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MULTIPLE PERIODIC SOLUTIONS OF AUTONOMOUS SEMILINEAR WAVE EQUATIONS

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Dedicated to Jean Leray

1. Introduction

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(0) = 0$. We are concerned with the existence of nontrivial weak solutions u in $C^1([0,\pi] \times \mathbb{R})$ to the autonomous hyperbolic problem

$$(1.1) \qquad \begin{cases} u_{tt} - u_{xx} = g(u) & \text{in }]0,\pi[\times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x,t+T) = u(x,t) & \text{on } [0,\pi] \times \mathbb{R}, \end{cases}$$

where $T > 0$ is a rational multiple of π. We say that $u \in C^1([0,\pi] \times \mathbb{R})$ is a weak solution if the semilinear hyperbolic equation is satisfied in the distribution sense, while the boundary and periodicity conditions are satisfied in the classical sense.

Since the fundamental paper of Rabinowitz [20] concerning the superlinear case, several authors have treated this problem. Here we are interested in the case where g has linear growth at infinity. When the right hand side has the general form $g(x,t,u)$, the existence of at least one nontrivial solution, or even multiple solutions, has been established in [1], [2], [11], [12], [17]. In the autonomous

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case, the presence of a S^1-action allows one to prove multiplicity results of a different kind. The first paper in this direction is [3], where the right hand side of the form $g(x,u)$ is considered. The nonlinearity g is supposed to be monotone in the second variable. Moreover, suitable regularity is imposed on g with uniform bounds on $D_u g$, in order to perform a finite-dimensional reduction of the problem. When g is independent of x, and $T = 2\pi b/a$ with $a, b \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and b odd, the monotonicity assumption on g has been removed in [13]. Since the same finite-dimensional reduction is used, uniform bounds on $D_u g$ are still imposed. These conditions on $D_u g$ are relaxed in [4], [10], where the problem is directly treated in an infinite-dimensional setting, but only the case $sg(s) \leq s^2$ is considered, because of the difficulties created by the fixed point set of the S^1-action. Finally, let us point out that further results can be obtained when g is odd (see [3], [4], [8], [13], [16]).

Our aim is to prove a multiplicity result of the type of [4], including also the case $j^2 s^2 \leq sg(s) \leq (j+1)^2 s^2$. As in [4] and [13], we do not assume the monotonicity condition on g, but we restrict ourselves to the case where g is independent of x and the given period has the form $T = 2\pi b/a$ with b odd. On the other hand, we work directly in an infinite-dimensional setting, as in [4]. Actually, the only regularity we impose on g is just continuity.

To state our result, let us consider the set

$$\left\{ j^2 - \frac{4\pi^2}{T^2} k^2 : j \in \mathbb{N}^*,\ k \in \mathbb{N},\ j \text{ is odd and } k \text{ is even} \right\},$$

whose role will be explained in the third section. Since this set is unbounded from above and from below and has no finite accumulation point, we can denote by $\{\lambda_h : h \in \mathbb{Z}\}$ a strictly increasing enumeration of it.

We assume that

(A_1) $T = 2\pi b/a$ with $a, b \in \mathbb{N}^*$ and b odd;

(A_2) there exists $h \in \mathbb{Z}$ such that

$$\lambda_h < \liminf_{|s|\to\infty} \frac{g(s)}{s} \leq \limsup_{|s|\to\infty} \frac{g(s)}{s} < \lambda_{h+1};$$

(A_3) either $sg(s) \leq s^2$ for every $s \in \mathbb{R}$, or there exists $j \in \mathbb{N}^*$ such that $j^2 s^2 \leq sg(s) \leq (j+1)^2 s^2$ for every $s \in \mathbb{R}$.

Let us state our main result.

THEOREM 1.2. *Besides* (A_1), (A_2) *and* (A_3), *assume that either*

$$\liminf_{s\to 0} \frac{g(s)}{s} > \lambda_{h+1}$$

or

$$\limsup_{s\to 0} \frac{g(s)}{s} < \lambda_h.$$

In the first case denote by m the number of elements of

$$\left\{(j,k) \in \mathbb{N}^* \times \mathbb{N}^* : j \text{ is odd, } k \text{ is even and } \lambda_{h+1} \le j^2 - \frac{4\pi^2}{T^2}k^2 < \liminf_{s\to 0} \frac{g(s)}{s}\right\},$$

in the second case the number of elements of

$$\left\{(j,k) \in \mathbb{N}^* \times \mathbb{N}^* : j \text{ is odd, } k \text{ is even and } \limsup_{s\to 0} \frac{g(s)}{s} < j^2 - \frac{4\pi^2}{T^2}k^2 \le \lambda_h\right\}.$$

Then (1.1) *possesses at least m weak solutions of class $C^1([0,\pi]\times\mathbb{R})$, which are geometrically distinct and nonconstant in time. Moreover, if g is of class C^k, these solutions are of class $C^{k+1}([0,\pi]\times\mathbb{R})$.*

Let us point out that the possibility

$$\lim_{s\to 0} \frac{g(s)}{s} = -\infty$$

is not excluded. We then get the existence of infinitely many weak solutions.

In the next section we recall from [14] a critical point theorem which will play a crucial role in our argument. In the third section we prove Theorem 1.2.

2. A recall of critical point theory

Let X be a real Hilbert space on which S^1 acts by orthogonal transformations, let $\mathrm{Fix}\,(S^1)$ be the set of fixed points of the action and let S_r be the sphere centered at the origin of radius r.

Let $f : X \to \mathbb{R}$ be a functional of the form

$$f(x) = \frac{1}{2}(Lx|x) - \psi(x),$$

where $L : X \to X$ is linear, continuous, symmetric and equivariant, $\psi : X \to \mathbb{R}$ is of class C^1 and invariant and $\nabla\psi : X \to X$ is compact.

In [14] we have proved the following result.

THEOREM 2.1. *Assume there exist two closed invariant linear subspaces V, W of X and $r > 0$ with the following properties:*

a) *$V + W$ is closed and of finite codimension in X;*
b) *$\mathrm{Fix}\,(S^1) \subseteq V + W$;*
c) *$L(W) \subseteq W$;*
d) *$\sup_{S_r\cap V} f < +\infty$ and $\inf_W f > -\infty$;*
e) *$x \notin \mathrm{Fix}\,(S^1)$ whenever $\nabla f(x) = 0$ and $\inf_W f \le f(x) \le \sup_{S_r\cap V} f$;*
f) *f satisfies $(PS)_c$ whenever $\inf_W f \le c \le \sup_{S_r\cap V} f$.*

Then f possesses at least

$$\frac{1}{2}(\dim(V \cap W) - \operatorname{codim}_X(V + W))$$

distinct critical orbits in $f^{-1}([\inf_W f, \sup_{S_r \cap V} f])$.

The above theorem is related to similar results contained in [5]–[7], which were the basic tool used in [4]. The main difference consists in the fact that we require only $\operatorname{Fix}(S^1) \subseteq V + W$, while in those papers one must have either $\operatorname{Fix}(S^1) \subseteq V$ or $\operatorname{Fix}(S^1) \subseteq W$. Just this improvement allows us to treat the case $j^2 s^2 \leq sg(s) \leq (j+1)^2 s^2$.

3. Proof of the main result

By the change of variable $\tilde{u}(x,t) = u(x, t/\omega)$, $\omega = 2\pi/T$, problem (1.1) can be transformed into the equivalent problem

$$\begin{cases} \omega^2 u_{tt} - u_{xx} = g(u) & \text{in }]0,\pi[\times \mathbb{R}, \\ u(0,t) = u(\pi,t) = 0 & \text{on } \mathbb{R}, \\ u(x, t+2\pi) = u(x,t) & \text{on } [0,\pi] \times \mathbb{R}. \end{cases} \tag{3.1}$$

From [9], [15], [18] and [19], we recall some basic facts about the weak formulation of (3.1).

Let $Q =]0,\pi[\times]0, 2\pi[$. In $L^2(Q;\mathbb{C})$ we can consider the Hilbert basis given by

$$e_{jk}(x,t) = \frac{1}{\pi} \sin(jx) \exp(ikt), \qquad j \in \mathbb{N}^*,\ k \in \mathbb{Z}.$$

It is readily seen that $\omega^2 (e_{jk})_{tt} - (e_{jk})_{xx} = \lambda_{jk} e_{jk}$, with $\lambda_{jk} = j^2 - \omega^2 k^2$.

Let

$$E = \left\{ u \in L^2(Q;\mathbb{R}) \, : \, \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} (1 + |\lambda_{jk}|) |u_{jk}|^2 < +\infty \right\},$$

where

$$u_{jk} = \frac{1}{\pi} \int_Q u(x,t) \sin(jx) \exp(-ikt) \, dx \, dt.$$

The space E, endowed with the scalar product

$$(u|v) = \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} (1 + |\lambda_{jk}|) u_{jk} \overline{v_{jk}},$$

is a real Hilbert space continuously embedded in $L^2(Q;\mathbb{R})$.

Let $A : E \to E$ be the linear, continuous and symmetric operator defined by

$$(Au|v) = \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} \lambda_{jk} u_{jk} \overline{v_{jk}} = \sum_{j=1}^{+\infty} \sum_{k=-\infty}^{+\infty} (1 + |\lambda_{jk}|) \frac{\lambda_{jk}}{1 + |\lambda_{jk}|} u_{jk} \overline{v_{jk}},$$

so that $R(A) = \{u \in E : \lambda_{jk} = 0 \Rightarrow u_{jk} = 0\}$, and let $f : E \to \mathbb{R}$ be the functional defined by

$$f(u) = \frac{1}{2}(Au|u) - \int_Q G(u)\, dx\, dt,$$

where $G(s) = \int_0^s g(\sigma)\, d\sigma$. By assumption ($A_2$), it is readily seen that f is well defined and of class C^1. Moreover, if u is a critical point of f with $u \in R(A)$, then $u \in C^1([0, \pi] \times \mathbb{R})$ and u is a weak solution of (3.1). If, in addition, $g \in C^k(\mathbb{R})$, then $u \in C^{k+1}([0, \pi] \times \mathbb{R})$.

PROPOSITION 3.2. *Let $u \in E$ be a critical point of f with u independent of t. Then $f(u) = 0$.*

PROOF. Let us consider the case $j^2 s^2 \le sg(s) \le (j+1)^2 s^2$. Let $c : \mathbb{R} \to \mathbb{R}$ be the Borel function defined by

$$c(s) = \begin{cases} g(s)/s & \text{if } s \neq 0, \\ j^2 + j + 1/2 & \text{if } s = 0. \end{cases}$$

For every $v \in H_0^1(0, \pi)$ we have

$$\int_0^\pi (u_x v_x - c(u) u v)\, dx = 0.$$

Let $u = u_1 + u_2$, with $u_1, u_2 \in H_0^1(0, \pi)$, $\int_0^\pi (u_{1,x})^2\, dx \le j^2 \int_0^\pi u_1^2\, dx$ and $\int_0^\pi (u_{2,x})^2\, dx \ge (j+1)^2 \int_0^\pi u_2^2\, dx$. If we set $v = u_2 - u_1$ in the above equation, we find after easy calculations

$$\int_0^\pi (c(u) - j^2) u_1^2\, dx + \int_0^\pi ((j+1)^2 - c(u)) u_2^2\, dx \le 0.$$

It follows that $(c(u) - j^2) u_1^2 = ((j+1)^2 - c(u)) u_2^2 = 0$ a.e. in $]0, \pi[$. Since c is continuous on $\mathbb{R} \setminus \{0\}$, we deduce that $c(u(x)) \in \{j^2, (j+1)^2\}$ whenever $u(x) \neq 0$. In any case it follows that $u(x) g(u(x)) = 2G(u(x))$, so that

$$f(u) = 2\pi \int_0^\pi \left(\frac{1}{2} u g(u) - G(u) \right) dx = 0.$$

The case $sg(s) \le s^2$ can be treated in a similar way, with some obvious simplifications. □

The compact Lie group S^1 acts on E by means of time-translations, hence by orthogonal transformations. It is readily seen that A is equivariant and f is invariant.

Now, following [13], set

$$X = \{u \in E : u_{jk} = 0 \text{ whenever } j \text{ is even or } k \text{ is odd}\}.$$

Then X is a closed invariant linear subspace of E compactly embedded in $L^2(Q;\mathbb{R})$. Moreover, $A(X) \subseteq X$, $A : X \to X$ is an isomorphism and $\nabla f(X) \subseteq X$.

Therefore constrained critical points on X are in fact free critical points on E. Since $X \subseteq R(A)$, they are subjected to the regularity properties we have mentioned before. Moreover, by [3], distinct critical orbits give rise to geometrically distinct solutions. From now on f will denote the restriction of f to X.

It is easy to see that the set of eigenvalues of the problem

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times X, \\ (Au|v) = \lambda \int_Q uv\, dx\, dt, \qquad \forall v \in X, \end{cases}$$

is just the set $\{\lambda_h : h \in \mathbb{Z}\}$ of the introduction. Let $c_\infty = \frac{1}{2}(\lambda_h + \lambda_{h+1})$ and let $L : X \to X$ be the linear continuous operator such that

$$(Lu|v) = (Au|v) - c_\infty \int_Q uv\, dx\, dt.$$

Of course, L is symmetric, bijective and equivariant and it induces an orthogonal decomposition

$$X = X^-(L) \oplus X^+(L),$$

where $X^-(L)$ is the negative space of L and $X^+(L)$ is the positive space of L. Moreover, we have

$$\forall u \in X^-(L) : \ (Lu|u) \leq (\lambda_h - c_\infty) \int_Q u^2\, dx\, dt,$$

$$\forall u \in X^+(L) : \ (Lu|u) \geq (\lambda_{h+1} - c_\infty) \int_Q u^2\, dx\, dt$$

and there exists $\nu > 0$ such that

$$\forall u \in X^-(L) : \ (Lu|u) \leq -\nu\|u\|^2,$$

$$\forall u \in X^+(L) : \ (Lu|u) \geq \nu\|u\|^2.$$

We can write

$$f(u) = \frac{1}{2}(Lu|u) - \psi(u),$$

with

$$\psi(u) = \int_Q \left[G(u) - \frac{1}{2}c_\infty u^2\right] dx\, dt.$$

Since X is compactly embedded in L^2, the map $\nabla\psi : X \to X$ is compact.

LEMMA 3.3. *For every $c \in \mathbb{R}$ the functional f satisfies $(PS)_c$.*

PROOF. Let (u_h) be a sequence in X with $\nabla f(u_h) \to 0$ and $f(u_h) \to c$. Since L is an isomorphism and $\nabla\psi$ is compact, it is sufficient to show that (u_h) is bounded in X. By contradiction, assume that $\|u_h\| \to +\infty$. Take $\underline{\lambda}$, $\overline{\lambda} \in \mathbb{R}$ with

$$\lambda_h < \underline{\lambda} < \liminf_{|s|\to\infty} \frac{g(s)}{s} \le \limsup_{|s|\to\infty} \frac{g(s)}{s} < \overline{\lambda} < \lambda_{h+1}$$

and define $g_\infty : \mathbb{R} \to \mathbb{R}$ by

$$g_\infty(s) = g(s) - c_\infty s.$$

Set $\underline{\alpha} = \underline{\lambda} - c_\infty$ and $\overline{\alpha} = \overline{\lambda} - c_\infty$, so that

$$\underline{\alpha} < \liminf_{|s|\to\infty} \frac{g_\infty(s)}{s} \le \limsup_{|s|\to\infty} \frac{g_\infty(s)}{s} < \overline{\alpha}.$$

Let $g_\infty(s) = \eta_\infty(s) + \gamma_\infty(s)s$ with

$$\gamma_\infty(s) = \begin{cases} \min\{\max\{g_\infty(s)/s, \underline{\alpha}\}, \overline{\alpha}\} & \text{if } s \ne 0, \\ c_\infty & \text{if } s = 0. \end{cases}$$

Of course, γ_∞ is a Borel function with $\underline{\alpha} \le \gamma_\infty(s) \le \overline{\alpha}$ for every $s \in \mathbb{R}$ and $\eta_\infty \in C_c(\mathbb{R})$. Let $v_h = u_h/\|u_h\|$. We have, up to passing to a subsequence, $v_h \rightharpoonup v$ in X, $v_h \to v$ in $L^2(Q)$ and $\gamma_\infty(u_h) \overset{*}{\rightharpoonup} a$ in $L^\infty(Q)$ with $\underline{\alpha} \le a \le \overline{\alpha}$ a.e. in Q. Moreover,

$$\frac{\eta_\infty(u_h)}{\|u_h\|} \to 0 \quad \text{in } L^\infty(Q).$$

Let $P^+ : X \to X^+(L)$ and $P^- : X \to X^-(L)$ denote the orthogonal projections. Since $(P^+v_h - P^-v_h)$ is bounded in X, we have

$$(LP^+v_h|P^+v_h) - (LP^-v_h|P^-v_h) - \int_Q \frac{\eta_\infty(u_h)}{\|u_h\|}(P^+v_h - P^-v_h)\, dx\, dt$$
$$- \int_Q \gamma_\infty(u_h)v_h(P^+v_h - P^-v_h)\, dx\, dt \to 0.$$

Since $P^+v_h - P^-v_h \to P^+v - P^-v$ in L^2, we get

$$\nu \leq \int_Q av(P^+v - P^-v)\,dx\,dt,$$

hence $v \neq 0$. On the other hand, we also have

$$(Lv_h|P^+v - P^-v) - \int_Q \frac{\eta_\infty(u_h)}{\|u_h\|}(P^+v - P^-v)\,dx\,dt - \int_Q \gamma_\infty(u_h)v_h(P^+v - P^-v)\,dx\,dt \to 0,$$

so that

$$\begin{aligned}(LP^+v|P^+v) - (LP^-v|P^-v) - \int_Q a(P^+v)^2\,dx\,dt + \int_Q a(P^-v)^2\,dx\,dt \\ = (Lv|P^+v - P^-v) - \int_Q a(P^+v + P^-v)(P^+v - P^-v)\,dx\,dt = 0.\end{aligned}$$

It follows that

$$(\lambda_{h+1} - c_\infty - \overline{\alpha})\int_Q (P^+v)^2\,dx\,dt + (c_\infty + \underline{\alpha} - \lambda_h)\int_Q (P^-v)^2\,dx\,dt \leq 0,$$

which yields $P^+v = P^-v = 0$: a contradiction. □

LEMMA 3.4. *The functional f is bounded from below on $X^+(L)$ and from above on $X^-(L)$.*

PROOF. For some constant $\overline{c} \geq 0$, we have $G_\infty(s) \leq \frac{1}{2}\overline{\alpha}s^2 + \overline{c}$, where $G_\infty(s) = \int_0^s g_\infty(\sigma)\,d\sigma$. On the other hand, we also have

$$\forall w \in X^+(L): \quad (Lw|w) \geq (\lambda_{h+1} - c_\infty)\int_Q |w|^2\,dx\,dt = \frac{\lambda_{h+1} - \lambda_h}{2}\int_Q |w|^2\,dx\,dt.$$

Since $\overline{\alpha} < (\lambda_{h+1} - \lambda_h)/2$, the functional f is bounded from below on $X^+(L)$.

In a similar way, we show that f is bounded from above on $X^-(L)$. □

LEMMA 3.5. *Let $G_0 : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\inf_{s\in\mathbb{R}} \frac{G_0(s)}{1 + s^2} > -\infty, \quad \liminf_{s\to 0} \frac{G_0(s)}{s^2} \geq 0.$$

Then

$$\liminf_{\substack{u\to 0\\ u\in X}} \|u\|^{-2}\int_Q G_0(u)\,dx\,dt \geq 0.$$

PROOF. Let

$$\gamma_0(s) = \begin{cases} (G_0(s)/s^2)^- & \text{if } s \neq 0,\\ 0 & \text{if } s = 0.\end{cases}$$

Then $\gamma_0 : \mathbb{R} \to \mathbb{R}$ is bounded, continuous, with $\gamma_0(0) = 0$ and $G_0(s) \geq -\gamma_0(s)s^2$. If (u_h) is a sequence in X with $u_h \to 0$, then up to a subsequence, $u_h \to 0$ a.e. and $v_h := u_h/\|u_h\|$ is strongly convergent in $L^2(Q)$. Since

$$\|u_h\|^{-2} \int_Q G_0(u_h)\, dx\, dt \geq -\int_Q \gamma_0(u_h) v_h^2\, dx\, dt,$$

the assertion follows. □

Now we can prove the main result of the paper.

PROOF OF THEOREM 1.2. First of all, consider the case

$$\liminf_{s\to 0} \frac{g(s)}{s} > \lambda_{h+1}$$

and denote by c_0 the left hand side of the above inequality. Let $L_0 : X \to X$ be the linear continuous operator defined by

$$(L_0 u|v) = (Au|v) - c_0 \int_Q uv\, dx\, dt.$$

It is readily seen that L_0 is symmetric and equivariant. Consider the orthogonal decomposition

$$X = X^-(L_0) \oplus X^0(L_0) \oplus X^+(L_0)$$

into the negative space, the null space and the positive space of L_0. Moreover, set $G_0(s) = G(s) - \frac{1}{2}c_0 s^2$, so that

$$f(u) = \frac{1}{2}(L_0 u|u) - \int_Q G_0(u)\, dx\, dt.$$

By Lemma 3.5 we have

$$\liminf_{\substack{u\to 0\\ u\in X}} \|u\|^{-2} \int_Q G_0(u)\, dx\, dt \geq 0.$$

If we set $V = X^-(L_0)$ and $W = X^+(L)$, it follows that

$$\limsup_{\substack{u\to 0\\ u\in V}} \frac{f(u)}{\|u\|^2} < 0.$$

Therefore there exists $r > 0$ such that

$$\sup_{S_r \cap V} f < 0. \tag{3.6}$$

We want to apply Theorem 2.1 to the functional $f : X \to \mathbb{R}$. Of course V and W are closed invariant subspaces of X with $L(W) \subseteq W$. Since $c_\infty < c_0$, we have $V + W = X$. By (3.6) and Lemma 3.4, assumption d) of Theorem 2.1 is satisfied. Condition e) follows from Proposition 3.2 and (3.6), while hypothesis f) is implied by Lemma 3.3.

On the other hand (see [4]), $\frac{1}{2}\dim(V \cap W)$ is just the number m introduced in Theorem 1.2. By Theorem 2.1 the assertion follows.

In the case

$$\limsup_{s\to 0} \frac{g(s)}{s} < \lambda_h,$$

denote by c_0 the left hand side of the inequality if it is finite, otherwise let c_0 be any real number less than λ_h. We introduce L_0 and G_0 as in the previous case and set $V = X^+(L_0)$ and $W = X^-(L)$. Then we can apply the same argument to the functional $-f$ and the conclusion follows also in this case. □

REFERENCES

[1] H. AMANN, *Saddle points and multiple solutions of differential equations*, Math. Z. **169** (1979), 127–166.

[2] H. AMANN AND E. ZEHNDER, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 539–603.

[3] ———, *Multiple periodic solutions for a class of nonlinear autonomous wave equations*, Houston J. Math. **7** (1981), 147–174.

[4] N. BASILE AND M. MININNI, *Multiple periodic solutions for a semilinear wave equation with nonmonotone nonlinearity*, Nonlinear Anal. **9** (1985), 837–848.

[5] V. BENCI, *On critical point theory for indefinite functionals in the presence of symmetries*, Trans. Amer. Math. Soc. **274** (1982), 533–572.

[6] V. BENCI, A. CAPOZZI AND D. FORTUNATO, *On asymptotically quadratic Hamiltonian systems*, Equadiff 82, Würzburg, 1982, Lecture Notes in Math., vol. 1017, Springer-Verlag, Berlin-New York, 1983, pp. 83–92.

[7] ———, *Periodic solutions of Hamiltonian systems with superquadratic potential*, Ann. Mat. Pura Appl. (4) **143** (1986), 1–46.

[8] J. BERKOVITS AND V. MUSTONEN, *On multiple solutions for a class of semilinear wave equations*, Nonlinear Anal. **16** (1991), 421–434.

[9] H. BREZIS AND L. NIRENBERG, *Forced vibrations for a nonlinear wave equation*, Comm. Pure Appl. Math. **31** (1978), 1–30.

[10] A. CAPOZZI AND A. SALVATORE, *Nonlinear problems with strong resonance at infinity: an abstract theorem and applications*, Proc. Roy. Soc. Edinburgh Sect. A **99** (1985), 333–345.

[11] K. C. CHANG, *Solutions of asymptotically linear operator equations via Morse theory*, Comm. Pure Appl. Math. **34** (1981), 693–712.

[12] K. C. CHANG, S. P. WU AND S. LI, *Multiple periodic solutions for an asymptotically linear wave equation*, Indiana Univ. Math. J. **31** (1982), 721–731.

[13] J. M. CORON, *Periodic solutions of a nonlinear wave equation without assumption of monotonicity*, Math. Ann. **262** (1983), 273–285.

[14] M. DEGIOVANNI AND L. OLIAN FANNIO, *Multiple periodic solutions of asymptotically linear Hamiltonian systems*, Quaderni Sem. Mat. Brescia **8/93** (1993).

[15] L. DE SIMON AND G. TORELLI, *Soluzioni periodiche di equazioni a derivate parziali di tipo iperbolico non lineari*, Rend. Sem. Mat. Univ. Padova **40** (1968), 380–401.

[16] N. HIRANO, *On the existence of multiple solutions for a nonlinear wave equation*, J. Differential Equations **71** (1988), 334–347.

[17] S. LI AND A. SZULKIN, *Periodic solutions of an asymptotically linear wave equation*, Reports Dept. Math. Univ. Stockholm **15**, Stockholm, 1992.

[18] H. LOVICAROVÁ, *Periodic solutions of a weakly nonlinear wave equation in one dimension*, Czechoslovak Math. J. **19** (1969), 324–342.

[19] P. H. RABINOWITZ, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math. **20** (1967), 145–205.

[20] ———, *Free vibrations for a semilinear wave equation*, Comm. Pure Appl. Math. **31** (1978), 31–68.

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