2\,dx = \int_\Omega |u|^{2^*}\,dx$$

and $\overline{u} = u/\|u\|_{L^{2^*}}$ is a critical point for the functional

$$f(u) = \int_\Omega |Du|^2\,dx$$

constrained on the manifold

$$V(\Omega) = \left\{u \in H_0^{1,2}(\Omega) : \int_\Omega |u|^{2^*}\,dx = 1\right\}.$$

So, solving Problem $P(\Omega)$ is equivalent to looking for constrained critical points for f on $V(\Omega)$.

Since the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is only continuous but not compact, this functional does not satisfy the well known Palais-Smale compactness condition. So it is not possible to apply directly the classical variational methods to find constrained critical points for f on $V(\Omega)$. In particular, the minimum of f on $V(\Omega)$ does not exist. Indeed, the infimum

$$S = \inf\left\{\int_\Omega |Du|^2\,dx : u \in H_0^{1,2}(\Omega),\ \int_\Omega |u|^{2^*}\,dx = 1\right\} \tag{2.1}$$

is a well known positive constant (the best Sobolev constant for the embedding $H_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$), whose properties can be summarized as follows:

PROPOSITION 2.2 (see [6], [27], [30], [15], [17]). *Let S be the best Sobolev constant (see (2.1)). Then:*

a) *S is independent of $\Omega \subseteq \mathbb{R}^n$; it depends only on the dimension n;*
b) *the infimum in (2.1), which defines S, is never achieved when Ω is a bounded domain of $\mathbb{R}^n$;*
c) *only when $\Omega = \mathbb{R}^n$ is the infimum S achieved by the function*

$$\overline{u} = \frac{u}{\|u\|_{L^{2^*}}} \qquad \text{with } u(x) = \frac{1}{(1+|x|^2)^{(n-2)/2}};$$

moreover, all the minimizing functions are of the form

$$\overline{u}_{\sigma,x_0} = \frac{u_{\sigma,x_0}}{\|u_{\sigma,x_0}\|_{L^{2^*}}} \qquad \text{where } u_{\sigma,x_0}(x) = u\left(\frac{x-x_0}{\sigma}\right)$$

with $\sigma > 0$ and $x_0 \in \mathbb{R}^n$;

d) *if $\Omega = \mathbb{R}^n$ and $u \in H^{1,2}(\mathbb{R}^n)$, $u \geq 0$, is a critical point for the functional f constrained on $V(\mathbb{R}^n)$, then $u = \overline{u}_{\sigma,x_0}$ for suitable $\sigma > 0$ and x_0 in $\mathbb{R}^n$.*

Now, let us remind the notion of capacity.

DEFINITION 2.3. Let D be a domain in $\mathbb{R}^n$ and K be a subset of $\overline{D}$. We say that a function $u \in H^{1,2}(D)$ is *nonnegative on K ($u \geq 0$ on K) in the sense of* $H^{1,2}(D)$ if there exists a sequence $(u_i)_i$ in $C^1(\overline{D})$ such that $u_i \geq 0$ on K for all $i \in \mathbb{N}$ and $(u_i)_i$ converges to u in $H^{1,2}(D)$.

If $c \in \mathbb{R}$, we say that $u \geq c$ on K in the $H^{1,2}(D)$ sense if $u - c \geq 0$ on K in the $H^{1,2}(D)$ sense.

If the set

$$\{u \in H_0^{1,2}(D) : u \geq 1 \text{ on } K \text{ in the } H^{1,2}(D) \text{ sense}\}$$

is nonempty, then the *capacity of K with respect to D* ($\text{cap}_D K$) is the following number:

$$\text{cap}_D K = \inf\left\{\int_D |Du|^2\, dx : u \in H_0^{1,2}(D),\ u \geq 1 \text{ on } K \text{ in the } H^{1,2}(D) \text{ sense}\right\}.$$

Moreover, we shall put $\text{cap}_D \emptyset = 0$.

When $D = \mathbb{R}^n$ we shall write simply $\text{cap}\, K$, instead of $\text{cap}_{\mathbb{R}^n} K$.

It is well known that, if the convex closed set

$$\{u \in H_0^{1,2}(D) : u \geq 1 \text{ on } K \text{ in the } H^{1,2}(D) \text{ sense}\}$$

is nonempty, then there exists a unique function $u_K \in H_0^{1,2}(D)$ such that

$$\int_D |Du_K|^2\, dx = \text{cap}_D K;$$

moreover, $u_K = 1$ on K in the sense of $H^{1,2}(D)$.

DEFINITION 2.4. Let $\widetilde{\Omega}$ be a subset of $\mathbb{R}^n$ and H, Ω two subsets of $\widetilde{\Omega}$. We say that *H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω* if does not exist a continuous function $h : H \times [0,1] \to \widetilde{\Omega}$ such that

$$h(x,0) = x \qquad \text{and} \qquad h(x,1) \in \Omega \qquad \forall x \in H.$$

THEOREM 2.5. *Let $\widetilde{\Omega}$ be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 3$ and H be a closed subset contained in $\widetilde{\Omega}$. Then there exists $\varepsilon > 0$ such that the following assertion holds: if Ω is a smooth domain contained in $\widetilde{\Omega}$, with $\text{cap}(\widetilde{\Omega} \backslash \Omega) < \varepsilon$, such that H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω (see Def. 2.4), then Problem $P(\Omega)$ has at least one solution u_Ω.*

PROOF. For the proof, see 2.12.

COROLLARY 2.6. *Let $\widetilde{\Omega}$ be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 3$ and H be a closed subset contained in $\widetilde{\Omega}$ and noncontractible in $\widetilde{\Omega}$. Then there exists $\varepsilon > 0$ such that the following assertion holds: if Ω is a smooth domain contained in $\widetilde{\Omega}$ and contractible in $\widetilde{\Omega}$ and if $\operatorname{cap}(\widetilde{\Omega}\backslash\Omega) < \varepsilon$, then Problem $P(\Omega)$ has at least one solution u_Ω.*

For the proof it suffices to observe that H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω (see Definition 2.4), because Ω is contractible in $\widetilde{\Omega}$ while H is not.

Examples 2.17 show some possible applications of Theorem 2.5 and Corollary 2.6.

Let us recall the following results of P. L. Lions, which will be used in the proof of Theorem 2.5.

THEOREM 2.7 (P. L. Lions [17]). *Let $(u_i)_i$ be a minimizing sequence for the Sobolev constant S, that is*

$$u_i \in H^{1,2}(\mathbb{R}^n), \qquad \int_{\mathbb{R}^n} |u_i|^{2^*}\,dx = 1 \qquad \forall i \in \mathbb{N},$$

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} |Du_i|^2\,dx = S.$$

Then there exist two sequences, $(y_i)_i$ in $\mathbb{R}^n$ and $(\sigma_i)_i$ in $\mathbb{R}^+$, such that the sequence $(\widetilde{u}_i)_i$ in $H^{1,2}(\mathbb{R}^n)$ defined by

$$\widetilde{u}_i(x) = \sigma_i^{-n/2^*} u_i\left(\frac{x+y_i}{\sigma_i}\right)$$

is relatively compact in $L^{2^*}(\mathbb{R}^n)$. Of course, $(\widetilde{u}_i)_i$ is also a minimizing sequence and, if $(\widetilde{u}_i)_i \to \widetilde{u}$ in $L^{2^*}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |D\widetilde{u}|^2\,dx = S.$$

In particular, when the functions u_i in Theorem 2.7 are zero outside a bounded domain in $\mathbb{R}^n$, it is easy to deduce the following proposition:

PROPOSITION 2.8. *If the minimizing sequence $(u_i)_i$ in Theorem 2.7 is in $H_0^{1,2}(\Omega)$, with Ω a bounded domain in $\mathbb{R}^n$, then the sequence $(\sigma_i)_i$ (see Theorem 2.7) satisfies $\lim_{i\to\infty} \sigma_i = +\infty$. Consequently, one can find a subsequence $(u_{i_j})_j$ and a point $\overline{x}$ in $\overline{\Omega}$ such that*

$$\lim_{j\to\infty} \int_\Omega v|u_{i_j}|^{2^*}\,dx = v(\overline{x})$$

for every function v which is continuous in $\overline{\Omega}$.

In order to obtain a suitable compactness property for the functional f constrained on $V(\Omega)$, we shall use a well known result of M. Struwe which, in our notations, can be stated as follows.

THEOREM 2.9 (M. Struwe [29]). *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$. Let $(u_i)_i$ be a Palais-Smale sequence for the functional f constrained on $V(\Omega)$, that is*

$$\sup_{i\in\mathbb{N}} f(u_i) < +\infty \quad \text{and} \quad \operatorname{grad} f_{|V(\Omega)}(u_i) \to 0 \text{ in } H^{-1,2}(\Omega).$$

Then one of the following two cases can happen: either the sequence $(u_i)_i$ is relatively compact in $H_0^{1,2}(\Omega)$, or there exist k solutions $\overline{u}_1, \dots, \overline{u}_k$ $(k \geq 1)$ of the problem

$$\begin{cases} \Delta u + |u|^{2^*-2}u = 0, & \text{in } \mathbb{R}^n, \\ u \in H^{1,2}(\mathbb{R}^n), \quad u \not\equiv 0 & \text{in } \mathbb{R}^n, \end{cases}$$

and a solution $\overline{u}_0$ of the problem

$$\begin{cases} \Delta u + |u|^{2^*-2}u = 0 & \text{in } \Omega, \\ u \in H_0^{1,2}(\Omega) \end{cases}$$

such that $(u_i)_i$ (or a subsequence) has the following properties:

$$u_i \longrightarrow \overline{u}_0 \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |\overline{u}_j|^{2^*} dx\right]^{-1/2^*} \qquad \text{weakly in } H_0^{1,2}(\Omega);$$

$$\lim_{i\to\infty} \int_\Omega |Du_i|^2 dx = \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |D\overline{u}_j|^2 dx\right] \left[\sum_{j=0}^{k} \int_{\mathbb{R}^n} |\overline{u}_j|^{2^*} dx\right]^{-2/2^*}$$

(here we consider $\overline{u}_0$ extended by zero in $\mathbb{R}^n \backslash \Omega$).

Now we can prove the following Palais-Smale condition for the functional f constrained on $V(\Omega)$.

PROPOSITION 2.10. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and $(u_i)_i$ be a sequence in $V(\Omega)$ such that:*

$$\begin{cases} \lim_{i\to\infty} f(u_i) \in]S, 2^{2/n}S[, \\ \operatorname{grad} f_{|V(\Omega)}(u_i) \to 0 & \text{in } H^{-1,2}(\Omega). \end{cases}$$

Then $(u_i)_i$ is relatively compact in $H_0^{1,2}(\Omega)$.

PROOF. Assume, by contradiction, that the sequence $(u_i)_i$ is not relatively compact in $H_0^{1,2}(\Omega)$; then there exist functions $\overline{u}_0, \overline{u}_1, \dots, \overline{u}_k$ with the properties described in Theorem 2.9.

First, let us observe that

$$\int_{\mathbb{R}^n} |D\overline{u}_j|^2 dx = \int_{\mathbb{R}^n} |\overline{u}_j|^{2^*} dx \qquad \forall j = 0, 1, \dots, k$$

(because $\overline{u}_j$ solves the equation $\Delta u + |u|^{2^*-1}u = 0$).

It follows that for every $j = 0, 1, \dots, k$ we have

$$\left[\sum_{j=0}^{k}\int_{\mathbb{R}^n}|D\overline{u}_j|^2\,dx\right]\left[\sum_{j=0}^{k}\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx\right]^{-2/2^*}$$
$$= \left[\sum_{j=0}^{k}\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx\right]^{1-2/2^*} \geq \left[\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx\right]^{1-2/2^*}$$
$$= \left[\int_{\mathbb{R}^n}|D\overline{u}_j|^2\,dx\right]^{1-2/2^*}.$$

Consequently, all the functions $\overline{u}_0, \overline{u}_1, \dots, \overline{u}_k$ have constant sign: otherwise, if $\overline{u}_j^+ \not\equiv 0$ and $\overline{u}_j^- \not\equiv 0$, we should have, by the properties of the Sobolev constant S (see Proposition 2.2)

$$\int_{\mathbb{R}^n}|\overline{u}_j^{\pm}|^{2^*}\,dx = \int_{\mathbb{R}^n}|D\overline{u}_j^{\pm}|^2\,dx \geq S\left(\int_{\mathbb{R}^n}|\overline{u}_j^{\pm}|^{2^*}\,dx\right)^{2/2^*},$$

which implies

$$\left[\int_{\mathbb{R}^n}|\overline{u}_j^{\pm}|^{2^*}\,dx\right]^{1-2/2^*} \geq S,$$

that is, $\int_{\mathbb{R}^n}|\overline{u}_j^{\pm}|^{2^*}\,dx \geq S^{n/2}$; therefore $\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}dx \geq 2S^{n/2}$, in contradiction with the assumption that $\lim_{i\to\infty} f(u_i) < 2^{2/n}S$.

Thus, property d) of Proposition 2.2 implies that, for every $j = 1, 2, \dots, k$,

$$\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx = \int_{\mathbb{R}^n}|D\overline{u}_j|^2\,dx = S\left[\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx\right]^{2/2^*},$$

that is, $\int_{\mathbb{R}^n}|\overline{u}_j|^{2^*}\,dx = S^{n/2}$, which implies

$$\lim_{i\to\infty} f(u_i) = \left[\int_{\mathbb{R}^n}|\overline{u}_0|^{2^*}\,dx + kS^{n/2}\right]^{2/n}.$$

Now, if $\overline{u}_0 = 0$, then $\lim_{i\to\infty} f(u_i) = k^{2/n}S$ with $k \in \mathbb{N}$; if $\overline{u}_0 \not\equiv 0$, we have

$$\left[\int_{\mathbb{R}^n}|\overline{u}_0|^{2^*}\,dx + kS^{n/2}\right]^{2/n} > (k+1)^{2/n}S \qquad \text{with } k \geq 1.$$

In any case we have a contradiction with the assumption

$$\lim_{i\to\infty} f(u_i) \in \left]S, 2^{2/n}S\right[.$$

□

PROPOSITION 2.11. *Let $u \in H_0^{1,2}(\Omega)$ be a critical point for the functional f constrained on $V(\Omega)$. If $f(u) < 2^{2/n}S$, then the function u has a constant sign.*

PROOF. Let us remark that the function u solves the equation

$$\Delta u + \mu |u|^{2^*-2}u = 0 \qquad \text{in } \Omega, \text{ with } \mu = f(u).$$

Assume, by contradiction, that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$; then we have

$$f(u)\int_\Omega |u^\pm|^{2^*}\,dx = \int_\Omega |Du^\pm|^2\,dx \geq S\Big(\int_\Omega |u^\pm|^{2^*}\,dx\Big)^{2/2^*}.$$

We infer that

$$\int_\Omega |u^\pm|^{2^*}\,dx \geq (S/f(u))^{n/2},$$

which implies $1 \geq 2(S/f(u))^{n/2}$, that is, $f(u) \geq 2^{2/n}S$, in contradiction with our assumption. □

2.12. Proof of Theorem 2.5. Let $\beta : V(\widetilde{\Omega}) \to \mathbb{R}^n$ be the "barycentre" function defined by

$$\beta(u) = \int_{\widetilde{\Omega}} x|u(x)|^{2^*}\,dx \qquad \forall u \in V(\Omega)$$

(every function of $H_0^{1,2}(\Omega)$ will be extended in $\widetilde{\Omega}\setminus\Omega$ by zero).

Since $\widetilde{\Omega}$ is a smooth bounded domain in $\mathbb{R}^n$, there exists $\widetilde{r} > 0$ such that $\widetilde{\Omega}$ is a deformation retract of the domain

$$\widetilde{\Omega}^+ = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \widetilde{\Omega}) < \widetilde{r}\}.$$

Using Theorem 2.7 and Proposition 2.8, it is easy to verify that

$$\inf\{f(u) : u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\} > S \tag{2.13}$$

(here we put $\inf \emptyset = +\infty$, if $\{u \in V(\Omega) : \beta(u) \notin \widetilde{\Omega}^+\} = \emptyset$).

By the properties of the Sobolev constant S (see Proposition 2.2) it follows that for every $\mu > S$ there exists $\varphi \in C_0^\infty(B(0,1))$ such that

$$\int_{B(0,1)} |\varphi|^{2^*}\,dx = 1 \qquad \text{and} \qquad \int_{B(0,1)} |D\varphi|^2\,dx < \mu.$$

In particular, we can choose $\mu > S$ such that $\mu < 2^{2/n}S$ and (see (2.13))

$$\mu < \inf\{f(u) : u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\}.$$

For every $\sigma > 0$ and $y \in \mathbb{R}^n$, set

$$\varphi_{\sigma,y}(x) = \varphi\left(\frac{x-y}{\sigma}\right)$$

(here we consider φ extended by zero outside $B(0,1)$).

Since H is a closed set contained in $\widetilde{\Omega}$, there exists $\widetilde{\sigma} > 0$ such that

$$\varphi_{\widetilde{\sigma},y} \in H_0^{1,2}(\widetilde{\Omega}) \qquad \forall y \in H.$$

Put $T_y(\varphi) = \varphi_{\widetilde{\sigma},y}/\|\varphi_{\widetilde{\sigma},y}\|_{L^{2^*}}$ (notice that $f(T_y(\varphi)) = \int_{B(0,1)} |D\varphi|^2\, dx$ for all $y \in H$).

For every $\Omega \subseteq \widetilde{\Omega}$, let $z_\Omega \in H^{1,2}(\mathbb{R}^n)$ be the nonnegative function such that $z_\Omega = 1$ on $\widetilde{\Omega}\backslash\Omega$ in the sense of $H^{1,2}$ and

$$\int_{\mathbb{R}^n} |Dz_\Omega|^2\, dx = \operatorname{cap}(\widetilde{\Omega}\backslash\Omega)$$

(see Definition 2.3). So, we have

$$(1 - z_\Omega) \cdot T_y(\varphi) \in H_0^{1,2}(\Omega) \qquad \forall y \in H$$

and, moreover, for every $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\operatorname{cap}(\widetilde{\Omega}\backslash\Omega) < \varepsilon \Rightarrow \sup_{y\in H} \|z_\Omega T_y(\varphi)\|_{H^{1,2}(\Omega)} < \delta.$$

Consequently,

$$\|(1 - z_\Omega) \cdot T_y(\varphi)\|_{L^{2^*}(\Omega)} \neq 0 \qquad \forall y \in H$$

if $\operatorname{cap}(\widetilde{\Omega}\backslash\Omega)$ is small enough; moreover, if we set

$$\Phi_\Omega(y) = \frac{(1 - z_\Omega) \cdot T_y(\varphi)}{\|(1 - z_\Omega) \cdot T_y(\varphi)\|_{L^{2^*}}},$$

then there exists $\varepsilon > 0$ such that

$$\operatorname{cap}(\widetilde{\Omega}\backslash\Omega) < \varepsilon \Rightarrow \sup\{f \circ \Phi_\Omega(y) : y \in H\} < \mu. \tag{2.14}$$

Notice that $\beta \circ \Phi_\Omega(y) \in B(y, \widetilde{\sigma}) \subseteq \widetilde{\Omega}$ for all $y \in H$; so, the map $\beta \circ \Phi_\Omega : H \to \widetilde{\Omega}$ is homotopically equivalent, in $\widetilde{\Omega}$, to the identity map.

Let Ω be a smooth bounded domain contained in $\widetilde{\Omega}$, with $\operatorname{cap}(\widetilde{\Omega}\backslash\Omega) < \varepsilon$; let $r \in]0, \widetilde{r}[$ be small enough such that Ω is a deformation retract of the domain

$$\Omega^+ = \{x \in \mathbb{R}^n \, : \, \operatorname{dist}(x, \Omega) < r\}.$$

From Theorem 2.7 and Proposition 2.8 we infer that

$$\inf\{f(u) : u \in V(\Omega),\ \beta(u) \notin \Omega^+\} > S. \tag{2.15}$$

Moreover, we have

$$\inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \Omega^+\} \leq \sup\{f \circ \Phi_\Omega(y) \,:\, y \in H\},$$

otherwise the continuous map $k : H \times [0,1] \to \widetilde{\Omega}$ defined by

$$k(y,t) = (1-t)y + t\beta \circ \Phi_\Omega(y) \qquad \forall y \in H,\ \forall t \in [0,1]$$

would be a deformation of H in a subset of Ω^+; since Ω is a deformation retract of Ω^+, it follows that H would be deformable in $\widetilde{\Omega}$ into a subset of Ω, in contradiction with our assumption.

Thus, if $\Omega \subset \widetilde{\Omega}$ and $\operatorname{cap}(\widetilde{\Omega}\backslash\Omega) < \varepsilon$, the topology of the sublevels of f on $V(\Omega)$ can be described by the following inequalities:

$$\begin{aligned} S &< \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \Omega^+\} \\ &\leq \sup\{f \circ \Phi_\Omega(y) \,:\, y \in H\} < \mu \\ &< \inf\{f(u) \,:\, u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\} \\ &\leq \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \widetilde{\Omega}^+\}, \end{aligned}$$

where $\mu < 2^{2/n}S$.

In particular, these inequalities imply that, if c_1 and c_2 are two constants satisfying

$$\begin{aligned} c_1 &< \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \Omega^+\}, \\ c_2 &= \sup\{f \circ \Phi_\Omega(y) \,:\, y \in H\}, \end{aligned}$$

then the sublevel

$$f^{c_1} = \{u \in V(\Omega) \,:\, f(u) \leq c_1\}$$

cannot be a deformation retract of

$$f^{c_2} = \{u \in V(\Omega) \,:\, f(u) \leq c_2\}.$$

In fact, if, by contradiction, there exists a deformation $\vartheta : f^{c_2} \times [0,1] \to f^{c_2}$ such that

$$\begin{cases} \vartheta(u,0) = u & \forall u \in f^{c_2}, \\ \vartheta(u,1) \in f^{c_1} & \forall u \in f^{c_2}, \end{cases}$$

then it is possible to obtain a deformation $h : H \times [0,1] \to \widetilde{\Omega}$ such that

$$h(x,0) = x \qquad \text{and} \qquad h(x,1) \in \Omega \qquad \forall x \in H,$$

in contradiction with the assumptions of Theorem 2.5.

The deformation h can be defined as follows:

$$h(x,t) = (1-3t)x + 3t\beta \circ \Phi_\Omega(x) \qquad \forall x \in H, \ \forall t \in [0,1/3]$$

(notice that $h(x,t) \in B(x,\widetilde{\sigma}) \subset \widetilde{\Omega}$ for all $t \in [0,1/3]$);

$$h(x,t) = \widetilde{\rho} \circ \beta \circ \vartheta(\Phi_\Omega(x), 3t-1) \qquad \forall x \in H, \ \forall t \in [1/3, 2/3],$$

where $\widetilde{\rho} : \widetilde{\Omega}^+ \to \widetilde{\Omega}$ is a continuous map such that $\widetilde{\rho}(x) = x$ for all $x \in \widetilde{\Omega}$ (notice that $\beta(u) \in \widetilde{\Omega}^+$ for every $u \in f^{c_2}$, because $c_2 < \inf\{f(u) \ : \ u \in V(\Omega), \ \beta(u) \notin \widetilde{\Omega}^+\}$).

Since $\vartheta(\Phi_\Omega(x), 1) \in f^{c_1}$ for all $x \in H$ and

$$c_1 < \inf\{f(u) : u \in V(\Omega), \ \beta(u) \notin \Omega^+\},$$

it follows that

$$\beta \circ \vartheta(\Phi_\Omega(x), 1) \in \Omega^+ \qquad \forall x \in H.$$

Therefore, for a fixed continuous function $\rho : \Omega^+ \times [0,1] \to \Omega^+$ such that

$$\rho(x,0) = x \qquad \text{and} \qquad \rho(x,1) \in \Omega \qquad \forall x \in \Omega^+,$$

we can define

$$h(x,t) = \widetilde{\rho} \circ \rho\{\beta \circ \vartheta[\Phi_\Omega(x), 1], 3t-2\} \qquad \forall x \in H, \ \forall t \in [2/3, 1].$$

Now, let us recall that the functional f constrained on $V(\Omega)$ satisfies the Palais-Smale condition in $f^{-1}(\,]S, 2^{2/n}S[\,)$ (see Proposition 2.10) and

$$\begin{aligned} S &< \inf\{f(u) \ : \ u \in V(\Omega), \ \beta(u) \notin \Omega^+\} \\ &\le \sup\{f \circ \Phi_\Omega(y) \ : \ y \in H\} < \mu < 2^{2/n}S. \end{aligned}$$

Therefore, the change of topology between the sublevels f^{c_1} and f^{c_2} implies the existence of a constrained critical value in $[c_1, c_2]$.

Moreover, since c_1 is an arbitrary constant such that

$$c_1 < \inf\{f(u) : u \in V(\Omega), \ \beta(u) \notin \Omega^+\},$$

we infer that there exists a constrained critical point u_Ω satisfying

$$\inf\{f(u) : u \in V(\Omega),\ \beta(u) \notin \Omega^+\} \leq f(u_\Omega) \\ \leq \sup\{f \circ \Phi_\Omega(y) : y \in H\} < \mu < 2^{2/n}S.$$

Finally, notice that u_Ω is a function having constant sign, because $f(u) < 2^{2/n}S$ (see Proposition 2.11). □

REMARK 2.16. In the proof of Theorem 2.5 we can choose the constant $\mu > S$ arbitrarily close to S and then we must take $\Omega \subset \widetilde{\Omega}$ with $\text{cap}(\widetilde{\Omega}\backslash\Omega)$ sufficiently small.

Since the constrained critical point u_Ω satisfies $f(u_\Omega) < \mu$, this shows that

$$f(u_\Omega) \to S \quad \text{as} \quad \text{cap}\,(\widetilde{\Omega}\backslash\Omega) \to 0.$$

So, $u_\Omega \rightharpoonup 0$ weakly in $H_0^{1,2}$ and concentrates near a point, as described in Theorem 2.7 and Proposition 2.8.

2.17. Examples. Here we show some simple situations where Theorem 2.5 and Corollary 2.6 can be applied.

In particular, by using Corollary 2.6, it is easy to obtain some bounded contractible domains, without any symmetry property, where Problem $P(\Omega)$ has a solution (notice that, on the contrary, a rotational symmetry assumption on the domains plays a very important rôle in the proof of the results exposed in [18]).

One can remark that the domains considered in the following examples do not have a smooth boundary, as required in Theorem 1.5. However, one can obviously obtain smooth nearby domains, with the same geometrical properties, satisfying all the assumptions of Theorem 2.5.

On the other hand, let us remark that some technical modifications in the proof of this theorem allow one to state an analogous existence result in domains having only a "piecewise smooth" boundary, like the ones considered in the following examples.

EXAMPLE 1. Choose x_0, x_1 in $\mathbb{R}^n$ with $n \geq 3$ and r_0, r_1 in $\mathbb{R}^+$ such that

$$\overline{B(x_1, r_1)} \subset B(x_0, r_0)$$

and put

$$\widetilde{\Omega} = B(x_0, r_0) \setminus \overline{B(x_1, r_1)}.$$

Let us fix $y_1 \in B(x_1, r_1)$ and $y_0 \notin \overline{B(x_0, r_0)}$, and define

$$[y_0, y_1] = \{(1-\lambda)y_0 + \lambda y_1 : \lambda \in [0,1]\},$$

$$\Omega_\varepsilon = \{x \in \widetilde{\Omega} : \operatorname{dist}(x, [y_0, y_1]) > \varepsilon\}$$

(notice that $\lim_{\varepsilon \to 0^+} \operatorname{cap}(\widetilde{\Omega} \setminus \Omega_\varepsilon) = 0$).

Then Problem $P(\Omega_\varepsilon)$ has at least one solution u_ε for $\varepsilon > 0$ small enough.

In fact the assumptions of Corollary 2.6 are satisfied with $H = \partial B(x_1, r)$ for a suitable $r > r_1$ (H is not contractible in $\widetilde{\Omega}$, while Ω_ε is contractible in $\widetilde{\Omega}$ for every $\varepsilon > 0$).

EXAMPLE 2. Let x_0, x_1, x_2 in $\mathbb{R}^n$ with $n \geq 3$ and r_0, r_1, r_2 in $\mathbb{R}^+$ be chosen in such a way that

$$\overline{B(x_1, r_1)} \subset B(x_0, r_0), \ \overline{B(x_2, r_2)} \subset B(x_0, r_0), \ \overline{B(x_1, r_1)} \cap \overline{B(x_2, r_2)} = \emptyset.$$

Set

$$\widetilde{\Omega} = B(x_0, r_0) \setminus \bigcup_{i=1}^{2} \overline{B(x_i, r_i)}.$$

Let $[y_1, y_2]$ be the segment joining $y_1 \in B(x_1, r_1)$ and $y_2 \in B(x_2, r_2)$, and put

$$\Omega_\varepsilon = \{x \in \widetilde{\Omega} : \operatorname{dist}(x, [y_1, y_2]) > \varepsilon\}.$$

Then $\lim_{\varepsilon \to 0^+} \operatorname{cap}(\widetilde{\Omega} \setminus \Omega_\varepsilon) = 0$ and the assumptions of Theorem 2.5 are satisfied with $H = \partial B(x_1, r)$ for a suitable $r > r_1$ (H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω_ε).

Therefore Problem $P(\Omega_\varepsilon)$ has at least one solution u_ε if $\varepsilon > 0$ is small enough.

EXAMPLE 3. Let $n \geq 4$, and set

$$C = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 = 1, \ x_i = 0 \text{ for } i = 3, \dots, n\};$$
$$\Sigma = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 \leq 1, \ x_i = 0 \text{ for } i = 3, \dots, n\}$$

(notice that $\operatorname{cap} \Sigma = 0$ if $n \geq 4$);

$$T = \{x \in \mathbb{R}^n : \operatorname{dist}(x, C) \leq 1/3\};$$
$$\widetilde{\Omega} = B(0, 2) \setminus T;$$
$$\Omega_\varepsilon = \{x \in \widetilde{\Omega} : \operatorname{dist}(x, \Sigma) > \varepsilon\}.$$

Then Problem $P(\Omega_\varepsilon)$ has at least one solution u_ε for $\varepsilon > 0$ small enough.

In fact, we have

$$\lim_{\varepsilon\to 0^+} \operatorname{cap}(\widetilde{\Omega}\backslash\Omega_\varepsilon) = 0$$

and, if we set

$$H = \{x \in \mathbb{R}^n : \operatorname{dist}(x, C) = 1/2\},$$

the conditions of Theorem 2.5 are satisfied (H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω_ε for every $\varepsilon > 0$).

EXAMPLE 4. Let $3 \le k \le n$ and set

$$C_k = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^k x_i^2 = 1;\ x_i = 0 \text{ for } i = k+1, \dots, n\right\},$$

$$T_k = \{x \in \mathbb{R}^n : \operatorname{dist}(x, C_k) < 1/2\},$$

$$\Sigma_k = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0 \text{ for } i = 0, \dots, k-1;\ 0 \le x_k \le 2;\ \sum_{i=k+1}^n x_i^2 \le 4\right\}$$

(notice that $\operatorname{cap}\Sigma_k = 0$, since $k \ge 3$).

Now, let us define $\widetilde{\Omega} = T_k$ and

$$\Omega_\varepsilon = \{x \in T_k : \operatorname{dist}(x, \Sigma_k) > \varepsilon\}.$$

Then, Problem $P(\Omega_\varepsilon)$ has at least one solution u_ε for $\varepsilon > 0$ small enough: in fact, for every $\varepsilon > 0$, the set $H = C_k$ cannot be deformed in $\widetilde{\Omega} = T_k$ into a subset of Ω_ε and, moreover, $\lim_{\varepsilon\to 0^+} \operatorname{cap}(\widetilde{\Omega}\backslash\Omega_\varepsilon) = 0$.

Let us remark that in all the previous examples the domain $\widetilde{\Omega}$ has nontrivial topology in the sense of Bahri-Coron [1]; so Problem $P(\widetilde{\Omega})$ has at least one solution $\widetilde{u}$; but $\widetilde{u}$ is not the limit of the solutions u_ε of $P(\Omega_\varepsilon)$ given by Theorem 2.5 (in fact, as $\varepsilon \to 0^+$, the functions u_ε converge weakly to zero in $H^{1,2}$, as pointed out in Remark 2.16, and concentrate near some point, as described in Theorem 2.7 and Proposition 2.8).

Therefore, it is very plausible that Problem $P(\Omega_\varepsilon)$ also has another solution $\widetilde{u}_\varepsilon$, which converges to $\widetilde{u}$ as $\varepsilon \to 0^+$.

3. A multiplicity result

In this section we show that certain perturbations of the domain, which change its topology by taking away some subsets having small capacity, can give the existence of more than one solution.

The main result is stated in Theorem 3.1; simple examples of application are shown in 3.8.

THEOREM 3.1. *Let $A, \widetilde{A}$ be two smooth bounded domains in $\mathbb{R}^n$, with $n \geq 3$, and H a subset of A such that*

$$H \subset A \subset \widetilde{A} \quad \textit{and} \quad \operatorname{dist}(H, \partial A) > 0.$$

Moreover, assume that H is contractible in $\widetilde{A}$. Then there exists $\delta > 0$ such that, if $\widetilde{\Omega}$ is a smooth bounded domain with

$$\overline{A} \subset \widetilde{\Omega} \subset \widetilde{A} \quad \textit{and} \quad \operatorname{cap}_{\widetilde{A} \backslash \overline{A}}(\widetilde{A} \backslash \widetilde{\Omega}) < \delta,$$

then

I) *if H is noncontractible in $\widetilde{\Omega}$, then Problem $P(\widetilde{\Omega})$ has at least one solution;*

II) *there exists $\varepsilon > 0$ such that the following assertion holds: if Ω is a smooth bounded domain, with $\Omega \subset \widetilde{\Omega}$ and $\operatorname{cap}(\widetilde{\Omega} \backslash \Omega) < \varepsilon$, and if H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω (see Definition 2.4), then Problem $P(\Omega)$ has at least two solutions.*

PROOF. Choose $\sigma > 0$ small enough in such a way that $\sigma < \operatorname{dist}(H, \partial A)$ and moreover the subset

$$\widetilde{A}^- = \{x \in \widetilde{A} : \operatorname{dist}(x, \partial \widetilde{A}) > \sigma\}$$

is a deformation retract of $\widetilde{A}$ (so, H is contractible in $\widetilde{A}^-$ too).

Let $\varphi \in C_0^\infty(B(0, \sigma))$ be a nonnegative radial function such that $\varphi = 1$ in a neighbourhood of zero.

Set $\overline{\psi}_\lambda(x) = \varphi(x)/(\lambda^2 + |x|^2)^{(n-2)/2}$ and define $\psi_\lambda = \overline{\psi}_\lambda / \|\overline{\psi}_\lambda\|_{L^{2^*}}$. It is well known (see [6]) that $\lim_{\lambda \to 0} f(\psi_\lambda) = S$. In particular, there exists $\lambda_0 > 0$ such that

$$f(\psi_\lambda) < 2^{2/n} S \qquad \forall \lambda \in [0, \lambda_0].$$

Let $T_{\lambda_0} : \widetilde{A}^- \to V(\widetilde{A})$ be the function defined by

$$T_{\lambda_0}(y)[x] = \psi_{\lambda_0}(x - y) \qquad \forall y \in \widetilde{A}^-, \ \forall x \in \widetilde{A}$$

(we consider ψ_{λ_0} extended by zero outside $B(0, \sigma)$).

Let $\widetilde{z} \in H_0^{1,2}(\widetilde{A} \backslash \overline{A})$ be the nonnegative function such that $\widetilde{z} = 1$ on $\widetilde{A} \backslash \widetilde{\Omega}$ in the sense of $H^{1,2}(\widetilde{A} \backslash \overline{A})$ and

$$\int_{\widetilde{A} \backslash \overline{A}} |D\widetilde{z}|^2 \, dx = \operatorname{cap}_{\widetilde{A} \backslash \overline{A}}(\widetilde{A} \backslash \widetilde{\Omega}).$$

Arguing as in [3], one can find $\delta > 0$ small enough such that, if $\text{cap}_{\widetilde{A}\setminus\overline{A}}(\widetilde{A}\setminus\widetilde{\Omega}) < \delta$, then

$$\|(1-\widetilde{z})T_{\lambda_0}(y)\|_{L^{2^*}(\widetilde{\Omega})} \neq 0 \qquad \forall y \in \widetilde{A}^-$$

and moreover, if we set

$$\Phi_{\lambda_0}(y) = \frac{(1-\widetilde{z})T_{\lambda_0}(y)}{\|(1-\widetilde{z})T_{\lambda_0}(y)\|_{L^{2^*}(\widetilde{\Omega})}},$$

we have $\Phi_{\lambda_0} : \widetilde{A}^- \to V(\widetilde{\Omega})$ and

$$\sup\{f \circ \Phi_{\lambda_0}(y) \ : \ y \in \widetilde{A}^-\} < 2^{2/n}S.$$

Choose $\widetilde{r} > 0$ so small that the smooth domain $\widetilde{\Omega}$ is a deformation retract of the domain

$$\widetilde{\Omega}^+ = \{x \in \mathbb{R}^n : \text{dist}(x, \widetilde{\Omega}) < \widetilde{r}\}.$$

Let $\beta : V(\widetilde{A}) \to \mathbb{R}^n$ be the "barycentre" function defined by

$$\beta(u) = \int_{\widetilde{A}} x|u(x)|^{2^*}\, dx$$

(every function of $H_0^{1,2}(\widetilde{\Omega})$ will be extended by zero outside $\widetilde{\Omega}$).

We infer from Theorem 2.7 and Proposition 2.8 that

$$\inf\{f(u) \ : \ u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\} > S.$$

Therefore, since $\lim_{\lambda\to 0} f(\psi_\lambda) = S$, there exists $\lambda_1 \in]0, \lambda_0[$ such that

$$f(\psi_{\lambda_1}) < \inf\{f(u) \ : \ u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\}.$$

Now, let us remark that

$$f[\Phi_{\lambda_1}(y)] = f(\psi_{\lambda_1}) \qquad \forall y \in H$$

(because $\widetilde{z} = 0$ on A); so we have

$$\sup\{f \circ \Phi_{\lambda_1}(y) : y \in H\} < \inf\{f(u) \ : \ u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\}.$$

Let us put

$$c_1 = \inf\{f(u) \ : \ u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\}$$

and denote by c_2 the maximum of

$$\sup\{f \circ \Phi_{\lambda_0}(y) : y \in \widetilde{A}^-\} \quad \text{and} \quad \sup\{f(\psi_\lambda) : \lambda \in [\lambda_1, \lambda_0]\}.$$

Notice that $c_1 \leq c_2$: in fact, there exists $\overline{y} \in \widetilde{A}$ such that $\beta \circ \Phi_{\lambda_0}(\overline{y}) \notin \widetilde{\Omega}^+$ (so we have $c_1 \leq f \circ \Phi_{\lambda_0}(\overline{y}) \leq c_2$); otherwise we should have

$$\beta \circ \Phi_{\lambda_0} : \widetilde{A}^- \to \widetilde{\Omega}^+, \quad \text{with} \quad \beta \circ \Phi_{\lambda_0}(y) = y \ \ \forall y \in H$$

(because $T_{\lambda_0}(y)$ has radial symmetry with respect to y and $\widetilde{z} = 0$ in $B(y,\sigma) \subset \overline{A}$); consequently, the subset H, which is contractible in $\widetilde{A}^-$, would be contractible in $\widetilde{\Omega}^+$ (and also in $\widetilde{\Omega}$, which is a deformation retract of $\widetilde{\Omega}^+$), in contradiction with our assumption.

Thus $S < c_1 \leq c_2 < 2^{2/n}S$. We shall prove that in $[c_1, c_2]$ there exists a critical value for the functional f constrained on $V(\widetilde{\Omega})$.

In fact, assume, by contradiction, that $[c_1, c_2]$ does not contain any critical value for f on $V(\widetilde{\Omega})$.

Since the Palais-Smale condition is satisfied in $]S, 2^{2/n}S[$ (see Proposition 2.10), the nonexistence of critical values in $[c_1, c_2]$ implies that there exists a constant c_1' with the following properties:

a) $f(\psi_{\lambda_1}) < c_1' < c_1$;
b) no critical value lies in $[c_1', c_2]$;
c) the sublevel

$$f^{c_1'} = \{u \in V(\widetilde{\Omega}) : f(u) \leq c_1'\}$$

is a deformation retract of

$$f^{c_2} = \{u \in V(\widetilde{\Omega}) : f(u) \leq c_2\}.$$

In particular, there exists a continuous function $R : f^{c_2} \to f^{c_1'}$ such that $R(u) = u$ for every $u \in f^{c_1'}$.

Let $\Gamma : H \times [0,1] \to \widetilde{A}^-$ be a continuous function such that

$$\Gamma(y,0) = y, \quad \Gamma(y,1) = a_0 \qquad \forall y \in H \tag{3.2}$$

for a suitable $a_0 \in \widetilde{A}^-$ (notice that such a function exists, because H is contractible in $\widetilde{A}^-$).

Let $\widetilde{\rho} : \widetilde{\Omega}^+ \to \widetilde{\Omega}$ be a continuous function such that $\widetilde{\rho}(y) = y$ for every $y \in \widetilde{\Omega}$.

Now we can define a continuous function $h : H \times [0,1] \to \widetilde{\Omega}$ in the following way:

$$h(y,t) = \widetilde{\rho} \circ \beta \circ R \circ \Phi_{\lambda_t}(y) \ \forall y \in H, \quad \forall t \in [0, 1/2]$$

where $\lambda_t = (1-2t)\lambda_1 + 2t\lambda_0$ and

$$\Phi_\lambda(y) = \psi_\lambda(x - y) \qquad \forall y \in H, \ \forall x \in \widetilde{\Omega};$$

$$h(y,t) = \widetilde{\rho} \circ \beta \circ R \circ \Phi_{\lambda_0} \circ \Gamma(y, 2t-1) \qquad \forall y \in H, \ \forall t \in [1/2, 1].$$

It is easy to verify that the function h is well defined: indeed, $\Phi_{\lambda_t}(y)$ and $\Phi_{\lambda_0} \circ \Gamma(y, 2t-1)$ lie in f^{c_2}, where R is defined; R has values in $f^{c_1'}$ and $\beta(u) \in \widetilde{\Omega}^+$ (where $\widetilde{\rho}$ is defined) for every $u \in f^{c_1'}$.

Moreover, h is a continuous function (since $\Gamma(y,0) = y \ \forall y \in H$) and satisfies

$$h(y,1) = \widetilde{\rho} \circ \beta \circ R \circ \Phi_{\lambda_0}(a_0) \quad \text{and} \quad h(y,0) = y \qquad \forall y \in H$$

(notice that $h(y,0) = y$ for $y \in H$ because $\Phi_{\lambda_1}(y) \in f^{c_1'}$ for $y \in H$, $R(u) = u$ for $u \in f^{c_1'}$ and $\beta(\Phi_{\lambda_1}(y)) = y$ for $y \in H$, since $\Phi(y)$ has radial symmetry with respect to y). But this is in contradiction with the assumption that H is not contractible in $\widetilde{\Omega}$: so part I of the theorem is proved.

In order to prove part II, let us remark that

$$\begin{aligned}(3.3) \qquad \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \widetilde{\Omega}^+\} \\ > \inf\{f(u) \,:\, u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\},\end{aligned}$$

because $\Omega \subset \widetilde{\Omega}$.

Let $z \in H^{1,2}(\mathbb{R}^n)$ be a nonnegative function such that $z = 1$ on $\widetilde{\Omega} \backslash \Omega$ in the sense of $H^{1,2}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |Dz|^2 \, dx = \operatorname{cap}(\widetilde{\Omega} \backslash \Omega).$$

Using the same arguments as above, one can find $\varepsilon > 0$ small enough such that, if $\operatorname{cap}(\widetilde{\Omega} \backslash \Omega) < \varepsilon$, then:

a) $\|(1-z)\Phi_{\lambda_0}(y)\|_{L^{2^*}(\Omega)} \neq 0$ for all $y \in \widetilde{A}^-$
(notice that $(1-z)\Phi_{\lambda_0}(y) \in H_0^{1,2}(\Omega)$ for all $y \in \widetilde{A}^-$);

b) if we set $\Phi'_{\lambda_0}(y) = (1-z)\Phi_{\lambda_0}(y) / \|(1-z)\Phi_{\lambda_0}(y)\|_{L^{2^*}(\Omega)}$, we have

$$(3.4) \qquad \sup\{f \circ \Phi'_{\lambda_0}(y) \,:\, y \in \widetilde{A}^-\} < 2^{2/n} S;$$

c) $\|(1-z)\Phi_\lambda(y)\|_{L^{2^*}(\Omega)} \neq 0$ for all $y \in H$ and $\lambda \in [\lambda_1, \lambda_0]$, and if we set

$$\Phi'_\lambda(y) = \frac{(1-z)\Phi_\lambda(y)}{\|(1-z)\Phi_\lambda(y)\|_{L^{2^*}(\Omega)}},$$

then

$$(3.5) \qquad \begin{aligned} &\sup\{f \circ \Phi'_\lambda(y) \,:\, y \in H,\ \lambda \in [\lambda_1, \lambda_0]\} < 2^{2/n} S, \\ &\sup\{f \circ \Phi'_{\lambda_1}(y) \,:\, y \in H\} < \inf\{f(u) \,:\, u \in V(\widetilde{\Omega}),\ \beta(u) \notin \widetilde{\Omega}^+\},\end{aligned}$$

which implies (by (3.3))

$$(3.6) \qquad \sup\{f \circ \Phi'_{\lambda_1}(y) \,:\, y \in H\} < \inf\{f(u) : u \in V(\Omega),\ \beta(u) \notin \widetilde{\Omega}^+\}.$$

Let us choose $r > 0$ small enough such that the smooth bounded domain Ω is a deformation retract of the domain

$$\Omega^+ = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < r\}.$$

Theorem 2.7 and Proposition 2.8 imply that

$$\inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \Omega^+\} > S. \tag{3.7}$$

Let us put:

$$\begin{aligned}
\bar{c}_1 &= \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \Omega^+\},\\
\bar{c}_2 &= \sup\{f \circ \Phi'_{\lambda_1}(y) \,:\, y \in H\},\\
\bar{c}_3 &= \inf\{f(u) \,:\, u \in V(\Omega),\ \beta(u) \notin \widetilde{\Omega}^+\}.
\end{aligned}$$

Moreover, let us denote by $\bar{c}_4$ the maximum of

$$\sup\{f \circ \Phi'_{\lambda_0}(y) \,:\, y \in \widetilde{A}^-\} \quad \text{and} \quad \sup\{f \circ \Phi'_{\lambda}(y) \,:\, y \in H,\ \lambda \in [\lambda_1, \lambda_0]\}.$$

If $\operatorname{cap}(\widetilde{\Omega}\setminus\Omega) < \varepsilon$, then

$$S < \bar{c}_1 \le \bar{c}_2 < \bar{c}_3 \le \bar{c}_4 < 2^{2/n} S.$$

The first inequality is just (3.7).

In order to verify that $\bar{c}_1 \le \bar{c}_2$, let us remark that, since the subset H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω, and since the domains $\widetilde{\Omega}, \Omega$ are deformation retracts of $\widetilde{\Omega}^+$, Ω^+ respectively, it follows that H cannot be deformed in $\widetilde{\Omega}^+$ into a subset of Ω^+.

In particular, this implies that there exists $\bar{y} \in H$ such that $\beta \circ \Phi'_{\lambda_1}(\bar{y}) \notin \Omega^+$: otherwise the continuous map $k : H \times [0,1] \to \widetilde{\Omega}^+$ defined by

$$k(y,t) = (1-t)y + t\beta \circ \Phi'_{\lambda_1}(y) \qquad \forall y \in H,\ \forall t \in [0,1]$$

would be a deformation of H in $\widetilde{\Omega}^+$ into a subset of Ω^+ (notice that $k(y,t) \in B(y,\sigma) \subset \widetilde{\Omega}$ for all $y \in H$ and $t \in [0,1]$).

Therefore, we have

$$\bar{c}_1 \le f \circ \Phi'_{\lambda_1}(\bar{y}) \le \bar{c}_2.$$

The inequality $\bar{c}_2 < \bar{c}_3$ is just (3.6).

In order to prove that $\bar{c}_3 \le \bar{c}_4$, let us remark that the subset H, which is contractible in $\widetilde{A}$, is noncontractible in $\widetilde{\Omega}$, otherwise it would be deformable in $\widetilde{\Omega}$ into a subset of Ω (here we assume, for simplicity, that $\widetilde{\Omega}$ is a connected domain).

Since $\widetilde{A}^-$ and $\widetilde{\Omega}$ are deformation retracts of $\widetilde{A}$ and $\widetilde{\Omega}^+$ respectively, we also see that H is contractible in $\widetilde{A}^-$ but noncontractible in $\widetilde{\Omega}^+$.

It follows that there exists $\overline{y} \in \widetilde{A}^-$ such that $\beta \circ \Phi'_{\lambda_0}(\overline{y}) \notin \widetilde{\Omega}^+$: otherwise we should have $\beta \circ \Phi'_{\lambda_0}(\overline{y}) : \widetilde{A}^- \to \widetilde{\Omega}^+$ and we can define the continuous function $\overline{k} : H \times [0,1] \to \widetilde{\Omega}^+$ by

$$\overline{k}(y,t) = (1-2t)y + 2t\beta \circ \Phi'_{\lambda_0}(y) \qquad \forall y \in H,\ \forall t \in [0,1/2],$$
$$\overline{k}(y,t) = \beta \circ \Phi'_{\lambda_0} \circ \Gamma(y, 2t-1) \qquad \forall y \in H,\ \forall t \in [1/2,1],$$

where Γ is a continuous map with the properties (3.2) (which exists because H is contractible in $\widetilde{A}^-$); but this is a contradiction because H is noncontractible in $\widetilde{\Omega}^+$.

Thus, $\overline{c}_3 \le f \circ \Phi'_{\lambda_0}(\overline{y}) \le \overline{c}_4$.

The last inequality $\overline{c}_4 < 2^{2/n} S$ follows by (3.4) and (3.5).

We shall prove that both intervals $[\overline{c}_1, \overline{c}_2]$ and $[\overline{c}_3, \overline{c}_4]$ (which are disjoint) contain a critical value for the functional f constrained on $V(\Omega)$. So, we obtain two distinct solutions of Problem $P(\Omega)$, corresponding to two distinct critical values.

Let us assume, by contradiction, that $[\overline{c}_1, \overline{c}_2]$ does not contain any critical value for f constrained on $V(\Omega)$. In this case, since the Palais-Smale condition is satisfied in $]S, 2^{2/n}S[$ (see Proposition 2.10), there exists $\overline{c}'_1 \in]S, \overline{c}_1[$ such that no critical value lies in $[\overline{c}'_1, \overline{c}_2]$ and the sublevel

$$f^{\overline{c}'_1} = \{u \in V(\Omega) \ : \ f(u) \le \overline{c}'_1\}$$

is a deformation retract of

$$f^{\overline{c}'_2} = \{u \in V(\Omega) \ : \ f(u) \le \overline{c}_2\}.$$

In particular, there exists a continuous function $\overline{\vartheta} : f^{\overline{c}_2} \times [0,1] \to f^{\overline{c}_2}$ such that

$$\begin{cases} \overline{\vartheta}(u,0) = u & \text{for } u \in f^{\overline{c}_2}, \\ \overline{\vartheta}(u,1) \in f^{\overline{c}'_1} & \text{for } u \in f^{\overline{c}_2}. \end{cases}$$

Thus, we can define a continuous function $\overline{h} : H \times [0,1] \to \widetilde{\Omega}$ in the following way:

$$\overline{h}(y,t) = (1-3t)y + 3t\beta \circ \Phi'_{\lambda_1}(y) \qquad \forall y \in H,\ \forall t \in [0,1/3]$$

(notice that $\overline{h}(y,t) \in B(y,\sigma) \subset \widetilde{\Omega}$ for all $y \in H$ and $t \in [0,1/3]$);

$$\overline{h}(y,t) = \widetilde{\rho} \circ \beta \circ \overline{\vartheta}[\Phi'_{\lambda_1}(y), 3t-1] \qquad \forall y \in H,\ \forall t \in [1/3,2/3],$$

where $\widetilde{\rho} : \widetilde{\Omega}^+ \to \widetilde{\Omega}$ is a continuous function such that $\widetilde{\rho}(x) = x$ for all $x \in \widetilde{\Omega}$ (let us remark that $\beta(u) \in \widetilde{\Omega}^+$ for all $u \in f^{\overline{c}_2}$ because $\overline{c}_2 < \overline{c}_3$);

$$\overline{h}(y,t) = \widetilde{\rho} \circ \rho[\beta \circ \overline{\vartheta}[\Phi'_{\lambda_1}(y), 1], 3t-2] \qquad \forall y \in H, \ \forall t \in [2/3, 1],$$

where $\rho : \Omega^+ \times [0,1] \to \Omega^+$ is a continuous function such that

$$\rho(x,0) = x \quad \text{and} \quad \rho(x,1) \in \Omega \qquad \forall x \in \Omega^+$$

(notice that $\beta \circ \overline{\vartheta}[\Phi'_{\lambda_1}(y), 1] \in \Omega^+$ for all $y \in H$ because $\overline{\vartheta}[\Phi'_{\lambda_1}(y), 1] \in f^{\overline{c}'_1}$ for all $y \in H$ and $\overline{c}'_1 < \overline{c}_1$).

One can easily verify that the function $\overline{h} : H \times [0,1] \to \widetilde{\Omega}$ is well defined, is continuous and has the following properties:

$$\overline{h}(y,0) = y \quad \text{and} \quad \overline{h}(y,1) \in \Omega \qquad \forall y \in H,$$

in contradiction with our assumption that H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω. So, there exists in $[\overline{c}_1, \overline{c}_2]$ a critical value for the functional f constrained on $V(\Omega)$.

Now, we look for a constrained critical value in $[\overline{c}_3, \overline{c}_4]$. Let us assume, by contradiction, that no critical value lies in $[\overline{c}_3, \overline{c}_4]$. In this case, since the Palais-Smale condition holds in $]S, 2^{2/n}S[$ (see Proposition 2.10), there exists $\overline{c}'_3 \in]\overline{c}_2, \overline{c}_3[$ such that the sublevel

$$f^{\overline{c}'_3} = \{u \in V(\Omega) \ : \ f(u) \le \overline{c}'_3\}$$

is a deformation retract of

$$f^{\overline{c}_4} = \{u \in V(\Omega) \ : \ f(u) \le \overline{c}_4\}.$$

In particular, there exists a continuous function $\overline{R} : f^{\overline{c}_4} \to f^{\overline{c}'_3}$ such that $\overline{R}(u) = u$ for all $u \in f^{\overline{c}'_3}$.

Let $\Gamma : H \times [0,1] \to \widetilde{A}^-$ be a continuous function with the properties (3.2) (notice that H is contractible in $\widetilde{A}^-$).

Let us define a continuous function $\overline{h}_1 : H \times [0,1] \to \widetilde{\Omega}$ in the following way:

$$\overline{h}_1(y,t) = (1-3t)y + 3t\beta \circ \Phi'_{\lambda_1}(y) \qquad \forall y \in H, \ \forall t \in [0, 1/3]$$

(notice that $\overline{h}_1(y,t) \in B(y,\sigma) \subset \widetilde{\Omega}$ for $y \in H$ and $t \in [0, 1/3]$);

$$\overline{h}_1(y,t) = \widetilde{\rho} \circ \beta \circ \overline{R} \circ \Phi'_{\lambda_t}(y) \qquad \forall y \in H, \ \forall t \in [1/3, 2/3],$$

where $\lambda_t = (2-3t)\lambda_1 + (3t-1)\lambda_0$ and $\widetilde{\rho} : \widetilde{\Omega}^+ \longrightarrow \widetilde{\Omega}$ is a continuous function such that $\widetilde{\rho}(y) = y$ for all $y \in \widetilde{\Omega}$;

$$\overline{h}_1(y,t) = \widetilde{\rho} \circ \beta \circ \overline{R} \circ \Phi'_{\lambda_0} \circ \Gamma(y, 3t-2) \qquad \forall y \in H,\ \forall t \in [2/3, 1].$$

One can easily verify that the function $\overline{h}_1$ is well defined: in fact, $\Phi'_{\lambda_t}(y)$ and $\Phi'_{\lambda_0} \circ \Gamma(y, 3t-2)$ lie in $f^{\overline{c}_4}$ (by the definition of $\overline{c}_4$) and $\overline{R} : f^{\overline{c}_4} \to f^{\overline{c}'_3}$; moreover, since $\overline{c}'_3 < \overline{c}_3$, we have $\beta(u) \in \widetilde{\Omega}^+$ (where $\widetilde{\rho}$ is defined) for every $u \in f^{\overline{c}'_3}$.

In order to verify the continuity of the function $\overline{h}_1$, it suffices to remark that:

a) $\overline{R} \circ \Phi'_{\lambda_1}(y) = \Phi'_{\lambda_1}(y)$ because $\overline{R}(u) = u$ for all $u \in f^{\overline{c}'_3}$ and $\Phi'_{\lambda_1}(y) \in f^{\overline{c}_2}$ for all $y \in H$, with $\overline{c}_2 < \overline{c}'_3$;

b) $\widetilde{\rho} \circ \beta \circ \Phi'_{\lambda_1}(y) = \beta \circ \Phi'_{\lambda_1}(y)$ for all $y \in H$, because $\beta \circ \Phi'_{\lambda_1}(y) \in \widetilde{\Omega}$ and $\widetilde{\rho}(y) = y$ for all $y \in \widetilde{\Omega}$;

c) $\Gamma(y, 0) = y$ for all $y \in \widetilde{A}^-$.

Moreover, the continuous function $\overline{h}_1 : H \times [0,1] \to \widetilde{\Omega}$ has the following properties:

$$\overline{h}_1(y,0) = y \quad \text{and} \quad \overline{h}_1(y,1) = \widetilde{\rho} \circ \beta \circ \overline{R} \circ \Phi'_{\lambda_0}(a_0) \qquad \forall y \in H,$$

in contradiction with our assumption that H cannot be deformed in $\widetilde{\Omega}$ into a subset of Ω (here we assume that $\widetilde{\Omega}$ is a connected domain).

Finally, let us remark that all the solutions we obtained correspond to critical values in $]S, 2^{2/n}S[$. Therefore, they have constant sign, by Proposition 2.11.

So, the proof of Theorem 3.1 is complete. □

3.8. Examples. Let $B(\overline{x}, \overline{r})$ be a ball in $\mathbb{R}^n$, with $n \geq 3$, such that $|\overline{x}| < \overline{r}$.

Let k be a positive integer such that $0 \leq k \leq n-3$, and assume that $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$ with $\overline{x}_i = 0$ for $i > n-k$.

For every $\delta \in]0, \overline{r} - |\overline{x}|[$, let us define $\widetilde{\Omega}^k_\delta = B(\overline{x}, \overline{r}) \setminus C^k_\delta$, where

$$C^k_\delta = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n-k} x_i^2 \leq \delta^2 \right\}.$$

Let us put

$$\Sigma_k = \Bigg\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=n-k+1}^{n} x_i^2 \leq 4\overline{r}^2,\ 0 \leq x_{n-k} \leq 2\overline{r};$$
$$x_i = 0 \text{ for } i = 1, \dots, n-k-1 \Bigg\}$$

and define

$$\Omega^k_{\delta,\varepsilon} = \{ x \in \widetilde{\Omega}^k_\delta \ : \ \mathrm{dist}(x, \Sigma_k) > \varepsilon \}.$$

Notice that $\operatorname{cap}\Sigma_k = 0$ (because $k \le n-3$) and $\lim_{\varepsilon\to 0} \operatorname{cap}(\widetilde{\Omega}_\delta^k \setminus \Omega_{\delta,\varepsilon}^k) = 0$.

Moreover, if we choose $\overline{\delta} \in]\delta, \overline{r} - |\overline{x}|[$, the subset

$$H = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n-k} x_i^2 = \overline{\delta}^2,\ x_i = 0 \text{ for } i > n-k \right\}$$

(which is contractible, of course, in $B(\overline{x}, \overline{r})$) is noncontractible in $\widetilde{\Omega}_\delta^k$ for every $\delta > 0$ and cannot be deformed in $\widetilde{\Omega}_\delta^k$ into a subset of $\Omega_{\delta,\varepsilon}^k$ for every $\varepsilon > 0$.

Thus, Theorem 3.1 allows to state that

I) there exists $\delta' > 0$ such that Problem $P(\widetilde{\Omega}_\delta^k)$ has at least one positive solution u_δ for every $\delta \in]0, \delta'[$;

II) for every $\delta \in]0, \delta'[$ there exists $\varepsilon(\delta) > 0$ such that Problem $P(\Omega_{\delta,\varepsilon}^k)$ has at least two positive solutions $u_{\delta,\varepsilon}$, $\overline{u}_{\delta,\varepsilon}$ for every $\varepsilon \in]0, \varepsilon(\delta)[$.

Moreover (see Remark 3.11),

$$\lim_{\delta\to 0} f\left(\frac{u_\delta}{\|u_\delta\|_{L^{2^*}}} \right) = S,$$

$$\lim_{\varepsilon\to 0} f\left(\frac{\overline{u}_{\delta,\varepsilon}}{\|\overline{u}_{\delta,\varepsilon}\|_{L^{2^*}}} \right) = S \qquad \forall \delta \in]0, \delta'[\,.$$

Therefore, the solutions u_δ (as $\delta \to 0$) and $\overline{u}_{\delta,\varepsilon}$ (as $\varepsilon \to 0$) converge weakly to zero in $H_0^{1,2}$ and concentrate near a point, as described in Theorem 2.7 and Proposition 2.8.

On the contrary, the solution $u_{\delta,\varepsilon}$, as $\varepsilon \to 0$, converges in $H_0^{1,2}(\widetilde{\Omega}_\delta^k)$ to the solution u_δ for every $\delta \in]0, \delta'[$ (here we consider $u_{\delta,\varepsilon}$ extended by zero in $\widetilde{\Omega}_\delta^k \setminus \Omega_{\delta,\varepsilon}^k$).

Let us remark that the domains considered in this example have a boundary which is only "piecewise smooth", while Theorem 3.1 concerns smooth domains. However, one can easily obtain smooth nearby domains which satisfy all the assumptions of Theorem 3.1; on the other hand, some technical modifications in the proof allow one to obtain an analogous existence and multiplicity result in domains with "piecewise smooth" boundary, like the ones introduced above.

REMARK 3.9. Part I of Theorem 3.1 gives, as a particular case, the well known existence result of Coron, concerning a domain with a "small hole" (see [11]); notice that the domain $\widetilde{\Omega}_\delta^k$ introduced in Examples 3.8 is just a domain of this type when $k = 0$.

However, it is evident that the assumptions of Theorem 3.1 allow domains with "holes of small capacity", having a more general shape than the ones of [11] (the holes are not necessarily contained in small spheres as in [11]).

REMARK 3.10. As a particular case of part II of Theorem 3.1 one can also obtain the result of Ding (see [13]).

In fact, if $k = 0$ and $\overline{x} = 0$, the domain $\Omega^k_{\delta,\varepsilon}$ introduced in Examples 3.8 is just the one considered by Ding in [13].

However, let us remark that (if $\delta > 0$ is small enough and $\varepsilon \in]0, \varepsilon(\delta)[$) in [13] the existence of only one solution of Problem $P(\Omega^0_{\delta,\varepsilon})$ is proved; this solution corresponds to our solution $u_{\delta,\varepsilon}$ (see Example 3.8). Moreover, in [13] a complete result is stated only for $n \geq 4$ (a weaker result is obtained in the case $n = 3$).

On the contrary, we obtain one positive solution of Problem $P(\Omega^0_{\delta,\varepsilon})$, for $\varepsilon > 0$ small enough, without the assumption that δ is small (see Theorem 2.5, Corollary 2.6, and 2.17 Example, 1); if, in addition, we assume that δ is small enough, then Theorem 3.1 ensures the existence of *two* positive solutions of Problem $P(\Omega^0_{\delta,\varepsilon})$ (for $0 < \delta < \delta'$ and $\varepsilon \in]0, \varepsilon(\delta)[$).

Moreover, the method used here holds for $n \geq 4$ as well as for $n = 3$.

REMARK 3.11. In analogy with Remark 2.16, we can also describe the asymptotic behaviour for the solutions given by Theorem 3.1.

The proof of this theorem shows that the solution $\widetilde{u}$ of $P(\widetilde{\Omega})$ obtained in part I satisfies

$$f\left(\frac{\widetilde{u}}{\|\widetilde{u}\|_{L^{2^*}}}\right) \to S \quad \text{as} \quad \operatorname{cap}_{\widetilde{A}\setminus\overline{A}}(\widetilde{A}\setminus\widetilde{\Omega}) \to 0;$$

likewise, among the two solutions ($\overline{u}$ and u) of Problem $P(\Omega)$, obtained in part II, the one corresponding to the lower critical value (say $\overline{u}$) satisfies

$$f\left(\frac{\overline{u}}{\|\overline{u}\|_{L^{2^*}}}\right) \to S \quad \text{as} \quad \operatorname{cap}(\widetilde{\Omega}\setminus\Omega) \to 0.$$

Therefore, the solutions $\widetilde{u}$ (as $\operatorname{cap}_{\widetilde{A}\setminus\overline{A}}(\widetilde{A}\setminus\widetilde{\Omega}) \to 0$) and $\overline{u}$ (as $\operatorname{cap}(\widetilde{\Omega}\setminus\Omega) \to 0$) converge to zero weakly in $H^{1,2}_0$ and concentrate near a point, as described in Theorem 2.7 and Proposition 2.8.

On the contrary, the proof of Theorem 3.1 suggests that the solution u of Problem $P(\Omega)$, which corresponds to an upper critical value, converges (as $\operatorname{cap}(\widetilde{\Omega}\setminus\Omega) \to 0$) to the solution $\widetilde{u}$ of Problem $P(\widetilde{\Omega})$ obtained in part I of the theorem.

Notice that the assumption that $\operatorname{cap}_{\widetilde{A}\setminus\overline{A}}(\widetilde{A}\setminus\widetilde{\Omega})$ is sufficiently small has been used only to prove the existence of the solutions $\widetilde{u}$ (of Problem $P(\widetilde{\Omega})$) and u (of Problem $P(\Omega)$); on the contrary, the existence of the solution $\overline{u}$ of $P(\Omega)$ can also be deduced from Theorem 2.5, and so it does not require this assumption, but only that $\operatorname{cap}(\widetilde{\Omega}\setminus\Omega)$ is small enough.

REMARK 3.12. Theorems 2.5 and 3.1 show that the existence of positive solutions for Problem $P(\Omega)$ is related to certain perturbations of small capacity, which modify the topological properties of the domain. Let us point out that several independent perturbations can give rise to several positive solutions.

One can consider, for example, domains Ω having k "holes" of suitable size in such a way that there exist k positive solutions.

Moreover, if we connect these "holes" by means of "tunnels" which are thin enough, the number of positive solutions increases further on.

In order to give a concrete example, consider k distinct points $x_1, \dots, x_k$ in $B(0,1)$ and h segments $\Sigma_1, \dots, \Sigma_h$, pairwise disjoint, whose extremities lie in $\{x_1, \dots, x_k\}$ or in $\mathbb{R}^n \setminus \overline{B(0,1)}$. Let us put

$$T(\Sigma_i, \varepsilon_i) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Sigma_i) \le \varepsilon_i\}$$

and

$$\Omega = B(0,1) \setminus \left\{ \left[\bigcup_{i=1}^{k} \overline{B(x_i, r_i)} \right] \cup \left[\bigcup_{j=1}^{h} T(\Sigma_j, \varepsilon_j) \right] \right\}.$$

Then it is possible to choose the positive numbers $r_1, \dots, r_k, \varepsilon_1, \dots, \varepsilon_h$ in such a way that Problem $P(\Omega)$ has at least $h+k$ distinct solutions (and a very plausible conjecture is that the number of positive solutions is at least $2^{(k+h)}-1$, in analogy with a well known result of Rey [28]).

It is evident that in this way we can obtain domains Ω, without any symmetry property, having a very complex shape but trivial topology (contractible domains, for instance), where the number of positive solutions is arbitrarily large. All these remarks will be developed and proved in a paper in preparation.

REFERENCES

[1] A. BAHRI AND J. M. CORON, *On a nonlinear elliptic equation involving the Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253–294.

[2] V. BENCI AND G. CERAMI, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal. **11** (1991), 79–93.

[3] V. BENCI, G. CERAMI AND D. PASSASEO, *On the number of positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis: A Tribute in Honour of G. Prodi (A. Ambrosetti and A. Marino, eds.), Quaderni Scuola Norm. Sup. Pisa, 1991, pp. 93–107.

[4] H. BREZIS, *Elliptic equations with limiting Sobolev exponents — The impact of topology*, Proceedings 50th Anniv. Courant Inst., Comm. Pure Appl. Math. **39** (1986), 517–539.

[5] H. BREZIS AND E. LIEB, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[6] H. BREZIS AND L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[7] ———, *A minimization problem with critical exponent and non zero data*, Symmetry in Nature, Scuola Norm. Sup., Pisa, 1989, pp. 129–140.

[8] A. CARPIO RODRIGUEZ, M. COMTE AND R. LEVANDOWSKI, *A nonexistence result for a nonlinear equation involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire (to appear).

[9] G. CERAMI AND D. PASSASEO, *Existence and multiplicity of positive solutions for nonlinear elliptic problems in exterior domains with rich topology*, Nonlinear Anal. **18** (1992), 109–119.

[10] ———, *Existence and multiplicity results for semilinear elliptic Dirichlet problems in exterior domains*, Nonlinear Anal. (to appear).

[11] J. M. CORON, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), 209–212.

[12] E. N. DANCER, *A note on an equation with critical exponent*, Bull. London Math. Soc. **20** (1988), 600–602.

[13] W. DING, *Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains*, J. Partial Differential Equations **2** (1989), 83–88.

[14] E. N. FADELL, *Lectures in cohomological index theories of G-spaces with applications to critical points theory*, Rend. Sem. Dip. Mat. Univ. Calabria **6** (1985).

[15] B. GIDAS, W. M. NI AND L. NIRENBERG, *Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$*, Part A, Mathematical Analysis and Applications (L. Nachbin, ed.), Academic Press, Orlando, Fla., 1981, pp. 370–401.

[16] J. KAZDAN AND F. WARNER, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.

[17] P. L. LIONS, *The concentration-compactness principle in the calculus of variations: the limit case*, Rev. Mat. Iberoamericana **1** (1985), 145–201; 45–121.

[18] D. PASSASEO, *Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains*, Manuscripta Math. **65** (1989), 147–166.

[19] ———, *Problemi ellittici con esponente critico. Forma del dominio e molteplicità di soluzioni positive*, preprint, Dip. Mat. Pisa **564** (1990).

[20] ———, *Esistenza e molteplicità di soluzioni positive per l'equazione $-\Delta u + a(x)u = u^{2^*-1}$ in domini limitati*, preprint, Dip. Mat. Pisa **563** (1990).

[21] ———, *Su alcune successioni di soluzioni positive di problemi ellittici con esponente critico*, Rend. Mat. Acc. Lincei **3** (1992), 15–21.

[22] ———, *Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains*, J. Funct. Anal. **114** n. 1 (1993), 97–105.

[23] ———, *Esistenza e molteplicità di soluzioni positive per equazioni ellittiche con nonlinearità sopracritica*, Rend. Acc. Naz. Scienze detta dei XL. Memorie di Matematica 110 **XVI**, 2 (1992), 77–98.

[24] ———, *Molteplicità di soluzioni positive per l'equazione $\Delta u + \lambda u + u^{2^*-1} = 0$ in aperti non contrattili*, preprint, Dip. Mat. Pisa **668** (1992).

[25] ———, *Categoria relativa e molteplicità di soluzioni positive per l'equazione $\Delta u + u^{2^*-1} = 0$*, preprint, Dip. Matem. Pisa **701** (1992).

[26] ———, *Elliptic equations with critical nonlinearity. The effect of the domain shape on the number of positive solutions*, in preparation.

[27] S. I. POKHOZHAEV, *Eigenfunctions of the equation $\Delta u + lf(u) = 0$*, Soviet Math. Dokl. **6** (1965), 1408–1411.

[28] O. REY, *Sur un problème variationnel non compact: l'effet de petits trous dans le domaine*, C. R. Acad. Sci., Paris; Série I **308** (1989), 349–352.

[29] M. STRUWE, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **187** (1984), 511–517.

[30] G. TALENTI, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

Manuscript received November 15, 1993

DONATO PASSASEO
Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti 2, 56127 Pisa, ITALY