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ANALYSIS ON FRACTALS, LAPLACIANS ON SELF-SIMILAR SETS, NONCOMMUTATIVE GEOMETRY AND SPECTRAL DIMENSIONS

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Dedicated to Professor Jean Leray

1. Introduction

In this paper, we address the question of finding a suitable analogue of Weyl's asymptotic formula for the eigenvalue distribution of Laplacians *on* (certain classes of self-similar) fractals. We propose, in particular, an analogue of the notion of Riemannian volume on fractals and establish, in the process, some connections between analysis on fractals, spectral geometry, and aspects of Connes' noncommutative geometry.

1.1. Weyl's asymptotic formula and drums with fractal boundary. Before stating our main problem more precisely, we briefly recall Weyl's formula for the spectral distribution of Laplacians on (possibly irregular) bounded open sets of Euclidean space and on (smooth, compact) Riemannian manifolds.

Let Ω be a bounded open set in $\mathbb{R}^n$, with boundary $\Gamma = \partial\Omega$. We consider the following variational eigenvalue problem (P):

$$\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

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with Dirichlet (resp., Neumann) boundary conditions

$$u = 0 \text{ on } \Gamma \qquad (\text{resp., } \partial u/\partial n = 0 \text{ on } \Gamma). \tag{1.2}$$

Here, $\Delta = -\sum_{k=1}^{n} \partial^2/\partial x_k^2$ denotes the (negative) Laplacian in $\mathbb{R}^n$. [For notational simplicity, we use throughout this paper the geometer's convention for the Laplacian (denoted by Δ here but by $-\Delta$ in most analysis papers and in [La1, KiLa1], in particular), so that the spectrum of Δ is contained in $[0, +\infty)$.] We stress that for rough boundaries, the problem (P) must be interpreted in the *variational sense*; namely, λ is an *eigenvalue* of (P) (with associated eigenfunction u) if $u \neq 0$ is a weak (distributional) solution of (1.1): $u \in \mathcal{F}$ and

$$\mathcal{E}(u,v) := \int_\Omega \nabla u \nabla v \, dx = \lambda \int_\Omega uv \, dx, \qquad \text{for all } v \in \mathcal{F}, \tag{1.3}$$

where dx denotes n-dimensional Lebesgue measure on Ω and $\mathcal{F} := H_0^1(\Omega)$ (resp., $H^1(\Omega)$) for the Dirichlet (resp., Neumann) problem. [We use here the French notation for the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. Hence $H^1(\Omega)$ is the Hilbert space of all functions u in $L^2(\Omega)$ with distributional gradient ∇u in $(L^2(\Omega))^N$, equipped with the norm $\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{(L^2(\Omega))^N})^{1/2}$, while $H_0^1(\Omega)$ denotes the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, the space of smooth functions with compact support contained in Ω.]

The spectrum of (P) is discrete and consists of an infinite sequence of eigenvalues, written in increasing order according to multiplicity:

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \to +\infty, \qquad \text{as } j \to \infty. \tag{1.4}$$

(This is always true for the Dirichlet problem and is also true for the Neumann problem under suitable hypotheses; for example, either (i) if Ω has a (locally) Lipschitz boundary or (ii) if Ω has the "extension property", e.g., if $\Omega \subset \mathbb{R}^2$ is a quasidisk; see [La1, §4.2.B, pp. 509–511] and references therein. In the sequel, for the Neumann problem, we shall assume that either (i) or (ii) holds.)

Define the *eigenvalue counting function*

$$N(\lambda) = \#\{j \geq 1 : \; \lambda_j \leq \lambda\}, \tag{1.5}$$

the number of (positive) eigenvalues (counted according to multiplicity) not exceeding $\lambda > 0$.

Then Weyl's classical asymptotic formula [We] states that

$$N(\lambda) \sim c_n \operatorname{Vol}_n(\Omega) \, \lambda^{n/2}, \qquad \text{as } \lambda \to +\infty, \tag{1.6}$$

where $\operatorname{Vol}_n(\Omega)$ denotes the n-dimensional volume (i.e., Lebesgue measure) of Ω and c_n is an explicit positive constant depending only on n; namely, $c_n = (2\pi)^{-n}\mathcal{B}_n$, where $\mathcal{B}_n$ is the volume of the unit ball in $\mathbb{R}^n$. (Here and thereafter, the symbol $\sim$ means that the ratio of the left and right sides of (1.6) tends to 1 as $\lambda \to +\infty$.)

REMARK 1.7. Formula (1.6) was first established by Weyl [We] for piecewise smooth boundaries. For the Dirichlet Laplacian, it was then extended by Birman and Solomyak [BiSo1]–[BiSo2] for arbitrary bounded open sets, and by Rozenblyum [Ro1–2] for open sets with finite volume. For further extensions, see, e.g., [BiSo2, Ro2, Me, FlLa1–2, La1] and the references therein.

Now, let $d = d_M \in [n-1, n]$ be the *Minkowski dimension* of the boundary Γ and assume that the d-dimensional upper Minkowski content of Γ is finite. (See, e.g., [La1, Definition 2.1 and §3].) Then the author [La1–2] has shown that if Γ is fractal (i.e., $d \neq n-1$), the following Weyl formula *with sharp remainder estimate* holds:

$$N(\lambda) = c_n \mathrm{Vol}_n(\Omega)\lambda^{n/2} + O(\lambda^{d/2}) \qquad \text{as } \lambda \to +\infty. \tag{1.8}$$

(See [La1, Theorems 2.1–2.3, Corollaries 2.1 and 2.2, pp. 479–483, and Theorem 4.1, pp. 510–511], specialized to the Dirichlet or Neumann Laplacian.)

REMARK 1.9. (a) Formula (1.8) yields a partial resolution of the Weyl-Berry conjecture ([We], [Be1–2]). We refer to [La1, §1] (as well as [La3,5]) and the references therein (including [BrCa]) for a detailed account of problems and results related to the conjectures of H. Weyl ([We], case of a smooth boundary) and M. V. Berry ([Be1–2], case of a fractal boundary).

(b) The Minkowski (-Bouligand) dimension $d = d_M$ (also called box dimension in the literature on fractal geometry) provides a measure of the roughness of the boundary Γ; in particular, the larger d, the more irregular Γ. Of course, if Γ is smooth (say, of class C^1), then $d_M = n-1$, the topological dimension of Γ. Further, for a nice self-similar boundary (such as the snowflake curve), we have $d_M = d_H$, the Hausdorff dimension of Γ. However, in general, one *cannot* substitute d_H for d_M in (1.8), as was first pointed out in [BrCa]. (See also [La1, Examples 5.1 and 5.1′, pp. 512–515].) Actually, it is shown in [La1] that one *cannot* replace the Minkowski dimension d_M (in (1.8)) by any notion of dimension that is associated with a (*countably additive*) measure rather than a (finitely subadditive) content. (See, in particular, [La1, Remark 5.2(b), p. 514].)

(c) The results of [La1] are also valid for (suitable) positive elliptic operators of order $2m$; in this case, the error term in (1.8) is $O(\lambda^{d/2m})$ and the leading term is expressed by means of an integral (in "phase space") involving the principal symbol of the operator. One should keep this fact in mind when trying to exploit the analogy (drawn in [La5]) between *"drums with fractal boundary"* and *"drums with fractal membrane"*.

The case of Riemannian manifolds. We have discussed so far Weyl's formula (1.6) in the context of open subsets Ω of Euclidean space $\mathbb{R}^n$. However, the counterpart of (1.6) also holds for the (Dirichlet or Neumann) Laplacian on a (smooth compact, n-dimensional) Riemannian manifold M (with boundary). In this case, the analogue of $\mathrm{Vol}_n(\Omega)$ in (1.6) is $\mathrm{Vol}_n(M)$, the (n-dimensional) volume of M, and the exponent of $\lambda^{1/2}$ in (1.6) is simply n, the dimension of the manifold. (See, e.g., [Ho] and the references therein.)

1.2. Laplacians on fractals and drums with fractal membrane. The question (à la Mark Kac [Kc]) *Can one hear the shape of a fractal drum?* has been the object of intense investigation by the author and his collaborators [La1–6, LaFl, LaPo1–3, LaMa1–2, KiLa2, LaPa, LNRG]. In particular, in the case of fractal strings (i.e., $n = 1$ and thus $\Omega \subset \mathbb{R}^1$), it was shown to be intimately connected with the Riemann hypothesis. [See ([LaMa1–2], joint with Helmut Maier), which builds upon ([LaPo1–2], joint with Carl Pomerance).] (We refer to [La5] for a recent survey of the main results and problems in this area, as well as for many other papers on this and related subjects. See also [La3] for an earlier expository article.) In this paper, however, we shall primarily be interested in the vibrations of *drums with fractal membrane* rather than *"drums with fractal boundary"* (in the terminology of [La5]). More precisely, we will look for an analogue of Weyl's asymptotic formula (1.6) for Laplacians *on* a (suitable, self-similar) fractal F, rather than on an open set Ω with "fractal" boundary $\Gamma = \partial\Omega$.

Several questions arise naturally in this context:

$\mathbf{Q_0}$. *What is a (self-similar) fractal?*

$\mathbf{Q_1}$. *What is a Laplacian on a fractal?*

$\mathbf{Q_2}$. *Is there an analogue of Weyl's formula for Laplacians on a fractal F? If so, what is its form?*

With some optimism and with the obvious notation, we may expect that for some exponent $d_S > 0$, we have

$$N(\lambda) \asymp \lambda^{d_S/2} \qquad \text{as } \lambda \to +\infty \tag{1.10}$$

(i.e., $0 < \underline{\lim}\, \lambda^{-d_S/2} N(\lambda) \leq \overline{\lim}\, \lambda^{-d_S/2} N(\lambda) < +\infty$), or even

$$N(\lambda) \sim \mathcal{C} \lambda^{d_S/2} \qquad \text{as } \lambda \to +\infty \tag{1.11}$$

(i.e., $\lim \lambda^{-d_S/2} N(\lambda) = \mathcal{C}$, for some positive constant $\mathcal{C} = \mathcal{C}(F)$).

$\mathbf{Q_3}$. *What is the value of the "spectral exponent" d_S? Further, what is the geometrical (or analytical) interpretation of d_S?*

Moreover, it is natural to ask whether d_S is equal to the Hausdorff (or Minkowski) dimension of F. (Note that this question may not make sense since, *a priori*, F need not be equipped with any metric.) Even if $F \subset \mathbb{R}^n$ for some integer n, examples from the physics [Dh, AO, RT, HHW, ...] and mathematics (esp., probabilistic) [Ku1, BP, Li, Fu2, FuSh, ...] literature show that the answer to this question is *no* in general. However, a general theorem established in our joint work [KiLa1] (and discussed later on in this paper) will provide a (reasonably) satisfactory answer to questions $\mathbf{Q_2}$ and $\mathbf{Q_3}$ for Laplacians on a certain class of fractals; namely, "finitely ramified" (i.e., p.c.f.) self-similar sets [Ki2]. We shall see that there are many possible Laplacians on F, each giving rise to a *"spectral exponent"* d_S. The largest value of these exponents—denoted by d_S^* and called the *"spectral dimension"* in [KiLa1]—can in fact be related (thanks to the results of [KiLa1] and then [Ki4]) to the Hausdorff (= Minkowski, in this case) dimension of F, equipped with a suitable ("intrinsic") metric.

Comparing (1.10) and (1.6), we are also led to ask the following question:

$\mathbf{Q_4}$. *Is there an analogue of the notion of "Riemannian volume" on a "fractal"?*

Finally, we mention a question that motivates many of the results and conjectures in [La5] and in the present work (see esp. [La5, §6]), although we will not attempt to answer it here.

$\mathbf{Q_5}$. *What is a suitable analogue of the notion of "geodesic flow" on a "fractal"?*

In this paper, we shall address or revisit several of these questions and propose possible answers and/or conjectures. In §2, we shall provide a formal definition of an (analytical) self-similar fractal F, thereby addressing questions $\mathbf{Q_0}$ and $\mathbf{Q_1}$. It will depend, in particular, on the geometrical, dynamical, measure-theoretic and analytical structures of F. (Our point of view will be slightly different from that adopted in [Ki2] and [KiLa1], for example.) Then, in §3, we shall recall, in particular, recent (joint) work [KiLa1] in which we partly answer questions $\mathbf{Q_2}$ and $\mathbf{Q_3}$ for a certain class of "fractals"; namely, the "finitely ramified" (i.e., p.c.f.) self-similar sets (formally) introduced in [Ki2]. We shall also suggest some possible extensions of these results to more general self-similar fractals. Moreover, in §4, using the main results of [KiLa1] and the notion of Dixmier trace—which is a basic tool in aspects of noncommutative geometry [Co4–5]—we shall construct a positive measure which we propose to be a suitable substitute on p.c.f. fractals for the notion of Riemannian volume. (In fact, according to a result of Connes [Co3; Co5, §VI.1], an analogous construction applied to the Dirac operator on a (spin) Riemannian manifold M yields the usual Riemannian volume on M.) This will enable us to complete the results of [KiLa1] and obtain a more precise counterpart of Weyl's asymptotic formula in this context, by reinterpreting them in terms of the total mass of this *"volume measure"*. In §5, we shall propose a specific conjecture regarding the nature of the positive measure constructed in §4 (for the case of the *"spectral dimension"*, that is, of the maximum "spectral exponent"). In the process, we shall establish contact with recent work of Connes and Sullivan ([CoSu], [Co5, §IV.3]) on "quantized calculus" and limit sets of quasi-Fuchsian groups. Two questions then arise naturally: Can the above *"volume measure"* be related to Hausdorff measure with respect to some suitable ("intrinsic") metric on F? Is there a suitable notion of *"quantized calculus"* on (analytical) self-similar fractals, arising from an appropriate analogue of the Dirac operator?

In closing, we mention that the above quest for a suitable notion of "volume" for "*drums with fractal membrane*" may shed some light on the corresponding (but even more delicate) problem for *"drums with fractal boundary"*, studied in particular in [La1–3,5] and [LaPo1–3].

For the simplicity of exposition, we will mostly work within the framework of [KiLa1], with only a few excursions in more complicated settings. It is clear, however, that more general results can be obtained by using similar methods, once certain difficulties (that we shall try to point out) have been circumvented.

Our goal is to help develop further (geometric) analysis on "fractals", as well as to establish further connections between this exciting new subject and aspects of Connes' noncommutative geometry. The results obtained in this paper constitute a modest step towards this latter goal but will hopefully raise enough questions to stimulate further research in this direction.

2. Self-similar fractals and energy functionals

In this section, we address Questions $\mathbf{Q_0}$ and $\mathbf{Q_1}$ regarding the definition of a self-similar (s.s., for short) fractal F, as well as of Laplacians on F, both from a topological, dynamical and analytical points of view. For more details, we refer to ([Ki2], [KiLa1, §1], [La5, Part II]), and the relevant references therein. Our treatment will differ somewhat from those references in that it is a bit more systematic and stresses the crucial role played by suitable *"self-similar energy functionals"*; namely, s.s. Dirichlet forms. In fact, by contrast to some other approaches, the existence of a (suitable) self-similar "energy functional" on F is an integral part (rather than a consequence) of the definitions. (See, in particular, Definition 2.24 in §2.4 below.)

2.1. Topological self-similar fractals. Let F be a compact topological space. Given an integer $N \geq 2$, we consider the *alphabet* in N letters, $A = \{1, \ldots, N\}$, and $\Sigma = A^{\mathbb{N}}$, the (one-sided) sequence space (or set of "words" over A). A *word* ω in Σ is denoted by $\omega = \omega_1\omega_2\ldots\omega_k\ldots$, with $\omega_k \in \{1, \ldots, N\}$. Further, a *finite* word of length m, $\omega = \omega_1\ldots\omega_m$, is one for which the above sequence terminates. We let $\Sigma_m = A^m$ denote the set of words of length m.

The following definition provides a convenient "symbolic dynamical" description of a s.s. fractal.

DEFINITION 2.1 (Topological s.s. fractal). Assume that F is compact and metrizable. Further, for each $i = 1, \dots, N$, let $W_i : F \to F$ be a continuous one-to-one map. Then $\mathcal{S} = (A, \{W_i\}_{i=1}^N)$ is a *self-similar structure* on F if there exists a continuous, onto map $\Pi : \Sigma \to F$ such that for each $i = 1, \dots, N$, the following diagram is commutative (i.e., $W_i \circ \Pi = \Pi \circ \sigma_i$):

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Pi} & F \\ \downarrow \sigma_i & & \downarrow W_i \\ \Sigma & \xrightarrow{\Pi} & F \end{array}$$

where the i-th shift σ_i is defined by $\sigma_i(\omega_1\omega_2\dots) = i\omega_1\omega_2\dots$. Moreover, F, equipped with $\mathcal{S}$, is called a *topological self-similar set* (or fractal).

REMARK 2.2. (a) The space F can be viewed as a purely topological object, without any *a priori* choice of metric. In fact, F is the quotient space of Σ by the equivalence relation induced by the "projection" Π; see [Ka, Ku2] for related (and more general) definitions. For example, in [Ka, Ku2], the maps W_i need not be injective and the topological space F need not be Hausdorff (and hence is not assumed to be metrizable).

(b) It can be checked [Ki2] that Π is uniquely determined by $\mathcal{S}$; indeed, given $\omega \in \Sigma$, we have $\{\Pi(\omega)\} = \bigcap_{m=1}^{\infty} W_{\omega_1} \circ \dots \circ W_{\omega_m}(F)$, the intersection of a nested sequence of compact subsets of F.

(c) It follows from Definition 2.1 that

$$F = \bigcup_{i=1}^{N} W_i(F); \tag{2.3}$$

i.e., F is "invariant under the transformations W_i". (For convenience, a proof of (2.3) is provided at the end of this remark.) Note, however, that contrary to the standard definitions of "self-similar fractals" ([Mo], [Hu], [Fc, §9]), F is *not* assumed to be embedded in some Euclidean space $\mathbb{R}^n$ and the mappings W_i are *not* assumed to be similarity transformations (with respect to the Euclidean metric) or even contractions (with respect to some metric on F). Actually, even if—as is the case of most of the standard "fractals" [Ma]— F is embedded in some linear space $\mathbb{R}^n$, it will be quite important in the following to "forget" this fact and to view F (and the associated mathematical structures on F) *intrinsically*. In this sense, analysis on "fractals" is essentially *"nonlinear"*.

(Here is a proof of (2.3): Let $x \in F$. Since Π is onto, we have $x = \Pi(\omega)$, for some $\omega = \omega_1\omega_2\dots$ in Σ; write $\omega_1 = i$, say. Thus $x = \Pi(i\omega_2\omega_3\dots) = \Pi \circ \sigma_i(\omega_2\omega_3\dots) = W_i \circ \Pi(\omega_2\omega_3\dots)$, and so $x \in W_i(F)$. Hence $F \subset \bigcup_{j=1}^N W_j(F)$, which is all that needs to be proved.)

(d) To avoid extreme situations, we will assume in the following that F is *connected.* (Otherwise, F would have to be totally disconnected, such as a Cantor set, for example.)

NOTATION 2.4. Given a finite word $\omega = \omega_1 \dots \omega_m$ in Σ_m, we let $W_\omega = W_{\omega_1} \circ \dots \circ W_{\omega_m}$ and write $F_\omega = W_\omega(F)$; in particular, we write $F_i = W_i(F)$ for $i = 1, \dots, N$.

REMARK 2.5. (a) Intuitively, each F_ω can be thought of as a "scaled copy" of F. Moreover, by (2.3), the subsets F_i ($i = 1, \dots, N$) constitute the basic "building blocks" of F. Note that there may be some overlap between the F_i's. Roughly speaking, F is said to satisfy the "open set condition" ([Mo], [Hu], [Fc, §9]) if this overlap is "small", and to be *"finitely ramified"* if it is finite; namely, $\cup_{i \neq j}(F_i \cap F_j)$ is finite. For example, the Sierpiński gasket is "finitely ramified" whereas the Sierpiński carpet is not.

(b) We shall assume, most often implicitly, that F satisfies the "topological open set condition" (in the sense of [Ki3]), an abstract version of the standard open set condition used in textbooks on fractal geometry (e.g., [Fc]). This condition is always satisfied if F is a p.c.f. self-similar set, a mathematical version of the notion of "finitely ramified" fractal.

DEFINITION 2.6 (p.c.f. self-similar set; [Ki2]). Given a self-similar structure $\mathcal{S} = (A, \{W_i\}_{i=1}^N)$ on F, we let $\mathcal{C}_r := \Pi^{-1}(\bigcup_{i \neq j}(F_i \cap F_j))$ (the "critical set") and $\mathcal{P} := \bigcup_{m=1}^\infty \sigma^m(\mathcal{C}_r)$ (the "post-critical set"), where $\sigma : \Sigma \to \Sigma$ is the shift map defined by $\sigma(\omega_1\omega_2\omega_3 \dots) = \omega_2\omega_3 \dots$ and σ^m is the m-th iterate of σ (so that $\sigma^m(\omega_1\omega_2 \dots) = \omega_{m+1}\omega_{m+2} \dots$). Then $\mathcal{S}$ is said to be *post-critically finite* (in short, p.c.f.) if $\mathcal{P}$ is a finite set. Further, F equipped with this structure is called a p.c.f. self-similar set (or fractal).

REMARK 2.7. The above terminology and definition is inspired, in particular, by the work of Sullivan, Thurston (and many other researchers) related to hyperbolic dynamical systems. Note, however, that we are dealing here with a *one-sided* (rather than two-sided) shift space, and that the dynamics is determined by a *semigroup* (spanned by $\{\sigma_i\}_{i=1}^N$) rather than a group. We shall return to this point later on when we discuss possible connections with the theory of operator algebras. (See esp. §5.2.)

For various examples of p.c.f. self-similar sets, we refer to [Ki2, §8] and [KiLa1, §3]. These include, in particular, the "nested fractals" (roughly, highly symmetric s.s. sets) introduced by Lindstrøm in [Li]; for instance, the Sierpiński gasket and its generalizations to higher dimensional spaces, as well as the modified Koch curve. Another example is Hata's fractal tree, a kind of self-similar tree introduced in [Ha].

(2.8) In the following, it will be useful to keep in mind that *a p.c.f. self-similar set can be viewed in a natural way as the limit of an increasing sequence of finite graphs.* Namely, if we set $V_0 = \Pi(\mathcal{P})$ and for $m \geq 0$, $V_m = \bigcup_{\omega\in\Sigma_m} W_\omega(\Pi(\mathcal{P}))$, then each V_m is finite,

$$V_0 \subset V_1 \subset V_2 \subset \dots \quad \text{and} \quad F = cl\left(\bigcup_{m=0}^{\infty} V_m\right). \tag{2.9}$$

(See, e.g., [KiLa1, Fig. 1, p. 95] for an illustration of this fact in the simple case of the Sierpiński gasket.) Hence various analytical objects on F, such as Green's function, Laplacians, Dirichlet forms, can be defined (when possible) as suitably renormalized limits of their discrete counterpart on the finite graphs V_m.

REMARK 2.10. In certain special cases, this idea was already exploited in the physics literature (e.g., [AO, RT]) as well as in the probabilistic literature (e.g., [Ku1, Go, BP, Li]), where for the Sierpiński gasket, say, "Brownian motion" on F is defined as a limit of rescaled random walks on the approximating sets.

2.2. Self-similar Dirichlet forms. In order to work with suitable "energy functionals" on F, we need to choose an appropriate measure on F.

Let μ be a Borel probability measure on F such that

(2.11a) support$(\mu) = F$ (i.e., $\mu(U) > 0$ for all nonempty open subsets U of F),

(2.11b) μ is diffuse (i.e., $\mu(\Lambda) = 0$ for all finite subsets Λ of F).

REMARK 2.12. As is observed in [Co4, p. 31], a remarkable (but insufficiently well-known) theorem of Oxtoby and Ulam [OxUl] implies that (2.11a) and (2.11b) are the only compatibility conditions between Lebesgue measure and the topology of Euclidean space (or, more generally, between the volume measure and the topology of a differentiable (oriented) manifold). Indeed, by [OxUl], given two Borel probability measures μ_1, μ_2 satisfying (2.11) on a topological manifold X, there exists a homeomorphism of X that maps μ_1 onto μ_2. Note, of course, that a (topological) s.s. fractal—such as the Sierpiński gasket, for example—is in general far from being a topological manifold (i.e., locally homemorphic to an open set of Euclidean space). It would be natural, however, to wonder what form the Oxtoby-Ulam theorem might take in this context.

Let $\mathcal{E}$ be a Dirichlet form on $L^2(\mu) = L^2(F,\mu)$, with domain $\mathcal{F}$. (See, e.g., [KiLa1, Definition 4.1, p. 115].) Essentially, $\mathcal{E}$ is a closed, nonnegative unbounded quadratic form on $L^2(\mu)$ which satisfies the "Markov property"; namely, for all $u \in \mathcal{F}$, $\mathcal{E}(\widetilde{u},\widetilde{u}) \leq \mathcal{E}(u,u)$, where $\widetilde{u}(x) := 0$ (resp., 1) if $u(x) < 0$ (resp., > 1), and $\widetilde{u}(x) := u(x)$ otherwise. In the sequel, we will not distinguish between the quadratic form $\mathcal{E}(\cdot)$ and the associated bilinear form $\mathcal{E}(\cdot,\cdot)$. For the general theory of Dirichlet forms, we refer to Fukushima's book [Fu1].

The following definition is inspired by results in [Fu2] extended in [KiLa1]. (We adopt the terminology of [La5].)

DEFINITION 2.13 (Self-similar Dirichlet form). Let $(F, \mathcal{S})$ be a (topological) s.s. fractal. Let c_i $(i = 1, \dots, N)$ be positive constants < 1. We say that the Dirichlet form $\mathcal{E}$ is self-similar (with respect to $\mathcal{S}$) with "harmonic constants" $\{c_i\}_{i=1}^N$ if $\mathcal{E} = \sum_{i=1}^N c_i^{-1} \mathcal{E} \circ W_i^{-1}$; i.e., for all $u \in \mathcal{F}$, we have $u \circ W_i \in \mathcal{F}$ for $i = 1, \dots, N$ and

$$\mathcal{E}(u,v) = \sum_{i=1}^N c_i^{-1} \mathcal{E}(u \circ W_i, v \circ W_i), \tag{2.14}$$

for all $u, v \in \mathcal{F}$.

REMARK 2.15. (a) In general, it is not clear whether there always exists a s.s. Dirichlet form on F; this is so even if F is assumed to be p.c.f. In that situation, by [Ki2, KiLa1], the answer is affirmative provided that a certain *nonlinear* operator, called the "*renormalization operator*" admits a positive eigenvalue α (called the "*renormalization constant*" in [La5]); i.e., provided that there exists a (regular) "*harmonic structure*" on F. The standard fixed point theorems from topology or nonlinear analysis do not seem to guarantee the existence of such an eigenvalue. It would be interesting to address this problem.

In the special case of "nested fractals", however, the renormalization procedure always works—as was shown in [Li] by means of the Lipschitz fixed point theorem—and so (by [Fu2]) a s.s. Dirichlet form can be constructed on F (with equal "harmonic constants" $c_1 = \dots = c_N = 1/\alpha$, due to the high symmetry of F).

(b) For a p.c.f. self-similar fractal F (equipped with a regular "harmonic structure"), the Dirichlet form $\mathcal{E}$ constructed in [Ki2]—as a renormalized limit of discrete Dirichlet forms on the approximating graphs—is shown in [KiLa1, Lemma 6.1, p. 119] to be self-similar (in the sense of Definition 2.13). Moreover, the domain $\mathcal{F}$ of $\mathcal{E}$ (and also $\mathcal{E}$ itself) is independent of the measure μ and $\mathcal{F} \subset C(F)$, where $C(F)$ denotes the space of all real-valued continuous functions on F. Heuristically, we propose here to interpret the inclusion $\mathcal{F} \subset C(F)$ as a counterpart of the standard *one-dimensional* Sobolev embedding theorem in Euclidean space. (This fact will explain several special features of the present situation.)

In this paper, we will mostly be interested in Dirichlet forms obtained in this manner and we will call them *regular* (s.s. Dirichlet forms). In particular, a *regular* p.c.f. self-similar set $(F, \mathcal{S}, \mathcal{E})$ will be one that is equipped with such a regular s.s. form $\mathcal{E}$. (With our present convention, this last definition is equivalent to that used in [Ki2, KiLa1].)

2.3. Self-similar measures. Let $(F, \mathcal{S})$ be a topological s.s. fractal. For our present purpose, we shall first work with special types of measures compatible with the topology of F, in the sense of (2.11); namely, the "Bernoulli measures". Recall that the *Bernoulli measure* with weights $\{\mu_i\}_{i=1}^N (\mu_i \geq 0, \sum_{i=1}^N \mu_i = 1)$ is the unique Borel probability measure μ on F such that (with the notation of (2.4))

$$\mu(F_\omega) = \mu_{\omega_1} \dots \mu_{\omega_m},$$

for all $\omega = \omega_1 \dots \omega_m \in \Sigma_m$ and all $m \in \mathbb{N}$; so that, in particular, $\mu(F_i) = \mu_i$, for $i = 1, \dots, N$.

In [La5, §5], we have used the terms "Bernoulli measure" and "self-similar measure" interchangeably. When F is p.c.f., this will be justified by Theorem 2.18 below. First, however, we recall the definition of a s.s. measure that was introduced by Hutchinson [Hu] for s.s. fractals in Euclidean (or metric) spaces and further studied, in particular, by Strichartz [St1–2] from the point of view of harmonic analysis.

DEFINITION 2.16 (Self-similar measure). Let $(F, \mathcal{S})$ be a topological self-similar fractal and let μ be a Borel probability measure on F. Then μ is said to be <u>self-similar</u> (with respect to $\mathcal{S}$) with "<u>weights</u>" $\{b_i\}_{i=1}^N$ $(b_i \geq 0, \sum_{i=1}^N b_i = 1)$ if $\mu = \sum_{i=1}^N b_i \mu \circ W_i^{-1}$; i.e.,

$$\int_F f \, d\mu = \sum_{i=1}^N b_i \int_F f \circ W_i \, d\mu, \tag{2.17}$$

for all continuous (and hence, for all bounded measurable) functions f on F.

The following simple result—which adapts [Hu, Theorems 4.4.1 and 4.4.4, p. 733] to the present setting—appears to be new in this context and clarifies the relationships between some of the concepts involved. It will also motivate and help us to formulate some of our conjectures later on in the paper (see §5.1).

THEOREM 2.18. *Let $(F, \mathcal{S})$ be a regular p.c.f. self-similar set, equipped with a (regular) s.s. Dirichlet form $\mathcal{E}$. Then, given b_i $(i = 1, \dots, N)$ with $b_i \geq 0$ and $\sum_{i=1}^N b_i = 1$, there exists a unique s.s. measure μ with weights $\{b_i\}_{i=1}^N$ (i.e., satisfying (2.17)). Moreover, μ coincides with the Bernoulli measure with weights $\{\mu_i := b_i\}_{i=1}^N$.*

REMARK 2.19. (a) In other words, if $\mu_\#$ is the pull-back measure of μ on $\sum = A^{\mathbb{N}}$ by the continuous map $\Pi : \Sigma \to F$ (given in Definition 2.1), then $\mu_\#$ is just the infinite product measure of the same copy of the standard probability measure on $A = \{1, \dots, N\}$ with weights $\{\mu_i\}_{i=1}^N$.

(b) Using analytical results in [KuZh, Ki5], one could extend Theorem 2.18 to more general s.s. fractals, such as the (two-dimensional) Sierpiński carpet, for example, first studied probabilistically in [BB1].

(c) We do not know whether one can obtain an analogous theorem concerning the existence and uniqueness of a (regular) s.s. Dirichlet form with given "harmonic constants" $c_i \in (0,1)$, $i = 1, \dots, N$. However, we would expect this to be true under suitable hypotheses. (See also Remark 2.15(a) above.)

PROOF OF THEOREM 2.18. Let $P(F)$ denote the set of probability measures on F. Consider (as in [Hu]) the map $T : P(F) \to P(F)$ defined by

$$T(\mu) = \sum_{i=1}^{N} b_i \mu \circ W_i^{-1}. \tag{2.20}$$

Then we claim that T is a *contraction* with respect to the complete metric $\delta_.$ on $P(F)$ given by

$$\delta_.(\mu_1, \mu_2) = \sup\{|\mu_1(u) - \mu_2(u)| : u \in \mathcal{F}, \sqrt{\mathcal{E}(u)} \leq 1\}, \tag{2.21}$$

where $\mu_j \in P(F)$ and $\mu_j(u) := \int_F u \, d\mu_j$ $(j = 1, 2)$. (This metric differs from that used in the setting of [Hu].) Hence, by the Banach fixed point theorem, T has a unique fixed point μ, which (by Definition 2.16) is the desired s.s. measure with weights $\{b_i\}_{i=1}^N$. (Further, μ is obtained by the method of successive approximations.) The rest of the proof follows easily from the uniqueness statement because, by Definition 2.1, a Bernoulli measure is clearly self-similar (with the same weights).

To see that T is a contraction (with Lipschitz constant $\kappa = \max_{1 \leq i \leq N} \sqrt{c_i} < 1$), observe that by (2.14) (and with the notation $\mathcal{E}(u) = \mathcal{E}(u, u)$, for $u \in \mathcal{F} = \mathrm{Dom}(\mathcal{E})$),

$$\mathcal{E}(u \circ W_i) \leq c_i \mathcal{E}(u)$$

and hence $\sqrt{\mathcal{E}(u \circ W_i)} \leq \kappa \sqrt{\mathcal{E}(u)}$, for $i = 1, \dots, N$ and $u \in \mathcal{F}$. Thus by (2.21), we have for $u \in \mathcal{F}$,

$$\left|\mu_1(u \circ W_i) - \mu_2(u \circ W_i)\right| \leq \kappa \delta_.(\mu_1, \mu_2) \sqrt{\mathcal{E}(u)}$$

and so, by (2.20),

$$\begin{aligned} \left|T(\mu_1)(u) - T(\mu_2)(u)\right| &\leq \sum_{i=1}^{N} b_i \left|\mu_1(u \circ W_i) - \mu_2(u \circ W_i)\right| \\ &\leq \sum_{i=1}^{N} \left(b_i \kappa \delta_.(\mu_1, \mu_2) \sqrt{\mathcal{E}(u)}\ \right) = \kappa \delta_.(\mu_1, \mu_2) \sqrt{\mathcal{E}(u)}; \end{aligned}$$

from which we deduce that

$$\delta_.(T(\mu_1), T(\mu_2)) \leq \kappa \delta_.(\mu_1, \mu_2), \qquad \text{with } \kappa < 1.$$

The fact that $\delta_.$ is a metric on $P(F)$ and that it is complete follows (using results in [KiLa1, §5] and [Ki4]) because F is regular and p.c.f.

Note that (formally) $\delta_.$ is the restriction to the compact space $P(F)$ of the "dual metric" on $C(F)'$ to δ, where

$$\delta(x,y) := \sup\{|u(x) - u(y)| : u \in \mathcal{F}, \sqrt{\mathcal{E}(u)} \leq 1\} \tag{2.22}$$

for $x, y \in F$. (Recall from Remark 2.15(b) that $\mathcal{F} \subset C(F)$ in the present situation.) Further, according to [Ki4], δ is a complete metric on F, with *finite diameter.* □

REMARK 2.23. (a) One could also use the same metric on $P(F)$ as in [Hu], defined as in (2.21) except that the supremum is now taken over all continuous functions on F with Lipschitz constant ≤ 1, with respect to the metric δ in (2.22). However, the above metric is better suited for our later purposes.

(b) The fact that δ is well-defined (and finite) is closely related to the "one-dimensional" nature of p.c.f. fractals, as suggested in Remark 2.15(b). However, in more general situations (such as the three-dimensional Sierpiński carpet, for example), one should use a Connes-type metric. We will return to this point later on. (See esp. §5.)

2.4. Analytical self-similar fractals. We are now ready to provide a formal definition of an analytical s.s. fractal. At this point, the reader may wish to review Definitions 2.13 and 2.16 of a s.s. Dirichlet form and a s.s. measure, respectively.

DEFINITION 2.24. An analytical self-similar set (or fractal) is a quadruple $(F, \mathcal{S}, \mu, \mathcal{E})$, where F is a compact metrizable topological space, $\mathcal{S}$ a s.s. structure on F, μ is a s.s. measure (with respect to $\mathcal{S}$) on F, and $\mathcal{E}$ is a s.s. Dirichlet form (with respect to $\mathcal{S}$) on $L^2(F, \mu)$. [When no confusion may arise, we shall refer to it as F or as (F, μ) (if $\mathcal{S}$ and $\mathcal{E}$ are fixed as will always be the case later on), as the need may be.]

Further, F is p.c.f. if $(F, \mathcal{S})$ is p.c.f. (in the sense of Definition 2.6) and it is regular if the energy functional $\mathcal{E}$ is regular (in the sense of Remark 2.15(b)).

REMARK 2.25. (a) Obviously, if the s.s. set $(F, \mathcal{S}, \mu, \mathcal{E})$ is an analytical s.s. set, in the above sense, then $(F, \mathcal{S})$ is a topological s.s. set (in the sense of Definition 2.1).

(b) Several variants of Definition 2.24 are possible. For instance, F could be allowed to be non-Hausdorff, as in [Ka, Ku2] (see Remark 2.2(a)). Further, the measure μ could be allowed to be Bernoulli instead of self-similar. (See, however, Theorem 2.18 and Remark 2.19(b) above.)

The framework studied in [Ki5]—where s.s. fractals are viewed as (renormalized) limits of electrical networks—should be useful in providing various examples of analytical s.s. fractals. In particular, the two-dimensional Sierpiński carpet, equipped with the s.s. Dirichlet form constructed in [KuZh], is such an example.

(Note that it is clearly *not* p.c.f.) This is of interest given the "universality" of the Sierpiński carpet among all one-dimensional continua in the plane. Indeed, the carpet contains a homeomorphic image of any given (Jordan) planar curve, according to a beautiful (but not so well-known) topological theorem discovered by Sierpiński in [S] (and recalled in [PJS, §2.7]).

3. Laplacians on fractals, Weyl's formula and spectral dimensions

In this section, we recall our recent joint work with Jun Kigami [KiLa1] in which we obtain a partial analogue of Weyl's asymptotic formula (1.6) for the spectral distribution of (Dirichlet or Neumann) Laplacians on a p.c.f. self-similar fractal F. (See Theorem 3.13 below, which partly addresses Question $\mathbf{Q_2}$ from the end of §1.) Further, we discuss the notion of *"spectral dimension"*—defined as in [KiLa1] as the largest possible "*spectral exponent*" occurring in our counterpart of (1.6)—and express it in terms of geometric and analytic data (thereby addressing Question $\mathbf{Q_3}$ from §1). In the process, we identify a specific self-similar measure μ^* on F; namely, the s.s. measure with "maximal spectral exponent", which we will also call here the *"natural s.s. measure"* on F. (See Theorem 3.22 below, from [KiLa1], and its corollaries.) These results will be key to our later work in this paper. (See esp. §4.2 and §5.1.)

We will also propose a conjecture that would extend these results to a broader setting. (See Conjecture 3.37 below.) We note that our results in [KiLa1] are stated somewhat differently. However, they will be rephrased here in terms of the terminology introduced in §2 above. This will enable us, in particular, to state the aforementioned conjecture in a concise manner.

3.1. Dirichlet and Neumann Laplacians. We first briefly explain how to define the Dirichlet and Neumann Laplacians on F, as well as the associated (variational) eigenvalue problems.

DEFINITION 3.1. Given a Borel measure μ on F and $\mathcal{E}$ a Dirichlet form on $L^2(F,\mu) = L^2(\mu)$, with domain $\mathcal{F}$, we say that λ is an eigenvalue of $\mathcal{E}$ (with associated eigenfuction u) if there exists a nonzero $u \in \mathcal{F}$ such that

$$\mathcal{E}(u,v) = \lambda(u,v)_{L^2(\mu)}, \qquad \text{for all } v \in \mathcal{F}, \tag{3.2}$$

where $(\cdot,\cdot)_{L^2(\mu)}$ denotes the inner product of $L^2(\mu)$.

Let $(F, \mathcal{S}, \mu, \mathcal{E})$ be a regular, analytical p.c.f. self-similar fractal, as in Definition 2.24, and let $\mathcal{F}$ denote the domain of $\mathcal{E}$. Let $\Delta = \Delta_\mu$ be the associated Laplacian on F, with domain $\mathcal{D}_\mu \subset L^2(\mu)$. Recall that $\mathcal{D}_\mu \subset \mathcal{F} \subset C(F)$ and that Δ_μ can be defined as a renormalized limit of discrete Laplacians (i.e., finite difference operators) on the finite graphs V_m approximating F (see (2.9) above).

In the following, the finite set $V_0 = \Pi(\mathcal{P})$ defined in (2.8) will play the role of a (Poisson-type) "boundary".

REMARK 3.3. (a) The operator Δ_μ is *"local"*, in the sense that if $u \in \mathcal{D}_\mu$ vanishes identically in a neighborhood of $x_0 \in F$, then so does $\Delta_\mu u$.

(b) For the standard Sierpiński gasket F, for example, V_0 consists of three points; namely, the vertices of the initial triangle in the construction of the gasket. (See, e.g., [KiLa1, Fig. 1, p. 95].) Moreover, for the standard Hausdorff measure μ ($b_1 = b_2 = b_3 = 1/3$) and Dirichlet form $\mathcal{E}$ ($c_1 = c_2 = c_3 = 3/5$), $\Delta_\mu u$ is defined by

$$(3.4) \qquad (\Delta_\mu u)(x) = \lim_{m\to\infty} 5^m \sum_y [u(x) - u(y)], \qquad \text{for } x \in V_m,$$

where the sum is extended over all neighbors y of x in the finite graph V_m. (See [Ki1], [KiLa1, pp. 94–95].)

The following two propositions ([KiLa1, Propositions 5.1 and 5.2, p. 117]) justify the definition of the Neumann and Dirichlet Laplacians on F.

PROPOSITION 3.5 (Neumann Laplacian). *The (nonnegative) self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(F, \mu)$ is called the Neumann Laplacian on F and denoted by $\Delta_1 = \Delta_{1,\mu}$. In particular,* (3.2) *holds if and only if $u \in \mathcal{D}_\mu$ and*

$$\begin{cases} \Delta_\mu u = \lambda u, \\ \partial u|_{V_0} = 0, \end{cases}$$

where $(\partial u)(x)$ is a (suitably defined) "Neumann derivative" of u at $x \in V_0$, and $\partial u|_{V_0}$ denotes the restriction of ∂u to V_0.

Next, let $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_{V_0} = 0\}$ and let $\mathcal{E}_0$ be the restriction of $\mathcal{E}$ to $\mathcal{F}_0$.

PROPOSITION 3.6 (Dirichlet Laplacian). *$(\mathcal{E}_0, \mathcal{F}_0)$ is a (local) Dirichlet form on $L^2(F, \mu)$. Moreover, the (nonnegative) self-adjoint operator associated with $(\mathcal{E}_0, \mathcal{F}_0)$ on $L^2(F, \mu)$ is called the Dirichlet Laplacian on F and denoted by $\Delta_0 = \Delta_{0,\mu}$. In particular,*

$$(3.7) \qquad \mathcal{E}_0(u, v) = \lambda(u, v)_{L^2(\mu)}, \qquad \text{for all } v \in \mathcal{F}_0,$$

if and only if $u \in \mathcal{D}_0 := \mathcal{D}_\mu \cap \mathcal{F}_0$ and

$$\begin{cases} \Delta_\mu u = \lambda u. \\ u|_{V_0} = 0. \end{cases}$$

Of course, $\Delta = \Delta_\mu$ and $\Delta_i = \Delta_{i,\mu}$ depend not only on μ, but also on $\mathcal{S}$ and $\mathcal{E}$ (which will be assumed to be fixed).

The proof of Propositions 3.5 and 3.6 above (given in [KiLa1, p. 117]) makes use of a suitable form of Green's formula in this context, as well as of the fact that $\mathcal{F} = \mathcal{F}_0 + \mathfrak{H}$, where $\mathfrak{H}$ is the space of *"harmonic functions"* on $(F, \mathcal{E})$ (i.e., $\mathfrak{H}$ is the kernel of Δ_μ, which is finite-dimensional and independent of μ, because F is p.c.f.).

Since $\mathcal{F}$ is compactly embedded in $L^2(\mu)$ [KiLa1, Lemma 5.4, p. 118], the spectrum of the Dirichlet or Neumann Laplacian is discrete and consists of a sequence of (positive) eigenvalues, as in (1.4). (We ignore the zero eigenvalue for Δ_1, which has finite multiplicity.)

3.2. Spectral exponents and Weyl's formula on fractals. Denote, as before, by $\{b_i\}_{i=1}^N$ the weights of the s.s. measure μ and by $\{c_i\}_{i=1}^N$ the "harmonic constants" of the s.s. Dirichlet form $\mathcal{E}$. Recall that $N \geq 2, \sum_{i=1}^N b_i = 1, b_i \geq 0$ and $0 \leq c_i < 1$, for all $i = 1, \ldots, N$. Further, recall that an additive subgroup of $\mathbb{R}$ is either discrete or dense in $\mathbb{R}$.

DEFINITION 3.8. Let

$$\gamma_i = \sqrt{b_i c_i}, \qquad \text{for } i = 1, \ldots, N. \tag{3.9}$$

Consider the additive subgroup of $\mathbb{R}$ defined by

$$G = \sum_{i=1}^{N} (\ln \gamma_i)\mathbb{Z}, \tag{3.10}$$

the set of all integral linear combinations of the logarithms of $\gamma_1, \ldots, \gamma_N$. Then the <u>*nonlattice case*</u> is that when G is dense in $\mathbb{R}$, whereas the <u>*lattice case*</u> is that when G is discrete.

Moreover, let d_S be the unique positive number such that

$$\sum_{i=1}^{N} \gamma_i^{d_S} = 1. \tag{3.11}$$

Then $d_S = d_S(\mathcal{S}, \mu, \mathcal{E})$ is called the <u>spectral exponent</u> of the analytical fractal $(F, \mathcal{S}, \mu, \mathcal{E})$.

REMARK 3.12. (a) When $N = 2$, for example, we are in the lattice case if and only if there exist positive integers p_1, p_2 such that $\gamma_1^{p_1} = \gamma_2^{p_2}$.

(b) The lattice case is also referred to as the "arithmetic case" in (probabilistic) renewal theory [Fel].

(c) The number d_S is well-defined since by definition, $\gamma_i \ (= \sqrt{b_i c_i} \leq \sqrt{c_i}) < 1$ for all $i = 1, \ldots, N$, and thus the function $\varphi(t) := \sum_{i=1}^N \gamma_i^t$ is (strictly) decreasing for $t > 0$ (since $\varphi(0) = N \geq 2$). Further, since $\varphi(2) < \sum_{i=1}^N b_i = 1$, we must have $d_S < 2$.

(d) With the notation of [KiLa1] and [La5], we have for $i = 1, \ldots, N, b_i = \mu_i$ and $c_i = r_i/\alpha$, where $(r_1, \ldots, r_n)$ are defined by the *"harmonic structure"* giving rise to $\mathcal{E}$, and α (denoted by λ in [KiLa1]) is the *"renormalization constant"*, a positive eigenvalue of the nonlinear "renormalization operator". (See Remark 2.15(a) above.)

We are now ready to restate the main results of [KiLa1]. [Note that in rephrasing these results, we use the terminology of §2, as well as Theorem 2.18 above (which enables us to replace the expression "Bernoulli measure" by "s.s. measure").]

THEOREM 3.13 (A partial analogue of Weyl's formula on fractals; [KiLa1, Theorem 2.4 and Corollary 2.5, pp. 104–105]). *Let $(F, \mathcal{S}, \mu, \mathcal{E})$ be a regular p.c.f. analytical self-similar fractal* (as in Definition 2.24). *In particular, μ is a s.s. measure with weights $\{b_i\}_{i=1}^N$ and $\mathcal{E}$ is a s.s. Dirichlet form with harmonic constants $\{c_i\}_{i=1}^N$, relative to the s.s. structure $\mathcal{S}$. Let $\Delta_i = \Delta_{i,\mu}$ be the associated Dirichlet $(i = 0)$ or Neumann $(i = 1)$ Laplacian on F (as defined in Propositions* 3.5 *and* 3.6), *and let $N(\lambda)$ denote the corresponding eigenvalue counting function. Then*

$$N(\lambda) \asymp \lambda^{d_S/2} \qquad \text{as } \lambda \to +\infty \tag{3.14}$$

(i.e., there exist positive constants c_1, c_2 such that $c_1\lambda^{d_S/2} \leq N(\lambda) \leq c_2\lambda^{d_S/2}$, for all sufficiently large λ). Here, $d_S = d_S(\mathcal{S}, \mu, \mathcal{E})$ is the spectral exponent defined by (3.11)*: $\sum_{i=1}^N \gamma_i^{d_S} = 1$.*

More precisely, we have the following dichotomy (as in Definition 3.8 above)*:*

(i) (*Nonlattice case: $G := \sum_{i=1}^N (\ln \gamma_i)\mathbb{Z}$ is dense in $\mathbb{R}$). Then*

$$N(\lambda) \sim C\lambda^{d_S/2} \qquad \text{as } \lambda \to +\infty \tag{3.15}$$

(i.e., $N(\lambda) = \lambda^{d_S/2}(C + o(1))$, where $o(1)$ vanishes as $\lambda \to +\infty$), for some positive constant $C = C(\mathcal{S}, \mu, \mathcal{E})$.

(ii) (*Lattice case: $G = \sum_{i=1}^N (\ln \gamma_i)\mathbb{Z}$ is discrete; say, of the form $G = h\mathbb{Z}$, with $h > 0$). Then*

$$N(\lambda) = \lambda^{d_S/2}\left(g\left(\frac{\ln\lambda}{2}\right) + o(1)\right), \quad \text{as } \lambda \to +\infty, \tag{3.16}$$

for some bounded positive periodic function g of period T, the "positive generator" of G (i.e., T is the smallest $h > 0$ such that $G = h\mathbb{Z}$). Further, g is right-continuous, measurable, and bounded away from zero on $[0, +\infty)$.

Moreover, the constant C in (i) *and the periodic function g in* (ii) *are independent of the boundary conditions. Hence, in particular, the Dirichlet and Neumann Laplacians have the same leading spectral asymptotics.*

Theorem 3.13 proves, specifies and corrects, for p.c.f. self-similar fractals, an earlier conjecture of the author for Laplacians on s.s. fractals. (See esp. [La5, Conjecture 5, p. 190 and Remark 5.16(b), p. 197].) This conjecture is a natural counterpart in this setting of the author's conjecture for Laplacians on open sets with self-similar fractal boundary [La5, Conjectures 2 and 3, p. 159 and pp. 163–164]—motivated in part by his earlier work on the Weyl-Berry conjecture (esp. [La1–3, LaPo1–3]), as well as by [L]. The proof of Theorem 3.13—provided in [KiLa1, §§2,4–6]—combines, in particular, techniques from symbolic dynamics, variational methods from [Me, La1] used in [La1] to deal with "drums with fractal boundaries", results in [Ki2], and an application of a suitable form of the Renewal Theorem [Fel] (given in [KiLa1, p. 108]). (An additional step [KiLa1, Lemma 6.1, p. 119] consists in showing that natural energy functionals on F give rise to self-similar Dirichlet forms, in the sense of Definition 2.13 above, which satisfy the hypotheses of Theorem 3.13.)

We refer to §3 (and the appendix) of [KiLa1] for many examples illustrating Theorem 3.13. See, in particular, [KiLa1, Examples 3 and 4, pp. 111–113]—discussing the "modified Sierpiński gasket" ([KiLa1, Fig. 2, p. 112]) and "Hata's self-similar fractal tree" ([KiLa1, Fig. 3, p. 113])—for natural examples of "nonlattice case".

COROLLARY 3.17. *For fixed S and $\mathcal{E}$, the following limit*

$$\lim_{\lambda\to+\infty} \lambda^{-d_S/2} N(\lambda) \tag{3.18}$$

exists (and is nontrivial) for a "generic" choice of μ (i.e., for Lebesgue-almost all $(b_i)_{i=1}^N$ in the standard $(N-1)$-simplex of $\mathbb{R}^N$).

PROOF. The nonlattice case is clearly generic in the above sense. □

REMARK 3.19. (a) In the next sections, we will address the problem—directly connected with Question $\mathbf{Q_2}$ from §1—of finding a more complete analogue of Weyl's formula (1.6). We will obtain, for example, in the nonlattice case, a geometric interpretation of the proportionality constant C in (3.15) in terms of a suitably defined "*volume measure*" on F. (See esp. Theorem 4.41 and Corollary 4.45.)

(b) We conjecture in [KiLa1, Remark 2, p. 105] that in the lattice case and when d_S is not an integer (i.e., $d_S \neq 1$), the periodic function g in (3.16) is nonconstant; that is, the limit in (3.18) does *not* exist. (Heuristically, these oscillations in the leading asymptotics should be due to the large symmetry group of the analytical fractal that gives rise to eigenvalues with large multiplicity.) In view of the results of Fukushima and Shima [FuSh], this is true for the standard N-Sierpiński gasket (with $b_1 = \ldots = b_N = 1/N$ and $c_1 = \ldots c_N = N/(N+2)$). More generally, it has recently been established for the "nested fractals" of [Li] by Barlow and Kigami (personal communication). (Note that we are clearly in the lattice case since now, $b_1 = \ldots = b_N = 1/N$ and $c_1 = \ldots = c_N = 1/\alpha$, where α is the "renormalization constant".) Furthermore, a preprint of Shima—received after the completion of the present work—shows that this is also the case for a certain class of "symmetric" p.c.f. fractals (for the same choice of b_i's and c_i's). (Note that the standard Koch curve, equipped with its natural Hausdorff measure, is isomorphic to $[0,1]$, equipped with Lebesgue measure; and so $d_S = 1$ but g is constant in this situation.)

(c) In the special case of the "modified Koch curve" (which is "nested" and hence p.c.f.), it would be interesting to express the (nonconstant, by [M1–2]) periodic function g in (3.16) in terms of the dynamical functions studied rigorously by Malozemov in [M1–2] (and inspired in part by the physical work of Rammal and Toulouse ([RT, R] and references therein) on the Sierpiński gasket).

(d) As was observed by Professors Alain Connes and Dennis Sullivan when they were presented by the author with these results (as well as with the results of §4–5 below), the dichotomy between the nonlattice case and lattice case in Theorem 3.13 above is very reminiscent of that between von Neumann algebras of type III_1 (the "generic" case) and type III_λ (with $0 < \lambda < 1$). (See, e.g., [Co5, Chap. 5].)

3.3. Spectral and fractal dimensions.

DEFINITION 3.20. Let $\mathcal{E}$ be a s.s. Dirichlet form with "harmonic constants" $\{c_i\}_{i=1}^N$ ($c_i < 1, i = 1, \ldots, N$) as above. Let S be the unique positive number such that

$$\sum_{i=1}^{N} c_i^S = 1. \tag{3.21}$$

Then S is called the <u>similarity dimension</u> of $\mathcal{E}$ (or of the Dirichlet space $(F, \mathcal{E})$).

In the following simple theorem, we single out, in particular, a distinguished s.s. measure μ^* on F; namely, *the s.s. measure with maximal spectral exponent*, the *"spectral dimension"* of $(F, \mathcal{E})$. It will play an important role in the rest of this paper. We will fix ($\mathcal{S}$ and) $\mathcal{E}$, and let the s.s. measure μ vary, so that we simply write $d_S = d_S(\mu)$.

THEOREM 3.22 (Spectral dimension; [KiLa1, Theorem A.2, p. 121]). *Assume the same hypotheses as in Theorem 3.13. Then the following maximum exists and is given by*

$$d_S^* := \max\{d_S(\mu) : \mu \text{ is a s.s. measure on } F\} = \frac{2S}{S+1}, \tag{3.23}$$

where S is the "similarity dimension" of $(F, \mathcal{E})$, defined by (3.21). Further, this maximum is achieved by a unique s.s. measure, denoted by μ^*, and characterized by the weights

$$b_i := c_i^S, \qquad i = 1, \ldots, N. \tag{3.24}$$

The positive number

$$d_S^* = d_S(\mu^*) = \frac{2S}{S+1} \tag{3.25}$$

is called the <u>*spectral dimension*</u> *of the s.s. Dirichlet form $\mathcal{E}$ (or of the Dirichlet space $(F, \mathcal{E})$).*

PROOF. We can calculate the maximum of the function

$$d_S(\mu) = d_S(b_1, \ldots, b_N),$$

subject to the constraints $\sum_{i=1}^N b_i = 1$, $b_i \geq 0 \quad (i = 1, \ldots, N)$, by applying the method of Lagrange multipliers. (Note that we must use Theorem 2.18 above in order to identify the s.s. measure μ with the N-tuple $(b_1, \ldots, b_N)$ in the standard $(N-1)$-simplex of $\mathbb{R}^N$.) □

REMARK 3.26. (a) We propose to call the unique s.s. measure μ^* of maximal spectral exponent d_S^*, obtained in Theorem 3.22, the "<u>*natural s.s. measure*</u>" (relative to $\mathcal{S}$) on $(F, \mathcal{E})$.

(b) It would be interesting to establish connections between our present work and Strichartz's work [St1–2] on self-similar measures in Euclidean spaces (or Riemannian manifolds, say).

The value of the "*spectral dimension*"—defined as above—coincides with that calculated earlier in special cases by physicists (e.g., [Dh, AO, RT, HHW, ...]) and, rigorously but from a different point of view, by probabilists (e.g., [Ku1, BP, Li, Fu2, ...]). (In physics, following Alexander and Orbach [AO], the "spectral dimension" is also called "fracton dimension".)

For examples of calculations of S and d_S^*, we refer to [KiLa1, Appendix, pp. 121–122]. For instance, for the standard N-Sierpiński gasket in $\mathbb{R}^{N-1}$ [KiLa1, Example 2, p. 122] (also studied analytically in [FuSh]), we have

$$S = \frac{\ln N}{\ln(N+2) - \ln N} \quad \text{and} \quad d_S^* = \frac{2 \ln N}{\ln(N+2)}. \tag{3.27}$$

[Note that for $N > 2, d_S^*$ is *not* equal to the Hausdorff dimension of F (with respect to the Euclidean metric), $\ln N / \ln 2$, and that $d_S^* \to 2^-$, as $N \to \infty$. Further, for the usual Sierpiński gasket in $\mathbb{R}^2$, we have $N = 3$ and so $d_S^* = \ln 9 / \ln 5$, whereas for the unit interval in $\mathbb{R}^1$, we have $N = 2$ and so $d_S^* = 1$.]

More generally, for "nested fractals" [KiLa1, Example 5, p. 122], the natural s.s. measure μ^* is determined by $b_1 = \ldots = b_N = 1/N$ and the "natural" s.s. Dirichlet form is given by $c_1 = \ldots = c_N = 1/\alpha$, where α is the renormalization constant. Consequently, by (3.21), the similarity dimension of $(F, \mathcal{E})$ is given by

$$S = \frac{\ln N}{\ln \alpha} \tag{3.28}$$

while by Theorem 3.22, the spectral dimension of $(F,\mathcal{E})$ is given by

$$d_S^* = \frac{2\ln N}{\ln(N\alpha)}, \tag{3.29}$$

in agreement with [Li] and [Fu2]. (Note that in this case, μ^* is nothing but the (normalized) Hausdorff measure on $F \subset \mathbb{R}^n$, with respect to the Euclidean metric.)

In particular, for the "modified Koch curve" [M1–2], we may choose $N = 5$ and $\alpha = 8/3$, so that

$$S = \frac{\ln 5}{\ln 8 - \ln 3} \quad \text{and} \quad d_S^* = \frac{2\ln 5}{\ln(40/3)} = \frac{2\ln 5}{\ln 40 - \ln 3}. \tag{3.30}$$

The next corollary, an immediate consequence of Remarks 3.12(c), has physical and probabilistic significance.

COROLLARY 3.31. *Let d_S^* be the spectral dimension defined in Theorem* 3.22. *Then $d_S^* < 2$.*

As was pointed out in the appendix of [KiLa1], the definition of the *similarity dimension* of $(F,\mathcal{E})$ is very analogous to that of standard (nonoverlapping) self-similar fractals [Ma; Fc, §9.2] embedded in Euclidean spaces. (See also [KiLa1, Remark 3, pp. 105–106].) We shall now specify this analogy. Indeed, in a subsequent work [Ki4], Kigami uses in particular some of the results in [KiLa1] to interpret S in Theorem 3.22 as the actual "similarity dimension" (as well as the Hausdorff dimension $d_H = d_H(\delta_1)$) of the p.c.f. self-similar set F, with respect to a suitably chosen (bounded, complete) *"intrinsic metric"*, $\delta_1 = \delta_{1,\mathcal{E}}$, called the "effective resistance metric" and depending only on the energy functional $\mathcal{E}$:

$$\begin{aligned}\delta_1 = \delta_{1,\mathcal{E}} &:= \sup\{|u(x)-u(y)|^2 : u \in \mathcal{F}, \mathcal{E}(u) \le 1\} \\ &= \max\left\{|u(x)-u(y)|^2/\mathcal{E}(u) : u \in \mathcal{F},\ u \text{ nonconstant}\right\},\end{aligned} \tag{3.32}$$

for $x, y \in F$. (Note that $\delta_1 = \delta^2$, where $\delta = \delta_{\mathcal{E}}$ is the metric defined by (2.22). Further, in (3.32) or (2.22), it is implicitly understood that $u(x) \neq u(y)$.)

Moreover, it is not difficult to check—by using the self-similarity of $\mathcal{E}$ much as in the proof of Theorem 2.18 above—that for each $i = 1, \dots, N$, the map W_i is a contraction on (F, δ_1) with Lipschitz constant c_i $(= r_i/\alpha)$. (We stress, however, that the W_i's, are in general genuinely *nonlinear* even when F is embedded in some Euclidean space $\mathbb{R}^n$. This is the case, for example, for the Sierpiński gasket even though the W_i's are (restrictions to F of) similarities of $\mathbb{R}^2$ with respect to the Euclidean metric. We will return to this point in §5.1 below.) It then follows from an extension to the present abstract setting [Ki3,4] of the corresponding results in [Mo, Hu] that

$$S = d_H(\delta_1) = d_M(\delta_1), \tag{3.33}$$

where $d_H = d_H(\delta_1)$ (resp., $d_M = d_M(\delta_1)$) denotes the Hausdorff (resp., Minkowski or "box") dimension of (F, δ_1). (The second equality in (3.33), although not explicitly stated, can easily be deduced from the above references. Furthermore, the reader may wish to compare (3.21) and (3.33) with, e.g., [Fc, §9.2, esp. Theorem 9.3, p. 118].)

We can now state the following corollary of Theorem 3.22 [KiLa1, Theorem A.2] and of [Ki4].

COROLLARY 3.34 (Spectral and fractal dimensions). *The spectral dimension* $d_S^* = d_S(\mu^*)$ *defined (and calculated) in Theorem* 3.22 *is also given by*

$$d_S^* = \frac{2S}{S+1} = \frac{2d_H}{1+d_H} = \frac{2d_M}{1+d_M}, \tag{3.35}$$

where $d_M = d_H(\delta_1)$ *(resp.,* $d_M = d_M(\delta_1)$*) denotes the Hausdorff (resp., Minkowski) dimension of* (F, δ_1)*. Here,* $\delta_1 = \delta_{1,\mathcal{E}}$ *is the (complete and bounded) metric defined by* (3.32).

PROOF. This follows by combining (3.25) and (3.33). □

REMARK 3.36. (a) We could work instead with the metric $\delta = \sqrt{\delta_1}$, given by (2.22). However, we would then have to replace (3.33) by

$$S = \frac{1}{2} d_H(\delta) = \frac{1}{2} d_M(\delta) \tag{3.33'}$$

and hence (3.35) would have to be changed accordingly. For some other purposes, δ is more convenient to use, as it is akin to a Connes metric (see (b) and §5.2 below). Hence, depending on the situation, we will work wi' either δ_1 or δ in the rest of this paper (see §5).

(b) The "intrinsic metric" $\delta_1 = \delta_1(\mathcal{E})$ can be thought of ([La5]) as the *"capacitary metric"* associated with the energy functional $\mathcal{E}$. Here, it is well-defined due to the "one-dimensional" nature of p.c.f. self-similar fractals. (See Remarks 2.15(b) and 2.23(b) above.) In more general situations, however, definition (3.32) will be meaningless and we will propose instead to work with Connes-type metrics [Co3–5] (see §5). An example when this may be appropriate is the three-dimensional Sierpiński carpet, studied from a very different (probabilistic) point of view in [BB2].

3.4. Conjecture. We close this section by stating a conjecture that would enable us, in particular, to extend the above results (in §3.2 and §3.3) from "finitely ramified" (i.e., p.c.f.) to certain "infinitely ramified" s.s. fractals, such as the two-dimensional Sierpiński carpet first studied probabilistically by Barlow and Bass in [BB1] and later analytically by Kusuoka and Zhou in [KuZh]. If proved, it would also enable us to extend to a broader class of self-similar fractals the main results of the present paper given in §4.2 below. (See §5.3.)

CONJECTURE 3.37 (Spectral distribution of Laplacians on analytical s.s. fractals). *Let* $(F, \mathcal{S}, \mu, \mathcal{E})$ *be a regular (but not necessarily p.c.f.) analytical self-similar fractal, in the sense of Definition* 2.24, *satisfying the "topological open set condition"* (see Remark 2.5(b)). *Then the analogue of Theorems* 3.13 *and* 3.22, *as well as their corollaries (Corollaries* 3.17, 3.31 *and* 3.34) *hold in this situation.*

Moreover, in the lattice case (part (ii) *of the analogue of Theorem* 3.13), *the periodic function g occurring in the counterpart of* (3.16) *is nonconstant when* $d_S = d_S(\mu)$ *is nonintegral.*

REMARK 3.38. (a) A more detailed statement of Conjecture 3.37—but using somewhat different teminology—is provided in [La5, Conjecture $\hat{5}$, pp. 195–197].

(b) Implicit in the statement of Conjecture 3.37 (regarding the analogue of Corollary 3.36 from §3.3) is that the "intrinsic" (or "capacitary") metric $\delta_1 = \delta_{1,\mathcal{E}}$ given by (3.32) can still be well-defined and that (3.33) continues to hold. (Compare with Remark 3.36(b) above.) The results of [Ki5] should be useful to establish this. Furthermore, for the important case of the (two-dimensional) Sierpiński carpet, we may use to tackle Conjecture 3.37 the s.s. Dirichlet form constructed in [KuZh].

(c) As discussed in [La5, Remarks 5.16(a), (b), p. 197], when the Dirichlet space $(F, \mathcal{E})$ is endowed with the natural (intrinsic) metric δ_1 and s.s. measure μ^*, it is easy to deduce from Conjecture 3.37 that the dichotomy "lattice/nonlattice case" corresponds to (the additive subgroup) $G' := \sum_{i=1}^{N} (\ln c_i)\mathbb{Z}$ being discrete or dense in $\mathbb{R}$, where the "harmonic constants" c_i are the Lipschitz constants (or "scaling ratios") of the maps W_i ($i = 1, \dots, N$) on the metric space (F, δ_1), in exact analogy with the author's original conjecture for "drums with self-similar fractal membrane" [La5, Conjecture 5, p. 190]. (Note, however, that we have now replaced the Euclidean metric on F ($\subseteq \mathbb{R}^n$) by the intrinsic metric δ_1.)

[Here is a proof of the above statement: By (3.9) and (3.24), we have $\gamma_i = \sqrt{b_i c_i}$ with $b_i = c_i^S$; so that $\gamma_i = c_i^{(S+1)/2}$ and hence $\ln \gamma_i = ((S+1)/2) \ln c_i$, for $i = 1, \dots, N$. It obviously follows that the groups $\sum_{i=1}^{N} (\ln \gamma_i)\mathbb{Z}$ and $\sum_{i=1}^{N} (\ln c_i)\mathbb{Z}$ are discrete (or dense in $\mathbb{R}$) at the same time.]

(d) Of course, in the light of the results recalled in §3, Conjecture 3.37 is true for (analytical) p.c.f. (i.e., "finitely ramified") self-similar fractals. In view of (c) above, this provides, *a posteriori*, a formal justification (and extension) of our original conjecture for the spectral distribution of Laplacians on (suitable) s.s. fractals [La5, Conjecture 5, p. 190].

4. Dixmier trace and volume measures on fractals

We recall in §4.1 the notion of "Dixmier trace" [Di, Co4–5] from the theory of operator algebras and then use it in §4.2 in order to construct an analogue of *"volume measures"* for certain classes of fractals. This will enable us, in particular, to obtain a more precise counterpart than in [KiLa1] of Weyl's formula for the spectral distribution of (Dirichlet or Neumann) Laplacians on p.c.f. self-similar fractals. (See Theorems 4.27, 4.41, 4.49, and esp. Corollary 4.45.) In the process, we use and complete our earlier joint results in [KiLa1] recalled in §3 above, as well as further address Question $\mathbf{Q_2}$ and partially answer Question $\mathbf{Q_4}$ of §1.

It will be clear that these methods can be applied in many related settings where results analogous to [KiLa1] can be obtained. We will discuss in §5 possible extensions of this work as well as the (conjectured) properties of a distinguished "volume measure" which we propose to be *the analogue of Riemannian volume* in this context.

A construction similar to that used in §4.2—but applied to the Dirac operator on a (spin) Riemannian manifold (instead of the square-root of the Laplacian on a s.s. fractal)—had enabled A. Connes [Co3; Co5, §VI.1] to recover the usual "Riemannian volume measure" from purely operator-theoretic data, within the framework of noncommutative geometry. We postpone to §5 further discussion of the possible connections between our present work and aspects of noncommutative geometry.

4.1. Dixmier trace and Maçaev ideal.

4.1.1. Operator ideals. In order to define the Dixmier trace, we first need to recall some facts about certain ideals of operators, notably the Maçaev ideals. For more details, we refer to ([Co2,4–5], [Si], [Vo]). In the following, all ideals considered will be *two-sided.*

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be the ideal of compact linear operators on $\mathcal{H}$. Given $R \in \mathcal{K}$, let $\rho_j = \rho_j(R)$ denote the characteristic values of R (i.e., the eigenvalues of $|R| = \sqrt{R^*R}$, the absolute value of R), written in nonincreasing order and according to multiplicity. (Recall that $\rho_j(R) \to 0$ as $j \to \infty$. Of course, if R is of finite rank, we set $\rho_j(R) = 0$ for all j sufficiently large.) Then we set for $J = 1, 2, \ldots,$

$$A_J(R) = \sum_{j=1}^{J} \rho_j(R). \tag{4.1}$$

For $p \in [1, \infty)$, we consider the *Schatten ideal* of order p

$$\mathcal{L}^p = \mathcal{L}^p(\mathcal{H}) = \left\{ R \in \mathcal{K} : \sum_{j=1}^{\infty} \rho_j(R)^p < \infty \right\}, \tag{4.2}$$

as well as the possibly less familiar *Maçaev ideal* of order p

$$\mathcal{L}^{p+} = \mathcal{L}^{p+}(\mathcal{H}) = \left\{ R \in \mathcal{K} : A_J(R) = O\left(\sum_{j=1}^{J} j^{-1/p} \right) \right\}. \tag{4.3}$$

(The "O" in (4.3), or in (4.4) below, is as $J \to \infty$, while in (4.5) below, it is as $j \to \infty$.) For $p = \infty$, we simply set $\mathcal{L}^{\infty+} = \mathcal{L}(\mathcal{H})$, the set of all bounded linear operators on $\mathcal{H}$. ($\mathcal{L}^{p+}$ is also denoted by $\mathcal{L}^{(p,\infty)}$ in the literature.)

A simple argument shows that

$$\mathcal{L}^{1+} = \{R \in \mathcal{K} : A_J(R) = O(\ln J)\} \tag{4.4}$$

with $A_J(R)$ defined by (4.1), whereas for $p \in (1, \infty)$,

$$\mathcal{L}^{p+} = \{R \in \mathcal{K} : \rho_j(R) = O(j^{-1/p})\}. \tag{4.5}$$

We have $\mathcal{L}^p \subset \mathcal{L}^{p+} \subset \mathcal{L}^t$ for $p < t$. In addition, there is a well-developed duality and interpolation theory for these (and related) ideals, but we will not need to go into that here. (See, e.g., [Co2] or [Co5, §IV.2], as well as [Vo].)

REMARK 4.6. (a) Note that $\mathcal{L}^1$ is just the ideal of trace class operators while by (4.4), $\mathcal{L}^{1+}$ is the ideal of bounded operators whose *"trace diverges logarithmically"*.

(b) Intuitively, $\mathcal{L}^p$ can be thought of as a noncommutative analogue of the usual Lebesgue L^p-space of p-summable functions, and $\mathcal{L}^{p+}$ as an analogue of the "weak L^p-space", of frequent use in harmonic analysis and (real) interpolation theory.

4.1.2. Dixmier trace. The "Dixmier trace", Tr_w, was introduced in 1966 by Jean Dixmier [Di] as the first example of a trace on the C^*-algebra $\mathcal{L}(\mathcal{H})$ which is *"non-normal"* (i.e., which is not proportional to the usual trace of operators, on the ideal where it is finite). Since the mid-eighties, it has been further studied and used extensively by Alain Connes (and his collaborators) in developing several aspects of noncommutative differential geometry and topology. (See, e.g., [Co1–3], [Co4, Chap. 5], [Co5, Chaps. IV and VI], as well as [CoSu].)

Roughly speaking, for $R \in \mathcal{L}^{1+}$, the "Dixmier trace" of R is well-defined and is given by

$$\mathrm{Tr}_w(R) = \mathrm{Lim}_w \frac{1}{\ln J} \sum_{j=1}^{J} \rho_j(R), \tag{4.7}$$

where "Lim_w" is a suitable notion of limit (of arbitrary bounded sequences of real numbers) with nice *scaling* properties.

More precisely, let w be an invariant mean on $\mathbb{R}_+^*$, the multiplicative group of positive real numbers (i.e., w is a positive linear functional on $L^\infty(\mathbb{R}_+^*)$, invariant by homothety and such that $w(1) = 1$).

Let ℓ^∞ be the linear space of bounded real sequences. Given $s = (s_j)_{j=1}^\infty$ in ℓ^∞, let

$$\text{Lim}_w s := w(\varphi_s), \tag{4.8}$$

where $\varphi_s \in L^\infty(\mathbb{R}_+^*)$ is defined by $\varphi_s(t) = s_j$ for $t \in (j-1, j]$, $j \geq 1$.

Then "Lim_w" is a linear functional on ℓ^∞ such that

$$\text{Lim}_w 1 = 1, \tag{4.9a}$$

$$\text{Lim}_w s \geq 0 \text{ if } s \geq 0, \tag{4.9b}$$

$$\text{Lim}_w s = L \text{ if } s = (s_j)_{j=1}^\infty \text{ converges to } L \tag{4.9c}$$

(actually, $\underline{\lim}\ s \leq \text{Lim}_w s \leq \overline{\lim}\ s$, from which (4.9) follows); and the following key *scale-invariance* property holds:

$$\text{Lim}_w(s_1, s_2, \dots) = \text{Lim}_w(s_1, s_1, s_2, s_2, \dots). \tag{4.10}$$

REMARK 4.11. It is stressed in ([Co4, §5.2, pp. 189–194], [Co5, §IV.2]) that Lim_w is a more "concrete" and usable notion of limit than it appears to be at first sight. For example, according to a result of Gabriel Mokobodski, it is compatible (and so can be interchanged) with very general barycentric means of sequences.

DEFINITION 4.12 (Dixmier trace). Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}^{1+} = \mathcal{L}^{1+}(\mathcal{H})$ be the Maçaev ideal defined by (4.4). For $R \geq 0, R \in \mathcal{L}^{1+}(\mathcal{H})$, we set

$$\text{Tr}_w(R) = \text{Lim}_w \frac{1}{\ln J} \sum_{j=1}^{J} \rho_j(R). \tag{4.13}$$

The (nonnegative) number $\text{Tr}_w(R)$ is called the ***Dixmier trace*** of R (associated with the mean w).

(Note that by (4.1) and (4.4), the sequence $\{\frac{1}{\ln J} \sum_{j=1}^{J} \rho_j(R)\}_{J=1}^\infty$ is *bounded* since $R \in \mathcal{L}^{1+}$; so that $\text{Tr}_w(R)$ is well-defined and finite. Further, since (R is self-adjoint and) $R \geq 0$, $\rho_j(R)$ is just the jth *eigenvalue* of R, written in nonincreasing order.)

It can be shown [Di, Co4–5] (by using in particular the variational characterization of $\rho_j(R)$, and its consequences, esp. the Rayleigh-Ritz inequalities) that Tr_w is *additive*:

$$\text{Tr}_w(R_1 + R_2) = \text{Tr}_w(R_1) + \text{Tr}_w(R_2),$$

for $R_j \geq 0$, $R_j \in \mathcal{L}^{1+}$ $(j = 1, 2)$. Hence Tr_w extends (uniquely) by *linearity* to all of $\mathcal{L}^{1+}$. It then suffices to let $\text{Tr}_w(R) = +\infty$ for $R \geq 0, R \notin \mathcal{L}^{1+} (R \in \mathcal{L}(\mathcal{H}))$, to obtain a (*non-normal*) *trace*, Tr_w, on the C^*-algebra $\mathcal{L}(\mathcal{H})$.

The following proposition ([Di], extended in [Co5, §IV.2 and §VI.1]) summarizes some of the basic properties of the Dixmier trace.

PROPOSITION 4.14.

(i) (Positivity). $\mathrm{Tr}_w(R) \geq 0$ *for* $R \geq 0$.

(ii) (Finiteness). $|\,\mathrm{Tr}_w(R)| < \infty$ *for* $R \in \mathcal{L}^{1+}$.

(iii) (Covariance). $\mathrm{Tr}_w(RV) = \mathrm{Tr}_w(VR)$ *for* $R \in \mathcal{L}^{1+}$ *and* $V \in \mathcal{L}(\mathcal{H})$.

In particular, $\mathrm{Tr}_w(URU^{-1}) = \mathrm{Tr}_w(R)$ *for all* $R \in \mathcal{L}^{1+}$ *and* U *unitary (or more generally, for* U *bounded and invertible); so that* Tr_w *is independent of the choice of the inner product compatible with the topology of* $\mathcal{H}$.

(iv) ("Locality"). $\mathrm{Tr}_w(R) = 0$ *for all* $R \in \mathcal{L}_0^{1+}$ *(and in particular, for all trace class operators).*

Here, the Maçaev ideal $\mathcal{L}_0^{1+} = \mathcal{L}_0^{1+}(\mathcal{H})$ *is the closure in* $\mathcal{L}^{1+}$ *of the ideal of finite rank operators with respect to the natural norm on* $\mathcal{L}^{1+}$, $\|R\|_{1+} := \sup_J \frac{1}{\ln J} A_J(R)$. *For* $\mathcal{H}$ *infinite-dimensional, we have* $\mathcal{L}^1 \not\subset \mathcal{L}_0^{1+} \not\subset \mathcal{L}^{1+}$. *Further,* $\mathcal{L}_0^{1+}$ *is characterized just as* $\mathcal{L}^{1+}$ *in* (4.4), *except with "O" replaced by "o".*

REMARK 4.15. (a) Property (iii) implies that Tr_w is a *unitary trace* on the two-sided ideal $\mathcal{L}^{1+}$ of $\mathcal{L}(\mathcal{H})$.

(b) As is explained in detail in [Co5], it is the vanishing property (iv) which enables one to use the Dixmier trace to capture only the "leading (spectral) asymptotics" and neglect all trace class perturbations of $R \in \mathcal{L}^{1+}$, in other words, to retain the *"semiclassical information"* contained in R.

In general, Tr_w may depend on the choice of the limiting procedure w. The following proposition (a simple extension of [Co2, §II.6, p. 67] or [Co5, §IV.2]) provides a useful example when this is not the case. (Actually, much weaker conditions than (1)–(3) below are sufficient.)

Let $R \in \mathcal{L}(\mathcal{H}), R \geq 0$, be such that $R \in \mathcal{L}^p$ for all $p > 1$. (This is the case, in particular, if $\mathcal{R} \in \mathcal{L}^{1+}, R \geq 0$.) Then, clearly, the associated *"spectral zeta-function"*

$$\zeta_R(s) = \mathrm{Trace}(R^s) = \sum_{j=1}^{\infty} \rho_j(R)^s \tag{4.16}$$

is well-defined for $s \in \mathbb{C}$, Re $s > 1$.

PROPOSITION 4.17. *Let* $R \in \mathcal{L}^{1+}, R \geq 0$ *and let* $L \in \mathbb{R}$. *Then:*

(i) *Conditions* (1) *and* (2) *below are equivalent:*

$$(s-1)\zeta_R(s) \to L \qquad \text{as } s \to 1+ \ (\text{i.e., } s \to 1,\ s \in \mathbb{R},\ s > 1). \tag{1}$$

$$\frac{1}{\ln J}\sum_{j=1}^{J} \rho_j(R) \to L \qquad \text{as } J \to \infty. \tag{2}$$

(ii) *Further, if in addition, $\rho_j(R) \asymp 1/j$ as $j \to \infty$, then condition* (3) *below is equivalent to* (2) *(and hence also to* (1)*):*

$$\frac{1}{\ln \Lambda}\int_1^\Lambda \lambda^{-1} n(\lambda)\frac{d\lambda}{\lambda} \to L \qquad \text{as } \Lambda \to +\infty, \tag{3}$$

where

$$n(\lambda) := n(\lambda; R) = \#\{j \geq 1 : \rho_j(R) \geq \lambda^{-1}\}. \tag{4.18}$$

(iii) *Moreover, if any of the above conditions* (1), (2) *and (under the assumptions of* (ii)) (3) *is satisfied, then $\mathrm{Tr}_w(R) = L$, and in particular, $\mathrm{Tr}_w(R)$ is independent of the limiting procedure w used in defining it.*

PROOF. (i) (and thus (iii), by (4.13) and (4.9c)) follows from the aforementioned result in [Co2,5] (and from (ii)). It is a consequence of the Hardy-Littlewood Tauberian Theorem.

(ii) For the reader's convenience, we provide a proof of (ii). We write $\rho_j = \rho_j(R)$ for simplicity. Since $\{\rho_j\} \downarrow 0$, we may assume without loss of generality that $\rho_1 < 1$. Given $\Lambda > 0$, let $J = J(\Lambda)$ be the largest positive integer such that $\rho_J \geq \Lambda^{-1}$, so that $1 < \rho_1^{-1} \leq \ldots \leq \rho_J^{-1} \leq \Lambda < \rho_{J+1}^{-1}$. Then, breaking up the integral and calculating the resulting cancellating sum, we obtain for $\Lambda > 1$:

$$\begin{aligned}\frac{1}{\ln \Lambda}\int_1^\Lambda \lambda^{-1} n(\lambda)\frac{d\lambda}{\lambda} &= \frac{1}{\ln \Lambda}\left(\rho_1 + \cdots + \rho_J - \frac{J}{\Lambda}\right)\\ &= \frac{\ln J}{\ln \Lambda}\left(\frac{1}{\ln J}\sum_{j=1}^{J}\rho_j\right) - \frac{J}{\Lambda \ln \Lambda}.\end{aligned}$$

The equivalence of conditions (2) and (3) (under the assumptions of (ii)) is now apparent since $J(\Lambda) \asymp \Lambda$ (and hence also $\ln J/\ln \Lambda) \to 1$) as $\Lambda \to +\infty$. □

REMARK 4.19. (a) Condition (3) is not explicitly stated in [Co2,5]. It will be useful, however, to deal with the lattice case in §4.2 below.

(b) The hypothesis made in (ii) will fit exactly our situation when we apply Proposition 4.17 in §4.2. (See esp. the proof of Theorem 4.27.) However, weaker assumptions are clearly possible.

(c) I am grateful to Ms. Christina He for a comment about Proposition 4.17(ii).

(d) It is shown in [Co1,5] that this generalized notion of residue coincides—in the case of pseudodifferential operators on a compact manifold—with the *"noncommutative residue"* of Adler, Manin, Wodzicki, and Guillemin [Ad, Man, Wo, Gu].

(e) Proposition 4.17 and the results of §4.2 below suggest how to reinterpret some of the formulas in [La3–5, LaPo1–3, LaMa1–2] involving certain "generalized residues" of spectral zeta-functions in the case of "drums with fractal boundaries". We intend to develop this remark in a later work.

4.2. Construction of volume measures via the Dixmier trace. We now return to the setting of §3. Hence $F = (F, \mathcal{S}, \mu, \mathcal{E})$ is a (regular, analytical) p.c.f. self-similar fractal. In particular, in the terminology of §2, $\mathcal{E}$ is a (regular) s.s. energy functional on F (with harmonic constants $\{c_i\}_{i=1}^N$) and μ is an arbitrary, but *fixed*, s.s. measure on F (with weights denoted by $\{b_i\}_{i=1}^N$).

It will be clear that the construction below—which is inspired by [Co1,3; Co5, §VI.1] and relies on the results of [KiLa1] recalled in §3 above—can be adapted to a variety of situations involving "elliptic differential operators on fractals".

For the sake of simplicity, we will first deal with the Dirichlet Laplacian and then indicate how to extend our results to Neumann boundary conditions. In fact, it will turn out that the *"volume measure"* $\nu = \Phi(\mu)$ constructed in Theorem 4.41 below *is independent of the choice of boundary conditions.* (See Theorem 4.49.)

Let $\Delta_0 = \Delta_{0,\mu}$ be the Dirichlet Laplacian on F, defined as in §3.1. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ denote its eigenvalues, written in nondecreasing order according to multiplicity as in (1.4), and let

$$N(\lambda) = N(\lambda; \Delta_0) = \#\{j \geq 1 : \lambda_j \leq \lambda\}, \tag{4.20}$$

as in (1.5). Set $\mathcal{H} = L^2(\mu) = L^2(F, \mu)$. Recall from §3.1 that Δ_0 is a positive, unbounded self-adjoint operator on $\mathcal{H}$. Let

$$B = \Delta_0^{1/2} \tag{4.21}$$

denote the (positive) square-root of Δ_0, defined via the spectral theorem for unbounded self-adjoint operators. (See, e.g., [ReSi, Chap. VII].)

PROPOSITION 4.22. (a) *The positive, unbounded self-adjoint operator $B = \Delta_0^{1/2}$ is invertible, with compact inverse*

$$Q := B^{-1} = \Delta_0^{-1/2}. \tag{4.23}$$

(b) *Moreover,*

$$\inf\{p : Q \in \mathcal{L}^p\} = \sup\{p : Q \notin \mathcal{L}^p\} = d_S,$$

where $d_S = d_S(\mu)$ is the spectral exponent given by (3.11) *(i.e., $\sum_{i=1}^N \gamma_i^{d_S} = 1$, with $\gamma_i := \sqrt{b_i c_i}$ for $1 \leq i \leq N$).*

PROOF. (a) This follows from [KiLa1, esp. §5].

(b) Since $Q \geq 0$, we have (with the notation of §4.1.1), $\rho_j(Q) = \lambda_j^{-1/2}$. Further, by (3.14) (which follows from [KiLa1, Theorem 2.4]), we have $N(\lambda) \asymp \lambda^{d_S/2}$ as $\lambda \to +\infty$, and so

$$\lambda_j \asymp j^{2/d_S} \qquad \text{as } j \to \infty. \tag{4.24}$$

Thus $\rho_j(Q) \asymp j^{-1/d_S}$. Hence $\rho_j(Q)^p \asymp j^{-p/d_S}$ and so $\sum_{j=1}^{\infty} \rho_j(Q)^p < \infty$ if and only if $p > d_S$. □

REMARK 4.25. Note that $Q \in \mathcal{L}^{d_S+}$ but $Q \notin \mathcal{L}^{d_S}$. (We allow $p > 0$ in the definition of $\mathcal{L}^p$ or $\mathcal{L}^{p+}$. Actually, it can be shown that, at least for the spectral dimension $d_S = d_S^*$, we have $d_S \geq 1$.)

Next, let

$$R := Q^{d_S} = \Delta_0^{-d_S/2}, \tag{4.26}$$

defined via the spectral theorem.

At this point, the reader may wish to review the statement of Theorem 3.13 above [KiLa1, Theorem 2.4, pp. 104–105], as well as the definition of the Dixmier trace recalled in §4.1.2.

THEOREM 4.27. *Let R be the nonnegative, compact self-adjoint operator defined by* (4.26). *Then $R \in \mathcal{L}^{1+}$* (but R is *not* of trace class), *so that its Dixmier trace $\mathrm{Tr}_w(R)$ is well-defined and finite. Moreover, $\mathrm{Tr}_w(R) > 0$ and $\mathrm{Tr}_w(R)$ is independent of the choice of the limiting procedure w. More precisely, with the terminology and notation of Theorem* 3.13, *we have:*

(i) (*Nonlattice case: $G := \sum_{i=1}^{N} (\ln \gamma_i)\mathbb{Z}$ is dense in $\mathbb{R}$*). *Then*

$$\mathrm{Tr}_w(R) = \mathcal{C}, \tag{4.28}$$

where $\mathcal{C}$ is the positive constant defined by (3.15).

(ii) (*Lattice case: $G = \sum_{i=1}^{N} (\ln \gamma_i)\mathbb{Z}$ is discrete, say $G = T\mathbb{Z}$, with $T > 0$, the positive generator of G*). *Then*

$$\mathrm{Tr}_w(R) = \frac{1}{T} \int_0^T g(t)\, dt, \tag{4.29}$$

the mean-value of g, where g is the positive T-periodic function given by (3.16). (Recall that g is locally integrable on $\mathbb{R}$ and $0 < c_1 \leq g(t) < c_2 < \infty$ on $[0, +\infty)$, for some constants c_1, c_2.)

PROOF. (1) We first show that $R \in \mathcal{L}^{1+}$. Since $R \geq 0$, we have in view of (4.26),

$$\rho_j(R) = \lambda_j^{-d_S/2}; \tag{4.30}$$

so that by (4.24),

$$\rho_j(R) \asymp \frac{1}{j} \qquad \text{as } j \to \infty. \tag{4.31}$$

Hence the sequence $\frac{1}{\ln J}\sum_{j=1}^{J}\rho_j(R)$ ($\asymp 1$) is bounded. Thus $R \in \mathcal{L}^{1+}$ (but $R \notin \mathcal{L}^1$, since $\sum_j \rho_j(R)$ diverges). It follows from the definition of the Dixmier trace that $\mathrm{Tr}_w(R)$ is well-defined and $0 \leq \mathrm{Tr}_w(R) < \infty$.

(2) We will now use (part (iii) of) Proposition 4.17 above to calculate $\mathrm{Tr}_w(R)$ and to show that it does not vanish and is independent of w.

(i) In the *nonlattice case*, we have (by (3.15)) $N(\lambda) \sim \mathcal{C}\lambda^{d_S/2}$ as $\lambda \to +\infty$, and so (by (4.20) and a standard Tauberian argument) $\lambda_j \sim \mathcal{C}^{-2/d_S} j^{2/d_S}$. Thus, in view of (4.30),

$$\rho_j(R) \sim \frac{\mathcal{C}}{j} \qquad \text{as} \quad j \to \infty. \tag{4.32}$$

Hence

$$\frac{1}{\ln J}\sum_{j=1}^{J}\rho_j(R) \to \mathcal{C} \quad \text{as} \quad J \to \infty.$$

Therefore condition (2) of Proposition 4.17 is satisfied and so $\mathrm{Tr}_w(R) = \mathcal{C}$ (and is independent of w), as desired. This yields (4.28) and shows that $\mathrm{Tr}_w(R) > 0$.

(ii) In the *lattice case*, we will find it convenient to verify condition (3) of Proposition 4.17. By (3.16), we have

$$\lambda^{-d_S/2}N(\lambda) = g\left(\frac{\ln\lambda}{2}\right) + \varepsilon(\lambda), \tag{4.33}$$

where ε is a bounded (measurable) real-valued function on $[0, +\infty)$ such that $\varepsilon(\lambda) \to 0$ as $\lambda \to +\infty$. Set, as in (4.18),

$$n(\lambda) = n(\lambda; R) = \#\{j \geq 1 : \ \rho_j(R) \geq \lambda^{-1}\}.$$

Then, by (4.20) and (4.30), we have

$$n(\lambda) = N(\lambda^{2/d_S}), \qquad \text{for all } \lambda > 0. \tag{4.34}$$

Now, set for $\Lambda > 1$,

$$\theta(\Lambda) = \frac{1}{\ln\Lambda}\int_1^{\Lambda}\lambda^{-1}n(\lambda)\frac{d\lambda}{\lambda} \tag{4.35a}$$

and

$$\Theta(\Lambda) = \frac{1}{\ln\Lambda}\int_1^{\Lambda}\lambda^{-d_S/2}N(\lambda)\frac{d\lambda}{\lambda}. \tag{4.35b}$$

Then, in view of (4.34), the simple change of variables $u = \lambda^{2/d_S}$, $\frac{du}{u} = \frac{2}{d_S}\frac{d\lambda}{\lambda}$ yields

$$\theta(\Lambda) = \Theta(\Lambda^{2/d_S}). \tag{4.36}$$

Next we claim that

$$\Theta(\Lambda) \to \frac{1}{T}\int_0^T g(t)\,dt \qquad \text{as } \Lambda \to +\infty. \tag{4.37}$$

Indeed, by (4.35b) and (4.33),

$$\begin{aligned}\Theta(\Lambda) &= \frac{1}{\ln\Lambda}\int_1^\Lambda g\left(\frac{\ln\lambda}{2}\right)\frac{d\lambda}{\lambda} + \frac{1}{\ln\Lambda}\int_1^\Lambda \varepsilon(\lambda)\frac{d\lambda}{\lambda} \\ &= \Theta_1(\Lambda) + \Theta_2(\Lambda), \quad \text{say.}\end{aligned} \tag{4.38}$$

Since ε is bounded and tends to 0 at $+\infty$, we clearly have $\Theta_2(\Lambda) \to 0$ as $\Lambda \to +\infty$. Further, the change of variables $u = \frac{\ln\lambda}{2}$, $du = \frac{d\lambda}{2\lambda}$ yields

$$\begin{aligned}\Theta_1(\Lambda) &= \frac{1}{\ln(\Lambda^{1/2})}\int_0^{\ln(\Lambda^{1/2})} g(u)\,du \\ &\to \frac{1}{T}\int_0^T g(t)\,dt \qquad \text{as } \Lambda \to +\infty,\end{aligned}$$

since g being T-periodic (and locally integrable),

$$\frac{1}{\tau}\int_0^\tau g(u)\,du \to \frac{1}{T}\int_0^T g(t)\,dt \qquad \text{as } \tau \to +\infty.$$

Thus (4.37) follows from (4.38).

We can now conclude the proof of Theorem 4.27 as follows. In light of (4.35a), (4.36) and (4.37),

$$\theta(\Lambda) = \frac{1}{\ln\Lambda}\int_1^\Lambda \lambda^{-1}n(\lambda)\frac{d\lambda}{\lambda} \to \frac{1}{T}\int_0^T g(t)\,dt \qquad \text{as } \Lambda \to +\infty.$$

Thus condition (3) of Proposition 4.17 is satisfied and we deduce that $\mathrm{Tr}_w(R) = \frac{1}{T}\int_0^T g(t)\,dt$ (and is independent of w), as desired. This yields (4.29). Finally, we note that $\mathrm{Tr}_w(R) > 0$ since the (positive) function g is bounded away from zero. □

REMARK 4.39. (a) It follows from the above proof and Proposition 4.17 that, under the hypotheses of Theorem 4.27,

$$\mathrm{Tr}_w(R) = \mathrm{Res}_{s=1}\zeta_R(s) := \lim_{s\to 1+}(s-1)\zeta_R(s), \tag{4.40}$$

where the spectral zeta-function ζ_R is defined by (4.16).

(b) Intuitively, the operator $R = R_\mu$ defined by (4.26) can be thought of as an analogue of an *"elliptic pseudodifferential operator"* on F, of negative order $-d_S$. (Compare with [Co1].)

We now proceed—by analogy with [Co3; Co5, §VI.1]—with the construction of a suitable *"volume measure"*, $\nu = \Phi(\mu)$, on the fractal $F = (F, S, \mu, \mathcal{E})$.

First, recall that (in Bourbaki's terminology [Ch, Chap. 3]) a *positive Radon measure* on the compact (Hausdorff) space F is a positive linear functional ν

on $C(F)$, the space of continuous real-valued functions on F. (Here, positivity of ν simply means that $\nu(f) \geq 0$ for all $f \geq 0, f \in C(F)$.) Of course, ν is then automatically continuous on $C(F)$, equipped with the topology of uniform convergence on F associated with the norm $\|f\|_\infty := \max_{x \in F} |f(x)|$. Further, by the Riesz Representation Theorem, it induces a unique regular, positive Borel measure (still denoted by ν) on F, such that $\nu(f) = \int_F f \, d\nu$ for all $f \in C(F)$, and with (nonnegative and finite) total mass $\nu(F) = \nu(1) = \int_F 1 \, d\nu$. (See, e.g., [Ch] or [Ru, Theorem 2.14, pp. 40–41].)

We can now state one of our main results (see also Theorem 4.27 above and Corollary 4.45 below):

THEOREM 4.41 ("Volume measures" on fractals). *Let $F = (F, \mathcal{S}, \mu, \mathcal{E})$ be a (regular, analytical) p.c.f. self-similar fractal. Let $R = R_\mu = \Delta_0^{-d_S/2}$, as in* (4.26), *where $\Delta_0 = \Delta_{0,\mu}$ denotes the Dirichlet Laplacian on F (acting on $L^2(F, \mu)$), and $d_S = d_S(\mu)$ is the associated spectral exponent, given by* (3.11). *Let $\nu = \Phi(\mu)$ be defined by*

$$\nu(f) = \int_F f \, d\nu := \mathrm{Tr}_w(fR) = \mathrm{Tr}_w(f\Delta_0^{-d_S/2}), \tag{4.42}$$

for all $f \in C(F)$.

Then ν is a well-defined, positive Radon measure on the compact (metrizable) space F. Moreover, ν is nonzero and its total mass $\nu(F) = \nu(1)$ (or that of the associated Borel measure) is given by

$$\nu(F) = \int_F 1 \, d\nu = \mathrm{Tr}_w(R), \tag{4.43}$$

the Dixmier trace of R. Hence, by Theorem 4.27, *$\nu(F)$ is independent of the limiting procedure w and is given by* (4.28) *or* (4.29), *in the nonlattice case or in the lattice case, respectively.*

PROOF. As we will see, this follows from our previous results and from the basic properties of the Dixmier trace $\mathrm{Tr}_w(\cdot)$.

(a) We first clarify the notation used in (4.42). Let $f \in C(F)$. Since f is bounded (and real-valued), it is well-known that it induces a bounded (self-adjoint) *multiplication operator* M_f on $\mathcal{H} = L^2(F, \mu)$, as follows:

$$(M_f\varphi)(x) := f(x)\varphi(x), \qquad \text{for } \varphi \in \mathcal{H} \text{ and } \mu\text{-a.e. } x \in F. \tag{4.44}$$

Further, M_f has operator norm $\|M_f\| = \|f\|_\infty$. Hence, strictly speaking, we should really write, instead of (4.42),

$$\nu(f) = \int_F f \, d\nu := \mathrm{Tr}_w(M_f R), \qquad \text{for all } f \in C(F). \tag{4.42$'$}$$

However, following common practice, we identify the function $f \in C(F)$ and the (multiplication) operator $M_f \in \mathcal{L}(\mathcal{H})$, in (4.42).

(b) Next, fix $f \in C(F)$. Since by Theorem 4.27, $R \in \mathcal{L}^{1+}$, we deduce that $RM_f \in \mathcal{L}^{1+}$ (because $M_f \in \mathcal{L}(\mathcal{H})$ and $\mathcal{L}^{1+}$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$). Thus $\mathrm{Tr}_w(RM_f)$ is well-defined and is a real number. Hence the map $\nu : C(F) \to \mathbb{R}$ is well-defined.

(c) Further, ν is clearly linear because $\mathrm{Tr}_w(\cdot)$ is linear on $\mathcal{L}^{1+}$.

(d) Moreover, if $f \in C(F), f \geq 0$, we claim that $\nu(f) \geq 0$.

In fact, we clearly have $M_{\sqrt{f}}RM_{\sqrt{f}} \geq 0$ (as a bounded self-adjoint operator on $\mathcal{H}$) because for all $g \in \mathcal{H}$,

$$(M_{\sqrt{f}}RM_{\sqrt{f}}g, g) = (R(M_{\sqrt{f}}g), M_{\sqrt{f}}g) \geq 0$$

since ($M_{\sqrt{f}}$ is self-adjoint and) $R \geq 0$. (We denote here by $(\cdot, \cdot)$ the inner product in $\mathcal{H} = L^2(\mu)$.)

Therefore, by the positivity of the Dixmier trace (Proposition 4.14(i))

$$\nu(f) = \mathrm{Tr}_w(M_f R) = \mathrm{Tr}_w(M_{\sqrt{f}}M_{\sqrt{f}}R) = \mathrm{Tr}_w(M_{\sqrt{f}}RM_{\sqrt{f}}) \geq 0,$$

as desired. Note that we have used Proposition 4.14(iii) in the last equality above.

(e) We conclude from (b)–(d) that ν is a positive Radon measure on F. Let ν also denote the associated positive Borel measure. Then, by letting $f \equiv 1$ in (4.42), we have

$$\nu(F) = \int_F 1\, d\nu = \nu(1) = \mathrm{Tr}_w(R),$$

and thus the remaining assertions in Theorem 4.41 follow from Theorem 4.27.□

The following corollary (of Theorems 3.13, 4.27 and 4.41) provides a more precise analogue than in [KiLa1]—in the present context of Laplacians on p.c.f. self-similar fractals—of Weyl's original formula (1.6) for Laplacians on bounded domains in Euclidean space. We thereby complete our earlier (joint) results obtained in [KiLa1], as well as further address Question $\mathbf{Q_2}$ and answer in part Question $\mathbf{Q_4}$ of §1. (Compare with Theorem 3.13 above [KiLa1, Theorem 2.4].)

COROLLARY 4.45 (Weyl's formula on fractals, revisited). *Let $\nu = \Phi(\mu)$ be the "volume measure" constructed in Theorem 4.41. Let $d_S = d_S(\mu)$ be the spectral exponent given by (3.11), and let $N(\lambda)$ be the eigenvalue counting function of the Dirichlet Laplacian $\Delta_0 = \Delta_{0,\mu}$, as in (4.20).*

Then the following limit exists and the total mass $\nu(F)$ of ν, or "volume" of $(F, \mathcal{S}, \mu, \mathcal{E})$, is the finite and positive number given by:

(i) *In the nonlattice case,*

$$\nu(F) = \lim_{\lambda \to +\infty} \lambda^{-d_S/2} N(\lambda) = \mathcal{C}, \tag{4.46}$$

where $\mathcal{C}$ is the positive constant defined by (3.15).

(ii) *In the lattice case,*

$$\nu(F) = \lim_{\Lambda\to+\infty} \frac{1}{\ln \Lambda} \int_1^\Lambda \lambda^{-d_S/2} N(\lambda) \frac{d\lambda}{\lambda} = \frac{1}{T}\int_0^T g(t)\,dt, \tag{4.47}$$

the mean-value of g, where g is the bounded, positive T-periodic function occurring in (3.16).

PROOF. This follows by combining Theorems 3.13, 4.41, and the proof of Theorem 4.27. Note that by (4.35b) and (4.37), the limit in (4.47) exists and equals the mean-value of g. □

REMARK 4.48. (a) Of course, the operators B, Q and R in (4.21), (4.23) and (4.26), respectively, and the measure $\nu = \Phi(\mu)$ defined in (4.42), not only depend on the s.s. measure μ, but also on the energy functional $\mathcal{E}$ (and the s.s. structure $\mathcal{S}$).

(b) We do not know whether the "volume measure" $\nu = \Phi(\mu)$ constructed in Theorem 4.41 (see (4.42)) is independent of the choice of the limiting procedure w. [A similar difficulty occurs in the recent work of Connes and Sullivan ([CoSu], [Co5, §IV.3]).] However, we do know from Theorem 4.27 that its total mass is. We conjecture that this is also the case of $\nu = \Phi(\mu)$ itself, for an arbitrary s.s. measure μ. If this is correct, then we propose to call $\nu = \Phi(\mu)$ (resp., its total mass $\nu(F)$) *the "volume measure"* (resp., *the "volume"*) of (the p.c.f., analytical s.s. fractal) $(F, \mathcal{S}, \mu, \mathcal{E})$. Actually, when $\mu = \mu^*$, the self-similar measure of maximal spectral exponent $d_S = d_S(\mu^*) = d_S^*$ defined in Theorem 3.22, we will make a more precise conjecture in §5.1; namely, that $\nu^* = \Phi(\mu^*)$ is *"approximately self-similar"*, with "weight functions" Ψ_i $(i = 1, \dots, N)$ depending only on the energy functional $\mathcal{E}$, and hence that ν^* is independent of w. (See esp. Conjecture 5.10.) If that is the case, we propose to call $\nu^* = \Phi(\mu^*)$ (resp., its total mass $\nu^*(F)$) *the "natural volume measure"* (resp., the *"natural volume"*) of F. Thus ν^* could then be viewed as *an analogue of Riemannian volume* (measure) on F. (See (c) below.)

(c) Let $N = N^n$ be a smooth n-dimensional (spin) Riemannian manifold. Then a counterpart of formula (4.42)—with d_S replaced by n and $B = \Delta_0^{1/2}$ replaced by the Dirac operator on N (or equivalently, as can be checked, by its absolute value)—was used in [Co3; Co5, §VI.1] to reinterpret Weyl's classical formula and recover the usual n-dimensional *Riemannian volume* (measure) on $N = N^n$, from purely operator-theoretic data. In §5, we shall further discuss the possible connections between our work and aspects of noncommutative geometry.

(d) The reader may ponder the following wrong claim and proof.

False claim: The measure $\nu = \Phi(\mu)$ is absolutely continuous with respect to μ. Hence, by the Radon-Nikodym Theorem, there exists a function $\varphi \geq 0$ and

μ-integrable such that

$$\int f\,d\nu = \int \varphi f\,d\mu, \qquad \text{for all } f \text{ bounded and measurable.}$$

(*Wrong proof*: Let A be a Borel subset of F such that $\mu(A) = 0$. Let $f = 1_A$ be the characteristic function of A; then $f = 0$ μ-a.e. and so $M_f = 0$ (in $\mathcal{L}(\mathcal{H})$). It thus follows from (4.42′) (or (4.42)) and Proposition 4.14(iii) that

$$\nu(A) = \nu(f) = \int f\,d\nu = \mathrm{Tr}_w(M_f R) = \mathrm{Tr}_w(RM_f) = \mathrm{Tr}_w(0) = 0.$$

Hence ν is absolutely continuous with respect to μ, as claimed.)

[*Hint*: The use of (4.42′) is *not* legitimate here since $f = 1_A$ need not be in $C(F)$.]

We close this section by considering the case of Neumann boundary conditions. We show, in particular, that the "volume measure" $\nu = \Phi(\mu)$ (and hence also $\mathrm{Tr}_w(R) = \nu(F)$)—constructed via the Neumann instead of the Dirichlet Laplacian—is *independent of the boundary conditions.*

THEOREM 4.49 (Neumann boundary conditions). *Let $\Delta_1 = \Delta_{1,\mu}$ be the Neumann Laplacian on $F = (F, \mathcal{S}, \mu, \mathcal{E})$, defined as in Proposition* 3.5. *We define the operators B, Q and R exactly as in* (4.21), (4.23) *and* (4.26), *respectively, except that we replace Δ_0 by $\Delta_1 + \beta$, where β is a fixed but arbitrary positive constant. So that, for example,*

$$R = (\Delta_1 + \beta)^{-d_S/2}. \tag{4.50}$$

Then:

(a) *The analogues of Proposition* 4.22, *Theorems* 4.27 *and* 4.41, *as well as of Corollary* 4.45, *holds without change in this situation.*

(b) *Moreover, the "volume measure" $\nu = \Phi(\mu)$ is the same as for the Dirichlet problem* (and is independent of $\beta > 0$). *Hence the same is true of the Dixmier trace of R, $\mathrm{Tr}_w(R) = \nu(F)$ (which is therefore independent of both the limiting procedure w and the boundary conditions).*

PROOF. (a) In view of the results of [KiLa1] recalled in §3 above, the proof of (a) parallels that of the corresponding statement for Dirichlet boundary conditions. (See esp. Proposition 3.5 and Theorem 3.13.)

(b) Fix $\beta > 0$. Let $R_0 := (\Delta_0)^{-d_S/2}$ and $R_1 := (\Delta_1 + \beta)^{-d_S/2}$. Then it follows from [KiLa1] that

$$R_0 - R_1 \in \mathcal{L}_0^{1+}, \tag{4.51}$$

where $\mathcal{L}_0^{1+}$ denotes the Maçaev ideal defined in Proposition 4.14(iv). (We will comment on (4.51) below.)

Now, for a given $f \in C(F)$ and with the bounded multiplication operator M_f defined by (4.44), we also have

$$M_f R_0 - M_f R_1 = M_f(R_0 - R_1) \in \mathcal{L}_0^{1+}, \tag{4.52}$$

since $\mathcal{L}_0^{1+}$ is a two-sided ideal in $\mathcal{L}(\mathcal{H})$, where $\mathcal{H} = L^2(F,\mu)$. Hence Proposition 4.14 (iv) and the linearity of the Dixmier trace imply that

$$\mathrm{Tr}_w(M_f R_0) = \mathrm{Tr}_w(M_f R_1), \qquad \text{for all } f \in C(F);$$

from which (b) follows since by (4.42′), the associated "volume measures" are therefore the same for Dirichlet or Neumann boundary conditions.

Finally, we briefly explain how to derive (4.51). Write $R_0 - R_1 = A_2 + A_3$, where

$$A_2 = (\Delta_0)^{-d_S/2} - (\Delta_0 + \beta)^{-d_S/2} \quad \text{and} \quad A_3 = (\Delta_0+\beta)^{-d_S/2} - (\Delta_1+\beta)^{-d_S/2}.$$

We claim that both A_2 and A_3 are in the ideal $\mathcal{L}_0^{1+}$, and hence their sum $R_0 - R_1$ is also in $\mathcal{L}_0^{1+}$.

In fact, if λ_j denotes the jth eigenvalue of Δ_0, we have

$$\begin{aligned}\rho_j(A_2) &= \lambda_j^{-d_S/2}(1 - (1+\beta/\lambda_j))^{-d_S/2}\\ &= \beta\frac{d_S}{2}\lambda_j^{-(1+d_S/2)}(1+o(1)) \asymp j^{-(1+2/d_S)},\end{aligned}$$

by (4.24). Thus $\sum_{j=1}^{\infty}\rho_j(A_2) < \infty$; so A_2 is of trace class and is *a fortiori* in $\mathcal{L}_0^{1+}$.

To see that $A_3 \in \mathcal{L}_0^{1+}$, we use the results of [KiLa1] recalled in §3.1. They imply, for example, that the resolvents of Δ_0 and Δ_1 (are compact and) differ by a *finite rank* operator. (Indeed, given $v \in \mathcal{H}$, let $u_i := (\Delta_i+\beta)^{-1}v$, for $i = 0$ or 1; so that $(\Delta_i + \beta)u_i = v$. Then, by Propositions 3.5 and 3.6, $u_i \in \mathcal{D}_\mu$, the domain of the Laplacian $\Delta = \Delta_\mu$, and $\Delta(u_0 - u_1) = 0$; hence $u_0 - u_1$ is a harmonic function on $(F, \mathcal{E})$. The result follows because the space of harmonic functions on a p.c.f. (analytical) fractal is *finite-dimensional.*) □

5. Discussion: Conjectures, open problems and extensions

We conclude this paper by discussing possible extensions of this work, as well as proposing conjectures and open problems that may help point to the direction of future research in this area. In the process, we suggest further connections with aspects of Connes' noncommutative geometry as well as with the recent work of Connes and Sullivan ([CoSu], [Co5, §IV.3]) on "quantized calculus" and "Dirac operators" on limit sets of certain Fuchsian groups. (See esp. the comments in (5.23) below as well as in §5.2.)

As will be clear to the reader, much of the present section is of a rather speculative nature.

For clarity, we mostly work in §5.1 and §5.2 within the framework of "finitely ramified" (i.e., p.c.f.) self-similar fractals, for which we now have the most precise information. We then briefly discuss in §5.3 possible extensions to other classes of s.s. fractals. (We postpone to a later work the consideration of more general "fractals" not necessarily assumed to be self-similar.)

We begin in §5.1 by discussing the possible properties of the "natural volume measure" or analogue of "Riemannian volume measure", $\nu^* = \Phi(\mu^*)$, constructed in §4.2 above. (See Remark 4.48(b), (c).)

5.1. An analogue of Riemannian volume on fractals. We return to the setting of §4.2 and more precisely, of §3.3. Let $F = (F, \mathcal{S})$ be a (topological) *p.c.f.* self-similar set, as in Definition 2.6. Recall from Definition 2.6 that $\mathcal{S}$ is denoted by $\mathcal{S} = (A, \{W_i\}_{i=1}^N)$, with $A := \{1, \dots, N\}$. Further, let $\mathcal{E}$ be a regular s.s. Dirichlet form on F, with "harmonic constants" $\{c_i\}_{i=1}^N (0 < c_i < 1)$, as in Definition 2.13 and Remark 2.15(b). In the following, the self-similar structure $\mathcal{S}$ and the energy functional $\mathcal{E}$ will be fixed and we will often simply write F or $(F, \mathcal{E})$ to refer to the regular p.c.f. self-similar set $(F, \mathcal{S}, \mathcal{E})$.

At this point, some readers may wish to briefly review §2.3 and §3.3, and in particular, Definitions 2.16, 3.20 and Theorem 3.22.

Let $\mu = \mu^*$ be the *"natural self-similar measure"* (with respect to $\mathcal{S}$) on $(F, \mathcal{E})$; that is, the unique s.s. measure on F with *maximal spectral exponent* $d_S = d_S(\mu^*) = d_S^*$, the *spectral dimension* of the Dirichlet space $(F, \mathcal{E})$. (See Remark 3.26(a) above.) Recall from Theorem 3.22 that μ^* is the self-similar measure (with respect to $\mathcal{S}$) with weights $\{b_i := c_i^S\}_{i=1}^N$, where S is the *"similarity dimension"* of $(F, \mathcal{E})$; namely, S is the unique positive number such that $\sum_{i=1}^N c_i^S = 1$. Hence, by Definition 2.16, μ^* is the probability measure such that

$$\int_F f \, d\mu^* = \sum_{i=1}^N \int_F c_i^S (f \circ W_i) \, d\mu^*, \tag{5.1}$$

for all $f \in C(F)$.

Note that (for fixed $\mathcal{S}$) the weights of μ^* depend only on the energy functional $\mathcal{E}$.

Clearly, (5.1) implies by induction that for all integers $m \geq 1$,

$$\int_F f \, d\mu^* = \sum_{\omega \in \Sigma_m} \int_F c_\omega^S (f \circ W_\omega) \, d\mu^*, \tag{5.2}$$

for all $f \in C(F)$.

Here, as before, $\Sigma_m = A^m$ denotes the set of words of length m, with letters in the alphabet $A = \{1, \dots, N\}$. Furthermore, for $\omega = \omega_1 \dots \omega_m \in \Sigma_m$,

$$c_\omega := c_{\omega_1} \cdots c_{\omega_m} \tag{5.3}$$

and (as in (2.4))

$$W_\omega := W_{\omega_1} \circ \cdots \circ W_{\omega_m}. \tag{5.4}$$

We will next mimic this situation (first in Definition 5.5, and then more closely in Conjectures 5.10 and 5.14), except that we will allow *variable*—rather than constant—weight functions. (Compare with Definition 2.16 for a self-similar measure.)

DEFINITION 5.5 (Approximately self-similar measure). Let $(F, \mathcal{S})$ be a topological self-similar fractal and let κ be a Borel probability measure on F. Then κ is said to be *"approximately self-similar"* (with respect to $\mathcal{S}$) with *"weight functions"* $\{\Psi_i\}_{i=1}^N$ if there exist nonnegative and measurable functions Ψ_i $(i = 1, \ldots, N)$ on F such that

$$\int_F f\, d\kappa = \sum_{i=1}^N \int_F \Psi_i (f \circ W_i)\, d\kappa, \tag{5.6}$$

for all $f \in C(F)$ (and hence for all f bounded and measurable on F).

REMARK 5.7. (a) Of course, the functions Ψ_i must be κ-integrable and satisfy the following compatibility condition:

$$\sum_{i=1}^N \int_F \Psi_i\, d\kappa = 1. \tag{5.8}$$

(b) Clearly, an approximately self-similar (in short, a.s.s.) measure is s.s. if and only if each of its weight functions is constant.

(c) It is easy to find suitable sufficient conditions on the functions Ψ_i $(i = 1, \ldots, N)$ (for example, continuity) that guarantee the existence of an a.s.s. measure κ with weight functions $\{\Psi_i\}_{i=1}^N$. (This follows, for example, from the Schauder fixed-point theorem or else from a direct functional-analytic argument.) In general, however, such a measure need not be unique.

We now return to the main object of §5.1. Let μ^* be as above the *"natural s.s. measure"* on F. Let $\nu^* = \Phi(\mu^*)$ be the *nonzero* positive measure constructed in Theorem 4.4.1, and associated with the (regular) analytical p.c.f. self-similar fractal $F = (F, \mathcal{S}, \mu^*, \mathcal{E})$. (See esp. (4.42) or (4.42′).) As was mentioned in Remark 4.48(b), we propose to view ν^* as the *"natural volume measure"* or *"Riemannian volume (measure)"* in this context. Let ν_1 be the probability measure associated to ν^*; namely,

$$\nu_1 = \frac{\nu^*}{\nu^*(F)} = \frac{\Phi(\mu^*)}{\nu^*(F)}, \tag{5.9}$$

where $\nu^*(F)$, the total mass of ν^* or *"natural volume"* of $(F, \mathcal{E})$, is given by Theorem 4.41 and Corollary 4.45. (See esp. (4.46) or (4.47) in the "nonlattice" or "lattice" case, respectively.) Then ν_1 is called the *normalized "natural volume measure"* on $(F, \mathcal{E})$.

We are now ready to discuss the conjectured properties of ν^* or, equivalently, ν_1. (It should be clear from the context whether ν^* and ν_1 are viewed as Radon or Borel measures on the compact metrizable space F.)

CONJECTURE 5.10. *Let ν_1 be the probability measure on F given by* (5.9). *Then ν_1 is "approximately self-similar" (with respect to $\mathcal{S}$) with "weight functions" $\{\Psi_i\}_{i=1}^N$ depending only on the energy functional $\mathcal{E}$ (and hence, in particular, independent of the limiting procedure w used in defining $\nu^* = \Phi(\mu^*)$ in* (4.42)).

Thus by (5.6), *the measure ν_1 itself (and hence ν^* by Remark* 5.11 *below) is also independent of w and depends only on $\mathcal{E}$.*

REMARK 5.11. If ν_1 is independent of w, as stated in Conjecture 5.10, then so is $\nu^* = \nu^*(F)\nu_1$, in agreement with the conjecture made earlier in Remark 4.48(b). This is so because by Theorems 4.27 and 4.41, $\nu^*(F) = \mathrm{Tr}_w(R_{\mu^*})$ is independent of w.

We will next specify Conjecture 5.10 in several different ways. First recall from [Ki4] and §3.3 that "the" *"natural* (or *intrinsic*) *metric"* δ_1 on the Dirichlet space $(F, \mathcal{E})$ is given (as in (3.32)) by

$$\delta_1(x,y) = \max\{|u(x) - u(y)|^2 : u \in \mathcal{F},\ \mathcal{E}(u) \le 1\}, \qquad \text{for } x, y \in F. \tag{5.12}$$

(Here, as before, $\mathcal{F}$ denotes the domain of the Dirichlet form $\mathcal{E}$.) Further, recall that for every $i = 1, \ldots, N$, the mapping $W_i : F \to F$ is a contraction with respect to δ_1, with (global) Lipschitz constant $c_i \in (0,1)$. Moreover, it follows from [KiLa1] and [Ki4] that (as in (3.33) of §3.3) *the similarity dimension S of the Dirichlet space $(F, \mathcal{E})$ and the Hausdorff dimension $d_H = d_H(\delta_1)$ (as well as the Minkowski or "box" dimension $d_M = d_M(\delta_1)$) of the* (bounded, complete) *metric space (F, δ_1) coincide:*

$$S = d_H = d_M, \tag{5.13}$$

so that by (3.25), $d_S^* = 2S/(S+1) = 2d_H/(d_H+1) = 2d_M/(d_M+1)$. (See Corollary 3.34.) (The minor reason why we choose δ_1, given by (5.12), rather than the metric $\delta = \sqrt{\delta_1}$, given by (2.22), is that $d_H(\delta) = 2d_H(\delta_1) = 2S$.)

We next propose to relate the values $\Psi_i(x)$ of the weight function Ψ_i ($i = 1, \ldots, N$) of ν_1 to the *"local Lipschitz constant"*, denoted by $|W_i'(x)|$ in (5.15) below, of the contraction map W_i, with respect to the metric δ_1.

The following conjecture is motivated, in particular, by work in ([Pa], [Su1–3], [CoSu] and [Co5, §IV.3]), as will be further commented upon in (5.23) below.

CONJECTURE 5.14. *Assume that Conjecture* 5.10 *is true. Fix an arbitrary* $i \in \{1, \ldots, N\}$. *Then the following limit*

$$|W_i'(x)| := \lim_{y \to x, y \neq x} \frac{\delta_1(W_i(x), W_i(y))}{\delta_1(x,y)} \tag{5.15}$$

exists for ν_1-a.e. x *in* F. *Moreover, the "weight function"* Ψ_i *of the "approximately self-similar" measure* ν_1 *is given by*

$$\Psi_i = |W_i'|^S \qquad (= |W_i'|^{d_H} = |W_i'|^{d_M}, \text{ by (5.13)}). \tag{5.16}$$

More generally, we have for all $m \geq 1$,

$$\int_F f \, d\nu_1 = \sum_{\omega \in \Sigma_m} \int_F |W_\omega'|^S (f \circ W_\omega) \, d\nu_1, \tag{5.17}$$

for all $f \in C(F)$ (*and hence for all* f *bounded and measurable on* F).

Here, for $\omega = \omega_1 \ldots \omega_m \in \Sigma_m, W_\omega$ *is defined as in* (5.4) *and* $|W_\omega'|$ *as in* (5.15).

REMARK 5.18. (a) Note that, in view of (5.12) and (5.15), the "weight functions" Ψ_i given by (5.16) depend only on $\mathcal{E}$—and hence do *not* depend on the limiting procedure w used in defining ν_1 or ν^*, in agreement with Conjecture 5.10 above.

(b) Clearly, the measure $\nu^* = \nu^*(F)\nu_1$ also satisfies (5.17), except with ν^* instead of ν_1.

(c) Equation (5.17) obviously implies (but is *not* in general equivalent to) the simpler equation

$$\int_F f \, d\nu_1 = \sum_{i=1}^N \int_F |W_i'|^S (f \circ W_i) \, d\nu_1, \tag{5.19}$$

for all $f \in C(F)$.

(d) Since each map W_i is a contraction with Lipschitz constant c_i on (F, δ_1), we deduce from Conjecture 5.14 that ν_1-almost everywhere, $\Psi_i = |W_i'|^S \leq c_i^S$ (< 1), for $i = 1, \ldots, N$; so that (given that $\sum_{i=1}^N c_i^S = 1$), $\sum_{i=1}^N |W_i'|^S \leq 1$. By letting $f = 1$ in (5.19), we thus deduce that

$$\sum_{i=1}^N |W_i'|^S = 1, \qquad \nu_1\text{-a.e. on } F. \tag{5.20}$$

More generally, since $\sum_{\omega \in \Sigma_m} c_\omega^S = 1$, with c_ω defined as in (5.3), we deduce from (5.17) that for all $m \geq 1$,

$$\sum_{\omega \in \Sigma_m} |W_\omega'|^S = 1, \qquad \nu_1\text{-a.e. on } F. \tag{5.21}$$

(e) It follows by induction from (2.14) that for each $m \geq 1$, the s.s. Dirichlet form $\mathcal{E}$ satisfies

$$\mathcal{E}(u,v) = \sum_{\omega \in \Sigma_m} c_\omega^{-1} \mathcal{E}(u \circ W_\omega, v \circ W_\omega). \tag{5.22}$$

for all $u, v \in \mathcal{F}$. (This fact was essentially noted in [Ki4] by using [KiLa1, Lemma 6.1] which establishes that natural Dirichlet forms on F are self-similar, in the sense of our Definition 2.13.) Equation (5.22) should be helpful in attempting to prove Conjectures 5.10 and 5.14.

(5.23a) *The significance of Conjecture* 5.14 above (as well as of Conjectures 5.27 and 5.29 below) *is that* ν_1 *is* (at least in part) *the analogue*—in the present setting of p.c.f. self-similar fractals F—*of the "Patterson-Sullivan measure"* [Pa, Su1–3] for limit sets L of suitable (e.g., geometrically finite) Kleinian groups. (See the collective work [Bd]—and in particular the article [Ni]—for an excellent introduction to this beautiful subject; see also [Ba].) Originally, this measure was defined by Patterson in [Pa] for limit sets of Fuchsian groups and later constructed in the general case by Sullivan [Su1–3] for limit sets of arbitrary Kleinian groups. [Very roughly, a Fuchsian (resp., Kleinian) group is a suitable discrete group of isometries of the Poincaré hyperbolic plane (resp., of its higher dimensional analogue, a hyperbolic space of any dimension); see, e.g., [Ba; 7, Chap. 1]. Further, typically, the limit sets of such discrete groups have nonintegral "fractal" (Hausdorff, Minkowski, or packing) dimension and loosely speaking, exhibit some kind of "approximate self-similarity"; See, e.g., [La5, Part II, esp. §4.5] and the relevant references therein.]

(5.23b) In ([CoSu], reported on in [Co5, §IV.3]), Connes and Sullivan reconstruct by means of noncommutative geometry the Patterson-Sullivan measure for the case of limit sets L of quasi-Fuchsian groups. More specifically, their construction is based on the new notion of *"quantized (differential) calculus"*. Very briefly, their argument goes as follows:

(i) They first define a positive (Radon) measure τ on L by means of "integrals" involving the Dixmier trace of certain operators. (These "integrals" are of a somewhat different form than in [Co3], say, or in §4.2 above, due to the special nature of "quantized differentials" and/or to the existence of (an analogue of) the "Dirac operator". See, however, §5.2 below for further discussion of this point.)

(ii) Using the properties of "quantized differentials" (esp., the "chain rule" and the "change of variables formula"), as well as the unitarity of the Dixmier trace, they then show that the measure τ is *"q-conformal"*, in the sense of Sullivan [Su3], for some positive real number q. (Namely, that for all transformations β

in the group, $\tau \circ \beta^{-1} = |\beta'|^q \tau$ or equivalently,

$$\int_L (f \circ \beta)\, d\tau = \int_L |\beta'|^q f\, d\tau, \tag{5.24}$$

for all $f \in C(L)$.) They thus deduce from the uniqueness result in [Su3] that τ coincides with the (normalized) Patterson-Sullivan measure on the limit set L, which in this case coincides with the (q-dimensional) Hausdorff measure on $L \subset \mathbb{C}$. (In particular, q is equal to the Hausdorff dimension of L.)

REMARK 5.25. (a) When he had obtained the results in §4.2 above, the author was not aware of the work in [CoSu]—although he knew, of course, of the results in [Co3], for example. I am grateful to Professor Jean Bourgain for pointing out reference [CoSu] after I explained to him the work in §4.2.

(b) Very recently, after having (essentially) completed this paper, the author has received an interesting preprint by Mauldin and Urbański [MU] studying an analogue of the Patterson-Sullivan measure on *"conformal* (infinite) iterated function systems" or in the spirit of our present terminology, on *"conformal* (infinite) self-similar sets" F (with possibly an infinite number of conformal mappings W_i but also with F embedded in some Euclidean space $\mathbb{R}^n$). (See the relevant references in [MU] for the present case of finitely many W_i's.) A key technical notion in [MU] is that of *"q-semiconformal measure"*; i.e., a measure satisfying properties very analogous to those conjectured about ν_1 in (5.17) above, except with the "prime" now representing ordinary differentiation of (smooth) functions from $\mathbb{R}^n$ to itself. The authors show, in particular, that there exists exactly one "q-semiconformal measure" (with $q = d_H$, the Hausdorff dimension of $F \subset \mathbb{R}^n$ with respect to the Euclidean metric); that is, there exists exactly one probability (Borel or Radon) measure κ satisfying (5.17), with ν_1 replaced by κ (and with continuous weight functions). Actually, κ is an eigenmeasure (with eigenvalue 1) of the dual of the "Perron-Frobenius-Ruelle operator" [Ru, Bo, ...]. Moreover, under suitable assumptions, this unique "q-semiconformal measure" κ is necessarily "q-conformal"; namely (with our notation and ignoring technicalities), for every $i = 1, \dots, N$, $\kappa \circ W_i^{-1} = |W_i'|^q \kappa$ or equivalently,

$$\int_F (f \circ W_i)\, d\kappa = \int_F |W_i'|^q f\, d\kappa, \tag{5.26}$$

for all $f \in C(F)$. (Compare with (5.24) above.)

One possible strategy to prove Conjecture 5.14 above (as well as Conjectures 5.27 and 5.29 below) would be to extend and adapt the arguments in [MU] (and the relevant references therein) to the present more abstract (and somewhat different) situation.

We now continue to explore the possible properties of ν_1 (or ν^*).

CONJECTURE 5.27. *The "weight functions"* $\Psi_i = |W_i'|^S, i = 1, \ldots, N$ *(in* (5.16)) *can be chosen to be continuous. Furthermore,* ν_1 *is the unique probability* (Borel or Radon) *measure satisfying* (5.17). *Actually (up to measure equivalence),* ν_1 *is "uniquely ergodic" with respect to the action of the semigroup* $\mathcal{T}$ *spanned by* $\{W_i\}_{i=1}^N$*; namely,* $\mathcal{T} := \{W_\omega : \omega \in \Sigma_m$*, for some* $m \geq 1\}$.

Moreover, ν_1 *is a "Gibbs equilibrium measure"* [in a sense adapted from Ruelle [Ru] or Bowen [Bo], for example, except with "groups" (generated by a single map) replaced by "semigroups"].

REMARK 5.28. The "unique ergodicity" mentioned above will be viewed from a somewhat different perspective when we discuss "G-measures" at the end of this subsection. (See esp. Conjecture 5.35 below.)

The next conjecture, if true, would establish further ties with ([Su3], [CoSu], [Co5, §IV.3]). (See also §5.2 below where, in particular, further discussion of Connes metrics [Co3; Co4; Co5, Chap. VI] can be found.) We state it here in sufficient generality so as to allow maximum flexibility for extensions to other settings, for example. (See, e.g., ([Fc, Chaps. 2 and 3], [Su3]), and the references therein for the various notions of fractal dimensions and measures used below.)

CONJECTURE 5.29. *There exists a "Connes-type metric",* δ^**, on* F *with respect to which the following properties hold:*

(i) *The Hausdorff and Minkowski (box) [and hence also packing] dimensions coincide with* S*, the similarity dimension of* $(F, \mathcal{E})$*; namely, with the obvious notation, we have*

$$d_H(\delta^*) = d_M(\delta^*) = d_P(\delta^*) = S, \tag{5.30}$$

where S *is defined by* (3.21).

(ii) *The (*S*-dimensional) Hausdorff (resp., packing) measure of* F *is positive and finite.*

(iii) *The (*S*-dimensional) normalized Hausdorff and packing measures coincide with* ν_1*, the normalized "natural volume measure" on* $(F, \mathcal{E})$ *defined by* (5.9).

(iv) *More precisely, let* $\nu^* = \Phi(\mu^*) = \nu^*(F)\nu_1$ *be the (un-normalized) "natural volume measure" on* $(F, \mathcal{E})$*. Then* ν^* *is proportional to the (normalized) Hausdorff [and hence also packing, by* (iii)*] measure on* F*, with proportionality constant depending only on* $\mathcal{E}$*, and hence in particular independent of the limiting procedure* w *used in defining* ν^* *(in* (4.42) *of Theorem* 4.41).

Note that in the present setting, (iv) follows automatically from (iii), by Theorems 4.27 and 4.41.

REMARK 5.31. (a) Since we always have $d_H(\delta^*) \leq d_P(\delta^*) \leq d_M(\delta^*)$, the equality of $d_H(\delta^*)$ and $d_M(\delta^*)$ implies that the packing dimension $d_P(\delta^*)$ is equal to this common value.

(b) A natural candidate—*in the present case of p.c.f. self-similar (analytical) fractals*—for the above metric δ^* might be the metric $\delta = \delta_{\mathcal{E}}$ (given by (2.22)) or rather $\delta_1 = \delta_{1,\mathcal{E}}$ (given by (5.12)), so that $d_H(\delta_1) = d_M(\delta_1) = S$ by (5.13), and hence (i) holds by the previous remark. Further, by [Ki4], (ii) holds for the Hausdorff measure and we conjecture that it also holds for the packing measure. In fact, we also conjecture that the normalized (S-dimensional) packing and Hausdorff measures on (F, δ_1) coincide. The real question is to know whether they agree with $\nu_1 = \nu^*/\nu^*(F)$; that is, whether (iii) is true for this choice of metric $\delta^* := \delta_1$. Indeed, if this is so, then (as was noted above) (iv) follows from Theorem 4.41 because $\nu^*(F) = \mathrm{Tr}_w(R_{\mu^*})$, which by Theorem 4.27 depends only on $\mathcal{E}$ and is in particular independent of the limiting procedure w. One possible problem, however, is that Hausdorff measure on (F, δ_1) and μ^*, the original measure from which $\nu^* = \Phi(\mu^*)$ is constructed, are mutually absolutely continuous, by [Ki4]. This fact—in conjunction with (iii) and (iv)—is somewhat surprising since the *non-normality* of the Dixmier trace (typically) allows us to change the measure class, as was pointed out in ([CoSu], [Co5, §IV.3]). (It may not be a problem in the present case, however.) This raises the new and interesting question of possibly finding a Connes-type metric δ^* on F with respect to which Hausdorff measure and ν^* are *mutually singular*, while retaining all the other nice properties of δ_1. In the special case of the Sierpiński gasket, for example, the "Riemannian-type" metric constructed by Kusuoka in [Ku2] would be a natural substitute for δ_1. (In particular, the associated measures are mutually singular.)

(c) Conjecture 5.29, if correct, would also provide an interesting connection with Berry's original conjecture [Be1–2] for the spectral distribution of Laplacians *on* fractals. (Note, however, that even if $F \subseteq \mathbb{R}^n$, Hausdorff measure is defined here with respect to the metric δ^* rather than the Euclidean metric.)

We close §5.1 by a discussion of "*G-measures*" and of their relationship with the present work. (Our treatment will closely follow [BD2].) One is tempted to think of $|W_i'|$ (or more generally, $|W_\omega'|$) in (5.16) (or (5.17)) as a Radon-Nikodym derivative, say

$$\frac{d(\nu_1 \circ W_i^{-1})}{d\nu_1} \quad \left(\text{resp., } \frac{d(\nu_1 \circ W_\omega^{-1})}{d\nu_1}\right).$$

Instead, we will prefer to develop these possible connections with Radon-Nikodym derivatives (or cocyles) via the use of "G-measures" on the infinite product space $\Sigma = A^{\mathbb{N}}$, where $A = \{1, \ldots, N\}$.

The concept of "G-measure" was introduced by Keane in [Ke] (where it is called "g-measure") and extended to a more abstract setting by Brown and Dooley in [BD2]. It is useful, in particular, in order to determine how far a given (suitable) probability measure is from a true infinite product (i.e., Bernoulli) measure (on Σ, say). Intuitively, in favorable circumstances, "G-measures" can be viewed as "abstract Riesz products" or more generally, as weak limits of infinite product measures (on Σ, say). (See [BD1–2] and the relevant references therein.) They are also helpful to give a precise meaning to the notion of "*uniquely ergodic measure*" not just with respect to a single mapping (as in the traditional case), but rather with respect to an entire group of transformations. (Compare with Conjecture 5.27 above.)

We now briefly recall the definition of a "G-measure" on $\Sigma = A^{\mathbb{N}}$. (More general infinite product groups are allowed in [BD2].) First, we identify A with the additive group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ of relative integers modulo N. (Clearly, the name of the "letters" in the "alphabet" A is irrelevant for our previous arguments.) Accordingly, we identify Σ with the infinite product $\prod_{j=1}^{\infty} \mathbb{Z}_N$.

The abelian group $\Sigma = \prod_{j=1}^{\infty} \mathbb{Z}_N$ is acted upon by the group $\mathcal{B} = \bigoplus_{j=1}^{\infty} \mathbb{Z}_N$ of "finite coordinate changes" as follows. Given $\beta = (\beta_1, \ldots, \beta_m, 0, 0, \ldots) \in \mathcal{B}$ and $\omega = (\omega_1, \ldots, \omega_m, \omega_{m+1}, \ldots) \in \Sigma$, we have

$$\beta.\omega := (\beta_1 + \omega_1, \ldots, \beta_m + \omega_m, \omega_{m+1}, \ldots).$$

Next, given a probability measure κ on Σ, and a finite subset Λ of $\mathbb{N}$, we define the measure

$$\kappa_\Lambda := \frac{1}{\#(\mathcal{B}_\Lambda)} \sum_{\beta \in \mathcal{B}_\Lambda} \kappa \circ \beta^{-1}, \tag{5.32}$$

where $\mathcal{B}_\Lambda := \{\beta \in \mathcal{B} : \beta_j = 0, \text{ for all } j \notin \Lambda\}$. (Note that the identity belongs to $\mathcal{B}_\Lambda$ and hence $\kappa_\Lambda \geq \#(\mathcal{B}_\Lambda)^{-1}\kappa$, so that κ is always *absolutely continuous* with respect to κ_Λ.) Let $\{G_\Lambda\}$ ($\Lambda \subset \mathbb{N}, \Lambda$ finite) be the family of Radon-Nikodym derivatives of κ with respect to κ_Λ; namely, $G_\Lambda := d\kappa/d\kappa_\Lambda$. Then we assume that the G_Λ's are *normalized* (i.e., $\#(\mathcal{B}_\Lambda)^{-1} \sum_{\beta \in \mathcal{B}_\Lambda} G_\Lambda(\beta.\omega) = 1$) and, more importantly, *compatible*; i.e., for $\Lambda_1 \subset \Lambda_2$, finite subsets of $\mathbb{N}$, we have

$$G_{\Lambda_1}(\omega) G_{\Lambda_2}(\beta.\omega) = G_{\Lambda_1}(\beta.\omega) G_{\Lambda_2}(\omega), \qquad \text{for } \beta \in \Lambda_1. \tag{5.33}$$

Then, by definition, a "G-measure" is a Borel probability measure (on Σ) associated to such a compatible (and normalized) family $\{G_\Lambda\}$.

REMARK 5.34. (a) We refer to [BD2, p. 280 and §1.2, pp. 282–284] for the relationships between "G-measures" and "Radon-Nikodym cocycles".

(b) I am grateful to Professor Anthony Dooley for pointing out the possible relevance of the notion of "G-measure" after having heard me lecture on this subject at the University of New South Wales in Sydney in July 1993.

We now return to the setting of the rest of §5.1 above.

CONJECTURE 5.35. *Let τ_1 be the Borel probability measure on $\Sigma = A^{\mathbb{N}}$ obtained by pulling-back the rom(normalized) "natural volume measure" ν_1 (in (5.9)) by the continuous surjection $\Pi : \Sigma \to F$ (given by Definition 2.1). Then τ_1 is a "G-measure" on Σ. Furthermore, τ_1 can be given a continuous version (i.e., the associated Radon-Nikodym derivatives G_Λ can be chosen to be continuous, in the sense of* [BD2, §1.3]) *and more importantly, τ_1 is "uniquely ergodic" (with respect to the action of the group of "finite coordinate changes" $\mathcal{B}$), in the sense of "G-measures". (See* [BD2, *Proposition* 1.6, *p.* 287] *according to which, in particular, τ_1 is then the unique "G-measure" associated with the G_Λ's, and is therefore ergodic.)*

REMARK 5.36. Intuitively, Conjecture 5.35 says that $\tau_1 = \Pi_{\#}\nu_1$ is an "*approximate product measure*" on $\Sigma = A^{\mathbb{N}}$ and "hence" that ν_1 is "*approximately self-similar*", in agreement with Conjectures 5.10 and 5.14 above. (Compare with Theorem 2.18 and Remark 2.19(a).) Moreover, the fact that the G_Λ's can be chosen to be continuous and that τ_1 is "uniquely ergodic" is, of course, in agreement with Conjecture 5.27.

5.2. Towards a noncommutative fractal geometry? We suggest here further possible connections between the above work on analysis on fractals as well as spectral and fractal geometry on the one hand, and aspects of noncommutative geometry and "quantized calculus" (esp. [Co5, §IV.3 and §VI.1], [CoSu]) on the other hand. (See esp. §5.2.1 below.) Eventually, we would expect these subjects to give rise to a new subdiscipline, coined *"noncommutative fractal geometry"*.

A "fractal" will be viewed as a commutative or "classical" space in §5.2.1 and (very briefly) as a noncommutative or "quantum" space in §5.2.2.

5.2.1. Dirac operators, Connes metrics and quantized calculus. We first recall some basic definitions and examples. Our treatment will necessarily be fairly brief. (See, e.g., [Co1–5] and the references therein for further information.)

DEFINITION 5.37 (Fredholm modules). An unbounded Fredholm module (or K-cycle) is a quadruple $(\mathcal{A}, \rho, \mathcal{H}, D)$, where $\mathcal{A}$ is a (unital) C^*-algebra, $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a $*$-representation of $\mathcal{A}$ in the (complex) Hilbert space $\mathcal{H}$, and D is an unbounded self-adjoint operator on $\mathcal{H}$ such that

(i) the commutator $[D, \rho(a)]$ is bounded (i.e., lies in $\mathcal{L}(\mathcal{H})$) for all $a \in \mathcal{A}$,
(ii) $(1 + D^2)^{-1}$ exists and is compact (i.e., D has compact resolvents).

Moreover, the Fredholm module is said to be p^+-summable if $|D|^{-1} \in \mathcal{L}^{p^+}$ (i.e., the compact resolvents of D lie in $\mathcal{L}^{p^+}$), where $\mathcal{L}^{p^+}$ is the Maçaev ideal defined by (4.3) in §4.1.1 above.

REMARK 5.38. (a) The "commutator" $[D, \rho(a)]$ can also be viewed as an (*a priori*) unbounded quadratic form, $Q = Q_a$, on $\mathcal{H}$ (see, e.g., [Co5, §VI.1]); namely, $Q(u,v) = (\rho(a)u, Dv) - (Du, \rho(a^*)v)$, for u, v in the domain of D, and where $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}$.

(b) Following common usage, we will simply write $[D, a]$ instead of $[D, \rho(a)]$. Furthermore, we will ignore the representation ρ and write $(\mathcal{A}, \mathcal{H}, D)$ or simply $(\mathcal{H}, D)$ if no ambiguity may arise.

(c) We adopt here the definition of "p^+-summability" of a K-cycle given in [Co4, Definition 2, p. 188] rather than in [Co5], for example.

The following important example [Co1,3] is prototypical of the situation that we would like to mimic.

EXAMPLE 5.39 (Dirac operators on Riemannian manifolds). Let $N = N^n$ be an n-dimensional smooth compact (spin) Riemannian manifold. Let $\mathcal{A} = C(N)$ be the algebra of (complex) continuous functions on N, acting by multiplication operators on $\mathcal{H} := L^2(N; Sp)$, the fibre bundle of square-integrable spinors on N. Further, let D denote the classical Dirac operator on N. (Recall that D is a self-adjoint operator on $\mathcal{H}$ which is *neither* bounded from below *nor* from above.) Then $(\mathcal{A}, \mathcal{H}, D)$ is a p^+-summable Fredholm module (or K-cycle), with $p = n$, the dimension of the manifold N.

REMARK 5.40. (a) As is well-known, *the Dirac operator can be thought of as the square-root of the Laplacian, with a sign ambiguity.* We will pursue this analogy below (see esp. Conjecture 5.45). More precisely, the absolute value $|D|$ of D is equal to the square-root of the Laplacian, while the sign of D, $\mathcal{I} = \text{sgn}(D) = D|D|^{-1}$ (also defined through the spectral theorem for unbounded self-adjoint operators [ReSi, Chap. VII]) is an involution (i.e., $\mathcal{I}^2 = I$) which plays a central role in the corresponding abstract theory [Co4–5].

(b) More generally, let X be a compact Hausdorff space and let $(\mathcal{A}, \mathcal{H}, D)$ be a Fredholm module, where $\mathcal{A} = C(X)$, the space of continuous functions on X. Then, heuristically, D (resp., $D^2 = |D|^2$) can be thought of as an abstract analogue of the "Dirac operator" (resp., of the "Laplace operator") on X. Furthermore, in some sense, the involution $\mathcal{I} = \text{sgn}(D)$ may be viewed as an abstract analogue of the Hilbert transform (compare with [CoSu]).

Rather than introducing "Connes-type metrics" in the general (noncommutative) case, we will limit ourselves to the case of most immediate interest to us when the algebra $\mathcal{A}$ is commutative.

DEFINITION 5.41. Let X be a compact metrizable space, and let $(\mathcal{A}, \mathcal{H}, D)$ be a Fredholm module with $\mathcal{A} := C(X)$. Then δ^* defined by

$$\delta^*(x,y) = \sup\{|a(x) - a(y)| : a \in \mathcal{A}, \|[D,a]\| \leq 1\}, \tag{5.42}$$

for $x, y \in X$, is called the Connes metric on X associated with $(\mathcal{H}, D)$.

Here, $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(\mathcal{H})$. (See also Remark 5.38(a) above.)

REMARK 5.43. (a) In general, δ^* may only be a pseudo-metric (i.e., may take the value $+\infty$ but otherwise satisfies the remaining axioms of a metric). However, a very simple sufficient condition guarantees that δ^* is an actual metric on X; namely, the set $\{a \in \mathcal{A} : \|[D,a]\| \leq 1\}$ (modulo the constants) must be bounded. (See [Co3, Proposition 3, p. 209].)

(b) In the case of a Riemannian manifold $X = N$ studied in Example 5.39 above, δ^* coincides with the usual Riemannian metric on N. (See [Co3; Co4, p. 187; Co5, §VI.1].) That is, one recovers the Riemannian structure of N from the purely operator-theoretic data $(\mathcal{H}, D)$.

(c) If we identify points in X with states in the algebra $\mathcal{A} = C(X)$, then we may rewrite (5.42) in the following "dual way":

$$\delta_*(\varphi, \psi) := \sup\{|\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|[D,a]\| \leq 1\}, \tag{5.44}$$

for φ, ψ "states" on $\mathcal{A}$. Formula (5.44) is the basis of the noncommutative generalization of the notion of Connes metric. (See [Co3, Proposition 3, p. 209].) (Note the formal analogy between the definition of the metric δ given by (2.22) (resp., $\delta_.$ given by (2.21)) and formula (5.42) (resp., (5.44)) defining δ^* (resp., δ_*).

We now return to the setting of §5.1. In the process, we suggest further ties between our previous work and aspects of noncommutative geometry (in a commutative setting). We also specify, in particular, Conjecture 5.29 above.

CONJECTURE 5.45 (Dirac operators and Laplacians on fractals). *Let $F = (F, \mathcal{S}, \mu^*, \mathcal{E})$ be a p.c.f. (analytical) self-similar fractal, where μ^* is the "natural s.s. measure" (relative to $\mathcal{S}$) on $(F, \mathcal{E})$. (See* §5.1 *above for the precise setting.) Then there exists an unbounded Fredholm module $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{A} := C(F)$ and D a suitable analogue of the "Dirac operator" on F, having the following properties:*

(i) *The absolute value, $|D|$, of D is equal to $B := \sqrt{\Delta}$, the (positive) square-root of the Laplacian $\Delta = \Delta_{\mu^*}$ associated with the s.s. measure μ^*.*
(ii) *The K-cycle $(\mathcal{H}, D)$ is p^+-summable, with $p := d_S^* = d_S(\mu^*)$, the "spectral dimension" of $(F, \mathcal{E})$ defined in* (3.23) *and* (3.25). *(See Theorem* 3.22.)
(iii) *If δ^* denotes the Connes metric associated with $(\mathcal{H}, D)$ as in Definition* 5.41, *then Conjecture* 5.29 *from* §5.1 *holds for this choice of metric. In particular, $d_H(\delta^*) = S$, the similarity dimension of $(F, \mathcal{E})$ (see* (3.21)), *and we have $p := d_S^* = 2d_H(\delta^*)/(1 + d_H(\delta^*))$, where $d_H(\delta^*)$*

denotes the Hausdorf dimension of (F, δ^*). *Moreover, the (normalized, S-dimensional) Hausdorff measure of* (F, δ^*) *coincides with the (normalized) "natural volume measure"* ν_1 *on* $(F, \mathcal{E})$. *(See* (5.9) *and Theorem* 4.41.)

(iv) *The measure* $\nu^* = \nu^*(F)\nu_1$*—constructed in Theorem* 4.41 *via the Dixmier trace and by means of the (square-root of the) Laplace operator* $\Delta = \Delta_{0,\mu^*}$*—can also be recovered from the following formula (analogous to that in* [Co3]*):*

$$\nu^*(f) = \mathrm{Tr}_\omega(fD^{-d_S^*}), \qquad \text{for all } f \in C(F), \tag{5.46}$$

where we have substituted the "Dirac operator" D *for the square-root of the Laplacian in* (4.42).

REMARK 5.47. (a) Actually, (ii) follows from (i) and Proposition 4.22. More precisely, it follows from Proposition 4.22 (with $\mu := \mu^*$) and Remark 4.25 that $d_S^* = d_S(\mu^*)$ is the infimum of those p's for which $(\mathcal{H}, D)$ is p-summable (*i.e.*, for which $|D|^{-1} \in \mathcal{L}^{p^+}$). Hence, in some sense, the *spectral dimension* d_S^* should also be equal to the "*cohomological dimension*" ([Co4, Chap. 3], [Co5]) of the K-cycle $(\mathcal{H}, D)$.

(b) Implicit in the statement of part (iii) of Conjecture 5.45 is the fact that δ^* is a true metric on X; i.e., that the condition of [Co3] recalled in Remark 5.43(a) above is satisfied.

(c) When we write that $|D| = \sqrt{\Delta}$, we make a common abuse of language since these operators do not (*a priori*) act on the same Hilbert space $\mathcal{H}$. (Part of the problem is to extend Δ to "differential forms" rather than to just define it on functions on F.)

(d) Conjecture 5.45, if correct, would provide a framework for addressing several of the questions raised in [La5, Part II] concerning a suitable analogue of "Dirac operators" and other "elliptic pseudodifferential operators" on "fractals". (See esp. [La5, Question 5.$\hat{5}$, p. 179].)

(e) It would be interesting to establish connections between the present work (or conjectures) and that of Davies [Da] dealing, in particular, with Connes metrics and *discrete* Laplacians on *graphs*. (Recall that Laplacians on p.c.f. fractals are defined as suitable renormalized limits of discrete Laplacians on approximating finite graphs.)

We next state in a somewhat imprecise manner a problem in which we establish further contact with the recent work of Connes and Sullivan on "quantized calculus" (in one variable) on limit sets of quasi-Fuchsian groups ([CoSu], [Co5, §IV.3]). (Compare with comment (5.23b) above.)

PROBLEM 5.48. *Assume that Conjecture* 5.45 *is true and that its hypotheses hold. Let* $\mathcal{I} = \mathrm{sgn}(D) = |D|D^{-1}$ *denote the sign of* D*, defined via the functional* *calculus. Let Z be a "model map" for the "fractal" F. Extend the setting of Conjecture* 5.45 *as well as the "quantized calculus" of Connes and Sullivan so as to make sense, in particular, of the "quantized differential" of Z,*

$$dZ := [\mathcal{I}, Z] \tag{5.49}$$

(as well as of formula (5.50) *below).*

Then show that the "natural volume measure" $\nu^* = \Phi(\mu^*)$ on $(F, \mathcal{E})$ constructed in Theorem 4.41 *can also be defined by the following formula:*

$$\nu^*(f) = \int_F f\, d\nu^* := \mathrm{Tr}_w(f(Z)|dZ|^S), \tag{5.50}$$

for all $f \in C(F)$, where S denotes the similarity dimension of $(F, \mathcal{E})$ given by (3.21).

[Here, in (5.50), we assume implicitly that the operator dZ is compact and that $|dZ|^S \in \mathcal{L}^{1+}$. Furthermore, the bounded *operators* $f(Z)$ and $|dZ|^S$ are defined via the functional calculus. Of course, $|dZ|$ denotes the absolute value of dZ.]

REMARK 5.51. (a) Recall that ν^* is conjectured to be a natural counterpart of the "Patterson-Sullivan measure" in this context. (See Conjectures 5.14 and 5.29.) Hence (5.50) (together with Conjecture 5.45) would provide a more complete analogue of the results of [CoSu] in the present situation. (Compare with (5.23a) and (5.23b) above.)

(b) A natural candidate for the "model map" Z is the "projection" $\Pi : \Sigma \to F$, where $\Sigma = A^{\mathbb{N}}$ and $A = \{1, \ldots, N\}$. (See Definition 2.1 above.) Recall that Π is continuous and surjective. Given the "fractality" of F, however, we would not expect it to be "smooth" in any reasonable sense of the term (even if $F \subseteq \mathbb{R}^n$, for some n).

[The map Π would thus be the analogue in the present context of the boundary value $Z : S^1 \to L$ of a "Riemann map" or conformal equivalence, where (S^1 is the unit circle and) L is the limit set studied in ([CoSu], [Co5, §IV.3]).]

(c) We would expect the signum operator $\mathcal{I} = \mathrm{sgn}(D)$ to be the (pull-forward by Π) of a "dyadic martingale" (resp., or its obvious counterpart) on the abelian Cantor group $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ when $N = 2$ (resp., $N \geq 2$). (See, e.g., [SWS, Chap. 3] and the references therein.) Indeed, such "dyadic martingales" are natural analogues of the Hilbert transform or equivalently, of the "conjugate operator" in this context.

(d) Perhaps a "noncommutative" interpretation of Conjecture 5.45—in which, for example, $\mathcal{A}$ is a suitable Cuntz-Krieger algebra [CuKr], as in §5.2.2 below—would be better suited to formulate and address Problem 5.48.

5.2.2. Classical versus quantum spaces. So far, we have viewed (self-similar) fractals F only as "*classical spaces*" (i.e., commutative spaces) with an associated abelian algebra $\mathcal{A} := C(F)$. We now very briefly discuss the possibility of treating them (in the terminology of [Co4–5]) as "*quantum spaces*" (i.e., noncommutative spaces) for which the associated algebra $\mathcal{A}$ is nonabelian. This situation is probably very familiar to researchers working at the junction of operator algebras and dynamical systems.

There appears to be (at least) two natural noncommutative algebras that take into account (in somewhat different ways) not only the topological structure of (topological, s.s.) fractals $F = (F, \mathcal{S})$, but also their underlying dynamical structure. (See Definition 2.1 above and Remark 5.54(a) below.)

They are associated with the following two equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on F:

(i) $x\mathcal{R}_1 y$ if and only if there exist an integer m and finite words ω_i in Σ_{m_i} $(i = 1, 2)$ such that

$$W_{\omega_1} x = W_{\omega_2} y, \tag{5.52}$$

where W_{ω_i} is defined as in (2.4).

(ii) $x\mathcal{R}_2 y$ if and only if there exist an integer m_i and a finite word ω_i in Σ_{m_i} $(i = 1, 2)$ such that

$$W_{\omega_1} x = W_{\omega_2} y. \tag{5.53}$$

One typically associates with the equivalence relation $\mathcal{R}_1$ an AF (i.e., "approximately finite") algebra, defined as an inductive limit of a tower of finite-dimensional algebras, whereas one associates with $\mathcal{R}_2$ a (richer) Cuntz-Krieger algebra [CuKr]. (See, e.g., [MRS], [Co5] and the relevant references therein for further information.) Note that these algebras are defined in terms of the action of *semigroups* rather than of *groups* of transformations; see [MRS]. (Compare with Remark 2.7 and Conjecture 5.27 above.)

REMARK 5.54. (a) The author is grateful to Professor Colin Sutherland for a helpful conversation about the above subject after he lectured on the results of this paper at the University of New South Wales in July 1993.

(b) We have become recently aware of the work in [JoPe] in a related—but somewhat different context (s.s. measures on Euclidean space)—which makes use of certain types of Cuntz algebras.

5.3. Extensions. We close this paper by a brief discussion of some of the possible extensions of this work (as well as of the conjectures of §5.1 and §5.2) to other classes of self-similar fractals.

In a later work, we hope to deal (from a somewhat different point of view) with more general spaces of nonintegral "fractal" dimension, no longer assumed to be self-similar.

(i) (Analytical self-similar fractals). If Conjecture 3.37 of §3.4 is correct (and if the counterpart of the results recalled in §3.1 also hold in this setting), then the results of §4.2 can be extended (with essentially the same proof) to the class of (regular) analytical self-similar fractals. (See Definition 2.24.) Recall from §3.4 that *this class includes* the *"finitely ramified"* (i.e., p.c.f.) *fractals as well as* some interesting *"infinitely ramified fractals"*, such as the (universal) two-dimensional Sierpiński carpet. The analytical results of [KuZh, Ki5] should be useful to deal with the latter case. (See also [BB1] for a probabilistic approach.)

Hence, in particular, one could define "volume measures" as well as an analogue of the "Riemannian volume element" in this situation. Furthermore, the "volume" of F could be computed (much as in (4.46) or (4.47) in the nonlattice or lattice case, respectively) in terms of spectral data. Moreover, we would expect many of the conjectures of §5.1 and §5.2.1 to be formulated analogously. (Note that the regularity assumption implies that $d_S^* < 2$, as in Corollary 3.31.)

Finally, we discuss one last example, which is prototypical of the analytical difficulties remaining to be overcome in order to deal satisfactorily with more general classes of "infinitely ramified" s.s. fractals.

(ii) (The three-dimensional Sierpiński carpet.) In this case—as was previously mentioned—$\mathcal{F}$, the domain of the energy functional $\mathcal{E}$, is no longer contained in $C(F)$. (Conjecturally, this is akin to the (higher-dimensional) Sobolev Embedding Theorem for "fractals", when $d_S^* > 2$; compare with Remark 2.15(b) above and [BB2].) Further, $\mathcal{F}$ need not be independent of the choice of the s.s. measure μ. (Hence the definition of the associated eigenvalue problem must be adjusted accordingly by substituting $\mathcal{F} \cap L^2(\mu)$ for $\mathcal{F}$ in (3.2).)

Nevertheless, the recent probabilistic results of [BB2] can be used to define—via the Dixmier trace, much as in §4.2 and in [Co3]—an analogue of the Riemannian volume measure (but not necessarily to compute its total mass as in Corollary 4.45). [Of course, we may also consider here the n-dimensional carpet, with $n \geq 3$.]

Because of the aforementioned analytical difficulties, we would expect some of the tools of noncommutative geometry—such as Connes metrics, for example—to be particularly appropriate to deal with this situation.

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This work was presented at the Annual Meeting of the American Mathematical Society (San Antonio, Texas, January 1993) in the Special Session on "Eigenvalues in Riemannian Geometry", as well as, in particular, in seminars and/or colloquia at the California Institute of Technology in Pasadena, the University of California in Los Angeles (May 1993), and the University of New South Wales and Sydney University in Australia (July-August 1993). It was also discussed by the author in a plenary lecture at the Annual Meeting of the Australian Mathematical Society and at the Conference on "Harmonic Analysis and Partial Differential Equations" (Wollongong, Australia, July 1993). Finally, this work—together with some of its recent extensions—was presented at the Annual Meeting of the Mathematical Society (Cincinnati, Ohio, January 1994) in the Special Session on "Geometric Applications of Operator Algebras and Index Theory".

I would like to express my appreciation to the above institutions and, particularly to the University of Wollongong and the University of New South Wales in Sydney, where part of this paper was written during my visit to Australia in the summer of 1993. I also wish to thank Professors Rodney Nillsen and Tony Dooley for their very warm and kind hospitality during my stay in Wollongong and in Sydney, respectively.

Epilogue. After this work was completed, the author has received from Professor Alain Connes the final version of his forthcoming book [Co5] on "Non Commutative Geometry" (Preprint IHES/M/93/54, October 1993). At the end of the section (§IV.3) describing the work of Connes and Sullivan [CoSu], some of our earlier joint work with Carl Pomerance [LaPo1–2] on the vibrations of "fractal strings" (i.e., one-dimensional "drums with fractal boundary") is reinterpreted (and extended) in terms of the notion of "quantized calculus" (*ibid*, §IV.3 (ε)). In the case of certain Fuchsian groups (of the second kind), this result is then used to calculate explicitly (in our notation) the total mass, $\nu^*(F)$, of an *analogue* of the measure ν^* above. It would be interesting to obtain similar (but higher-dimensional) results in the context of the present paper; that is, for "drums with fractal membrane" rather than for "fractal strings".

REFERENCES

[Ad] M. ADLER, *On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg-de-Vries type equations*, Invent. Math. **50** (1979), 219–248.

[AO] S. ALEXANDER AND R. ORBACH, *Density of states on fractals: "fractons"*, J. Physique Lettres **43** (1982), L625–L631.

[BB1] M. T. BARLOW AND R. F. BASS, *Construction of Brownian motion on the Sierpiński carpet*, Ann. Inst. Henri Poincaré **25** (1989), 225–257.

[BB2] ———, *Coupling and Harnack inequalities for Sierpiński carpets* (1993) (preprint).

[BP] M. T. BARLOW AND E. A. PERKINS, *Brownian motion on the Sierpiński gasket*, Prob. Th. Rel. Fields **79** (1988), 542–624.

[Ba] A. F. BEARDON, *The Geometry of Discrete Groups*, Springer-Verlag, Berlin, 1983.

[Bd] T. BEDFORD, M. KEANE AND C. SERIES (EDS.), *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford Univ. Press, Oxford, 1991.

[Be1] M. V. BERRY, *Distribution of modes in fractal resonators*, Structural Stability in Physics (W. Güttinger and H. Eikemeier, eds.), Springer-Verlag, Berlin, 1979, pp. 51–53.

[Be2] ———, *Some geometric aspects of wave motion: wavefront dislocations, diffraction catastrophes, diffractals*, Geometry of the Laplace Operator, Proc. Symp. Pure Math., vol. 36, Amer. Math. Soc., Providence, 1980, pp. 13–38.

[BiSo1] M. S. BIRMAN AND M. Z. SOLOMYAK, *The principal term of spectral asymptotics for nonsmooth elliptic problems*, Funktsional. Anal. i Prilozhen. **4** (1970), 1–13.

[BiSo2] ———, *Spectral asymptotics of nonsmooth elliptic operators* I, Trans. Moscow Math. Soc. **27** (1972), 3–52; II, Trans. Moscow Math. Soc. **28** (1973), 3–34.

[Bo] R. BOWEN, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin, 1975.

[BrCa] J. BROSSARD AND R. CARMONA, *Can one hear the dimension of a fractal?*, Commun. Math. Phys. **104** (1986), 103–122.

[BD1] G. BROWN AND A. H. DOOLEY, *Ergodic measures are of weak product type*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 129–142.

[BD2] ———, *Odometer actions on G-measures*, Ergodic Theory and Dynamical Systems **11** (1991), 279–307.

[Ch] G. CHOQUET, *Lectures on Analysis, vol. I, Integration and Topological Vector Spaces*, W. A. Benjamin, Reading, 1969.

[Co1] A. CONNES, *The action functional in non-commutative geometry*, Commun. Math. Phys. **117** (1988), 673–683.

[Co2] ———, *Trace de Dixmier, modules de Fredholm et géométrie riemannienne*, Nuclear Phys. B (Proc. Suppl.) **58** (1988), 65–70.

[Co3] ———, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory and Dynamical Systems **9** (1989), 207–220.

[Co4] ———, *Géométrie Non Commutative*, InterEditions, Paris, 1990.

[Co5] ———, *Non Commutative Geometry* (preprint IHES /M/93/12, March 1993); Academic Press, New York (to appear).

[CoSu] A. CONNES AND D. SULLIVAN, *Quantized calculus on S^1 and quasi Fuchsian groups* (in preparation and [Co5, §IV.3]).

[CuKr] J. CUNTZ AND W. KRIEGER, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **63** (1981), 25–40.

[Da] E. B. DAVIES, *Analysis on graphs and noncommutative geometry* (1992) (preprint).

[Dh] D. DHAR, *Lattices of effectively nonintegral dimensionality*, J. Math. Phys. **18** (1977), 577–585.

[Di] J. DIXMIER, *Existence de traces non normales*, C. R. Acad. Sci. Paris **262** (1966), 1107–1108.

[Fc] K. FALCONER, *Fractal Geometry: Mathematical Foundations and Applications*, Wiley, Chichester, 1990.

[Fel] W. FELLER, *An Introduction to Probability Theory and its Applications, vol. II, Chap. XI*, J. Wiley, New York, 1966.

[FlLa1] J. FLECKINGER AND M. L. LAPIDUS, *Eigenvalues of elliptic boundary value problems with an indefinite weight function*, Trans. Amer. Math. Soc. **295** (1986), 305–324.

[FlLa2] ———, *Remainder estimates for the asymptotics of elliptic eigenvalue problems with indefinite weights*, Arch. Rational Mech. Anal. **98** (1987), 329–356.

[Fu1] M. FUKUSHIMA, *Dirichlet Forms and Markov Processes*, Kodansha and North-Holland, 1980.

[Fu2] ———, *Dirichlet forms, diffusion processes and spectral dimensions for nested fractals*, Proc. Conf. in Memory of Hoegh-Krohn, vol 1, Cambridge Univ. Press, Cambridge, 1993, pp. 151–161.

[FuSh] M. FUKUSHIMA AND T. SHIMA, *On a spectral analysis for the Sierpiński gasket*, Potential Analysis **1** (1992), 1–35.

[Go] S. GOLDSTEIN, *Random walks and diffusions on fractals*, IMA Math. Appl. (H. Kesten, ed.), vol. 8, Springer-Verlag, New York, 1987, pp. 121–129.

[Gu] V. GUILLEMIN, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. **55** (1985), 131–160.

[Ha] M. HATA, *On the structure of self-similar sets*, Japan J. Appl. Math. **3** (1985), 381–414.

[HHW] K. HATTORI, T. HATTORI AND H. WATANABE, *Gaussian field theories on general networks and the spectral dimensions*, Progr. Theoret. Phys. Suppl. **92** (1987), 108–143.

[Ho] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, vols. III & IV, Springer-Verlag, Berlin, 1985.

[Hu] J. E. HUTCHINSON, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[JoPe] P. E. T. JORGENSEN AND S. PEDERSEN, *Harmonic analysis and fractal limit-measures induced by representations of a certain C^*-algebra*, J. Funct. Anal. (to appear).

[Ka] A. KAMEYAMA, *Self-similar sets from the topological point of view*, Japan J. Indust. Appl. (to appear).

[Kc] M. KAC, *Can one hear the shape of a drum?*, Amer. Math. Monthly (Slaught Mem. Papers, No. 11) **73** (1966), 1–23.

[Ke] M. KEANE, *Strongly mixing g-measures*, Invent. Math. **16** (1972), 309–324.

[Ki1] J. KIGAMI, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. **6** (1989), 259–290.

[Ki2] ———, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Amer. Math. Soc. **335** (1993), 721–755.

[Ki3] ———, *Hausdorff dimensions of self-similar sets and shortest path metrics* (1992) (preprint).

[Ki4] ———, *Effective resistances for harmonic structures on p.c.f. self-similar sets*, Math. Proc. Cambridge Philos. Soc. (to appear).

[Ki5] ———, *Harmonic calculus on limits of networks and its applications to dendrites* (1992) (preprint).

[KiLa1] J. KIGAMI AND M. L. LAPIDUS, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Commun. Math. Phys. **158** (1993), 93–125.

[KiLa2] ———, *The vibrations of the Koch drum* (in preparation).

[Ku1] S. KUSUOKA, *A diffusion process on a fractal*, Probabilistic Methods in Mathematical Physics, Proc. of Taniguchi Intern. Symp. (K. Ito and N. Ikeda, eds.), Kinokuniya, Tokyo, 1987, pp. 251–274.

[Ku2] ———, *Dirichlet forms on fractals and products of random matrices*, Publ. RIMS **25** (1989), 659–680.

[KuZh] S. KUSUOKA AND X. Y. ZHOU, *Dirichlet forms on fractals: Poincaré constant and resistance*, Prob. Th. Rel. Fields **93** (1992), 169–196.

[L] S. P. LALLEY, *Packing and covering functions of some self-similar fractals*, Indiana Univ. Math. J. **37** (1988), 699–709.

[La1] M. L. LAPIDUS, *Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), 465–529.

[La2] ———, *Can one hear the shape of a fractal drum? Partial resolution of the Weyl-Berry conjecture*, Geometric Analysis and Computer Graphics, Proc. Workshop Differential Geometry, Calculus of Variations and Computer Graphics (MSRI, Berkeley, May 1988), Mathematical Sciences Research Institute Publications (P. Concus et al., eds.), vol. 17, Springer-Verlag, New York, 1991, pp. 119–126.

[La3] ———, *Spectral and fractal geometry: From the Weyl-Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function*, Differential Equations and Mathematical Physics, Proc. UAB Intern. Conf. (Birmingham, March 1990) (C. Bennewitz, ed.), Academic Press, New York, 1992, pp. 151–182.

[La4] ———, *Spectral zeta-functions, vibrating fractal strings and the Dirichlet divisor problem* (in preparation).

[La5] ———, *Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture*, Ordinary and Partial Differential Equations, vol. IV (B. D. Sleeman and R. J. Jarvis, eds.), Proc. Twelfth Intern. Conf. on the Theory of Ordinary and Partial Differential Equations (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Mathematics Series, vol. 289, Longman Scientific and Technical, London, 1993, pp. 126–209.

[La6] ———, *The vibrations of fractal drums and waves in fractal media*, Fractals in the Natural and Applied Sciences (A-41) (M. M. Novak, ed.), Proc. 2nd. IFIP Intern. Conf. Fractals 93 (London, England, UK, September 1993), Elsevier Science B. V. North-Holland, 1994, pp. 255–260.

[LaFl] M. L. LAPIDUS AND J. FLECKINGER-PELLÉ, *Tambour fractal: vers une résolution de la conjecture de Weyl-Berry pour les valeurs propres du laplacien*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), 171–175.

[LaMa1] M. L. LAPIDUS AND H. MAIER, *Hypothèse de Riemann, cordes fractales vibrantes et conjecture de Weyl-Berry modifiée*, C. R. Acad. Sci. Paris Sér. I Math. **313** (1991), 19–24.

[LaMa2] ———, *The Riemann hypothesis and inverse spectral problems for fractal strings*, J. London Math. Soc. (to appear).

[LNRG] M. L. LAPIDUS, J. W. NEUBERGER, R. RENKA AND C. A. GRIFFITH, *Snowflake harmonics and computer graphics* (in preparation).

[LaPa] M. L. LAPIDUS AND M. M. H. PANG, *Eigenfunctions of the Koch snowflake domain*, Commun. Math. Phys. (to appear).

[LaPo1] M. L. LAPIDUS AND C. POMERANCE, *Fonction zêta de Riemann et conjecture de Weyl-Berry pour les tambours fractals*, C. R. Acad. Sci. Paris Sér. I Math. **310** (1990), 343–348.

[LaPo2] ———, *The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums*, Proc. London Math. Soc. (3) **66** (1993), 41–69.

[LaPo3] ———, *Counterexamples to the modified Weyl-Berry conjecture on fractal drums*, Math. Proc. Cambridge Philos. Soc. (to appear).

[Li] T. LINDSTROM, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc. **420** (1990).

[M1] L. MALOZEMOV, *The integrated density of states for the difference Laplacian on the modified Koch graph*, Commun. Math. Phys. **156** (1993), 387–397.

[M2] ———, *Spectral theory of the differential Laplacian on the modified Koch curve*, Contemporary Math., Amer. Math. Soc., Providence (to appear).

[Ma] B. B. MANDELBROT, *The Fractal Geometry of Nature*, rev. and enl. ed., W. H. Freeman, New York, 1983.

[Man] YU I. MANIN, *Algebraic aspects of non-linear differential equations*, J. Soviet Math. **11** (1979), 1–122.

[MRS] M. H. MANN, I. RAEBURN AND C. E. SUTHERLAND, *Representations of compact groups, Cuntz-Krieger algebras and groupoid C^*-algebras* (1993) (preprint).

[MU] R. D. MAULDIN AND M. URBAŃSKI, *Dimensions and measures in infinite iterated functions systems* (1994) (preprint).

[Me] G. MÉTIVIER, *Valeurs propres de problèmes aux limites elliptiques irréguliers*, Bull. Soc. Math. France, Mém. **51–52** (1977), 158–219.

[Mo] P. A. P. MORAN, *Additive functions of intervals and Hausdorff measure*, Math. Proc. Cambridge Philos. Soc. **42** (1946), 15–23.

[Ni] P. J. NICHOLLS, *A measure on the limit set of a discrete group*, pp. 257–297 in reference [Bd] above.

[OxUl] J. C. OXTOBY AND S. M. ULAM, *Measure-preserving homomorphisms and metrical transitivity*, Ann. of Math. **42** (1941), 874–920.

[Pa] S. J. PATTERSON, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), 241–273.

[PJS] H.-O. PEITGEN, H. JÜRGENS AND D. SAUPE, *Chaos and Fractals*, Springer-Verlag, Berlin, 1992.

[R] R. RAMMAL, *Spectrum of harmonic excitations on fractals*, J. Physique **45** (1984), 191–206.

[RT] R. RAMMAL AND G. TOULOUSE, *Random walks on fractal structures and percolation clusters*, J. Physique Lettres **44** (1983), L13–L22.

[ReSi] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics*, vol. I, Functional Analysis, Academic Press, New York, 1978.

[Ro1] G. V. ROZENBLYUM, *On the distribution of eigenvalues of the first boundary value problem in unbounded regions*, Soviet Math. Dokl. **12** (1971), 1539–1541.

[Ro2] ———, *On the eigenvalues of the first boundary value problem in unbounded domains*, Math. USSR-Sb. **18** (1972), 235–248.

[Ru] W. RUDIN, *Real and Complex Analysis*, 3d ed., McGraw-Hill, New York, 1987.

[Rue] D. RUELLE, *Thermodynamic Formalism*, Addison-Wesley, Reading, 1978.

[SWS] F. SCHIPP, W. R. WADE AND P. SIMON, *Walsh Series: An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol, 1990.

[Sh] T. SHIMA, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets* (1994) (preprint).

[S] W. SIERPIŃSKI, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée*, C. R. Acad. Sci. Paris **162** (1916), 629–632.

[Si] B. SIMON, *Trace Ideals and their Applications*, London Math. Soc. Lecture Note Ser., vol. 35, Cambridge Univ. Press, 1979.

[St1] R. S. STRICHARTZ, *Fourier asymptotics of fractal measures*, J. Funct. Anal. **89** (1990), 154–187.

[St2] ———, *Self-similar measures and their Fourier transforms*, I–III; I, Indiana Univ. Math. J. **39** (1990), 797–817; II, Trans. Amer. Math. Soc. (to appear); III, Indiana Univ. Math. J. **42** (1993), 367–411.

[Su1] D. SULLIVAN, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. IHES **50** (1979), 171–202.

[Su2] ———, *Conformal dynamical systems*, Geometric Dynamics, Lecture Notes in Math., vol. 1007, Springer-Verlag, New York, 1983, pp. 725–752.

[Su3] ———, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153** (1984), 259–277.

[Vo] D. VOICULESCU, *Some results on norm ideal perturbations of Hilbert space operators*, I & II, J. Operator Theory **1** (1979), 3–37; **5** (1981), 77–100.

[We] H. WEYL, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen*, Math. Ann. **71** (1912), 441–479.

[Wo] M. WODZICKI, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), 143–178.

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