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THE CLASSIFICATION OF REVERSIBLE CUBIC SYSTEMS WITH CENTER

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Dedicated to Jean Leray

1. Introduction

One of the main problems in the qualitative theory of differential equations is the *center-focus problem* for singular points of planar vector fields (see [10], [11]). It asks for the conditions which must be imposed on a system to ensure the existence of a center. In the class of germs of smooth or analytic vector fields $V : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ the systems with center form a subset of infinite codimension and the problem is divided into an infinite number of steps. Given the n-th jet $j^nV(0)$ decide whether (i) 0 cannot be a center for any V' with $j^nV'(0) = j^nV(0)$ or (ii) otherwise. In the case (ii) we pass to $j^{n+1}V$ (see [1]). (The Lyapunov stability and the asymptotic stability problems are stated analogously and for planar vector fields they are equivalent to the center-focus problem.)

Therefore the core of the problem is to classify polynomial vector fields with center. The following classes of polynomial vector fields with center are known: integrable systems with a *Darboux integral* or with a *Darboux-Schwartz-Christoffel integral* or with a *Darboux hyperelliptic integral* and *rationally reversible* systems. Let us define what these names mean.

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The *Darboux integrable systems* have a first integral

$$H = \prod f_j^{a_j} e^{g_1/g_2}, \tag{1}$$

where $f_i(x,y)$, $g_j(x,y)$ are polynomials and a_j are constants.

If $M = \prod f_j^{a_j-1} e^{g_1/g_2} g_2^{-2}$ then the vector field $V_H = M^{-1} X_H$ (where X_H is the Hamiltonian vector field with the Hamilton function H) is a polynomial vector field and the critical points of H are centers for V_H. The centers are at the points of local extremum of H and are called *centers of Darboux type.*

The class of Darboux integrable systems includes also all limit cases of (1) and forms a closed subset of the set of all systems with center. For example, the first integral

$$H_a = y^a \left(x^2 + \frac{y^2}{a+2} - \frac{1}{a} \right) + \frac{1}{a}$$

of the system $\dot{x} = ax^2 + y^2 - 1$, $\dot{y} = -2xy$ tends to

$$H_0 = x^2 + \frac{y^2}{2} - \ln y \qquad \text{as } a \to 0.$$

The *Darboux-Schwartz-Christoffel integrable systems* (DSC) are polynomial systems with the first integral

$$H = \prod |(U - u_j)^{\mu_j}|^2 e^{g(U)} W + \int^U \prod |(u - u_j)^{\mu_j - 1}|^2 e^{g(u)} P(u)\, du,$$

where $u_j \in \mathbb{C} \setminus \mathbb{R}$, $\mu_j \in \mathbb{C}$, P is a polynomial, g is a rational function and $U = R/S$ and W are some rational functions. The centers under consideration are at $R = S = 0$. Some additional (non-algebraic) conditions must be imposed on the parameters to obtain the univalence of H in $(\mathbb{R}^2, 0)$ (the existence of a center). The centers (located in the indefiniteness locus of U) are called *centers of DSC type.*

In this paper we shall often meet systems with a first integral of a DSC type but without centers of DSC type. Notice also that the integral in the definition of H belongs to the class of Schwartz-Christoffel integrals justifying the name of H.

The *Darboux hyperelliptic integrable systems* (DHE) have a first integral of the form

$$H = \prod \left(\frac{R_i - \sqrt{S(X)}}{R_i + \sqrt{S(X)}} \right)^{a_i} \exp \left[T\sqrt{S(X)} + \int^X W(u)\sqrt{S(u)}\, du \right],$$

with $R_i = R_i(x,y)$, $X(x,y)$, $T(x,y)$ rational and $S(\cdot)$, $W(\cdot)$ rational. The critical points of H are centers of DHE type. Notice that the integral appearing in the definition of H belongs to the class of hyperelliptic integrals.

A *rationally reversible system* at a center O admits some rational non-invertible map $\Phi : \mathbb{R}P^2 \to \mathbb{R}P^2$ and a polynomial vector field V' on $\mathbb{R}P^2$ such that

(i) $\Phi_* V$ and $V' \circ \Phi$ are collinear,

(ii) the curve of non-invertibility Γ_Φ of Φ passes through O and there is a neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of O such that the boundary of $\Phi(\mathcal{U})$ contains a part of the curve $\Gamma' = \Phi(\Gamma_\Phi)$, the vector field V' is tangent to Γ' at $\Phi(O)$ from the outside of $\Phi(\mathcal{U})$ and $V'(\Phi(O)) \neq 0$.

Under such conditions the point O must be a center because the real trajectories of V are the preimages of compact pieces of trajectories of V' lying in $\Phi(\mathcal{U})$.

Here the *curve of non-invertibility* Γ_Φ is formed by the points near which the map Φ is not invertible. In particular, $\{\det(d\Phi) = 0\} \subset \Gamma_\Phi$. The components of Γ_Φ going through the center O form the so-called *fold curve.*

Probably the first examples of reversible systems were given by Poincaré [11]. There $\Phi(x, y) = (x^2, y)$ and the systems are invariant with respect to the reflection with respect to the axis $x = 0$ and reversion of time. They are known as *time-reversible systems* (see [13]).

REMARK. The above three types of first integrals have interesting monodromy groups, treated as the automorphisms groups of their Riemann surfaces (coverings over $\mathbb{C}P^2 \setminus \{\text{branching curves}\}$). They form exactly three types of solvable subgroups of the Möbius group $PSL(2, \mathbb{C})$ (defined by generators):

$$
\begin{array}{lll}
h \to \lambda_i h, & i = 1, \dots, r & \text{(Darboux);} \\
h \to \lambda_i h + \mu_i, & i = 1, \dots, r & \text{(DSC);} \\
h \to \lambda_i h, & i = 1, \dots, r, \ h \to \mu_j / h, \ j = 1, \dots, s & \text{(DHE).}
\end{array}
$$

The author thinks that the above examples form all the cases of center for polynomial systems and that the three types of first integrals represent all first integrals expressed in quadratures (see [12]). Probably the DHE type integral did not appear in the literature before.

Starting with the above examples we begin the classification of all cubic systems with center.

(Recall that quadratic systems with center were classified by Dulac [4] and by Kapteyn [8]. They are of the form $\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2$, where either $B = 0$ or $A = -1/2$, $B = 1$ or $A, B, C \in \mathbb{R}$ or $A = 2B = 2|C| = 2$, see also [16].)

The cubic systems with center have not yet been classified. There are only restricted results concerning some special subfamilies of the entire 20-parameter family $\mathcal{A}_3$ of all cubic vector fields (see [6], [7], [15], [17]). A simple answer is obtained only for systems with homogeneous non-linearity $\dot{z} = iz + Dz^3 + Ez^2\overline{z} + Fz\overline{z}^2 + G\overline{z}^3$ (see [17]): $\operatorname{Re} E = 3D + \overline{F} = 0$ or $\operatorname{Re} E = \operatorname{Im} DF = \operatorname{Re} D^2G = 0$ or $E = D - 3\overline{F} = 2|F| - |G| = 0$.

It is not difficult to show that there are no cubic centers with a DSC integral. (If the linear part j^1V at the center is an elementary rotation then the DSC integral is simply of Darboux type. If j^1V is nilpotent then after the resolution of the singularity it can be shown that either the DSC integral is of Darboux type or the system is reversible. If $j^1V = 0$ then also $j^2V = 0$ and homogeneous systems are Darboux integrable. The detailed proof will be given elsewhere.)

The first step towards a classification of Darboux integrals for cubic systems with center was made by Sokulski in [14].

The DHE integrals for cubic centers have not yet been investigated. Probably there are no such cases.

The present work forms the second step, the classification of cubic reversible centers.

When the author started the investigation of cubic centers the problem of classification of reversible centers seemed to be quite easy. As the reader will see it is not so. The consideration of many separate subcases has turned out very complicated. Moreover, some new methods of studying algebraic invariant curves were developed (see Lemmas 4–6 in Section 5).

2. The result

In Theorem 1 we present a complete classification of reversible cubic systems with center which are not integrable by means of Darboux or DSC or DHE integrals. We denote by $\mathcal{A}_3^R$ the set of such systems. To be precise, we have found all semi-algebraic families of cubic systems with center which are reversible and such that a generic system from such a family is not Darboux integrable. Often these families contain some infinite series of Darboux integrable systems but these series cannot be included in any continuous family of Darboux integrable systems.

In the classification we exhibit a rational map $\Phi : \mathbb{R}P^2 \to \mathbb{R}P^2$, $\Phi(x, y) = (X, Y)$, realizing the reversibility and a vector field

$$V' = F(X, Y)\partial_X + G(X, Y)\partial_Y$$

in the Φ-image described by means of the equation $dX/dY = F(X, Y)/G(X, Y)$.

THEOREM 1. *Any $V(x,y) \in \mathcal{A}_3^R$ is reversible by means of one of the following 17 pairs (Φ, V') (where we choose some special coordinates (x,y), $T_1 = x+y+c$ and $T_2 = ax^2+bxy+cy^2+dx+ey+1$):*

Case	(X,Y)	$\frac{dX}{dY}$
$CR_1^{(7)}$	(x^2, y)	$\frac{k+lX+mY+nY^2+pXY+qY^3}{r+sX+tY+uY^2}$
$CR_2^{(10)}$	$(x, \frac{y^2}{x+y})$	$\frac{k+lX+mX^2}{Y(n+pX)+qY^2}$
$CR_3^{(10)}$	$(x, \frac{y^2}{xy+ax^2+bx+1})$	$\frac{k+lX+mX^2}{n+pX+(q+rX)Y+sXY^2}$
$CR_4^{(8)}$	$(T_1 x, \frac{T_1}{y})$	$\frac{kX+lY+mXY}{Y(n+pY+qY^2)}$
$CR_5^{(8)}$	$(T_1 x, \frac{T_1}{y})$	$\frac{X(kX+lY+mXY)}{Y(kX+lY+nXY+pY^2+qXY^2)}$
$CR_6^{(7)}$	$(T_1 x, \frac{T_1}{y})$	$\frac{X(k+lY+mY^2)+nY^2}{Y(k+pY+qY^2+rY^3)}$
$CR_7^{(9)}$	$(T_1 x, \frac{T_1}{y})$	$\frac{X(k+lX)}{Y(k+mX+nY+pXY)}$
$CR_8^{(10)}$	$(T_1 x, \frac{T_1^2}{y})$	$\frac{X(k+lX)}{2Y(k+mX+nY)}$
$CR_9^{(10)}$	$(T_1 x, \frac{T_1^2}{y})$	$\frac{kX+lY+mXY}{2Y(k+pY)}$
$CR_{10}^{(10)}$	$(T_1 x, \frac{T_1^3}{y})$	$\frac{X(k+lX)}{3Y(k+mX+nY)}$
$CR_{11}^{(7)}$	$(\frac{T_1^2}{x}, \frac{T_1}{y})$	$\frac{2X(kY+lX+mXY)}{Y(kY+nX+pY^2+qXY+rXY^2)}$
$CR_{12}^{(7)}$	$(\frac{T_1^2}{x}, \frac{T_1}{y})$	$\frac{2X(kY+lX+mY^2+nXY+pXY^2)}{Y^2(k+qY+rY^2)}$
$CR_{13}^{(10)}$	$(\frac{T_1^3}{x}, \frac{T_1^2}{y})$	$\frac{3X(kY+lX)}{2Y(kY+mX+nY^2)}$
$CR_{14}^{(9)}$	$(\frac{T_1^3}{x}, \frac{T_1^2}{y})$	$\frac{3X^2(kY+lX+mY^2)}{2Y^2(kX+nXY+pY^2)}$
$CR_{15}^{(10)}$	$(\frac{T_1^4}{x}, \frac{T_1^2}{y})$	$\frac{X(kX+lY^2)}{Y^3(m+nY)}$
$CR_{16}^{(5)}$	$(\frac{T_2}{x}, \frac{T_2}{y})$	$\frac{X(kX+lY)+X^2(mX+nY)}{Y(kX+lY)+Y^2(pX+qY)}$
$CR_{17}^{(12)}$	$(\frac{T_1^3}{y}, \frac{T_1^2}{\eta})$	$\frac{3kX^2(1+3Y)}{kY(2X+3XY-9Y^2)}$

In the case $CR_3^{(10)}$ there are additional restrictions:

$$n = 4[(3a-b^2)bk + (b^2-2a)l - bm], \qquad p = 4a[(6a-b^2)k + bl - 2m],$$

$$q = 3bk - 2l, \qquad r = (4a-b^2)k + bl - 2m, \qquad s = k/2.$$

In the case $CR_{17}^{(12)}$ we have

$$T_1 = x, \qquad \eta = xy - 2ay^2 + 2x + 2(1+a)y + 1.$$

The upper index in the case labels above denotes the codimension in $\mathcal{A}_3$. It is calculated in the following way. Firstly, we fix the form Φ_0 of Φ given in the table (the normal form with respect to the changes $\Phi \to \Phi_2 \circ \Phi \circ \Phi_1$, $\Phi_{1,2}$ affine diffeomorphisms), depending on n_1 parameters. Then we count the dimension n_2 of the orbit of Φ_0. The number of parameters in V' is n_3. Then $n_1+n_2+n_3$ is the dimension of the corresponding stratum of the center manifold. We have the following.

In $CR_1^{(7)}$: $n_1 = 0$, $n_2 = 3$ (the position of $\Gamma = \{x = 0\}$ and the direction Ox), $n_3 = 10$.

In $CR_2^{(10)}$: $n_1 = 0$, $n_4 = 4$ (the position of $\Gamma = \{2x + y = 0\}$ and of the invariant line $y = 0$), $n_3 = 6$.

In $CR_3^{(10)}$: $n_1 = 1$ (a or b fixed), $n_2 = 6$ (the position of $\Gamma = \{y = 0\}$ and of an invariant quadratic curve), $n_3 = 3$, (k, l, m).

In $CR_4^{(8)}$–$CR_{15}^{(5)}$: $n_1 = 0$ ($c = 0, 1$), $n_2 = 6$ (the position of $T = 0$ and x, y), and n_3 depends on the case.

In $CR_{16}^{(5)}$: $n_1 = 3$ (d, e fixed), $n_2 = 6$, $n_3 = 6$.

In $CR_{17}^{(12)}$: $n_1 = 1$, $n_2 = 6$, $n_3 = 1$.

Usually we shall omit this index in referring to some cases of reversibility.

We have not written the formulas for the vector field $V(x, y) = \dot{x}\partial_x + \dot{y}\partial_y$ because they are rather messy. The reader will find them in Section 5.

The proof will be given in Sections 3, 4 and 5.

REMARK 1. The map Φ realizing the reversibility in the case CR_1 is the same as in the Poincaré example [11]. Here the fold curve forms the line $\Gamma = \{x = 0\} = \{\det(d\Phi) = 0\}$ and coincides with the curve of non-invertibility.

In the case CR_2 the curve of non-invertibility consists of two components: the invariant line $\{y = 0\}$ and the fold line $\Gamma = \{2x + y = 0\} \subset \{\det(d\Phi) = 0\}$. The vector field V has also two invariant lines given by the equation $k + lx + mx^2 = 0$.

The case CR_3 is characterized by the fold line $\Gamma = \{y = 0\}$ (one part of the curve of non-invertibility) and the invariant quadratic curve $\{xy + 2ax^2 + 2bx + 2 = 0\}$, the other component of the curve of non-invertibility. Here also the lines $k + lx + mx^2 = 0$ are invariant.

In the case CR_4 the fold curve is $\Gamma = \{2x + c = 0\}$. The other component of the curve of non-invertibility is $\{T_1 = x + y + c = 0\}$ transformed to a point; it is invariant for V. An equivalent form of the map Φ is $(T_1 x, T_1 y)$.

In the cases CR_5, CR_6, CR_7 the map realizing the invertibility is the same as in the case CR_4 with the same fold curve and the whole curve of non-invertibility. However, here the curve $T_1 = 0$ (transformed to a point) is not invariant for V.

In the case CR_5 the line $x = 0$ is invariant for V. In the case CR_6 the three lines given by the equation $ky^3 + pT_1y^2 + qT_1^2y + rT_1^3 = 0$ are invariant. In the case CR_7 the curves $x = 0$ and $k + lT_1x = 0$ are invariant.

In the cases CR_8 and CR_9 the fold curve is $\Gamma = \{2x - y + c = 0\}$ and the second component of the curve of non-invertibility $T_1 = 0$ is transformed to a point. The curve $T_1 = 0$ is not invariant for V. In the case CR_8 the curves $x = 0$ and $k + lT_1x = 0$ are invariant. In the case CR_9 the curve $ky + pT_1^2 = 0$ is invariant. An equivalent form of the map Φ is (T_1^2x, T_1y).

In the case CR_{10}, $\Gamma_\Phi = \Gamma \cup \{T_1 = 0\}$, $\Gamma = \{2x - 2y + c = 0\}$. The curve $T_1 = 0$ is not invariant for V but the curves $x = 0$ and $k + lT_1x = 0$ are. An equivalent form of the map Φ is (T_1^3x, T_1y).

In the cases CR_{11} and CR_{12}, $\Gamma_\Phi = \Gamma \cup \{T_1 = 0\}$, $\Gamma = \{x - c = 0\}$. The curve $T_1 = 0$ is not invariant for V. In the case CR_{11} the line $x = 0$ is invariant and in the case CR_{12} the two lines given by the equation $ky^2 + pT_1y + qT_1^2 = 0$ are invariant.

In the cases CR_{13} and CR_{14}, $\Gamma_\Phi = \Gamma \cup \{T_1 = 0\}$, $\Gamma = \{2x + y - c = 0\}$. The curve $T_1 = 0$ is not invariant for V. In the case CR_{13} the line $x = 0$ is invariant.

In the case CR_{15}, $\Gamma_\Phi = \Gamma \cup \{T_1 = 0\}$, $\Gamma = \{3x + y - c = 0\}$. The curve $T_1 = 0$ is invariant. So is also the curve $my + nT_1^2 = 0$.

In the case CR_{16}, $\Gamma_\Phi = \Gamma \cup \{T_2 = 0\}$, $T_2 = ax^2 + bxy + cy^2 + dx + ey + 1$, $\Gamma = \{ax^2 + bxy + cy^2 - 1 = 0\}$. The curve $T_2 = 0$ is not invariant for V but the two lines given by $qx^2 + (p - n)xy - my^2 = 0$ are.

Finally, in the case CR_{17}, $\Gamma_\Phi = \Gamma \cup \{T_1 = 0\} \cup \{y = 1\}$, $\Gamma = \{x - ay + 1 = 0\}$. Here $T_1 = 0$ is not invariant, $y = 0$ and $y = 1$ are invariant.

REMARK 2. It is interesting that the cases CR_2, CR_3, CR_4 were discovered in investigating possible forms of Darboux integrals for cubic systems. There appears an infinite series of Darboux integrals with degrees of factors going to infinity. Any such series turns out to be included in a continuous family of reversible systems.

EXAMPLES.

$$H = \frac{x^{k-\beta}(x+1)^\beta}{x^k + \sum \binom{\beta}{j} x^{k-j} + y^2} = \widetilde{H}(x, y^2),$$

$$H = \frac{x^{k-\beta}(x+1)^\beta y^2}{y^2(x^k + \sum \binom{\beta}{j} x^{k-j}) + ax + by + c} = \widetilde{H}\left(x, \frac{y^2}{ax + by + c}\right),$$

$$H = \frac{(y+x)^\alpha y^{k-\alpha}}{y^k + \binom{\alpha}{1} xy^{k-1} + \ldots + \binom{\alpha}{k} x^k + x^{k+1}P} = \widetilde{H}\left(\frac{x}{y}, xP\right),$$

where $P(x, y)$ is a linear polynomial. It is easy to check that such integrals give cubic systems (see [14]).

An example of a more complicated series in CR_3 is given in the proof of Proposition 15 in Section 5. Also some other cases contain analogous series.

3. General properties of reversible systems

Our aim is to classify all continuous semi-algebraic families of cubic systems with center which are reversible and are not integrable. (In fact, we do not need bother with DSC integrals because by their definition the center lies in the set of indefiniteness of $U = R/S$, $R = S = 0$, the first component of the map $\Psi = (U, W)$ by means of which the system can be pushed forward). More precisely, if Σ is such a semi-algebraic component of the variety $\mathcal{A}_3^R$ then it may also contain Darboux integrable systems, but a generic $V \in \Sigma$ should be non-integrable by means of a Darboux or DHE integral.

Assume that any $V(x, y) \in \Sigma$ is of the form

$$V = f \cdot \Phi_*^{-1} V' \circ \Phi,$$

where

$$\Phi(x, y) = (X, Y),$$

with X, Y rational functions,

$$V' = F(X, Y)\partial_X + G(X, Y)\partial_Y$$

is a polynomial vector field and f is a suitable rational factor. Of course, F and G are relatively prime, their greatest common factor (F, G) is 1.

Let us look more closely at the form of the function f. Assume for simplicity that X and Y are polynomials. Generally $f = \det(d\Phi)$ because along the curve $\det(d\Phi) = 0$ the vector field $\Phi_*^{-1} V' \circ \Phi$ has singularities (Φ_* is not invertible). However, it may happen that the polynomial vector field

$$\widehat{V} = \det(d\Phi) \cdot \Phi_*^{-1} V' \circ \Phi$$

has some curve $Z = 0$ of equilibrium points, so that the vector field $Z^{-d}\widehat{V}$, $d \in \mathbb{N}$, is polynomial but has a lower degree than $\widehat{V}$.

LEMMA 1. (a) *If the above happens for an irreducible Z then Z divides* $\det(d\Phi)$ *and either*

(i) *the curve $Z = 0$ is transformed by Φ to a point in $\mathbb{C}^2$ ($X = Z^\gamma X_1$, $Y = Z^\delta Y_1$ in some coordinates), or*

(ii) *there are local coordinates* $\widetilde{x}, \widetilde{y}$ *near a generic point of* $Z = 0$ *and linear coordinates* $\widetilde{X}, \widetilde{Y}$ *in the image of* Φ *such that* $\{Z = 0\} = \{\widetilde{y} = 0\}$ *and* $(\widetilde{X}, \widetilde{Y}) = (\widetilde{x}, \widetilde{y}^{k+1})$.

(b) *In the case* (i) *the curve* $Z = 0$ *may or may not be invariant for* V. *It is not invariant iff* $V' = (\gamma X \partial_X + \delta Y \partial_Y) K(X, Y) + V_1'$, *where* $K(X, Y)$ *is a quasi-homogeneous polynomial* (*with respect to the quasi-homogeneous filtration with degrees* $d(X) = \gamma$, $d(Y) = \delta$), *and* $d(V_1') > d(F_0)$ (*see below*).

(c) *In the case* (ii), $Z = 0$ *is an invariant curve for* V *and does not form a part of the fold curve* Γ.

We have $d(X^i Y^j \partial_X) = (i-1)\gamma + j\delta$, $d(X^i Y^j \partial_Y) = i\gamma + (j-1)\delta$ and $d(X^i Y^j) = i\gamma + j\delta$ in the *quasi-homogeneous filtration* with the degrees of generators (γ, δ). We shall call the degrees γ, δ the *indices*.

PROOF. (a) If $\Phi(\{Z = 0\})$ is a point then of course Z divides $\det(d\Phi)$.

If $\Phi(\{Z = 0\})$ is some curve and Φ is invertible near a generic point p of $Z = 0$ then V' vanishes along $\Phi(\{Z = 0\})$, but we have assumed that $(F, G) = 1$. If $\det(d\Phi) = Z^k(1 + \dots)$ then there are linear coordinates $\widetilde{X}, \widetilde{Y}$ such that $\widetilde{X}$ is regular near p, $\widetilde{X}(p) = 0$, and $\widetilde{Y} = Z^{k+1}(1 + \dots)$. We put $\widetilde{x} = \widetilde{X}$ and $\widetilde{y} = Z(1 + \dots)^{1/(k+1)}$.

(b) Let $X = Z^\gamma X_1$, $Y = Z^\delta Y_1$. If $Z = 0$ is not invariant for V and the trajectories ζ of V near a generic point of $Z = 0$ are given by $x = x(Z) = x_0 + \dots$ then the $\Phi(\zeta)$ are given by $Y^\gamma = cX^\delta + \dots$, phase curves of the system $\dot{X} = \gamma X + \dots$, $\dot{Y} = \delta Y + \dots$ Because V can change direction and can have points of contact with $Z = 0$ we have the factor K in (b). On the other hand, if the lowest order part of V' is $(\gamma X \partial_X + \delta Y \partial_Y) K$ then the trajectories of V cross the curve $Z = 0$.

(c) If $\Phi = (x, y^{k+1})$ then $\dot{x} = (k+1) y^k F(x, y^{k+1})$, $\dot{y} = G(x, y^{k+1})$ for $\widetilde{V}$. If the right hand sides are divisible by y then they are divisible by $y^{k+i(k+1)}$ for some i and then $\dot{x} = F'(x, y^{k+1}), \dot{y} = yG'(x, y^{k+1})$. □

REMARK 1. The assertions of Lemma 1 also hold when X or/and Y are rational.

REMARK 2. If $\Phi(\{Z = 0\}) = \{p\}$ then we can blow up the point p to a line and the composed map (now rational) transforms $Z = 0$ to a curve. We shall always do that when the curve $Z = 0$ passes through the center.

However, after resolution usually another curve is transformed to a point in $\mathbb{C}P^2$ (on the line at infinity). Notice that when we make the blowing-up $(X, Y) \to (X, Y/X)$ or $(X : Y : Z) \to (X^2 : YZ : XZ)$ then the line at infinity

$Z = 0$ goes to $(1:0:0)$. Therefore we cannot avoid the phenomenon of squeezing a curve to a point.

In the following lemma we present some situations which imply the Darboux integrability.

LEMMA 2. *In the situations listed below the systems $V(x, y)$ and $V'(X, Y)$ are Darboux integrable:*

(a) *V' is linear.*

(b) *V' is homogeneous.*

(c) *V' is quasi-homogeneous with respect to the quasi-homogeneous filtration with indices (γ, δ).*

(d) *$\dot{X} = F_1(X)F_2(Y)$, $\dot{Y} = G_1(X)G_2(Y)$ (separated variables).*

(e)

$$\dot{X} = F_0(X), \qquad \dot{Y} = G_0(X) + YG_1(X) \tag{2}$$

where $F_0 = (aX + b)^i$ and $G_1 = c(aX + b)^{i-1}$.

(f) *V' is of the form (2) and it has an invariant algebraic curve $Z' = 0$ such that $X|_{Z'=0} \neq \text{const}$. The curve $Z' = 0$ is then of the form $Y = Q(X)$, Q rational, and the first integral is of the form $e^{-\int G_1/F_0}(Y - Q(X))$. This situation happens when $\widehat{V}|_{Z=0} = 0$, where $Z = 0$ is transformed to a curve $Z' = 0$ and $X|_{Z=0} \neq \text{const}$. If $\Phi(\{Z = 0\})$ is not a point then $X|_{Z=0} = a = \text{const}$ iff $X - a = Z^2X_1$ (the latter means that $\deg \phi \geq 2 \deg Z$ or $\deg \omega \geq 2 \deg Z$ for $X = \phi/\omega$).*

(g) *$\dot{X} = F_0(X)$, $\dot{Y} = YG_1(X) + Y^nG_2(X)$ where either F_0 and G_1 are as in (e) or we have the same restrictions as in (f).*

(h) *$V' = V'_0 + V'_1$, where the V'_i are quasi-homogeneous, $d(V'_0) < d(V'_1)$ and either $V'_0 = (\gamma X\partial_X + \delta Y\partial_Y)K$ or $V'_1 = (\gamma X\partial_X + \delta Y\partial_Y)K_1$ and $U = X^\delta Y^{-\gamma} \neq \text{const}$ along $Z = 0$ (where $\widehat{V}|_{Z=0} = 0$). Here $U|_{Z=0} = a = \text{const}$ iff $X^\delta - aY^\gamma = Z^2W$. This implies that (when $Z = 0$ is not transformed to a point) either $\deg \phi > \deg Z$, $\deg \psi > \deg Z$ or $\deg \omega > \deg Z$, $\deg \eta > \deg Z$ for $X = \phi/\omega$, $Y = \psi/\eta$.*

REMARK 3. Notice that here we allow the situation when some of the μ_j are 0 or ∞. According to [12] in such situations the Darboux integral takes the form

$$g + \sum \mu_i \ln f_i,$$

with rational g, f_i or

$$e^g \prod f_i^{\mu_i}.$$

PROOF OF LEMMA 2. The statements (a), (b) and (d) are obvious.

(c) If V' is quasi-homogeneous then it is easy to check that the variables $U = X^{\delta'}Y^{-\gamma'}$, $\gamma' = \gamma/(\gamma,\delta)$, $\delta' = \delta/(\gamma,\delta)$ ((γ,δ) is the greatest common divisor) and $R = X^kY^{-l}$, $k\gamma' + l\delta' = 1$, are separating.

REMARK 4. However, we cannot expect a Darboux integral in the situation when the resolution $(S,W) \to (X,Y) = (S^\gamma, S^\delta W)$ gives a system with separated variables though the initial system is not quasi-homogeneous.

EXAMPLE.

$$\dot{X} = 2aX, \qquad \dot{Y} = aY + bX + cY^2,$$

$X = S^2$, $Y = SW$, $\dot{S} = 2a$, $\dot{W} = b + cW^2$ with

$$H = \sqrt{X} + d\ln\left(\frac{Y + \alpha\sqrt{X}}{Y - \alpha\sqrt{X}}\right)$$

(DSC and DHE integral but not Darboux).

The system (2) has a DSC integral

$$H = Ye^{-\int G_1/F_0} - \int e^{-\int G_1/F_0}\frac{G_0}{F_0}\,dX, \tag{3}$$

where $e^{-\int G_1/F_0} = \prod (X - X_i)^{a_i} e^{g(X)}$. We have to show that H is of Darboux form.

In the case (e) we can assume that $F_0 = X^i$. Then $e^{-\int G_1/F_0} = X^\lambda$, $\lambda \in \mathbb{R}$, and the second integral in (3) is of the form $X^{-\lambda-i+1}P(X)$ with polynomial P.

In the case (f) assume that the curve $Z' = 0$ is irreducible and $X|_{Z'=0} \neq$ const. The restriction of the system (3) to $Z' = 0$ gives

$$\frac{G_0}{F_0}(X) = \left(-Y\frac{G_1}{F_0} + \frac{dY}{dX}\right)\Big|_{Z'=0}$$

(we have also used the fact that $\frac{dY}{dX}|_{Z'=0} = (\frac{\partial Y}{\partial y}/\frac{\partial X}{\partial y})|_{Z'=0} = \frac{d}{dX}(Y|_{Z'=0})$). Therefore $e^{-\int G_1/F_0}G_0/F_0 = \frac{d}{dX}(e^{-\int G_1/F_0}Y|_{Z'=0})$ and $H = e^{-\int G_1/F_0}(Y - Y|_{Z'=0})$. Moreover, the branching points of the integral $\Omega = \int e^{-\int G_1/F_0}G_0/F_0$ are at $X = X_i$ and $\Omega = \text{const} + (X - x_i)^{a_i} \times$(ration. funct.) near $X = X_i$. Also from the behaviour near $g(X) = \infty$ we know that Ω contains the term e^g. Thus from $\Omega = \prod(X - X_i)^{a_i}e^g(Y|_{Z'=0})$ we see that $Z' = 0$ is rational of the form $Y = Q(X)$ with some rational Q. So H is of Darboux type.

If $X|_{Z=0} = a$ and the functions Y and Z form a coordinate system near a generic point of $Z = 0$ (here we use the assumption that $Z = 0$ is not transformed to a point), then $\partial X/\partial Z = Z \cdot X'$ and $X = a + O(Z^2)$.

In the case (g) we have the Bernoulli equation which reduces to (2).

In the case (h) the variables R, U from the proof of (c) give a system of the form (2). If $Z = 0$ is not transformed to a point and $U|_{Z=0} = a = \text{const}$ then $X|_{Z=0}$ is of the form χ^γ and $Y|_{Z=0}$ is of the form χ^δ for some rational function χ; $X = \chi^\gamma + Zv_1 + O(Z^2)$, $Y = \chi^\delta + Zv_2 + O(Z^2)$. Moreover, $\delta v_1 \chi^{-\gamma} \equiv \gamma v_2 \chi^{-\delta}$. From this the estimates for the degrees follow. □

NOTATIONS. In the proof of Theorem 1 we shall follow the notations presented below.

The map Φ is of the form

$$X = \frac{\phi}{\omega}, \qquad Y = \frac{\psi}{\eta},$$

with $\omega = \prod \omega_i^{\alpha_i}$, $\eta = \prod \eta_j^{\beta_j}$, $\phi, \psi, \omega_i, \eta_j$ polynomials. We have

$$\nabla X = M^{-1}(A, B), \qquad M = \prod \omega_i^{\alpha_i+1},$$
$$\nabla Y = N^{-1}(C, D), \qquad N = \prod \eta_j^{\beta_j+1}.$$

The vector field $V(x, y)$, which we denote by $\Phi^* V'$, is given by the formula

$$\Phi^* V' : \quad \dot{x} = [D(MF) - B(NG)]R, \quad \dot{y} = [A(NG) - C(MF)]R, \tag{4}$$

where R is a suitable rational factor.

We shall estimate the degrees of the terms $D(MF)R$, $B(NG)R$ and $A(NG)R$, $C(MF)R$. However, the highest order terms of these expressions can be cancelled in $(\dot{x}, \dot{y})$. This happens when the highest degree parts of X and Y are *functionally dependent* (e.g. $X = xy + \ldots$, $Y = (x^3y^3 + \ldots)/(xy + \ldots)$ or $\deg X = \deg Y = 0$). We shall eliminate such cases, which will allow us to estimate the distinguished terms independently. We shall estimate the degrees of the whole terms $(C, D)(MF)R$ and $(A, B)(NG)R$.

We set

$$|P| = \deg P$$

for a polynomial $P(x, y)$ and $|(W_1, \ldots, W_n)| = \max_i |W_i|$ for a rational vector-valued function $(W_1, \ldots, W_n)$. Here $|P/Q| = |P| - |Q|$ when P and Q are polynomials.

For the degrees of polynomials $K(X, Y)$ and vector fields $V_i'(X, Y)$ depending on the variables in the image we shall use the standard notation $\deg K$.

We write

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = WZ \prod T_i^{\mu_i},$$

where $\{W = 0\} = \Gamma$ is the fold curve (not invariant for V), the curve $Z = 0$ is transformed by Φ to a curve (it is invariant) and the curves $T_i = 0$ are transformed to points.

By S_j we shall denote the common irreducible factors of ω and η.

Notice that when $X = T^\gamma$, $Y = T^\delta Y_1$ then $\Phi = \Phi_2 \circ \Phi_1$, $\Phi_1 = (T, Y_1)$, $\Phi_2(x,y) = (x^\gamma, x^\delta y)$, and the center O does not belong to the curve $T = 0$ (because we have assumed that we blow up all such points $\Phi(\{T = 0\})$). So $\Phi_{2*}^{-1}V'$ has a center and V is reversible by means of Φ_1. Therefore we do not consider the cases when $X = T^\gamma$ or $Y = T^\delta$.

We denote by $d(\widetilde{V}')$ the degree of a monomial vector field $\widetilde{V}' = aX^iY^j\partial_{X,Y}$ in the quasi-homogeneous filtration. Often we shall have the situation when $V' = V_0' + V_1'$, where V_0' is quasi-homogeneous, and the monomial components $\widetilde{V}'$ of V_1' have greater degrees. In such cases we denote by

$$\Delta(\widetilde{V}') = d(\widetilde{V}') - d(V_0')$$

the difference between the degrees.

We denote by I' the degree of X in $\dot{Y}$, by I'' the degree of X in $\dot{X}$, by J' the degree of Y in $\dot{X}$ and by J'' the degree of Y in $\dot{Y}$. We also set

$$I = \max(I', I'' - 2), \qquad J = \max(J', J'' - 2).$$

By E_∞ we denote the line at infinity.

If $X = \phi/\omega$, $\omega \neq 1$, then from the form of V' we can see whether the curve $\omega = 0$ is invariant for V or not.

LEMMA 3. *If $Y = \psi/\eta$ with $(\psi, \omega) = (\eta, \omega) = 1$ then the curve $\omega = 0$ is invariant for V iff $I'' < I + 2$.*

PROOF. Near a generic point of a component $\xi = \omega_i = 0$ we can choose ξ and Y as local coordinates, $X = \xi^{-a}X_1$. Then $\dot{X} = \xi^{-a-1}(-aX_1 + \xi X_{1\xi})\dot{\xi} + \xi^{-a}X_{1Y}\dot{Y}$ and hence

$$(*) \qquad \begin{aligned} \dot{Y} &= (-aX_1 + \xi X_{1\xi})G(\xi^{-a}X_1, Y), \\ \dot{\xi} &= \xi^{a+1}F(\xi^{-a}X_1, Y) - \xi X_{1Y}G(\xi^{-a}X_1, Y) \end{aligned}$$

(after multiplication by a factor).

If $I = I' > I'' - 2$ then we have to multiply $(*)$ by $(\xi^a)^{I'}$ and then $\dot{\xi} = (\xi^a)^{I-I''+1}\xi F_1 + \xi G_1 = \xi H(\xi, Y)$ with analytic H.

If $I' \leq I'' - 2$ then we multiply $(*)$ by $\xi^{(I''-1)a-1}$ and $\dot{\xi}|_{\xi=0} \neq 0$. $\square$

4. The main part of the proof of Theorem 1

Here we present the general scheme of the proof of Theorem 1. It is divided into many subcases. Each of the subcases is solved in an appriopriate proposition. Many of the propositions are solved by estimating the degrees and by using Lemma 2. The more complicated propositions are put off to the next subsection.

We shall use the symbol $\Rightarrow$ to denote either an implication or a reference to another proposition. When the systems considered in some case belong to the list given in Lemma 2 then they are integrable and we indicate the corresponding statement of Lemma 2, e.g. "(Lemma 2(d))".

PROPOSITION 1. *The map* $\Phi=(X,Y)$, $X=\frac{\phi}{\omega}$, $Y=\frac{\psi}{\eta}$ *can be chosen so that:*

(i) $|X|\geq 0$, $|Y|\geq 0$.
(ii) $(\phi,\eta)=1$ *and* $(\psi,\omega)=1$.
(iii) X *and* Y *are functionally independent at infinity.*
(iv) *The curves* $\omega_i=0$ *and* $\eta_j=0$ *are not in the fold curve* Γ *and none of the curves* $T_i=0$ *(transformed to points) and of the curves* $S_j=0$ *(where the* S_j *are divisors of* (ω,η)*) goes through the center.*

Moreover, we have one of the following cases:

- $\dim\Phi^{-1}(\{p\})=0$, $(\omega,\eta)=1\Rightarrow$ *Proposition* 2,
- $(\phi,\psi)=T^{\gamma}$, $(\omega,\eta)=1$, $\dim\Phi^{-1}(\{p\})=0$ *for* $p\neq 0$ $\Rightarrow$ *Proposition* 16,
- $\dim\Phi(\{T_1=0\})=\dim\Phi(\{T_2=0\})=0$, $(\omega,\eta)=1$ $\Rightarrow$ *Proposition* 100,
- $\dim\Phi^{-1}(\{p\})=0$, $(\omega,\eta)=S^{\tau}$ $\Rightarrow$ *Proposition* 107,
- $(\phi,\psi)\neq 1$, $(\omega,\eta)\neq 1$ $\Rightarrow$*Proposition* 120,
- *the most general case* $\Rightarrow$ *Proposition* 125.

Often we shall use the points (f), (g) and (h) of Lemma 2 where the system reduces to (2). In such a case we write only $|Z|>0$, which will mean that the additional condition that $Y|_{Z=0}\neq$ const (or $X|_{Z=0}\neq$ const or $U|_{Z=0}\neq$ const) also holds.

No curves transformed to points.

PROPOSITION 2. *If* $\dim\Phi^{-1}(\{p\})=0$ *then* $R=\omega^I\eta^J Z^{-1}R_1$ *in* (4) *with* R_1 *a polynomial,*

$$\begin{aligned}&|Z|\leq|Z|_{\max}=|\phi|+|\psi|+\sum|\omega_i|+\sum|\eta_j|-3,\\ &3\geq|\Phi^*X^iY^j\partial_X|\geq(i-1)|X|+j|Y|+2+\nu,\\ &3\geq|\Phi^*X^iY^j\partial_Y|\geq i|X|+(j-1)|Y|+2+\nu,\\ &\nu=I|\omega|+J|\eta|+|Z|_{\max}-|Z|.\end{aligned}\tag{5}$$

We divide this case into subcases considered in Propositions 3–15. (Recall that $|X|$ denotes the degree of X.)

PROPOSITION 3. $\nu \geq 2 \Rightarrow \dot{X} = F(Y)$, $\dot{Y} = G(X)$ (*Lemma* 2(d)).

PROPOSITION 4. $\nu = 1$, $0 < |Y| = |X| \Rightarrow V'$ *linear* (*Lemma* 2(a)).

PROPOSITION 5. $\nu = 1$, $0 < |Y| \leq |X| \Rightarrow \dot{X} = kX + F_0(Y)$, $\dot{Y} = l + mY$ (*Lemma* 2(d)).

PROPOSITION 6. $\nu = 1$, $0 = |Y| < |X| \Rightarrow J|\eta| \leq 1$, $|X| \geq 2$, $\dot{X} = k + lX + mY$, $\dot{Y} = G(Y)$, *and either* $|Z| > 0$ (*Lemma* 2(f)), *or* $J = 0$, $m = 0$ (*Lemma* 2(d)).

In Propositions 7–15, $\nu = 0$, X is a polynomial, Y is a polynomial or a rational function with $J = 0$ ($\dot{X} = F(X)$ and the degree of Y in $\dot{Y}$ is ≤ 2). (Note that if X, Y are rational then $I = J = 0$, Lemma 2(d).)

PROPOSITION 7. $2 \leq |Y| \leq |X| < 2|Y| - 1 \Rightarrow V'$ *linear* (*Lemma* 2(a)).

PROPOSITION 8. $2 \leq |Y|$, $|X| > 2|Y| - 1 \Rightarrow \dot{X} = kX + F_0(Y)$, $\dot{Y} = l + mY$ (*Lemma* 2(d)).

PROPOSITION 9. $2 \leq |Y|$, $|X| = 2|Y| - 1 \Rightarrow |X| = 3$, $|Y| = 2$, *and* $\dot{X} = k + lX + mY + nY^2$, $\dot{Y} = p + qX + rY$ *and* $|Z| = 2$ (*for* $qn \neq 0$), $\deg \Phi(\{Z = 0\}) \geq 4$ (*see Lemma* 4 *in the next section*).

PROPOSITION 10. $|Y| = 1$, $|X| \geq 3 \Rightarrow \dot{X} = F_0(Y) + XF_1(Y)$, $\dot{Y} = G_0(Y)$, *and* $|Z| > 0$ (*Lemma* 2(f)).

PROPOSITION 11. $|Y| = 1$, $|X| = 2$, Y *a polynomial* $\Rightarrow V \in CR_1$ (*see the proof in the next section*).

PROPOSITION 12. $|Y| = 1$, $|X| \geq 2$, Y *rational* $\Rightarrow \dot{X} = k + lX$, $\dot{Y} = m + nX + pY + qY^2$, $|Z| > 0$ *and Darboux integrable* (*see the proof in the next section*).

PROPOSITION 13. $|Y| = 0$, $|X| \geq 2 \Rightarrow \dot{X} = k + lX$, $\dot{Y} = G(Y)$ (*Lemma* 2(d)).

PROPOSITION 14. $|X| = |Y| = 1 \Rightarrow Y$ *rational*, $\dot{X} = k + lX + mX^2$, $\dot{Y} = n + pX + qX^2 + Y(r + sX) + tY^2$ *and either* $V \in CR_2$ *or the system is Darboux integrable* (*see the proof in the next section*).

PROPOSITION 15. $|Y| = 0$, $|X| = 1 \Rightarrow \dot{X} = k + lX + mX^2$, $\dot{Y} = n + pX + (q + rX)Y + (s + tX)Y^2$, *and either* $V \in CR_3$ *or the system is Darboux integrable* (*see the proof in the next section*).

A curve transformed to a point.

PROPOSITION 16. *If* $X = T^\gamma X_1$, $Y = T^\delta Y_1$, $\dim \Phi^{-1}(\{p\}) = 0$ *for* $p \neq 0$, $X_1 = \phi_1/\omega$, $Y_1 = \psi/\eta$, $(\omega, \eta) = 1$ *then one of the two cases holds:*

- $T = 0$ *is invariant for* V $\Rightarrow$ *Proposition* 17,
- $T = 0$ *is not invariant* $\Rightarrow$ *Proposition* 44.

The curve $T = 0$ invariant.

PROPOSITION 17. $X = T^\gamma \phi_1/\omega$, $Y = T^\delta \psi_1/\eta$, $T = 0$ *invariant* $\Rightarrow$

$$R = \omega^I \eta^J T^{-\sigma} Z^{-1} R_1$$

in (4), *with* R_1 *a polynomial,*

$$\sigma = \min(\sigma_X + \delta - 1, \sigma_Y + \gamma - 1), \qquad \sigma_X = \min\{\gamma i + \delta j : X^i Y^j \in \dot{X}\},$$

$$|Z| \leq |Z|_{\max} = |T| + |\phi_1| + |\psi_1| + \sum |\omega_i| + \sum |\eta_j| - 3. \tag{6}$$

We have one of the three cases:

- $|X_1|, |Y_1| < 0$ $\Rightarrow$ *Proposition* 18,
- $|Y_1| < 0 \leq |X_1|$ $\Rightarrow$ *Proposition* 29,
- $0 \leq |X_1|, |Y_1|$ $\Rightarrow$ *Proposition* 37.

PROPOSITION 18. $X = T^\gamma X_1$, $Y = T^\delta Y_1$, $\underline{|X_1|, |Y_1|} < 0$ $\Rightarrow$

$$\begin{aligned} 3 &\geq |\Phi^* X^i Y^j \partial_X| \geq (i - I - 1)|X_1| + (j - J)|Y_1| + 2 + \Delta|T| + \nu, \\ 3 &\geq |\Phi^* X^i Y^j \partial_Y| \geq (i - I)|X_1| + (j - J - 1)|Y_1| + 2 + \Delta|T| + \nu, \end{aligned} \tag{7}$$

$$\nu = I|\phi_1| + J|\psi_1| + |Z|_{\max} - |Z|$$

for the monomial components in V'. *The right inequalities become equalities when* ω *and* η *have only simple factors* ($\alpha_i = \beta_j = 1$).

Recall that $I = \max(I', I'' - 2)$, where I', I'' are the degrees of X in $\dot{Y}, \dot{X}$ (analogously J is defined), and $\Delta(\cdot) = d(\cdot) - d(V_0')$ is the difference of degrees (with respect to the first term) in the quasi-homogeneous filtration with indices (γ, δ).

PROPOSITION 19. $|X_1| = |Y_1| \leq -2$, $|X| > 0$ $\Rightarrow$ $I, J \leq 1$,

$$\dot{X} = X^2(kX + lY + mXY), \qquad \dot{Y} = Y^2(nX + pY + qXY), \tag{8}$$

which becomes linear in the variables $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y$ (*Lemma* 2(a)).

In Propositions 20–30, $\underline{|X_1| = |Y_1| = -1}$.

PROPOSITION 20. $\gamma = \delta$, $\deg V_0' \le I + J \Rightarrow \gamma \ge 2$, $V' = V_0'$ *homogeneous* (*Lemma* 2(c)).

PROPOSITION 21. $\gamma > \delta$, $\deg V_0' \le I + J \Rightarrow X_1 = 1/x$, $Y_1 = 1/y$ *and one of the four cases holds:*

- $X^{I+2}Y^{J-2}\partial_X$ *in* $V_0' \Rightarrow$ *Proposition* 22,
- $X^I Y^{J-1}(kX\partial_X + lY\partial_Y)$ *in* $V_0' \Rightarrow$ *Proposition* 23,
- $X^{I-1}Y^J(kX\partial_X + lY\partial_Y)$ *in* $V_0' \Rightarrow$ *Proposition* 24.
- $X^{I-2}Y^{J+2}\partial_Y$ *in* $V_0' \Rightarrow V' = V_0'$ (*Lemma* 2(c)).

PROPOSITION 22. $\gamma > \delta$, $X^{I+2}Y^{J-2}\partial_X$ *in* $V_0' \Rightarrow$

$$\dot{X} = X^2(kX + lXY + mXY^2 + nY^2), \qquad \dot{Y} = Y^3(pY + qX + rXY),$$

$\Delta(X^2Y^2\partial_X) = \Delta(XY^3\partial_Y) = 2\delta - \gamma$, $\Delta(Y^4\partial_Y) = 3\delta - 2\gamma$ *and one of the six cases holds:*

(a) $2\gamma/3 \le \delta < \gamma$, $2\delta - \gamma > 1 \Rightarrow \delta \ge 2$, $l = m = n = q = r = 0$ (*Lemma* 2(a)).

(b) $\gamma/2 < \delta < 2\gamma/3 \Rightarrow \delta \ge 2$, $l = m = p = r = 0$ (*Lemma* 2(d)).

(c) $\delta \le 2\gamma/3$, $2\delta - \gamma = 1 \Rightarrow \gamma = 3$, $\delta = 2$, $l = m = r = 0$ *and either* $|T| = 1$ *and* $V \in CR_{14}$ (*subcase with* $T = 0$ *invariant,* $k = 0$ *in Theorem* 1), *or* $|T| > 1$ *and* $n = q = 0$ (*Lemma* 2(a)).

(d) $\gamma = 2\delta$, $\delta|T| > 2 \Rightarrow l = m = p = r = 0$ (*Lemma* 2(d)).

(e) $\gamma = 2\delta$, $\delta|T| = 2 \Rightarrow l = m = p = 0$ *and either* $X = T^4/x$, $Y = T^4/y$, $|T| = 1$, $V \in CR_{15}$, *or* $|Z| > 0$ (*Lemma* 2(g)).

(f) $\gamma = 2$, $\delta = 1$, $|T| = 1 \Rightarrow p = 0$, *the line* $y = 0$ *not invariant and the change* $(X', Y') = (X/Y^2, 1/Y) = (T'^2/x', T'/y')$ *with* $T' = y = 0$ *not invariant gives the case* CR_{12} (*see the proof of Proposition* 59(a)).

PROPOSITION 23. $\gamma > \delta$, $X^{I+1}Y^{J-1}\partial_X$ *in* $V_0' \Rightarrow \dot{X} = X(k + lY)$, $\dot{Y} = Y(m + nY + pY^2)$ (*Lemma* 2(d)).

PROPOSITION 24. $\gamma > \delta$, $X^I Y^J \partial_X$ *in* $V_0' \Rightarrow \dot{X} = kX$, $\dot{Y} = Y(l + mY)$ (*Lemma* 2(d)).

PROPOSITION 25. $\gamma = \delta$, $\deg V_0' = I + J + 1 \Rightarrow$ (8) (*see Proposition* 19).

PROPOSITION 26. $\gamma > \delta$, $\deg V_0' = I + J + 1 \Rightarrow$ *one of the three cases holds:*

- $X^{I+2}Y^{J-1}\partial_X$ *in* $V_0' \Rightarrow$ *Proposition* 27,
- $X^I Y^J(kX\partial_X + lY\partial_Y)$ *in* $V_0' \Rightarrow$ *Proposition* 28,
- $X^{I-1}Y^{J+2}\partial_Y$ *in* $V_0' \Rightarrow$ *Proposition* 29.

PROPOSITION 27. *$\gamma > \delta$, $X^{I+2}Y^{J-1}\partial_X$ in V_0' $\Rightarrow$ $\dot X = X^2(k+lY)$, $\dot Y = mY^3$ (Lemma 2(d)).*

PROPOSITION 28. *$\gamma > \delta$, $X^{I+1}Y^J\partial_X$ in V_0' $\Rightarrow$ $\dot X = X(kY + lX + mXY)$, $\dot Y = Y^2(n+pY)$, $\Delta(X^2\partial_X) = \gamma - \delta$ and one of the three cases holds:*

(a) *$\gamma > \delta + 1$ $\Rightarrow$ $l = 0$ (Lemma 2(d)).*

(b) *$\delta|T| > 1$ $\Rightarrow$ $m = p = 0$ (Lemma 2(d)).*

(c) *$\gamma = 2$, $\delta = 1$, $|T| = 1$ $\Rightarrow$ either $X = T^2/x$, $Y = T/y$ with the line $y = 0$ not invariant and the change $(X', Y') = (y^2/x, y/T)$ shows that $V \in CR_{12}$, or $\nu > 0$, $l = m = p = 0$ (Lemma 2(a)).*

PROPOSITION 29. *$\gamma > \delta$, $X^{I-1}Y^{J+2}\partial_Y$ in V_0' $\Rightarrow$ $\dot X = kX^2$, $\dot Y = Y(lY + mX + nXY)$ (Lemma 2(d)).*

PROPOSITION 30. *$\gamma \geq \delta$, $\deg V_0' \geq I + J + 2$ $\Rightarrow$ one of the two cases holds:*

(a) *$\gamma > \delta$, $X^{I+2}Y^J\partial_X$ in V_0' $\Rightarrow$ $V' = kX^{I+2}Y^J\partial_X$ (Lemma 2(a)).*

(b) *Otherwise $\Rightarrow$ $\dot X = kX^2$, $\dot Y = lY^2$ (Lemma 2(d)).*

In Propositions 31–34, $\underline{|Y_1| < |X_1| < 0}$.

PROPOSITION 31. *$|Y_1| < |X_1| < 0$ $\Rightarrow$ $J \leq 1$.*

PROPOSITION 32. *$J = 0$ $\Rightarrow$ $\dot X = F(X)$, $\dot Y = YX^{I-1}(m + nX) + Y^2G_2(X)$ and one of the three cases holds:*

(a) *$YX^{I-1}\partial_Y$ in V_0' $\Rightarrow$ $\dot X = X^I F_1(X)$ and $|Z| > 0$ (Lemma 2(g)).*

(b) *$YX^I\partial_Y$ in V_0' $\Rightarrow$ $m = 0$, $\dot X = X^{I+1}(k + lX)$ and either $|Z| > 0$ (Lemma 2(g)), or $|Z| = 0$, $|X| > 0$, $l = 0$ (Lemma 2(d)), or $|Z| = |X| = 0$, $G_2 = pX^I$ (Lemma 2(d)).*

(c) *$YX^{I-1}(m + nX)\partial_Y$ in V_1' $\Rightarrow$ $|Z| > 0$ (Lemma 2(g)).*

PROPOSITION 33. *$J = 1$ $\Rightarrow$ $|Y_1| = |X_1| - 1$ and one of the two cases holds:*

- *$|X_1| < -1$ $\Rightarrow$ $\dot X = X^3(k + lY), \dot Y = Y^2(mX + nY + pXY)$ (see Proposition 19).*
- *$|X_1| = -1$ $\Rightarrow$ Proposition 34.*

PROPOSITION 34. *$J = 1$, $|Y_1| = -2$, $|X_1| = -1$ $\Rightarrow$*

$$\dot X = X^3[kX^2 + Y(l + mX + nX^2)],$$
$$\dot Y = Y^2[X^2(p + qX) + Y(r + sX + tX^2 + uX^3)],$$

and one of the three cases holds:

(a) *$l^2 + p^2 \neq 0$ $\Rightarrow$ V_0' contains $X^2Y(lX\partial_X + pY\partial_Y)$, $\Delta(Y^3\partial_Y) = \delta - 2\gamma$, $\Delta(X^5\partial_X) = 2\gamma - \delta$ and either $\delta > 2\gamma$, $k = 0$, so $J = 0$, or $\delta = 2\gamma$, so $\gamma \geq 2$, $V' = V_0'$ (Lemma 2(c)), or $\delta < 2\gamma$, so $k = 0$ and $J = 0$.*

(b) $l = p = 0$, $X^3Y(mX\partial_X + qY\partial_Y)$ *in* V_0' $\Rightarrow$ $\Delta(Y^3\partial_Y) = \delta - \gamma$, $\Delta(X^5\partial_X) = \gamma - \delta$ *and either* $\gamma \geq 2$, *so* $u = 0$ ((8), *see Proposition* 19), *or* $\delta = \gamma = 1$, $|T| \geq 2$, $n = r = s = u = 0$ (*Lemma* 2(b)), *or* $\delta > \gamma = 1$, *so* $k = 0$ *and* $J = 0$.

(c) $Y^2X^2(p+qX)\partial_Y$ *not in* V_0' $\Rightarrow$ $l = m = p = q = 0$ (*Lemma* 2(d)).

PROPOSITION 35. *If* $|Y_1| < 0 \leq |X_1|$ *then the following estimates hold for the degrees of the monomial components of* V':

$$(9)\quad \begin{aligned} &3 \geq |\Phi^* X^iY^j\partial_X| \geq (i-1)|X_1| + (j-J)|Y_1| + 2 + \Delta|T| + \nu, \\ &3 \geq |\Phi^* X^iY^j\partial_Y| \geq i|X_1| + (j-J-1)|Y_1| + 2 + \Delta|T| + \nu, \end{aligned}$$

$$\nu = I|\omega| + J|\psi_1| + |Z|_{\max} - |Z|.$$

The right inequalities become equalities when $\alpha_i = \beta_j = 1$ (*see Propositions* 16 *and* 17).

This case is divided below into subcases considered in Propositions 36–42.

PROPOSITION 36. $|Y_1| \leq -2$, $|X_1| = 0$, $I = 0$ $\Rightarrow$ X *rational*, $J = 0$, $\dot{X} = F(X)$, $\dot{Y} = G(Y)$ (*Lemma* 2(d)).

PROPOSITION 37. $|Y_1| \leq -1$, $|X_1| = 0$, $I \geq 1$ $\Rightarrow$ $I|\omega| = 1$, $J = 0$, $\dot{X} = kX^2$, $\dot{Y} = lXY + Y^2(m+nX)$ (*Lemma* 2(d)).

PROPOSITION 38. $|Y_1| = -1$, $|X_1| = 0$, $I = 0$ $\Rightarrow$ $\dot{X} = F_0(Y) + XF_1(Y) + X^2F_2(Y)$, $\dot{Y} = G(Y)$, $F_0 \cdot F_2 \equiv 0$, *and either* $|Z| > 0$ (*Lemma* 2(f),(g)), *or* $\nu > 0$, $\dot{X} = F(X)$ (*Lemma* 2(d)).

PROPOSITION 39. $|Y_1| \leq -2$, $|X_1| \geq 1$, $I = 0$ $\Rightarrow$ *either* $\dot{X} = F_0(Y) + XF_1(Y)$, $\dot{Y} = G(Y)$, *and* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $F_0 = 0$, $F_1 = \mathrm{const}$, (*Lemma* 2(d)), *or* $\dot{X} = kX^2$, $\dot{Y} = G(Y)$ (*Lemma* 2(d)).

PROPOSITION 40. $|Y_1| \leq -2$, $|X_1| \geq 1$, $I \geq 1$ *one of the two cases holds:*

(a) V_0' *contains* $X^2Y^J\partial_X$ *or* $XY^{J+1}\partial_Y$ $\Rightarrow$ $\dot{X} = kX^2$, $\dot{Y} = lXY + Y^2G_1(X)$ (*Lemma* 2(d)).

(b) V_0' *contains* $X^iY^{J+2}\partial_Y$, $i \geq 1$ $\Rightarrow$ $\dot{X} = 0$ (*Lemma* 2(a)).

PROPOSITION 41. $|Y_1| = -1$, $|X_1| \geq 2$ $\Rightarrow$ $\dot{X} = F_0(Y) + XY^{J-1}(k + lY)$, $\dot{Y} = Y^J(m + nY + pY^2 + qXY^2)$ *and one of the two cases holds:*

(a) $q \neq 0$ $\Rightarrow$ $qXY^{J+2}\partial_Y$ *in* V_0', $k = l = m = n = 0$ (*Lemma* 2(d)).

(b) $q = 0$ $\Rightarrow$ *either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $k = m = 0$, $Y^J(lX\partial_X + nY\partial_Y$ *in* V_0', $F_0 = 0$ (*Lemma* 2(d)).

PROPOSITION 42. $|Y_1| = -1$, $|X_1| = 1 \Rightarrow$ *one of the six cases holds:*

(a) $X^2Y^J\partial_X$ *and* $XY^{J+1}\partial_Y$ *in* $V_0' \Rightarrow \dot{X} = kX^2$, $\dot{Y} = lXY + Y^2(m+nX)$ (*Lemma* 2(d)).

(b) $XY^{J-1}\partial_X$ *and* $Y^J\partial_Y$ *in* $V_0' \Rightarrow$

$$\dot{X} = kX + lY + mXY, \qquad \dot{Y} = Y(n + pY + qY^2),$$

and

- *if* $\gamma = \delta$ *then either* $X = Tx$, $Y = T/y$, $|T| = 1$ ($V \in CR_4$), *or* $|Z| > 0$ (*Lemma* 2(f)),
- $\gamma < \delta$ *then* $|Y| > 0$, $p = q = 0$ (*Lemma* 2(d)),
- $\gamma > \delta$ *then* $l = 0$ (*Lemma* 2(d)).

(c) $XY^J\partial_X$ *and* $Y^{J+1}\partial_Y$ *in* $V_0' \Rightarrow \dot{X} = kX$, $\dot{Y} = lY + mY^2$ (*Lemma* 2(d)).

(d) $Y^{J-2}\partial_X$ *in* $V_0' \Rightarrow \dot{X} = k + lY + mY^2$, $\dot{Y} = 0$ (*Lemma* 2(d)).

(e) $Y^{J-1}\partial_X$ *in* $V_0' \Rightarrow \dot{X} = k + lY$, $\dot{Y} = 0$ (*Lemma* 2(d)).

(f) $Y^J\partial_X$ *in* $V_0' \Rightarrow \dot{X} = k + lX$, $\dot{Y} = Y(m + nY)$ (*Lemma* 2(d)).

PROPOSITION 43. *If* $\underline{0 \le |X_1|, |Y_1|}$ *then the estimates for the degrees of the monomial components of* V' *are*

$$3 \ge |\Phi^* X^i Y^j \partial_{X,Y}| \ge i|X_1| + j|Y_1| + 2 + \nu + \Delta|T| - |X_1|, |Y_1|, \tag{10}$$

with $\nu = I|\omega| + J|\eta| + |Z|_{\max} - |Z|$. *The right inequalities become equalities iff* $\alpha_i = \beta_j = 1$.

We divide this case into subcases considered in Propositions 44–49.

PROPOSITION 44. $|X_1| = |Y_1| = 0 \Rightarrow X, Y$ *rational*, $\nu \ge 1$, $V' = V_0'$ (*Lemma* 2(c)).

PROPOSITION 45. $0 = |Y_1| < |X_1|$, $J = 0$, $\dot{Y}(0) \ne 0 \Rightarrow \dot{X} = F(X)$, $\dot{Y} = G(Y)$ (*Lemma* 2(d)).

PROPOSITION 46. $0 = |Y_1| < |X_1|$, $J = 0$, $\dot{Y}(0) = 0 \Rightarrow$ *one of the two cases holds:*

(a) $X\partial_Y$ *in* $V' \Rightarrow \dot{X} = kX$, $\dot{Y} = lX + mY + nY^2$, $|Z| > 0$ *and the system is Darboux integrable (see the proof of Proposition* 12).

(b) $\dot{X} = F(X)$, $\dot{Y} = YG_1(X) + Y^2G_2(X)$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $G_{1,2} = \mathrm{const}$ (*Lemma* 2(d)).

PROPOSITION 47. $0 = |Y_1| < |X_1|$, $J \geq 1 \Rightarrow J|\eta| \geq 1$, $\dot{X} = F_0(Y) + XF_1(Y)$, $\dot{Y} = G(Y)$ *and either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 1$, $F_1 = 0$ (*Lemma* 2(d)).

PROPOSITION 48. $1 \leq |X_1| = |Y_1| \Rightarrow \deg V_0' \leq 2$, $\deg V_1' \leq 1$ *and one of the two cases holds:*

(a) $\gamma = \delta \Rightarrow \deg V_0' \leq 1$, V' *linear* (*Lemma* 2(a)).

(b) $\gamma > \delta$, $\deg V_0' = 2 \Rightarrow \gamma = 2$, $\delta = 1$, $|X_1| = 1$,

$$\dot{X} = kX + lY^2, \qquad \dot{Y} = mY + nX, \tag{11}$$

where either $X = T^2x$, $Y = Ty$, $|T| = 1$, *which is equivalent to* $(X', Y') = (yT, y^2/x)$,

$$\dot{X}' = X'(nX' + mY'), \qquad \dot{Y}' = Y'(2nX' + pY' + qY'^2)$$

($V \in CR_7$, *see Proposition* 81(d)), *or* $|Z| > 0$, $\deg Z' \geq 4$, $\{Z' = 0\} = \Phi(\{Z = 0\})$ (*see Lemma* 4 *in the next section*).

PROPOSITION 49. $1 \leq |Y_1| < |X_1| \Rightarrow \dot{X} = F_0(Y) + XF_1(Y)$ *and one of the two cases holds:*

(a) $\dot{Y} = G(Y)$ *and either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $F_1 = 0$ (*Lemma* 2(d)).

(b) $\dot{Y}$ *depends on* $X \Rightarrow |X_1| = |Y_1| + 1$, $|Z| > 0$, $X\partial_Y$ *in* V_0', $\dot{X} = k + lY + mX + nY^2 + pY^3$, $\dot{Y} = qX + rY$, $q \neq 0$.

Here if $\gamma > \delta$ *then* $k = l = m = r = 0$, (*Lemma* 2(d)), *if* $\gamma < \delta$ *then* $n = p = 0$, (*Lemma* 2(a)), *and if* $\gamma = \delta$ *then* $k = p = 0$, $|X_1| = 2$, $|Y_1| = 1$, $|Z| > 0$, $\Phi(\{Z = 0\}) = \{Z' = 0\}$, $Z' \sim AY^3 + BX^2 + \ldots$ *invariant but* $Z' \neq AY^3 + BX^2$ (*see the proof in the next section*).

The curve $T = 0$ not invariant.

PROPOSITION 50. *If* $X = T^\gamma X_1$, $Y = T^\delta Y_1$, $T = 0$ *not invariant then*

$$R = \omega^I \eta^J T^{-\sigma-1} Z^{-1} R_1,$$

with R_1, σ, d, $\deg Z$ *the same as in Proposition* 17 (*the difference between the two* R*'s is in the power of* T). *Moreover,*

$$V' = V_0' + V_1', \qquad V_0' = (\gamma X\partial_X + \delta Y\partial_Y)K(X, Y),$$

$K(X, Y)$ *is quasi-homogeneous and* $d(V_1') > d(V_0') = d(K)$. *We have one of the three cases:*

- $|X_1|, |Y_1| < 0 \Rightarrow$ *Proposition* 51,
- $|Y_1| < 0 \le |X_1| \Rightarrow$ *Proposition* 71,
- $0 \le |X_1|, |Y_1| \Rightarrow$ *Proposition* 83.

In Propositions 51–99 the assumptions of Proposition 44 hold.

PROPOSITION 51. *If* $\underline{|X_1|, |Y_1| < 0}$ *then the components of* V_0' *are estimated as follows:*

$$3 \ge |\Phi^*(\gamma X \partial_X + \delta Y \partial_Y) X^i Y^j| \ge (i-I)|X_1| + (j-J)|Y_1| + 2 - |T| + \nu,$$

$$\nu = I|\phi_1| + J|\psi_1| + |Z|_{\max} - |Z|,$$

and the other components have the estimate

$$(12) \qquad \begin{aligned} 3 &\ge |\Phi^* X^i Y^j \partial_{X,Y}| \\ &\ge (i-I)|X_1| + (j-J)|Y_1| + 2 + (\Delta - 1)|T| + \nu - |X_1|, |Y_1|. \end{aligned}$$

The right inequalities become equalities when $\alpha_i = \beta_j = 1$. *Also, if* $X^i Y^j$ *is in* $K(X,Y)$ *then* $i \le I$, $j \le J$.

We divide this case into subcases considered in Propositions 52–70.

PROPOSITION 52. $\deg K \ge I + J \Rightarrow I = J = 0$ (*Lemma* 2(d)).

In Propositions 53–55, $\underline{|X_1| = |Y_1| \le -2}$.

PROPOSITION 53. $|X_1| = |Y_1| \le -2 \Rightarrow i + j > I + J$ *for* $X^i Y^j \partial_{X,Y}$ *in* V_1'.

PROPOSITION 54. $\deg V_1' = I + J + 1 \Rightarrow V_1'$ *quasi-homogeneous and* $|Z| > 0$ (*Lemma* 2(h)), $\nu > 0$, $V_1' = 0$ (*Lemma* 2(a)).

PROPOSITION 55. $\deg V_1' \ge I + J + 2 \Rightarrow (I + J - \deg K)(-|X_1|) \le |T| + 1$, $2|T| \le (\Delta(V_1') - 1)|T| \le -|X_1| + 1$, *so* $I + J - 1 \le \deg K \le I + J$ *and one of the three cases holds:*

(a) $K = kX^I Y^J \Rightarrow \dot X = \gamma X(k + lX)$, $\dot Y = \delta Y(k + mY)$ (*Lemma* 2(d)).

(b) $\gamma = \delta$, $\deg K = I + J - 1 \Rightarrow |X| = |Y| > 0$, (8) (*see Proposition* 19).

(c) $\gamma > \delta$, $K = kX^{I-1}Y^J \Rightarrow \dot X = \gamma X(k + lX)$, $\dot Y = \delta Y(k + mX + nY + pXY)$ *and* $|Z| > 0$ (*Lemma* 2(g)).

(d) $\gamma > \delta$, $K = kX^I Y^{J-1} \Rightarrow \dot X = \gamma X^2(k + lY)$, $\dot Y = \delta Y(kX + mXY + nY^2 + pXY^2)$, *which in the variables* $\widetilde X = 1/X, \widetilde Y = 1/Y$ *becomes*

$$\dot{\widetilde X} = \gamma \widetilde X(l + k\widetilde Y), \qquad \dot{\widetilde Y} = \delta(p + n\widetilde X + m\widetilde Y + k\widetilde Y^2),$$

a system equivalent to the system (18) *from Lemma* 6 *below. Here, if* $2\delta \leq \gamma$ *then* $n = 0$ (*Lemma* 2(d)), *if* $\delta > 2$ *or* $|T| > 1$ *or* $\nu > 0$ *then* $l = m = p = 0$ (*Lemma* 2(d)), *and if* $|X_1| = |Y_1| < -2$ *then* $K = 0$.

Therefore $X = x^3/\omega$, $Y = x^2/\eta$, $|\omega| = |\eta| = 2$. *Then* $\{\widetilde{Z} = 0\} = \widetilde{\Phi}(\{Z = 0\})$, $\widetilde{\Phi} = (\widetilde{X}, \widetilde{Y})$, *is an invariant algebraic curve of degree* ≥ 6 *and the system has a rational first integral* (*see Lemma* 6 *in the next section*).

In Propositions 56–66, $\underline{|X_1| = |Y_1| = -1}$.

PROPOSITION 56. $\gamma > \delta$, *K contains X^iY^j with* $i+j \leq I+J-3 \Rightarrow |X| > 0$, $|T| > 1$, $\Delta(V_1') = 1$, *the degrees of the components of V_1' are* $\geq I+J$, $K = kX^iY^j$, $\dot{X} = XF_1(Y) + F_0(Y)$, $\dot{Y} = G(Y)$ *and either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $F_0 = 0$ (*Lemma* 2(d)).

PROPOSITION 57. $\gamma > \delta$, $\deg K = I + J - 2 \Rightarrow \dot{X} = \gamma X(k + lY + mY^2) + nY^2$, $\dot{Y} = \delta Y(k + pY + qY^2 + rY^3)$ (*we consider the cases* $K = X^{I-2}Y^J, X^{I-1}Y^{J-1}, X^IY^{J-2}$), $\Delta(Y^2\partial_X) = 2\delta - \gamma$ *and one of the two cases holds:*

(a) $2\delta \geq \gamma + 1 \Rightarrow \delta > 1$, $l = m = p = q = r = 0$ (*Lemma* 2(d)).
(b) $2\delta \leq \gamma \Rightarrow n = 0$ (*Lemma* 2(d)).

PROPOSITION 58. $\gamma > \delta$, $K = kX^{I-1}Y^J \Rightarrow I = J = 1$,

$$\dot{X} = \gamma X(kY + lX + mXY), \qquad \dot{Y} = \delta Y(kY + nX + pY^2 + qXY + rXY^2),$$

$\Delta(X^2\partial_X) = \Delta(XY\partial_Y) = \gamma - \delta$, *and one of the six cases holds:*

(a) $(\gamma - \delta - 1)|T| \geq 2 \Rightarrow l = m = n = q = r = 0$ (*Lemma* 2(d)).
(b) $\gamma = 2$, $|T| = 1 \Rightarrow$ *either* $X = T^2/x, Y = T/y$ ($V \in CR_{11}$), *or* $\nu > 0$ *and hence* $l = m = n = q = r = 0$ (*Lemma* 2(d)).
(c) $\gamma = \delta + 1 = 3$, $|T| = 1 \Rightarrow m = q = r = 0$ *and either* $X = T^3/x$, $Y = T^2/y$ ($V \in CR_{13}$), *or* $\nu > 0$ *leading to* $l = n = 0$ (*Lemma* 2(d)).
(d) $\gamma = \delta + 1 > 3 \Rightarrow m = q = r = p = 0$ (*Lemma* 2(b)).
(e) $\gamma = 2$, $|T| > 1 \Rightarrow m = q = r = 0$, *$V_1'$ quasi-homogeneous and either* $|Z| > 0$ (*Lemma* 2(h)), *or* $\nu > 0$, $V_1' = 0$ (*Lemma* 2(a)).
(f) $\gamma = \delta + 1 = 3$, $|T| > 1 \Rightarrow m = p = q = r = 0$, *$V'$ homogeneous* (*Lemma* 2(c)).

PROPOSITION 59. $\gamma > \delta$, $K = kX^IY^{J-1} \Rightarrow$

$$\begin{aligned} \dot{X} &= \gamma X^2(kY + lX + mY^2 + nXY + pXY^2), \\ \dot{Y} &= \delta Y^2(kX + qXY + rY^2 + sXY^2), \end{aligned} \tag{13}$$

$\Delta(X^3\partial_X) = \gamma - \delta$, $\Delta(Y^4\partial_Y) = 2\delta - \gamma$, *and one of the four cases holds:*

(a) $\gamma = 2$, $\delta = 1 \Rightarrow r = 0$ *and either* $X = T^2/x$, $Y = T/y$, $|T| = 1$ ($V \in CR_{12}$), *or* $|Z| > 0$ (*Lemma* 2(g), *here* (13) *is divided by* X), *or* $\nu > 0$, $l = n = p = 0$ (*Lemma* 2(d)).

(b) $\gamma \geq 2\delta$, $\gamma \geq 3 \Rightarrow l = n = p = r = 0$ (*Lemma* 2(d)).

(c) $\gamma = 3$, $\delta = 2 \Rightarrow n = p = s = 0$ *and either* $X = T^3/x$, $Y = T^2/y$, $|T| = 1$ ($V \in CR_{14}$), *or* $\nu > 0$ *or* $|T| \geq 2$ *leading to* $m = q = 0$, $|Z| > 0$, V_1' *quasi-homogeneous* (*Lemma* 2(d)).

(d) $3 \leq \delta < \gamma < 2\delta \Rightarrow m = n = p = q = s = lr = 0$ (*Lemma* 2(d)).

PROPOSITION 60. $\gamma = \delta$, $\deg K \leq I + J - 3 \Rightarrow V' = V_0'$ (*Lemma* 2(a)).

PROPOSITION 61. $\gamma = \delta$, $\deg K = I + J - 2 \Rightarrow$ *one of the two cases holds:*

(a) $|T| = 1 \Rightarrow \gamma \geq 2$, $V' = V_0'$ (*Lemma* 2(a)).

(b) $|T| \geq 2 \Rightarrow V_1'$ *quasi-homogeneous and either* $|Z| > 0$ (*Lemma* 2(h)), *or* $\nu > 0$, $V_1' = 0$ (*Lemma* 2(a)).

PROPOSITION 62. $\gamma = \delta$, $|T| = 1$, $\deg K = I + J - 1$, $I = 0 \Rightarrow \gamma \geq 2$, $\dot{X} = X(k + lX + mY)$, $\dot{Y} = Y(k + nY)$ *and* $Y, U = X/Y$ *separating variables* (*Lemma* 2(d)).

PROPOSITION 63. $\gamma = \delta$, $|T| = 1$, $\deg K = I + J - 1$, $I, J > 0 \Rightarrow$

$$\begin{aligned} \dot{X} &= X(kX + lY) + X^2(mX + nY), \\ \dot{Y} &= Y(kX + lY) + Y^2(pX + qY), \end{aligned}$$

and either $X = T^2/x, Y = T^2/y$, $|T| = 1$ (*subcase of* CR_{16}), *or* $|Z| > 0$, V_1' *quasi-homogeneous* (*Lemma* 2(h)), *or* $\nu > 0$, $V' = V_0'$ (*Lemma* 2(a)), *or* $\gamma \geq 3$, $V' = V_0'$ (*Lemma* 2(a)).

PROPOSITION 64. $\gamma = \delta$, $|T| \geq 2$, $\deg K = I + J - 1$, $I = 0 \Rightarrow$

$$\dot{X} = X(k + lX + mY + nXY), \qquad \dot{Y} = Y(k + pY + qY^2)$$

and one of the two cases holds:

(a) $|Z| = 0 < |Z|_{\max}$ *or* $|T| > 2$ *or* $\gamma > 1 \Rightarrow n = q = 0$ *and* $Y, U = X/Y$ *separating variables* (*Lemma* 2(d)).

(b) *Otherwise* $\Rightarrow$

$$X = \frac{xy + ax + by}{cx + dy}, \qquad Y = \frac{xy + ax + by}{ex + fy},$$

$Z = y$, $\Gamma = \{x = 0\}$, $\Phi(\Gamma)$ *a point* (*see Proposition* 1(iv)).

PROPOSITION 65. *$\gamma = \delta$, $|T| \geq 2$, $\deg K = I + J - 1$, $I, J > 0$ $\Rightarrow$*

$$\dot{X} = X(kX + lY) + X^2(mX + nY) + pX^3Y,$$
$$\dot{Y} = Y(kX + lY) + Y^2(qX + rY) + sXY^3,$$

and one of the four cases holds:

(a) $|T| = 2$, $|Z| = 0 < |Z|_{\max}$, $\gamma = 1$ $\Rightarrow$ $p = s = 0$ *and either* $X = T/x, Y = T/y$ ($V \in CR_{16}$), *or* $|Z| > 0$ (*Lemma* 2(h)), *or* $\nu > 0$, $V' = V_0'$ (*Lemma* 2(a)).
(b) $|T| > 2$ $\Rightarrow$ $p = s = 0$ *end either* $|Z| > 0$ (*Lemma* 2(h)), *or* $\nu > 0$, $V' = V_0'$ (*Lemma* 2(a)).
(c) $\gamma \geq 2$ $\Rightarrow$ $V' = V_0'$ (*Lemma* 2(a)).
(d) $|T| = 2$, $\gamma = 1$, $|Z| > 0$ $\Rightarrow$ *Proposition* 64(b).

In Propositions 66–70, $\underline{|Y_1| < |X_1| < 0}$.

PROPOSITION 66. *$\deg K \leq I+J-3$ or degree of Y in $K \leq J-2$ $\Rightarrow$ $V' = V_0'$ (Lemma 2(a)).*

PROPOSITION 67. *$X^{I-2}Y^J$ in K $\Rightarrow$ $\dot{X} = F(X)$, $\dot{Y} = YG_1(X)+Y^2G_2(X)$, $\deg G_1 = 2$, $|X| = 0$, $|Y| > 0$, and either $|Z| > 0$ (Lemma 2(g)), or $\nu > 0$, $G_2 = 0$ (Lemma 2(d)).*

PROPOSITION 68. *$X^{I-1}Y^{J-1}$ in K $\Rightarrow$*

$$\dot{X} = \gamma X(k + lY + mXY + nX^2Y),$$
$$\dot{Y} = \delta Y(k + pY + qXY + rY^2 + sXY^2),$$

and one of the two cases holds:

(a) $|X| > 0$ $\Rightarrow$ $(\gamma + \delta - 1)|T| \geq 2$, $m = n = q = s = 0$ (*Lemma* 2(d)).
(b) $|X| = 0 < |Y|$ $\Rightarrow$ $\Delta(V_1') > 1$, $V' = V_0'$ (*Lemma* 2(a)).

PROPOSITION 69. *$X^{I-1}Y^J$ in K $\Rightarrow$ $|T| \geq -|X| - 1$,*

$$\dot{X} = \gamma X^I(kY + lX + mX^2 + nXY + pXY^2),$$
$$\dot{Y} = \delta X^{I-1}Y(kY + qX + rXY) + Y^3G_3(X), \tag{14}$$

$\Delta(X^{I+2}\partial_X) = 2\gamma - \delta$ and one of the five cases holds:

(a) $\gamma = \delta$ $\Rightarrow$ $l = q$, $G_3 = X^{I-1}(s + tX)$ *and after the change* $X \to 1/X + C_1$, $Y \to 1/Y + C_2$ *we get*

$$\dot{X} = \gamma[aX + bY + (kX + lY)X],$$
$$\dot{Y} = \delta[cX + dY + (kX + lY)Y],$$

where either $|Z| > 0$ (*Lemma* 2(h)), *or* $\nu > |T|$, $l = m = p = q = r = t = 0$ (*Lemma* 2(d)).

(b) $\gamma > \delta \;\Rightarrow\; l = q = n = p = r = 0$, $G_3 = sX^{I-1}$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $m = 0$ (*Lemma* 2(d)).

(c) $2\gamma \leq \delta$ *or* $\gamma \leq \delta < 2\gamma - 1$ *or* $|X_1| > |Y_1| + 1 \;\Rightarrow\; l = m = q = 0$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $G_3 = 0$ (*Lemma* 2(d)), *or* $\nu > 0$, $G_3 = sX^I$, $|Y| = 0$, $|X| > 0$, $n = p = r = 0$ (*Lemma* 2(g)).

(d) $|X_1| = |Y_1| + 1$, $\delta = 2\gamma - 1$ *and either* $\gamma \geq 3$ *or* $|T| > 1 \;\Rightarrow\; l = n = p = q = r = 0$ (*Lemma* 2(d)).

(e) $\gamma = 2$, $\delta = 3$, $|X_1| = |Y_1| + 1$, $|T| = 1 \;\Rightarrow\; l = p = q = 0$, $G_3 = 0$, *in the variables* $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y$ *we get*

$$\dot{\widetilde{X}} = 2(k\widetilde{X}^2 + m\widetilde{Y} + n\widetilde{X}), \qquad \dot{\widetilde{Y}} = 3(k\widetilde{X} + r)\widetilde{Y},$$

and either $\nu > 0$, $m = n = r = 0$ (*Lemma* 2(a)), *or* $X = x^2/y$, $Y = x^3/\eta$, $|\eta| = 2$, $|Z| = 1$, $\{\widetilde{Z} = 0\} = \widetilde{\Phi}(\{Z = 0\})$ *a cubic invariant curve* (*here* $\widetilde{\Phi} = (\widetilde{X}, \widetilde{Y})$). *By Lemma* 6 *below either* $V \in CR_{17}$ *or there is a rational first integral* (*see the proof in the next section*).

PROPOSITION 70. $X^I Y^{J-1}$ *in* K, $\gamma \neq \delta$ (*for the case* $\gamma = \delta$ *see Proposition* 69) $\Rightarrow$ $|Y_1| \geq -|T| - 1$,

$$\dot{X} = \gamma X^{I+1}(l + mX + nY + pXY),$$
$$\dot{Y} = \delta X^I Y(l + rY) + Y^3 G_3(X),$$

and one of the five cases holds:

(a) $\delta < \gamma < 2\delta \;\Rightarrow\; m = p = 0$, $G_3 = X^{I-1}(s + tX)$. *Next, if* $|Y_1| < -|T| - 1$ *then* $s = 0$ (*Lemma* 2(d)), *if* $|Y_1| = -|T| \leq -2$ *or* $\delta > 2$ *or* $|\phi_1| + |\psi_1| > 0$ *or* $|Z| < |Z|_{\max}$ *then* $n = r = t = 0$ (*Lemma* 2(d)). *So* $X = x^3/y$, $Y = x^2/\eta$, $|\eta| = 2$, $|Z| = 1$ *and in the chart* $\widetilde{\Phi} = (\widetilde{X}, \widetilde{Y}) = (1/X, 1/Y)$ *we get the system*

$$\dot{\widetilde{X}} = 3\widetilde{X}(l\widetilde{Y} + n), \qquad \dot{\widetilde{Y}} = 2(l\widetilde{Y}^2 + r\widetilde{Y} + s\widetilde{X} + t)$$

with an invariant curve $\widetilde{Z} = 0$ *of degree* ≤ 3. *Either there is a rational first integral or there is no center* (*see the proof in the next section*).

(b) $\gamma \geq 2\delta \;\Rightarrow\; m = p = 0$, $G_3 = tX^I$ (*Lemma* 2(d)).

(c) $1 < \gamma < \delta \;\Rightarrow\; m = n = p = r = 0$ (*Lemma* 2(d)).

(d) $\gamma = 1$, $\delta > 2 \;\Rightarrow\; n = p = r = 0$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $m = 0$ (*Lemma* 2(d)).

(e) $\gamma = 1$, $\delta = 2 \;\Rightarrow\; |T| > -|X_1| \geq 1$, $n = p = r = 0$ *and* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $m = 0$ (*Lemma* 2(d)).

PROPOSITION 71. *If* $\underline{|Y_1| < 0 \leq |X_1|}$ *then*

$$3 \geq |\Phi^*(\gamma X\partial_X + \delta Y\partial_Y)X^iY^j| \geq i|X_1| + (j-J)|Y_1| + 2 - |T| + \nu,$$

$$\nu = I|\omega| + J|\psi_1| + |Z|_{\max} - |Z|,$$

for terms in V_0' *and generally*

$$3 \geq |\Phi^* X^iY^j\partial_{X,Y}| \geq i|X_1| + (j-J)|Y_1| + 2 + \nu + (\Delta - 1)|T| - |X_1|, |Y_1|.$$

In particular, $I \leq 2$.

We divide this case into subcases considered in Propositions 72–82.

In Propositions 72–76, $\underline{|Y_1| < |X_1| = 0}$.

PROPOSITION 72. $I = 0$, $K = kY^j$, $j \leq J-2 \;\Rightarrow\; J = 2$, $\dot{X} = \gamma X(k + lY + mY^2) + nY + pY^2$, $\dot{Y} = G(Y)$ *and either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $n = p = 0$ (*Lemma* 2(d)).

PROPOSITION 73. $I = 0$, $K = kY^j$, $j \geq J-1 \;\Rightarrow\; \dot{X} = XF_1(Y) + kY + X^2F_2(Y), \dot{Y} = G(Y)$, $kF_2 = 0$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $k = F_2 = 0$ (*Lemma* 2(d)).

PROPOSITION 74. $I = 1$, $K = kY^j \;\Rightarrow\; \nu \geq I|\omega| \geq 1$ *and either* $\dot{X} = \gamma X(k + lY)$, $\dot{Y} = G(Y)$ $(j = J-1)$ (*Lemma* 2(d)), *or* $\dot{X} = \gamma X(k + lX)$, $\dot{Y} = \delta Y(k + mX + nY + pXY)$ *and* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu \geq 2$, $l = m = 0$ (*Lemma* 2(d)).

PROPOSITION 75. $I = 1$, K *depends on* $X \;\Rightarrow\; \nu \geq 1$,

$$\begin{aligned}\dot{X} &= \gamma X(kX + lY + mXY),\\ \dot{Y} &= \delta Y(kX + nY + pXY + qY^2 + rXY^2),\end{aligned}$$

$\Delta(Y^3\partial_Y) = 2\delta - \gamma$ *and one of the four cases holds:*

(a) $\delta < \gamma < 2\delta \;\Rightarrow\; l = n = 0$, $\delta \geq 2$, $m = p = r = 0$ (*Lemma* 2(d)).

(b) $2\delta \leq \gamma \;\Rightarrow\; l = n = q = 0$ (*Lemma* 2(d)).

(c) $\gamma < \delta \;\Rightarrow\; m = p = q = r = 0$ (*Lemma* 2(b)).

(d) $\gamma = \delta \;\Rightarrow\; n = l$ *and the system is equivalent to* (*here* $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y + C$)

$$\dot{\widetilde{X}} = \widetilde{X}(m + l\widetilde{X} + k\widetilde{Y}), \qquad \dot{\widetilde{Y}} = a\widetilde{X} + b\widetilde{Y} + (l\widetilde{X} + k\widetilde{Y}) \tag{15}$$

with $|Z| > 0$ (*Lemma* 2(h)).

PROPOSITION 76. $I \geq 2 \Rightarrow I|\omega| \geq 2$, $V' = V_0'$ (*Lemma* 2(a)).

In Propositions 77–82, $\underline{|Y_1| < 0 < |X_1|}$.

PROPOSITION 77. $I = 0$, $K = kY^j$, $j \leq J-3 \Rightarrow |T| \geq 2|Y_1|$, $\dot{X} = \gamma XK + F_0(Y)$, $\dot{Y} = \delta YK$ (*Lemma* 2(d)).

PROPOSITION 78. $I = 0$, $K = kY^{J-2} \Rightarrow$

$$\dot{X} = \gamma X(k + lY + mY^2) + nY + pY^2, \qquad \dot{Y} = \delta Y(k + qY + rY^2 + sY^3),$$

and one of the three cases holds:

(a) $\gamma = \delta = 1 \Rightarrow n = 0$ *and either* $X = Tx$, $Y = T/y$, $|T| = 1$ ($V \in CR_6$) *or* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $l = m = q = r = s = 0$ (*Lemma* 2(e)).

(b) $\delta \geq 2 \Rightarrow l = m = q = r = s = 0$ (*Lemma* 2(e)).

(c) $\gamma > \delta = 1 \Rightarrow n = p = 0$ (*Lemma* 2(d)).

PROPOSITION 79. $I = 0$, $K = kY^{J-1} \Rightarrow \dot{X} = \gamma X(k + lY) + mY$, $\dot{Y} = \delta Y(k + nY + pY^2)$ *and one of the four cases holds:*

(a) $\delta \leq \gamma \Rightarrow m = 0$ (*Lemma* 2(d)).

(b) $\gamma < \delta$, *either* $|Y_1| < -1$ *or* $|T| > 1$ *or* $|X_1| > 1 \Rightarrow$ *either* $|Z| > 0$ (*Lemma* 2(f)), *or* $\nu > 0$, $l = n = p = 0$ (*Lemma* 2(a)).

(c) $\delta > 2 \Rightarrow l = n = p = 0$ (*Lemma* 2(a)).

(d) $\delta = 2$, $\gamma = 1 \Rightarrow X = Tx$, $Y = T^2/y$, $|T| = 1$, $V \in CR_9$.

PROPOSITION 80. $I = 0$, $K = kY^J \Rightarrow \dot{X} = \gamma X(k + lX)$, $\dot{Y} = G(Y)$ (*Lemma* 2(d)).

PROPOSITION 81. $I = 1$, $K = kY^j \Rightarrow j = J = 0$,

$$\dot{X} = \gamma X(k + lX), \qquad \dot{Y} = \delta Y(k + mX + nY + pXY)$$

and one of the six cases holds:

(a) $\gamma > 1$ *or* $|X_1| > 1 \Rightarrow l = m = p = 0$ (*Lemma* 2(d)).

(b) $|Y_1| < -1$ *or* $|T| > 1$ *or* $X_1 \neq x$ *or* $Y_1 \neq 1/y \Rightarrow$ *either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $l = m = p = 0$ (*Lemma* 2(d)).

(c) $|\Phi^* Y^2 \partial_Y| = 2 + |Y_1| + (\delta - 1)|T| = \delta > 3 \Rightarrow n = p = 0$ (*Lemma* 2(d)).

(d) $X = Tx$, $Y = T/y$, $|T| = 1 \Rightarrow V \in CR_7$.

(e) $X = Tx$, $Y = T^2/y$, $|T| = 1 \Rightarrow p = 0$, $V \in CR_9$.

(f) $X = Tx$, $Y = T^3/y$, $|T| = 1 \Rightarrow p = 0$, $V \in CR_{10}$.

PROPOSITION 82. *$I = 1$, $K = kXY^j + \dots \Rightarrow$ either $j = J = 0$, $\dot{X} = \gamma k X^2$, $\dot{Y} = \delta kXY + Y^2(l + mX)$ (Lemma 2(d)), or $j = J - 1 = 0$,*

$$\begin{aligned} \dot{X} &= \gamma X(kX + lY + mXY), \\ \dot{Y} &= \delta Y(kX + nY + pXY + qY^2 + rXY^2), \end{aligned} \tag{16}$$

and one of the five cases holds:

(a) *$\gamma = \delta = 1 \Rightarrow l = n$ and either $X = Tx$, $Y = T/y$, $|T| = 1$ ($V \in CR_5$), or (16) equivalent to (15) ($\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y + C$) with $|Z| > 0$ (Lemma 2(h)), or $\nu > 0$, $m = p = r = 0$ and $X, U = Y/X$ separating variables (Lemma 2(d)).*

(b) *$\delta = \gamma + 1 = 2 \Rightarrow m = p = r = 0$, $|X_1| - |Y_1| - |T| \le 1$ and either $q = 0$ (Lemma 2(b)), or $q \ne 0$, $|Y_1| + 2|T| \le 1$ (so $|X_1| = |T| = 1$, $|Y_1| = -1$, $|\omega| = \psi_1| = 0$), and $X = Tx$, $Y = T^2/y$, $|T| = 1$ ($V \in CR_9$).*

(c) *$\gamma \ge 2\delta \Rightarrow l = n = q = 0$ (Lemma 2(d)).*

(d) *$\delta < \gamma < 2\delta \Rightarrow l = n = 0$, $\delta \ge 2$, $m = p = r = 0$ (Lemma 2(d)).*

(e) *$\delta = \gamma + 1 > 2$ or $\delta = \gamma \ge 2$ or $\delta > \gamma + 1 \Rightarrow m = p = q = r = 0$, V' homogeneous (Lemma 2(b)).*

PROPOSITION 83. *If $\underline{0 \le |X_1|, |Y_1|}$ then*

$$3 \ge |\Phi^*(\gamma X\partial_X + \delta Y\partial_Y)X^iY^i| \ge i|X_1| + j|Y_1| + 2 + \nu - |T|,$$
$$\nu = I|\omega| + J|\eta| + |Z|_{\max} - |Z|,$$

for the terms in V_0' and

$$3 \ge |\Phi^* X^iY^j\partial_{X,Y}| \ge i|X_1| + j|Y_1| + 2 + \nu + (\Delta - 1)|T| - |X_1|, |Y_1|$$

for the other terms.

We divide this case into subcases considered in Propositions 84–99.

PROPOSITION 84. *$|X_1| = |Y_1| = 0 \Rightarrow X, Y$ rational, $\nu \ge 1$, V_1' quasi-homogeneous and either $|Z| > 0$ (Lemma 2(h)), or $\nu \ge 2$, $V' = V_0'$ (Lemma 2(a)).*

In Propositions 85–90, $\underline{0 = |Y_1| < |X_1|}$, Y rational.

PROPOSITION 85. *$J = 0 \Rightarrow \dot{X} = F(X)$, $\dot{Y} = lX + YG_1(X) + Y^2G_2(X)$, $lG_2 = 0$ (if $l \ne 0$ then $\delta > \gamma$) and either $|Z| > 0$ (Lemma 2(d)), or $\nu > 0$, $F = k\gamma X$, $G_1 = k\delta$ (Lemma 2(g)).*

PROPOSITION 86. *$J=1$, $K=\mathrm{const}$ $\Rightarrow$ $\nu \ge J|\eta| \ge 1$, $\dot X = XF_1(Y)+F_0(Y)$, $\dot Y = G(Y)$ and $|Z|>0$ (Lemma 2(f)).*

PROPOSITION 87. *$J=1$, $K\ne\mathrm{const}$, $\delta = I\gamma$ $\Rightarrow$ $\dot X = \gamma X(kX^I+lY)$, $\dot Y = \delta Y(kX^I+lY+mY^2)$ and one of the two cases holds:*

(a) $\delta>1 \Rightarrow m=0$ (Lemma 2(a)).

(b) $\delta=1 \Rightarrow \gamma=1$ and $X, U=Y/X$ separating (Lemma 2(d)).

PROPOSITION 88. *$J=1$, $K\ne\mathrm{const}$, $\delta<I\gamma$ $\Rightarrow$ $\dot X=\gamma kX^{I+1}$, $\dot Y = \delta Y(kX^I+lY^2)$, V_1' quasi-homogeneous and either $|Z|>0$ (Lemma 2(h)), or $\nu\ge 2$, $l=0$ (Lemma 2(a)).*

PROPOSITION 89. *$J=1$, $K\ne\mathrm{const}$, $\delta>I\gamma$ $\Rightarrow$ $\dot X=\gamma X(kX^I+lY)$, $\dot Y=\delta Y(kX^I+mY)$, V_1' quasi-homogeneous and either $|Z|>0$ (Lemma 2(h)), or $\nu\ge 2$, $l=m=0$ (Lemma 2(a)).*

PROPOSITION 90. *$J\ge 2 \Rightarrow \nu\ge 2$, $V'=V_0'$ (Lemma 2(a)).*

In Propositions 91–99, $\underline{0<|Y_1|\le|X_1|}$.

PROPOSITION 91. *$1\le|Y_1|<|X_1|$, $K=\mathrm{const}$ $\Rightarrow$ $\dot X = XF_1(Y)+F_0(Y)$, $\dot Y=G(Y)$ and either $|Z|>0$ (Lemma 2(f)), or $\nu>0$, $F_1=k\gamma$, $G=k\delta Y$ (Lemma 2(d)).*

PROPOSITION 92. *$1\le|Y_1|<|X_1|$, $K\ne\mathrm{const}$, $\delta=I\gamma$ $\Rightarrow$ $\dot X=\gamma X(kX^I+lY)+Y^2F_0(Y)$, $\dot Y=\delta Y(kX^I+lY)$, $R=Y^{1/I}, U=X/R$ separating variables, and there is a Darboux first integral*

$$Y^{-1/I}[\gamma U(kU^I/(I+1)+l)-\Psi(Y)],$$
$$\Psi(Y)=R\int_0^R \tau^{I-2}F_0(\tau^I)\,d\tau.$$

PROPOSITION 93. *$1\le|Y_1|<|X_1|$, $K\ne\mathrm{const}$, $\delta<I\gamma$ $\Rightarrow$ $K=kX^I$, $\dot X=\gamma kX^{I+1}+Y^2F_0(Y)$, $\dot Y=\delta kYX^I$ (see Proposition 92).*

PROPOSITION 94. *$1\le|Y_1|<|X_1|$, $K\ne\mathrm{const}$, $I=0$ $\Rightarrow$ $\dot X = XF_1(Y)+F_0(Y)$, $\dot Y=G(Y)$ and either $|Z|>0$ (Lemma 2(f)), or $\nu>0$, $F_1=k\gamma$, $G=k\delta Y$ (Lemma 2(d)).*

PROPOSITION 95. *$1\le|Y_1|<|X_1|$, $K\ne\mathrm{const}$, $\delta>I\gamma>0$ $\Rightarrow$ $\dot X = \gamma X(kX^I+lY)+Y^2F_0(Y)$, $\dot Y=\delta Y(lX^I+mY)$, where the inequalities $3\ge|\Phi^*Y^2\partial_X|\ge 2+2|Y_1|-|X_1|+(2\delta-(I+1)\gamma-1)|T|\ge 2+2|Y_1|-|X_|+|T|$, $3\ge|\Phi^*X^{I+1}\partial_X|\ge 2+I|X_1|-|T|$ lead to either $F_0=0$ (Lemma 2(a)), or $I=\gamma=1$, $\delta=2$, $|Y_1|=1$, $|T|=1$, $|X_1|=2$, V' homogeneous (Lemma 2(b)).*

PROPOSITION 96. $2 \le |X_1| = |Y_1| \Rightarrow V' = V_0'$ (*Lemma* 2(a)).

PROPOSITION 97. $|X_1| = |Y_1| = 1 \Rightarrow \deg V_1' \le 2$.

PROPOSITION 98. $|X_1| = |Y_1| = 1$, $\gamma = \delta \Rightarrow$

$$\dot{X} = kX + F_2(X,Y), \qquad \dot{Y} = kY + G_2(X,Y),$$

with F_2, G_2 homogeneous quadratic polynomials. This system has an invariant line; assume that it is $X = 0$. We have one of the two cases:

(a) $\gamma = 1 \Rightarrow$ *either $X = Tx$, $Y = Ty$, $|T| = 1$, which is equivalent to $X' = X$, $Y' = X/Y$ with $T' = x = 0$ not invariant ($V \in CR_4$), or $|Z| > 0$ (Lemma 2(h)), or $\nu > 0$, $F_2 = G_2 = 0$ (Lemma 2(a)).*

(b) $\gamma \ge 2 \Rightarrow F_2 = G_2 = 0$ (*Lemma* 2(a)).

PROPOSITION 99. $|X_1| = |Y_1| = 1$, $\gamma > \delta \Rightarrow$

$$\dot{X} = \gamma X(k + lY), \qquad \dot{Y} = \delta kY + mX + nY^2$$

and one of the three cases holds:

(a) $\gamma = 2$, $\delta = 1 \Rightarrow$ *either $X = T^2x$, $Y = Ty$, $|T| = 1$ which is equivalent to $X' = Y$, $Y' = Y^2/X$ with $T' = y = 0$ not invariant, $\dot{X}' = mX' + kY' + nX'Y'$, $\dot{Y}' = 2mY' + 2(n-l)Y'^2$ ($V \in CR_9$), or $|Z| > 0$, V_1' quasi-homogeneous (Lemma 2(h)), or $\nu > 0$, $n = l = 0$ (Lemma 2(a)).*

(b) $\gamma > \delta + 1 \Rightarrow m = 0$ (*Lemma* 2(d)).

(c) $\delta < \gamma < 2\delta \Rightarrow \delta \ge 2$, $l = n = 0$ (*Lemma* 2(a)).

Two curves transformed to points.

PROPOSITION 100. $\Phi(\{T_1 = 0\}) = \{p_1\}$, $\Phi(\{T_2 = 0\}) = \{p_2\}$, $\dim \Phi^{-1}(\{p\}) = 0$, $p \ne p_{1,2} \Rightarrow$ *one of the three cases holds:*

(a) $p_1 \ne p_2$, $T_2|_{T_1=0} = \text{const} \Rightarrow T_2 = fT_1 + \mu$ *and choosing $p_1 = (0,1)$, $p_2 = (0,0)$ we have $X = gT_1T_2^\gamma$, $Y = T_2^\delta(\lambda + hT_1)$, $\lambda\mu^\delta = 1$, λ, μ parameters, f, g, h functions (possibly divided by T_1). In the limit $\lambda \to 0$ we get Proposition* 101.

(b) $p_1 \ne p_2$, $T_i|_{T_j=0} \ne \text{const} \Rightarrow$ *choosing $p_1 = (0,0)$, $p_2 = (1,0)$ we have $X = T_1\phi_1/(\lambda T_1\phi_1 + T_2\phi_2)$, $Y = T_1T_2Y_1$, $\lambda = 1$. In the limit $\lambda \to 0$ we get the situation as in Proposition* 101.

(c) $p_1 = p_2 \Rightarrow$ *Proposition* 101.

REMARK 1. In the first two situations we have taken the limit $\lambda \to 0$. But here V and V' can also depend on the parameter λ. One may suppose that for

$\lambda = 0$ the system V' becomes Darboux integrable, e.g. some of the coefficients of V' vanish. However, the form of V' is obtained by estimating the degrees of terms in V arising from monomials in V'. In the limit the degrees remain the same.

PROPOSITION 101.

$$X = T_1^{\gamma_1} T_2^{\gamma_2} X_1,\ Y = T_1^{\delta_1} T_2^{\delta_2} Y_1 \ \Rightarrow\ \det(d\Phi) = T_1^{\gamma_1+\delta_1-1} T_2^{\gamma_2+\delta_1-1}\Omega,$$

$\{T_1 = T_2 = 0\} \subset \{\Omega = 0\}$. *We consider the limit situation* $T_i \to T^{|T_i|}$, $|T| = 1$, $i = 1, 2$ (*see Remark* 1). *We have one of the two cases:*

(a) $X_1 \neq 1$, $Y_1 \neq 1$ $\Rightarrow$ *either* CR_{16} *with* $X = T_1T_2/x \to T^2/x, Y = T_1T_2/y \to T^2/y$, *or* CR_{15} (*with* $X = T_1^2T_2^2/x \to T^4/x$, $Y = T_1T_2/y \to T^2/y$, $\dot{X} = X(kX + lY^2)$, $\dot{Y} = Y^2(m + nY)$ *and either* $|Z| > 0$ (*Lemma* 2(g)), *or* $\nu > 0$, $n = 0$ (*Lemma* 2(e)), *or* CR_{13} *or* CR_{14}, *both with* $X \to T^3/x$, $Y \to T^2/y$, $V_0' = (2X\partial_X + 3Y\partial_Y)K$, $T = 0$ *invariant. The invariance property cannot hold for* $T_1 \neq T_2$.

(b) $Y_1 = 1$ $\Rightarrow$ *Proposition* 102.

PROPOSITION 102. $X_1 = \phi_1/\omega$, $Y_1 = 1$ $\Rightarrow$ *one of the two cases holds:*

(a) $|\phi_1| + |\omega| > 1$ $\Rightarrow$ *Proposition* 103.

(b) $|\phi_1| + |\omega| = 1$ $\Rightarrow$ *Proposition* 104.

PROPOSITION 103. $|\phi_1| + |\omega| > 1$ $\Rightarrow$ $\Phi_0 = \lim \Phi = (T^\gamma X_1, T^\delta) = \Phi_2 \circ \Phi_1$, $\Phi_2 = (X, Y^\delta)$, $\delta \geq 2$ (*otherwise take* $(X/Y^i, Y)$ *as new coordinates*) *and* Φ_1 *is equivalent to* $\widetilde{\Phi}_1 = (X_1, T)$. $V_0 = \lim V$ *is reversible by means of* $\widetilde{\Phi}_1$. *The line* $T = 0$ *is invariant for* V_0 (*because otherwise both curves* $T_{1,2} = 0$ *are non-invariant and* $\gamma_1/\delta_1 = \gamma_2/\delta_2$ *so we take* $T_3 = T_1^\alpha T_2^\beta$) *and in local coordinates* (z, T) *near* $T = 0$ *we have* $\dot{T} = Tf(z, T^\delta), \dot{z} = g(z, T^\delta)$, $\delta \geq 2$. *We have one of the three cases:*

(a) $V_0 \in CR_1$, $X_1 = x^2$, $T = y$, $\dot{x} = k + lx^2 + my + nx^2y + py^2 + qy^3$, $\dot{y} = 2x(r + sx^2 + ty + uy^2)$ $\Rightarrow$ $r = s = u = m = n = 0$, $\dot{X}_1 = F_0(T) + lX_1$, $\dot{T} = tT$ (*Lemma* 2(d)).

(b) $V_0 \in CR_2$, $T = x, X_1 = y^2/(x + y)$ $\Rightarrow$ $\dot{T} = lT$, $\dot{X}_1 = nX_1 + qX_1^2$ (*Lemma* 2(d)).

(c) $V \in CR_3$, $T = x$, $X_1 = y^2/(xy + ax^2 + bx + c)$ $\Rightarrow$ $\dot{T} = lT$, $\dot{X}_1 = G(X_1)$ (*Lemma* 2(d)).

PROPOSITION 104. *$|\phi_1| + |\omega| = 1$, $X = T_1^{\gamma_1}T_2^{\gamma_2}x$ or $X = T_1^{\gamma_1}T_2^{\gamma_2}/x$, $Y = T_1^{\delta_1}T_2^{\delta_2}$, $|T_i| = 1 \Rightarrow$*

(i) *We can assume that $\delta_1 > \gamma_1$ (if $\delta_i \leq \gamma_i$ then we can make γ_i smaller).*
(ii) *$(\delta_1, \delta_2) = 1$ (otherwise $\Phi = \Phi_2 \circ \Phi_1$, $\Phi_2 = (X, Y^{(\delta_1,\delta_2)})$).*
(iii) *$D_1 = \frac{|\delta_1\gamma_2 - \gamma_1\delta_2|}{(\gamma_1,\delta_1)(\gamma_2,\delta_2)} > 1$ (here $|\cdot|$ denotes the absolute value; if $D_1 \leq 1$ then $X' = (X^{\delta_1}Y^{-\gamma_1})^{1/(\gamma_1,\delta_1)} = x^{\pm a}$, $Y' = X^kY^{-l}$, $k\gamma_1 - l\delta_1 = (\gamma_1, \delta_1)$, are such that only one curve $x = 0$ is transformed to a point).*
(iv) *$\gamma_2/\delta_2 \notin \mathbb{N}$ (otherwise we take $(Y^{\gamma_2/\delta_2}/X, Y)$).*
(v) *We have two quasi-homogeneous filtrations $d_{1,2}$ with indices (γ_i, δ_i), $i = 1, 2$, with the lowest degree parts V'_{0i}. We define $\Delta_i = d_i - d_i(V'_{0i})$. If $T_i = 0$ are not invariant then $V'_{01} \neq V'_{02}$ (otherwise $D_1 = 0$, see (iii)).*

We have either $\gamma_1 + \delta_1 = 3, \gamma_2 + \delta_2 \geq 7$ or $\gamma_1 + \delta_1 \geq 4, \gamma_2 + \delta_2 \geq 5$. (The lowest degrees $\gamma_1, \delta_1, \gamma_2, \delta_2$ are $1, 2, 4, 3$ and $1, 3, 3, 2$.) In the estimates of Δ_i in the next proposition we strongly use the above properties; particularly (iii) is useful.

PROPOSITION 105. *If $X_1 = x$ then*

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq i + \Delta_1 + \Delta_2 + \Lambda + (-1, 0),$$

$\Lambda = 0$ iff $T_{1,2} = 0$ not invariant, $\Lambda = 1$ iff one of $T_i = 0$ invariant, and $\Lambda = 2$ iff both $T_{1,2} = 0$ invariant. Moreover,

$$\begin{aligned} \dot{X} &= F_0(Y) + XF_1(Y) + X^2F_2(Y) + X^3F_3(Y), \\ \dot{Y} &= G_0(Y) + XG_1(Y) + X^2G_2(Y), \end{aligned}$$

and one of the five cases holds:

(a) *$F_0(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta_1 + \Delta_2)(F_0\partial_X) + \Lambda \geq 5$.*
(b) *$F_0 = 0$, $(F_1^2 + G_0^2)(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta_1 + \Delta_2)(XF_1\partial_X + G_0\partial_Y) + \Lambda \geq 3$ and equality holds when $V'_{02} = (XF_1 + X^2F_2)\partial_X + (G_0 + XG_1)\partial_Y = (kY^{j_1} + lXY^{j_2})(\gamma_2X\partial_X + \delta_2Y\partial_Y)$, $j_1 - j_2 = \gamma_2/\delta_2$ (see (iv)).*
(c) *$F_3 = G_2 = 0$, $F_0(F_2^2 + G_1^2) \neq 0 \Rightarrow V'_{02} = (F_0 + XF_1)\partial_X + G_0\partial_Y$, $F_0 = kY^{j_0}$, $F_1 = lY^{j_1}$, $j_0 - j_1 = \gamma_2/\delta_2$ (see (iv)).*
(d) *$F_0 = F_3 = G_2 = 0 \Rightarrow F_1 = kY^{j_1}$, $F_2 = lY^{j_2}$, $G_0 = mY^{j_1+1}$, $G_1 = nY^{j_2+1}$, V' quasi-homogeneous (Lemma 2(c)).*
(e) *$F_0 = F_1 = G_0 \Rightarrow$ (c).*

PROPOSITION 106. *If* $X_1 = 1/x$ *then*

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq (I-i) + \Delta_1 + \Delta_2 + \Lambda + 1, 0,$$

with Λ *as in Proposition* 105. *Moreover,*

$$\dot{X} = XF_1(Y) + X^2F_2(Y) + X^3F_3(Y) + X^4F_4(Y),$$
$$\dot{Y} = G_0(Y) + XG_1(Y) + X^2G_2(Y),$$

and one of the four cases holds:

(a) $(F_1^2 + G_0^2)F_4 \neq 0 \Rightarrow (\Delta_1 + \Delta_2)(X^4F_4\partial_X) + \Lambda \geq 5$.

(b) $F_4 = 0$, $(F_1^2 + G_0^2)(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta_1 + \Delta_2)(X^3F_3\partial_X + X^2G_2\partial_Y) + \Lambda \geq 3$ *and equality holds when* $V_{02}' = (F_1 + XF_2)X\partial_X + (G_0 + XG_1)\partial_Y = (kY^{j_1} + lXY^{j_2})(\gamma_2 X\partial_X + \delta_2 Y\partial_Y)$, $j_1 - j_2 = \gamma_2/\delta_2$ (*see* (iv)).

(c) $F_3 = F_4 = G_2 = 0 \Rightarrow F_1 = kY^{j_1}$, $F_2lY^{j_2}$, $G_0 = mY^{j_1+1}$, $G_1 = nY^{j_2+1}$, V' *quasi-homogeneous* (*Lemma* 2(c)).

(d) $F_1 = F_4 = G_0 = 0 \Rightarrow$ *division by* X *gives one of the previous cases.*

Common denominator.

PROPOSITION 107. *If* $X = \phi/(S^\tau\omega_1) = S^{-\tau}X_1$, $Y = \psi/(S^\upsilon\eta_1) = S^{-\upsilon}Y_1$, $(\phi, \psi) = (\omega_1, \tau_1) = 1$ *then* $|X_1| \geq \tau|S|$, $|Y_1| \geq \upsilon|S|$, $|X_1| + |Y_1| > (\tau + \upsilon)|S|$ *and one of the two cases hold:*

- $S = 0$ *invariant* $\Rightarrow$ *Proposition* 108,
- $S = 0$ *not invariant* $\Rightarrow$ *Proposition* 112.

PROPOSITION 108. $X = S^{-\tau}X_1$, $Y = S^{-\upsilon}Y_1$, $S = 0$ *invariant* $\Rightarrow$

$$R = \omega_1^I \eta_1^J S^\rho Z^{-1} R_1,$$

$$\rho = \max(\rho_X - \upsilon - 1, \rho_Y - \tau - 1), \qquad \rho_X = \max\{i\tau + j\upsilon : X^iY^j \in \dot{X}\},$$

R_1 *a polynomial,*

$$|Z| \leq |Z|_{\max} = |S| + |\phi| + |\psi| + \sum |\omega_i| + \sum |\eta_j| - 3$$

and we have the following estimates for the monomial components of V':

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq i|X_1| + j|Y_1| + 2 + \nu + \Delta'|S| - |X_1|, |Y_1|,$$

$$\nu = I|\omega_1| + J|\eta_1| + |Z|_{\max} - |Z|,$$

where $\Delta'(\cdot) = d'(V_0'') - d'(\cdot) \geq 0$, d' *is the degree in the quasi-homogeneous filtration with indices* τ, υ *and* V_0'' *is the highest degree part.*

We divide this case into subcases considered in Propositions 109–111.

PROPOSITION 109. *$\nu \geq 1$, $|Y_1| < |X_1| \Rightarrow \dot{X} = kX + F_0(Y)$, $\dot{Y} = l + mY$ (Lemma 2(d)).*

PROPOSITION 110. *$|X_1| = |Y_1| \Rightarrow |X_1| > 1$, V' linear (Lemma 2(a)).*

PROPOSITION 111. *$\nu = 0, 1 < |Y_1| < |X_1| \Rightarrow \dot{X} = kX + F_0(Y)$, $\dot{Y} = l + mY$ (Lemma 2(d)).*

PROPOSITION 112. *$\nu = 0$, $1 = |Y_1| < |X_1| \Rightarrow Y = Y_1/S$, $|S| = 1$, $\tau \geq 2$, $|X_1| \geq 3$, $\dot{X} = kXY + lX + F_0(Y)$, $\dot{Y} = mY^2 + nY + p$ and $|Z| > 0$ (Lemma 2(f)).*

PROPOSITION 113. *$X = X_1 S^{-\tau}$, $Y = Y_1 S^{-\upsilon}$, $S = 0$ not invariant $\Rightarrow$*

$$\dot{X} = \tau X K + \dots, \qquad \dot{Y} = \upsilon Y K + \dots,$$

(where $K(X, Y)$ is quasi-homogeneous and the dots denote lower degree terms), R from Proposition 108 changes to R/S and

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq i|X_1| + j|Y_1| + 2 + \nu + (\Delta' - 1)|S| - |X_1|, |Y_1|.$$

We divide this case into subcases considered in Propositions 114–119.

PROPOSITION 114. *$|X_1| = |Y_1| > |S| + 1 \Rightarrow V'$ linear (Lemma 2(a)).*

PROPOSITION 115. *$|X_1| = |Y_1| = |S| + 1 \Rightarrow \dot{X} = \tau X(kX + lY) + mX + nY$, $\dot{Y} = \upsilon Y(kX + lY) + pX + qY$ (after translations of X, Y) and one of the four cases holds:*

(a) *$\tau = \upsilon \Rightarrow V' - V_0''$ quasi-homogeneous and either $|Z| > 0$ (Lemma 2(h)), or $\nu > 0$, $k = l = 0$ (Lemma 2(a)).*
(b) *$\tau > \upsilon$, $k \neq 0 \Rightarrow l = n = 0$ and either $|Z| > 0$ (Lemma 2(f)), or $\nu > 0$, $k = 0$ (Lemma 2(a)).*
(c) *$\tau \geq 2\upsilon$, $l \neq 0 \Rightarrow k = n = p = 0$ (Lemma 2(d)).*
(d) *$\upsilon < \tau < 2\upsilon$, $l \neq 0 \Rightarrow \upsilon \geq 2$, $k = n = q = 0$ (Lemma 2(d)).*

PROPOSITION 116. *$|S| < |Y_1| < |X_1| \Rightarrow \dot{X} = XF_1(Y) + F_0(Y)$, $\dot{Y} = G(Y)$ and either $|Z| > 0$ (Lemma 2(f)), or $\nu > 0$, $F_1 = k\tau$, $G = k\upsilon Y$ (Lemma 2(d)).*

PROPOSITION 117. *$|S| = |Y_1| < |X_1|$, X_1 a polynomial $\Rightarrow Y = Y_1/S$, $(X', Y') = (X, XY^{-\tau})$ (see Proposition 16).*

PROPOSITION 118. *$\nu \geq 1$, $|S| = |Y_1| < |X_1|$, X_1 rational $\Rightarrow \dot{X} = XF_1(Y) + F_0(Y)$, $\dot{Y} = G(Y)$ and either $|Z| > 0$ (Lemma 2(f)), or $\nu \geq 2$, $F_1 = k\tau$, $G = k\upsilon Y$ (Lemma 2(d)).*

PROPOSITION 119. *$\nu = 0$, $|S| = |Y_1| < |X_1|$, X_1 rational $\Rightarrow$ $I = 0$, $K = K(Y)$, $\dot{X} = XF_1(Y) + F_0(Y)$, $\dot{Y} = G(Y)$ and $|Z| > 0$ (Lemma 2(f)).*

Common denominator and a curve sent to a point.

PROPOSITION 120. *$X = \frac{T^\gamma}{S^\tau}X_1$, $Y = \frac{T^\delta}{S^\upsilon}Y_1$ $\Rightarrow$ in the limit $S \to 1$ (or $\{S = 0\} \to E_\infty$, the line at infinity), where Γ remains finite, we get the situation from Propositions 101, 102, 103. Conclusion: $X_1 = 1, x, 1/x$, $Y_1 = 1, y, 1/y$. We have one of the two cases:*

(a) *$X_1 \neq 1 \neq Y_1$ $\Rightarrow$ Proposition 121.*

(b) *$Y_1 = 1$ $\Rightarrow$ Proposition 122.*

PROPOSITION 121. *$X_1 \neq 1 \neq Y_1$ $\Rightarrow$ in the limit $S \to 1$ we get the cases CR_4–CR_{16} and considering the limits $T \to 1$ and $T \to S$ we get additional restrictions on τ, υ. We have one of the nine cases:*

(a) *$(Tx/S, Ty/S)$ $\Rightarrow$ equivalent to $(Tx/S, x/y)$.*

(b) *$(Tx/S, T^2y/S)$ $\Rightarrow$ equivalent to $(Ty/x, T^2y/S)$.*

(c) *$(Tx/S, T^2y/S^2)$ $\Rightarrow$ equivalent to $(Tx/S, x^2/y)$.*

(d) *$(Tx/S^2, T^2y/S)$ $\Rightarrow$ equivalent to $(T^3y/x, T^2y/S)$.*

(e) *$(Tx/S^2, T^2y/S^2)$ $\Rightarrow$ equivalent to $(Ty/x, T^2y/S^2)$.*

(f) *$(Tx/S, T^2/(yS))$ $\Rightarrow$ equivalent to $(x^2y/S, xy/T)$.*

(g) *$(T^3/(xS), T^2/(yS))$ $\Rightarrow$ equivalent to $(yT/x, y^2S/x^3) = (T'x'/S', T'^2y'/S'^3)$, which for $T' = 1$ does not appear in CR_4–CR_{15}.*

(h) *$(T^4/(xS), T^2/(yS))$ $\Rightarrow$ equivalent to $(y^2S/x, yS/T^2)$.*

(i) *$(T^4/(xS^2), T^2/(yS))$ $\Rightarrow$ equivalent to $(y^2/x, yS/T^2)$.*

PROPOSITION 122. *$Y_1 = 1$ $\Rightarrow$*

(i) *$X = T^\gamma S^{-\tau}x^{\pm 1}$, $Y = T^\delta S^{-\upsilon}$, $|T| = |S| = 1$ as in Proposition 104(b).*

(ii) *$\delta > \gamma$ or $\upsilon > \tau$ (otherwise we can make γ and τ smaller).*

(iii) *$\tau/\upsilon \notin \mathbb{N}$ (otherwise Φ equivalent to $(T^j x^{\pm 1}, T^\delta S^{-\upsilon})$).*

(iv) *$(\delta, \upsilon) = 1$ (otherwise $\Phi = \Phi_2 \circ \Phi_1$).*

(v) *$D_1 = (\upsilon\gamma - \tau\delta)/((\gamma, \delta)(\tau, \upsilon)) < -1$ (otherwise $X' = X^\upsilon Y^{-\tau})^{1/(\tau,\upsilon)}$, $Y' = X^k Y^{-l}$, $k\tau - l\upsilon = (\tau, \upsilon)$ of the form considered before).*

(vi) *$\delta > \upsilon > 1$ (if $\delta = \upsilon = 1$ then we take $(X/Y^i, Y)$).*

(vii) *We have two quasi-homogeneous filtrations d and d' with indices (γ, δ) and (τ, υ) and with distinguished parts V_0' (lowest) and V_0'' (highest). Define $\Delta = d - d(V_0')$ and $\Delta' = d'(V_0'') - d' \geq 0$. If neither $T = 0$ nor $S = 0$ is invariant then $V_0' \neq V_0''$ (by (v)).*

The above implies high values of the exponents (as in Proposition 104).

PROPOSITION 123. *If $X_1 = x$ then*

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq i - 1 + \Delta + \Delta' + 0, 1.$$

Moreover,

$$\dot{X} = F_0(Y) + XF_1(Y) + X^2F_2(Y) + X^3F_3(Y),$$
$$\dot{Y} = G_0(Y) + XG_1(Y) + X^2G_2(Y),$$

and one of the seven cases holds:

(a) $F_0(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta + \Delta')(X^3F_3\partial_X + X^2G_2\partial_Y) \geq 6.$

(b) $F_0 = 0$, $(F_1^2 + G_0^2)(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta + \Delta')(X^3F_3\partial_X + X^2G_2\partial_Y) \geq 4.$

(c) $F_3 = G_2 = 0$, $F_0(F_2^2 + G_1^2) \neq 0 \Rightarrow (\Delta + \Delta')(X^2F_2\partial_X + XG_1\partial_Y) \geq 4.$

(d) $F_0 = F_1 = G_0 \Rightarrow$ (e).

(e) $F_0 = F_3 = G_2 = 0 \Rightarrow F_2 = l$, $G_1 = nY$, $(\Delta + \Delta')(X^2F_2\partial_X + XG_1\partial_Y) \geq 2$ *and equality holds when* $F_1 = kY^j$, $G_0 = mY^{j+1}$ *and* V' *quasi-homogeneous* (*Lemma* 2(c)).

(f) $F_2 = F_3 = G_1 = G_2 = 0 \Rightarrow$ *one of* $T = 0$ *and* $S = 0$ *invariant,* $|\Phi^* XF_1\partial_X| \geq 3$, $|\Phi^* G_0\partial_Y| \geq 3$ *and equalities hold when* $F_1 = k$, $G_0 = lY$ (*Lemma* 2(d)).

(g) $F_1 = F_2 = F_3 = G_0 = G_1 = G_2 = 0 \Rightarrow \dot{Y} = 0$ (*Lemma* 2(a)).

PROPOSITION 124. *If $X_1 = 1/x$ then*

$$3 \geq |\Phi^* X^i Y^j \partial_{X,Y}| \geq (I - i) + \Delta + \Delta' + 1, 0.$$

Moreover,

$$\dot{X} = XF_1(Y) + X^2F_2(Y) + X^3F_3(Y) + X^4F_4(Y),$$
$$\dot{Y} = G_0(Y) + XG_1(Y) + X^2G_2(Y),$$

and one of the five cases holds:

(a) $(F_1^2 + G_0^2)F_4 \neq 0 \Rightarrow (\Delta + \Delta')(X^4F_4\partial_X) + \Lambda \geq 6.$

(b) $F_4 = 0$, $(F_1^2 + G_0^2)(F_3^2 + G_2^2) \neq 0 \Rightarrow (\Delta + \Delta')(X^3F_3\partial_X + X^2G_2\partial_Y) \geq 4.$

(c) $F_1 = G_0 = 0$, $F_4(F_2^2 + G_1^2) \neq 0 \Rightarrow$ (b).

(d) $F_3 = F_4 = G_2 = 0$, $(F_1^2 + G_0^2)(F_2^2 + G_1^2) \neq 0 \Rightarrow (\Delta + \Delta')(X^2F_2\partial_X + XG_1\partial_Y) \geq 2$ *and equality holds when* V' *quasi-homogeneous* (*Lemma* 2(c)).

(e) $F_1 = F_4 = G_0 = 0 \Rightarrow$ (d).

PROPOSITION 125. *The most general cases when (ω, η) has more than one factor and/or more than two curves are transformed to points reduce to the cases considered before by passing to the limit $T_i \to T^n$ or $S_i \to S^n$.*

5. The proofs of some propositions and additional lemmas

PROOF OF PROPOSITION 1. It is enough to show (i)–(iv).

(i) The changes $X \to 1/X$ and $Y \to 1/Y$ allow us to make $|X|, |Y| \geq 0$.

(ii) If $X = U^\alpha \phi_1/\omega, Y = \psi/(U^\beta \eta_1)$ then we define $X' = X$, $Y' = YX^i$, $i = [(\beta+1)/\alpha]$.

(iii) There are cancellations of highest degree terms in (4) in the following two situations:

(a) $|X| = |Y| = 0$,

(b) $|X| + |Y| > 0$, $X = R^m + \ldots$ and $Y = R^n + \ldots$ for some rational homogeneous R.

(Notice that these are exactly the cases when X and Y are functionally dependent at infinity.)

To show the above statement let us introduce the variables $x, u = y/x$. Then $X = x^k f(u) + \ldots$, $Y = x^l g(u) + \ldots$ Assume first that $\partial X/\partial x = X'_x \sim X'_y$ and $Y'_x \sim Y'_y$. There are cancellations of highest order terms in (4) iff $F/G \sim X'_y/Y'_y \sim X'_x/Y'_x$. This means that $|\det(d\Phi)| < |X|+|Y|-2$ or that $kfg' = lgf'$. Hence $f = \mathrm{const}\, g^{k/l}$ ($f = f_1^m$, $g = f_1^n$, $m/n = k/l$), or $k = l = 0$. If $|X'_y| < |X'_x|$ and there are cancellations then also $|Y'_y| < |Y'_x|$ and $X = y^m + \ldots, Y = y^n + \ldots$

In order to make X and Y independent at infinity we make the following changes. In the case (a) $C = \Phi(E_\infty) = \{P(X,Y) = 0\}$ is a rational curve (rationally isomorphic to $\mathbb{C}P^1$) whose field of rational functions $\mathbb{C}(C)$ is isomorphic to $\mathbb{C}(z)$. On the other hand, $\mathbb{C}(C)$ is the field of quotients of its structural ring $\mathbb{C}[X,Y]/(P)$. We choose two functions 0 and z from $\mathbb{C}(C)$. They are represented by some rational functions $X_1 = P(X,Y)$ and $X_2 = Q(X,Y)$. The map $\Phi_1 = (X_1, Y_1)$ is invertible because we can express X and Y as functions of X_1 and Y_1: $X|_C = w(z)$ so $X = w(Q) + PX_2$ and we do the same with X_2 etc. The map $\Phi' = \Phi_1 \circ \Phi = (X', Y')$ has the property that X' and Y' are independent at infinity.

In the case (b) we do the same but with the curve $\mathbb{C}P^1 \ni (R:1) \to (R^m, R^n)$.

(iv) To ensure the last property we make the blowing-up (in the image) when the center O belongs to the curve $(\phi, \psi) = 0$ or to the curve $(\omega, \eta) = 0$ and we apply the change $X \to 1/X$ when $\Gamma \subset \{\omega = 0 \neq \eta\}$. Next we apply the same transformations as in (i) and (ii). $\square$

PROOF OF PROPOSITION 2. We have $AD - BC = ZW$, where $\{W = 0\} = \Gamma$ and Z is the factor by which we divide $\widehat{V}$. Hence $|Z| \leq |Z|_{\max} = |(AD, BC)| - 1$. Because we classify cubic systems we have the bound 3 for the degrees of the corresponding components in (5). $\square$

LEMMA 4. *Consider the quadratic system*

$$\dot{x} = a + bx + cy + dy^2, \qquad \dot{y} = ex + fy. \tag{17}$$

If it has an invariant algebraic curve $Z(x, y) = 0$ of degree ≥ 3 and not containing any of the coordinate axes then either the system is Darboux integrable or $Z = 2y^3 - 3x^2$, $\dot{x} = 3\lambda x + y^2$, $\dot{y} = 2\lambda y + x$.

PROOF. If $d = 0$ then the system is linear (Lemma 2(a)), and if $e = 0$ then we can apply the proof of Lemma 2(f) (we have a DSC integral). Assume then that these parameters are non-zero and after rescaling we can put $d = e = 1$.

The system (17) has ≤ 3 critical points in $\mathbb{C}P^2$: two finite points p_1, p_2 and $p_3 = (1 : 0 : 0)$ at infinity. If $p_1 \neq p_2$ are real then one of them, say p_1, is a saddle and p_2 is an anti-saddle (node, focus or center). If $p_1 = p_2$ then it is either a saddle-node (for $p_1 = (0, 0)$, $\dot{x} = b(x + fy) + y^2, \dot{y} = x + fy$, $b + f \neq 0$), or a Bogdanov-Takens singularity ($b + f = 0$).

The point p_3 has a more complex structure. In the projective coordinates $z = 1/x, u = y/x$ we get

$$\begin{aligned}
\dot{u} &= z - u^3 + (b - f)uz - cu^2 z - auz^2, \\
\dot{z} &= -z(u^2 + bz + cuz + az^2).
\end{aligned}$$

Here we should apply the desingularization procedure. But instead of doing this we notice that when z is of order u^3 then the leading part is $\dot{u} = z - u^3, \dot{z} = -zu^2$. Putting $w = u^3$ we obtain $\dot{w} = 3(z - w), \dot{z} = -z$, a $3 : 1$ resonant sink. This shows that p_3 is a sink with separatrices $z = 0$ and $f_0 = 0$, $f_0 \sim 2u^3 - 3z + \dots$ Other trajectories near p_3 are either all analytic or all non-analytic (depending on the existense of resonant terms in the normal form of the $3 : 1$ resonant sink appearing in the desingularization of p_3). Their equations are $f_0 + Cz^3 + \dots = 0$ or $2y^3 - 3x^2 + C + \dots = 0$.

Therefore

$$Z = \prod_{i=1}^{r} (2y^3 - 3x^2 + C_i + o(1))(1 + o(1))$$

near infinity. On the other hand,

$$\dot{Z} = (k + lx + my)Z.$$

But

$$\begin{aligned}
\dot{Z} = \sum [&6x^2(x + fy) - 6x(y^2 + bx + cY + a) \\
&+ O(1)] \prod_{j \neq i} (2y^3 - 3x^2 + C_i + o(1)) \\
&+ \prod (2y^3 - 3x^2 + C_i + o(1)) O(1).
\end{aligned}$$

We see then that $l=m=0$, $c=a=0$, $f/b=2/3$, i.e. $\dot{x}=3\lambda x+y^2, \dot{y}=2\lambda y+x$ with invariant curve $2y^3=3x^2$.

Notice that in the invertible variables $X=y^3x^{-2}$, $Y=y/x$ we get $\dot{X}=X(3-2X)$, $\dot{Y}=Y(1-X)-\lambda Y^2$ with DSC integral

$$X^{-1}Y^{-1/3}(2Y-3)^{1/6}-\lambda\int^Y u^{-4/3}(2u-3)^{-5/6}du$$

(not Darboux or DHE for $\lambda\neq 0$). Near any singular point there is a local meromorphic first integral but there is no such global integral. □

PROOF OF PROPOSITION 11. If $|X|=2$, $|Y|=1$ and X and Y are polynomials then $Y=y$, $X=ax^2+bxy+cy^2+dx+ey+f=ax_1^2+P(Y)$ is equivalent to $X=x^2$, $Y=y$. Moreover,

$$\dot{X}=k+lX+mY+nY^2+pXY+qY^3,$$
$$\dot{Y}=r+sX+tY+uY^2,$$

$$CR_1: \quad \begin{aligned} \dot{x}&=k+lx^2+my+ny^2+px^2y+qy^3,\\ \dot{y}&=2x(r+sx^2+ty+qy^2). \end{aligned}$$

It remains to show that the latter system has a center and that it is not Darboux integrable and not DHE integrable.

The fold curve is $\Gamma=\{x=0\}$ and its image is $\Gamma'=\{X=0\}$. The tangency condition for V' means that $\dot{X}|_{X=0}=F(0,Y)$ changes sign in such a way that $F(0,Y_0)=0$, $F'_Y(0,Y_0)G(0,Y_0)<0$. We see that this is possible; to ensure it we must add some inequalities on the coefficients.

To show the non-integrability we put $s=0$ and get $\frac{dY}{dX}=\frac{F_0(Y)+XF_1(Y)}{G(Y)}$ with arbitrary quadratic G. The latter equation has a DSC integral which generally is not of Darboux or DHE type. □

PROOF OF PROPOSITION 12. The formula $\dot{X}=k+lX$, $\dot{Y}=m+nX+pY+qY^2$ follows from (5). V has two invariant algebraic curves $X=-k/l$ and $Z=0$ both of degree 2 (if $q\neq 0$). In the image we also have the invariant curves $X=-k/l$ and $\{Z'=0\}=\Phi(\{Z=0\})$ of degree ≥ 4. We can assume that $n\neq 0\neq q$ (otherwise we can apply Lemma 2(d) or Lemma 2(a)).

Let us look at the behaviour of the curve $Z'=0$ near infinity. There are two singular points at infinity: $p_1=(0:1:0)$, $p_2=(1:0:0)$. The point p_1 is a $1:1$ integrable node with trajectories of the form $X\sim C_1+C_2/Y$ as $Y\to\infty$.

The point p_2 is very complicated and no algebraic trajectory goes through it; one can check this trying $Y \sim CX^{\alpha}$ and $Y \to C$ as $X \to \infty$.

So, $\{Z' = 0\} \cap E_{\infty} = \{p_1\}$ or $Z' = X^i + \ldots$ But then any local component of $Z' = 0$ is of the form $Y \sim AX^{\beta}$, $\beta > 0$, which contradicts the behaviour near p_1. Therefore such a Z' does not exist. $\square$

PROOF OF PROPOSITIONS 14 AND 15. By the assumptions of Proposition 14, $|X| = |Y| = 1$, $\nu = 0$ and $Y = \psi/\eta$ is rational, so we can put $X = x$. Then $J = 0$,

$$\dot{X} = k + lX + mX^2, \qquad \dot{Y} = n + pX + qX^2 + Y(r + sX) + tY^2$$

and $|Z| \geq 2|\eta| - 1 > 0$ (otherwise V' is linear).

Under the assumptions of Proposition 15, $|Y| = 0$, $|X| = 1$, $\nu = 2$ we have $X = x$, $Y = \psi/\eta$ and from (5) we get

$$\dot{X} = k + lX + mX^2, \qquad \dot{Y} = n + pX + Y(q + rX) + Y^2(s + tX).$$

If $|\psi| = |\eta| = 1$ then Φ is invertible, so $|\psi| = |\eta| \geq 2$ and $|Z| \geq 2$.

In both cases V' forms a Riccati system with respect to Y,

$$(*) \qquad \frac{dY}{dX} = A(X) + B(X)Y + C(X)Y^2,$$

with A, B, C rational. We know that there is some invariant algebraic curve $\{Z' = 0\} = \Phi(\{Z = 0\})$, where $\{Z' = 0\} \not\subset \{M(X) = 0\}$.

LEMMA 5. *If the equation $(*)$ has an invariant algebraic curve $Z' = 0$ not parallel to the OY-axis then we have one of the three cases:*

(i) *$(*)$ has a DSC integral if $Z' = Q(X)Y - P(X)$ and there is no other invariant algebraic curve not parallel to the OY-axis;*

(ii) *$(*)$ has a Darboux integral if there are two invariant curves $Y = R_1(X)$ and $Y = R_2(X)$;*

(iii) *$(*)$ has a DHE integral if $Z' = P(X)Y^2 + Q(X)Y + T(X)$ and Z' is irreducible.*

(iv) *$(*)$ has a rational integral otherwise.*

PROOF. Let us choose a branch $Y = \Pi_1(X)$ of the curve $Z' = 0$. Then we get the following equation for $Y_1 = Y - \Pi_1(X)$:

$$\frac{dY_1}{dX} = (B + 2C\Pi_1)Y_1 + CY_1^2 = B_1Y_1 + CY_1^2$$

with first integral

$$H = Y_1^{-1} e^{\int B_1} + \int e^{\int B_1} C.$$

If $\{Z' = 0\} = \{Y = \Pi_1(X)\}$ with Π_1 rational and there are no other invariant algebraic curves then we have (i).

If $\{Z' = 0\} = \{Y = \Pi_1(X)\}$ with Π_1 rational and there is another invariant algebraic curve $Z'' = 0$ then we can apply Lemma 2(f). We find that H is of Darboux type and $\{Z'' = 0\} = \{Y = R_2(X)\}$ with R_2 rational.

Assume now that Z' is irreducible and has degree ≥ 2 with respect to Y. We can repeat the proof of Lemma 2(f) for the integral H and another branch $Y = \Pi_i(X)$ of the curve $Z' = 0$. We obtain

$$H = e^{\int B_1} \left(\frac{1}{Y_1} - \frac{1}{\Pi_j - \Pi_1} \right) = (\Pi_1 - \Pi_i)^{-1} e^{\int B_1} \frac{Y - \Pi_j}{Y - \Pi_1}.$$

Consider the discriminant locus of the algebraic function $Y(X)$, the points X where $\Pi_i(X) = \Pi_j(X)$. Then branching around such points means that Π_i is transformed to Π_k. On the other hand, the monodromy group of H forms a subgroup of the Möbius group $PSL(2, \mathbb{C})$. If the degree of Z' with respect to Y is greater than 2 then this monodromy group is finite and H is rational.

Let the degree of Y in Z' be 2. After possibly changing $Y \to Y + U(X)$ we can assume that $\{Z' = 0\} = \{Y^2 = R(X)\}$. Then $Y = \sqrt{R(X)}$ and $R'/(2\sqrt{R}) = A + B\sqrt{R} + CR$. So, $B = R'/(2R)$, $A = -CR$ and

$$H = \sqrt{R} e^{2\int C\sqrt{R}} / Y_1 + \int C\sqrt{R} e^{2\int C\sqrt{R}} = (-1/2) e^{2\int C\sqrt{R}} \frac{Y + \sqrt{R}}{Y - \sqrt{R}}.$$

This is a DHE integral which (generally) cannot be simplified. □

Let us come back to the map $\Phi = (x, \psi/\eta)$. We have $Y_y = \eta^{-2} WZ$, where $W(x, y)$ is a linear function and by Lemma 5, $Y|_{Z=0}$ is a rational function of $X = x$. Let us treat $Y = Y(y)$ as a function of y (with x as a parameter). Then all the critical values of $Y(y)$ corresponding to the critical points defined by $Z(y) = 0$ are equal. We can assume that this critical value is zero (change $Y \to Y + Q(x)$), and then we have

$$Y(y) = \frac{Z^2(y)\psi_1(y)}{\eta(y)}, \qquad Y_y = \frac{Z(2Z_y\psi_1\eta + Z\psi_{1y}\eta - Z\psi_1\eta_y)}{\eta^2}.$$

We know that $\lim_{y\to\infty} Y(y)$ is not a constant (depending on x) and hence the degree of y in $2Z_y\psi_1\eta + Z\psi_{1y}\eta - Z\psi_1\eta_y$ is $\deg Z(y) + \deg \psi_1(y) + \deg \eta(y) - 1 = \deg W(y) \leq 1$. We have three possibilities:

(a) $\deg Z(y) = 0$,

(b) $\deg \eta(y) = 0$,

(c) $\deg \psi_1(y) = 0$.

In the case (a), Φ is equivalent to $(x, \psi_1/\eta)$, with ψ_1, η linear in y, or (better) to $(x, \psi_2(x)/\eta(x,y)) \sim (x, \eta) \sim (x, y)$.

The possibility (b) means that Φ is equivalent to $(x, \psi(x,y))$ considered before (see Propositions 10 and 11).

Consider the case (c). We transform Y in such a way that the critical value of $Y(y)$ corresponding to $W(y) = 0$ is zero. Then $Y = W^2/\eta$.

Let the assumptions of Proposition 14 hold. Because $|W| = |\eta| = 1$ we have $|\eta| = |Z| = 1$. We can put $Z = y$ and $X = x$, $Y = y^2/(x+y)$, $W = 2x + y$. It is the case CR_2. The initial system reads

$$CR_2: \qquad \begin{aligned} \dot{x} &= (k + lx + mx^2)(2x + y), \\ \dot{y} &= y[(n + px)(x + y) + qy^2 - (k + lx + mx^2)]. \end{aligned}$$

V' has a true DSC integral, not Darboux or DHE, because all the parameters are arbitrary. Next, $\Gamma = \{2x+y = 0\}$, $\Gamma' = \{Y = -4X\}$ and $(Y+4X)\dot{}|_{Y=-4X} = A + BX + CX^2$. So some trajectory of V' can be tangent to Γ' from the outside and we have a center.

Let the assumptions of Proposition 15 hold. Here we put $W = y$, $X = x$, $Y = y^2/(xy + ax^2 + bx + c)$, $c \neq 0$ (otherwise we have the case CR_2). In the sequel we assume that $c = 1$. We obtain the case

$$CR_3: \qquad \dot{x} = (k + lx + mx^2)y,$$

$$\dot{y} = [(n + px)\eta^2 + (q + rx)y^2\eta + (s + tx)y^4 + y^2(y + 2ax + b)(k + lx + mx^2)]/Z,$$

$Z = xy + 2ax^2 + 2bx + 2$. We have $\eta|_{Z=0} = -\alpha(x) = -(ax^2 + bx + 1)$, $y|_{Z=0} = -2\alpha(x)/x$, $(y + 2ax + b)|_{Z=0} = -b - 2/x$. We obtain the condition $16(s + tx)\alpha^2(x) - 4(q + rx)x^2\alpha(x) + (n + px)x^4 + 4x(bx + 2a)(k + lx + mx^2) \equiv 0$. The constant term on the left hand side of this equation is $16s$. Hence $s = 0$ and one can easily express the parameters n, p, q, r, t as functions of a, b, c, k, l, m. The formulas are given in Theorem 1 (the case CR_3).

Now we have to show that the systems from CR_3 are not Darboux or DHE integrable and have a center. To prove the first property it is enough to show that CR_3 contains a series of systems with Darboux integrals $\prod f_j^{\alpha_j}$ with $\deg f_j$ going to infinity. Such integrals are given by

$$H = \frac{x^{k-\beta}(x+1)^\beta (xy + dy + 1)^2}{y^2 P(x) + yQ(x) + R(x)},$$

where

$$\begin{aligned} P(x) &= (x+d)^2 S(x) + \binom{\beta}{k+1} x + \binom{\beta}{k+2}, \\ Q(x) &= 2(x+d)S(x) + 2\binom{\beta}{k+1}, \\ R(x) &= S(x) + d', \\ S(x) &= \sum_{i=0}^{k} \binom{\beta}{i} x^{k-i}, \end{aligned}$$

with β, d, d' real constants, $k \in \mathbb{N}$. It is not difficult to check that H is an integral of a cubic system and that H can be represented as a function of

$$X = x \qquad \text{and} \qquad Y = \frac{(xy+dy+1)^2}{xy^2+\mu y^2+2y+\nu}$$

for some constants μ and ν. The map (X, Y) is equivalent to the map from CR_3. Indeed,

$$Y_y = 2(xy^2+\mu y^2+2y+\nu)^{-2}(xy+dy+1)(\nu x+(d-\mu)y+d\nu-1)$$

has the curve of non-invertibility consisting of two components $\Gamma = \{\nu x + (d-\mu)y + d\nu - 1 = 0\}$ (the fold curve) and $Z = xy + dy + 1 = 0$. The equivalence is realized in a series of transformations:

1) $x \to A(x+d)$, $y \to By$ gives $Y = \dfrac{(xy+1)^2}{xy^2+y^2+2y+\beta}$,
2) $Y \to Y' = x - Y$,
3) $Y' \to Y'' = Y'/(1-Y') = \dfrac{xy^2+\beta x-1}{y^2+2xy-\beta x+\beta+1}$,
4) $Y''' = x - Y'' = \dfrac{\beta x^2-2xy-x-1}{y^2+2y-\beta x+\beta+1}$,
5) $Y^{(\mathrm{iv})} = 1/Y''' + \beta = \dfrac{(y-\beta x+1)^2}{\beta x^2-2xy-x-1}$.

REMARK 2. Exactly in this way the third reversible case was discovered. One can also notice that when one applies the inverse transformation to the one described above then one obtains the Bernoulli system with a DSC integral. So, the case CR_3 is DSC integrable.

The existence of center is proved easily: $\Gamma = \{y = 0\}$, $\Gamma' = \{Y = 0\}$ and $\dot{Y}|_{Y=0} = n + pX$, where n and p are arbitrary. □

LEMMA 6. *Consider the quadratic system*

$$\dot{x} = bx + cy + 2dx^2, \qquad \dot{y} = ay + 3dxy, \qquad d \neq 0. \tag{18}$$

If it has an invariant algebraic curve not containing $y = 0$ then either it has a rational first integral, or $a = 3b \neq 0 \neq c$ and

$$Z = bd^2x^3 - c^2dy^2 + 3bcdxy + b^2cy = 0.$$

In the latter case the system is not Darboux or DHE integrable.

PROOF. If $c = 0$ then Lemma 2(d) works, so let $c \neq 0$. By coordinate changes we can assume that $d = c = 1$.

The crucial point of the proof is the existence of a meromorphic first integral near E_∞, the line at infinity. We seek it in the form

$$H = x^6y^{-4} + \sum_{i+i<2} a_{ij}x^iy^j = H_2 + H_1 + \dots ,$$

with H_j homogeneous of degree j. The above series is convergent as $(x, y) \to E_\infty$ along a generic ray. From $\dot{H} = 0$ we get the following system of equations:

$$\frac{\partial H_j}{\partial x}2x^2 + \frac{\partial H_j}{\partial y}3xy + \frac{\partial H_{j+1}}{\partial x}(bx + y) + \frac{\partial H_{j+1}}{\partial y}ay = 0,$$

or

$$(2i + 3j)a_{ij} + [(i + 1)b + ja]a_{i+1,j} + (i + 2)a_{i+2,j-1} = 0,$$

from which we can calculate a_{ij}'s inductively. The only obstacle is that $2i + 3j$ could vanish. But this does not happen.

In order to see this let us find what pairs of indices (i, j) appear in the expansions of H_k's. We have

$$\begin{aligned} &H_1 : \ (5, -4), \ (4, -3); \\ &H_0 : \ (4, -4), \ (3, -3), \ (2, -2); \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &H_{-m} : \ (4 - m, -4), \dots , (2 - 2m, -2 + m) \end{aligned}$$

We see that $i/j \neq -3/2$ for $H_1, \dots , H_{-3}$, and for H_{-m}, $m \geq 4$, if $i/j < 0$ then $i/j \leq -2(m - 1)/(m - 2) \leq -2$.

The existence of a meromorphic first integral near E_∞ can also be seen from the analysis of the system (18) in that domain. The system (18) has at most 5 singular points in $\mathbb{C}P^2$: two finite singular points p_1, p_2 on the invariant line $y = 0$, the finite real point $p_3 : x = -a/3$, $y = (3ab - 2a^2)/9$, and two singular points at infinity: $p_4 = (0 : 1 : 0)$ and $p_5 = (1 : 0 : 0)$.

To study the point p_4 we use the variables $z = 1/y$, $w = x/y$:

$$\dot{z} = -3zw - az^2, \qquad \dot{w} = -w^2 + z + bzw - azw^2.$$

The linear part is nilpotent but assuming $z \sim w^2$ we find two real separatrices $z = 0$ and $\gamma = \{2z + w^2 + \ldots = 0\}$, or $F = x^2 + 2y + \ldots = 0$ in the old variables. Besides γ we have the family of analytic trajectories $w \sim Cz^{1/3}$, or $x^3 \sim Cy^2$.

To study p_5 we use the variables $\widetilde{z} = 1/x$, $u = y/x$;

$$\dot{\widetilde{z}} = -\widetilde{z}(2 + \ldots), \qquad \dot{u} = u(1 + \ldots).$$

Hence p_4 is a saddle with separatrices $\widetilde{z} = 0$ and $y = 0$.

When one blows up the point p_4 $(v = z/w^2, w)$, then on the exceptional divisor $E_4 = \{w = 0\}$ there appear three singular points $p_4' = E_\infty \cap E_4$, p_4'' with the separatrix $\gamma = \{F = 0\}$ and $p_4''' : v = \infty$. One finds that the first integral H is critical on γ, i.e. $H = F^3 H_1$. This gives the representation

$$H = F^3/y^4,$$

if F is well defined (recall that the formula for γ is not unique). So, $H = 0$ is the equation of γ.

If an invariant algebraic curve $Z = 0$ contains a branch $F = 0$ then its equation is $Hy^4 = 0$. Hence Hy^4 is a polynomial and H is rational.

If an invariant algebraic curve $Z = 0$ has degree ≥ 4 then $\{Z = 0\} = \{(H - C)y^4 = 0\}$ and H is also rational.

Therefore it remains to consider the case $Z = x^3 - Cy^2 + \ldots$ Here the curve $H = C$ may have two components, one algebraic $Z = 0$ and the other transcendental. We have three possibilities:

(i) $Z = 0$ is smooth or has a double point (transversal self-intersection) outside the line $y = 0$ and $Z = 0$ has at most first order tangency with the line $y = 0$.

(ii) $Z = 0$ has a double point at $y = 0$;

(iii) $Z = 0$ is a cusp.

(iv) $Z = 0$ has double tangency with $y = 0$.

Consider the case (i). The following lemma is very useful for us. It was proved by Christopher (with the assumptions (i) and (ii), see [9]), and by the author [18] (independently).

LEMMA 7. *If a polynomial vector field $V(x, y)$ in $\mathbb{C}^2$ has invariant irreducible algebraic curves $S_1, \ldots, S_r$, $S_i = \{f_i = 0\}$, with the properties*

(a) *S_i are smooth,*

(b) *S_i and S_j intersect transversally,* *then*

$$V = \prod f_i \Big(W + \sum h_i X_{f_i}/f_i \Big),$$

where $X_f = \frac{\partial f}{\partial y}\partial_x - \frac{\partial f}{\partial x}\partial_y$ *is the Hamiltonian vector field,* W *is a polynomial vector field and* h_i *are polynomials.*

If instead of (a) *and* (b) *we assume*

(a′) *the only singular points of* S_i *are simple double points outside other* S_j*'s,*

(b′) *the intersections of* S_i *with* S_j *have at most first order tangencies,*

then

$$QV = \prod f_i \Big(W + \sum h_i X_{f_i}/f_i \Big),$$

where Q *is an arbitrary polynomial vanishing at the double points of* S_i *and at the points of non-transversal intersections* $S_i \cap S_j$ *with the condition* $Q = 0$ *transversal to* S_i.

We apply this lemma to our situation. Of course we can choose the factor Q in the form $Q = x + \lambda y + \mu$ such that the line $Q = 0$ passes through the double point of $Z = 0$ and through a possible tangency point of $Z = 0$ and $y = 0$. Then we get $QV = W(x,y)yZ + h_1(x,y)X_Z y + h_2(x,y)X_y Z$, or

$$(x + \lambda y + \mu)(3x^2 + y + bx) = W_1 yZ + h_1(-2Cy + \dots)y + h_2(x^3 - Cy^2 + \dots),$$

$$(x + \lambda y + \mu)(3xy + ey) = W_2 yZ + h_1(-3x^2 + \dots)y.$$

On the line $y = 0$ we have $3x^3 + \dots = h_2 x^3 + \dots$, so $h_2 = \text{const} + yh_2'$ and we can put $h_2 = \text{const}$. Along the curve $Z = 0$, $x \sim y^{2/3}$ and we get

$$\begin{aligned} C_1 y^2 + C_2 \lambda y^{2+1/3} + \dots &= C_3 h_1 y^2 + \dots, \\ C_4 y^{7/3} + C_5 \lambda y^{8/3} + \dots &= C_6 h_1 y^{7/3} + \dots \end{aligned}$$

Hence $\lambda = 0$, $h_1|_{Z=0} = \text{const}$ and we can put $h_1 = \text{const}$. But then we have $W_1 = W_2 = 0$ and $Z^{h_1} y^{h_2}$ is a first integral.

Consider the case (ii). Assuming that the double point is $x = y = 0$ we deduce that it is a $1:1$ resonant node with diagonal linear part. So, $c = 0$, $a = b$ and we have separated variables. Integration gives a rational first integral.

Consider the case (iii). We have $Z = (x - x_0)^3 - C(y + Ax + B)^2$, where we can put $x_0 = 0$. We have two possibilities:

(a) $B = 0$, $b/a = 3/2$, the wedge of the cusp at $y = 0$;

(b) $B \neq 0$ and by translation we get

$$\dot{x} = bx + y + 2x^2, \qquad \dot{y} = fx + 3xy.$$

Let us solve the equation $\dot{Z} = gZ$, $g = 6x + \rho$, $Z = x^3 - C(y + Ax)^2$ for $\dot{x} = bx + y + 2x^2$, $\dot{y} = 3xy + ay + fx$, where $a = 3b/2$, $f = 0$ in the case (a) and $a = 0$ in the case (b). We have

$$[3x^2 - 2AC(y + Ax)][2x^2 + y + bx] - 2C(y + Ax)(3xy + ay + fx) \\ = (6x + \rho)[x^3 - C(y + Ax)^2]$$

and equating the coefficients we find the following conditions:

(a) $AC = 3/4$, $\rho = 3b + 2A^2C$, $\rho = 3b + 2A$, so $A = 0$ (contradiction);
(b) $AC = 3/2$, $\rho = 3b+2A^2C$, $\rho = 2A$, $f = \rho A/2 - A^2 - Ab$, $f = \rho A/2 - Ab$ and hence $A = 0$.

Consider the case (iv). If the tangency point is $x = y = 0$ then $y \sim x^3$ on $Z = 0$. Therefore $(0,0)$ is a $3:1$ resonant node, $a = 3b$ and $Z = Ay+Bxy+Cy^2+Dy^3$. Invariance of $Z = 0$ means that $\dot{Z} = gZ$, $g = 6dx + 3b$. The calculation of the coefficients gives then the form of Z from the statement of Lemma 6.

To show the non-integrability of (18) for $c = d = 1$, $b \neq 0$ we study the singular point $p_3 = (-b, -b^2)$. It is a stable focus with eigenvalues $\lambda_{1,2} = \frac{1}{2}(-3 \pm i\sqrt{3})b$. If the system (18) had a Darboux or DHE integral then from its behaviour at infinity it would follow that the integral should be rational. This contradicts the existence of the focus. $\square$

REMARK 3 (General). Below we shall prove the propositions (from Section 4) which are connected with the cases CR_4–CR_{15} of rational reversibility described in Theorem 1. We shall be less concerned with the estimates of the degrees than with the proof that these cases are really new situations of reversibility. For this we shall need two things:

- non-integrability in the Darboux or DHE sense and
- existence of a reversible center.

The purpose of Remark 3 is to discuss the existence of a center here in order not to do it separately in each case.

We have either $X = T_1x$ or $X = T_{1,2}^i/x$, and $Y = T_{1,2}^j/y$, where T_k is a general polynomial of degree k.

If $T_1 = ax + by + c$ then we can assume that $a \neq 0$ and $b \neq 0$ because otherwise X is a function of x or Y is a function of y and $\Phi = (X,Y)$ can be decomposed as $\Phi = \Phi_2 \circ \Phi_1$, so V is reversible by means of a map which does not transform a curve to a point. This is either impossible or yields one of the cases CR_1–CR_3. Sometimes also $c \neq 0$ (but not always). Anyway we can normalize the variables x, y so that

$$T_1 = x + y + c, \qquad c = 0, 1, \qquad T_2 = ax^2 + bxy + cy^2 + dx + ey + f.$$

Let the variables x, y be fixed. Then the fold curve Γ is fixed too.

Next we make the changes $X \to \mu X$, $U \to \nu Y$. After these changes $\Gamma' = \Gamma'_{\mu,\nu} = \Phi(\Gamma)$ changes but the general form of V' remains the same. Assume that V' does not depend on μ, ν. For almost all μ, ν, $\Gamma'_{\mu,\nu}$ is not invariant for V' (otherwise V' would have a family of invariant algebraic curves and then a rational first integral).

If Γ' is not invariant then it remains to show that V' is not everywhere transversal to Γ' and that there is a tangency point of first order. If this point is (X_0, Y_0) and Γ' is given by $Y = Y_0 + A(X - X_0) + B(X - X_0)^2 + \ldots = \Psi(X)$ (we choose a generic point), then we have to prove that

$$\dot{Y} - (A + 2B(X - X_0))\dot{X}]|_{Y=\Psi(X)} = C + D(X - X_0) + \ldots, \qquad C = 0,\ D \neq 0$$

(with definite sign of D). But C and D depend on the coefficients of V'. In all the cases the number of coefficients is large enough to ensure that C and D are independent.

PROOF OF PROPOSITION 22(e). If $|X_1| = |Y_1| = -1$, $T = 0$ invariant, $\gamma = 2\delta$, $\delta|T| = 2$ and $X^{I+2}Y^{J-2}\partial_X$ in V'_0 then from (7) we get $I = 0$, $J = 2$ and

$$\dot{X} = X(kX + lY^2), \qquad \dot{Y} = Y^3(m + nY).$$

This system has a DSC integral and either $|Z| > 0$ (Lemma 2(g)), or $X = T^4/x$, $Y = T^2/y$, $T = x + y + c$, i.e. we have the case

$$CR_{15}: \qquad \begin{aligned} \dot{x} &= (2y - T)(ky^2 + lx) - 4x(my + nT^2), \\ \dot{y} &= (4x - T)(my + nT^2) - 2y(ky^2 + lx). \end{aligned}$$

Because the parameters k, l, m, n are arbitrary the DSC integral is not of Darboux or DHE type. $\square$

PROOF OF PROPOSITION 42(b). We have $X = T^\gamma X_1$, $Y = T^\gamma Y_1$, $|X_1| = 1$, $|Y_1| = -1$, $T = 0$ invariant and $XY^{J-1}\partial_X$ in V'_0. It is easy to check that if $\gamma = \delta$ then $\gamma = |T| = 1$,

$$\dot{X} = kX + lY + mXY, \qquad \dot{Y} = Y(n + pY + qY^2).$$

If $X \neq x$, $Y \neq 1/y$ then $|Z| > 0$ and we can apply Lemma 2(f).

We then have $X = Tx$, $Y = T/y$, $T = x + y + c$, i.e. the case

$$CR_4: \quad \begin{aligned} \dot{x} &= -(x+c)(kxy + l + mTx) - x(ny^2 + pTy + qY^2), \\ \dot{y} &= (2x + y + c)(ny^2 + pTy + qT^2) - y(kxy + l + mTx). \end{aligned}$$

Of course V' has a DSC integral which is not of Darboux or DHE type (because the coefficients k, l, m, n, p, q are independent). □

PROOF OF PROPOSITION 49(b). We need only consider the case $X = TX_1$, $Y = TY_1$, $|X|_1 = 2$, $|Y_1| = 1$. We have the system

$$\dot{X} = lY + mX + nY^2, \qquad \dot{Y} = qX + rY.$$

If $|T| \geq 2$ or X_1, Y_1 are not polynomials then $|Z| \geq 2$, $|Z'| \geq 4$ and Lemma 4 excludes this possibility ($\{Z' = 0\} = \Phi(\{Z = 0\})$).

Let then $|T| = |Z| = 1$. One can easily see that $Z' = AY^3 + BX^2 + \ldots$ By Lemma 4 there is no Darboux integral iff $Z' = AY^3 + BX^2$. But then $Y|_{Z=0}$ is a square and $X|_{Z=0}$ is a cube. So $X = T^2$, $Y = T^3$, which is impossible. □

PROOF OF PROPOSITION 58(b). We have the following situation: $X = T^2X_1$, $Y = TY_1$, $|T| = 1$, $|X_1| = |Y_1| = -1$, $T = 0$ not invariant and

$$\begin{aligned} \dot{X} &= 2X(kY + lX + mXY), \\ \dot{Y} &= Y(kY + nX + pY^2 + qXY + rXY^2). \end{aligned}$$

In the variables $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y$ it reads (without tildes)

$$\dot{X} = 2X(m + lY + kX), \qquad \dot{Y} = r + qY + pX + (nY + kX)Y. \tag{19}$$

If $|\phi_1| + |\psi_1| > 0$ then $I|\phi_1| + J|\psi_1| > 0$ and $l = m = n = q = r = 0$ (Lemma 2(d)).

If $X_1 = 1/x$, $Y_1 = 1/y$ then we have the case CR_{11}, where we have to show non-integrability. In the variables x, y the system is

$$CR_{11}: \quad \begin{aligned} \dot{x} =& 2x[-(p+k)x + (l-n)y + (m-l-q)yT - (m+r)T^2], \\ \dot{y} =& 2px^2 - kxy - pxT + 2(n-l)xy^2 \\ &+ 2(q-m)xyT - ny^2T + 2rxT^2 - qyT^2 - rT^3. \end{aligned}$$

The system (19) has 6 singular points: $p_{1,2}$ on the line $X = 0$, p_3 finite, $p_4 = (0:1:0)$, $p_5 = (1:0:0)$ and $p_6 = (n - 2l : k : 0)$ on the line at infinity.

If the system (19) has a Darboux or DHE integral for all values of the parameters then it has an invariant algebraic curve S. It is possible that $S = E_\infty$ or $S = \{X = 0\}$ is multiple but then some of the singular points on S must be resonant with resonant terms in the normal form.

Because the parameters n, k, l in (19) are independent we can achieve that the ratios $\lambda(p_6)$ and $\lambda(p_4)$ of the eigenvalues at p_6 and at p_4 are irrational (but $\lambda(p_5) = 2$ because there is a family of trajectories $X \sim CY^2$ and p_5 is analytically linearizable). The ratios $\lambda(p_i)$, $i = 1, 2, 3$, are governed by m, p, q, r and are independent. We choose all of them real and irrational. Then S cannot be $X = 0$ or E_∞. It has to be a separatrix of a singular point. But the phase portrait of (19) can be investigated. In particular, for generic parameters there is no separatrix connections and no new invariant algebraic curves.

The simplest way to see this is to consider a perturbation of an integrable system. (19) is equivalent to $\dot{x} = -2xy$, $\dot{y} = a + bx + cx^2 - y^2 + \varepsilon x(d + ey)$. For $\varepsilon = 0$ we have the integral $H = x^{-1}(y^2 + \alpha x^2 + \beta) + \gamma \ln x$ and for small ε we have the focus p_3 and at most one limit cycle. Next we make a continuous passage to the case when the focus becomes a node. $\square$

PROOF OF PROPOSITION 58(c). We have $X = T^3 X_1$, $Y = Y^2 Y_1$, $|X_1| = |Y_1| = -1$, $T = 0$ not invariant,

$$\dot{X} = 3X(kY + lX), \qquad \dot{Y} = 2Y(kY + nX + pY^2),$$

which after the transformation $X \to 1/X$, $Y \to 1/Y$ becomes

$$\dot{X} = 3X(kX + lY), \qquad \dot{Y} = 2[nX + (kX + mY)Y].$$

If $|\phi_1| + |\psi_1| > 0$ then $\nu > 0$ and $l = m = 0$ (Lemma 2(d)). If $|T| > 1$ then $p = 0$, V_1' is homogeneous (Lemma 2(c)).

If $|T| = 1$ then $X = T^3/x$, $Y = T^2/y$, i.e. we have the case

$$CR_{13}: \quad \begin{aligned} \dot{x} &= 3x[-kx + 2(l - m)y^2 - lyT - 2nxT], \\ \dot{y} &= 2[-kxy - my^2T + 3(m - l)xy^2 + 3nx^2T - nxT^2]. \end{aligned}$$

The non-integrability is proved in the same way as in the proof of Proposition 58(b), where the essential part of the perturbation of an integrable system reduces to $\dot{x} = -2xy, \dot{y} = ax + bx^2 + cy^2 + \varepsilon xy$, $H = x^c(\alpha y^2 + \beta x^2 + \gamma)$. $\square$

PROOF OF PROPOSITION 59(a). We have $X = T^2 X_1$, $Y = TY_1$, $|X_1| = |Y_1| = -1$ and one can check that

$$\dot{X} = 2X(kY + lX + mY^2 + nXY + pXY^2), \qquad \dot{Y} = Y^2(k + qY + rY^2).$$

If $|T| > 1$ or $|\phi_1| + |\psi_1| > 0$ then either $|Z| > 0$ and we apply Lemma 2(g), or $\nu > 0$, $l = n = p = 0$ (Lemma 2(d)).

If $X = T^2/x$, $Y = T/y$, $|T| = 1$ then we have the case

$$CR_{12}: \quad \begin{aligned} \dot{x} = {} & 2[(l-k-q)xy - (m+r)xT \\ & + ly^3 + (p-n)yT^2 + (n-l)y^2T - pT^3], \\ \dot{y} = {} & -ky^2 + 2(q-m)xy + 2rxT \\ & - pqT - rT^2 - 2ny^2T - 2pyT^2. \end{aligned}$$

V' has the DSC integral

$$H = X^{-1}Y^{\alpha}(Y-a)^{\beta}(Y-b)^{\gamma} - \int Y^{\alpha-1}(Y-a)^{\beta-1}(Y-b)^{\gamma-1}(l+nY+pT^2)dY$$

with the restriction $\alpha + \beta + \gamma = 0$. Therefore H is not of Darboux or DHE type. □

PROOF OF PROPOSITION 59(c). We have $X = T^3X_1$, $Y = T^2Y_1$, $|X_1| = |Y_1| = -1$ and one can check that

$$\dot{X} = 3X^2(kY + lX + mY^2), \qquad \dot{Y} = 2Y^2(kX + nXY + pY^2), \tag{20}$$

which after the change $X \to 1/X$, $Y \to 1/Y$ becomes

$$\dot{X} = 3(mX + lY^2 + kXY), \qquad \dot{Y} = 2(pX + nY + kY^2)$$

(the system from Lemma 6 with $1/2 < d = 2/3 < 1$). If $|T| > 1$ or $|\phi_1| + |\psi_1| > 0$ then there is an invariant curve $Z = 0$ which gives an invariant algebraic curve $\widetilde{Z} = 0$ of degree ≥ 4 for (20). By Lemma 6 we have a Darboux integral.

So $X = T^3/x$, $Y = T^2/y$, $|T| = 1$, i.e. we have the case

$$CR_{14}: \quad \begin{aligned} \dot{x} &= 3[-kxy - 2px^2 + 2ly^3 + 2(m-n)xyT - ly^2T - mxT^2], \\ \dot{y} &= 2[3px^2 - ky^2 - pxT - 3ly^3 + 3(n-m)xyT - nyT^2]. \end{aligned}$$

The Darboux and DHE non-integrability of the system (20) is proved in the same way as in the proof of Proposition 58(b) (there are irrational ratios $\lambda(p_i)$ and no separatrix connections because we have the perturbation $\dot{x} = -2xy + \varepsilon(y+a)^2$, $\dot{y} = b + cx - (4/3)y^2 + \varepsilon dy$, $H = x^{-4/3}(y^2 + \alpha x + \beta)$). □

PROOF OF PROPOSITION 66(a). If $|T| > 1$, $X = TX_1$, $Y = TY_1$, $|X_1| = |Y_1| = 1$ then $|Z|_{\max} > 1$ and

$$\begin{aligned} \dot{X} &= X(kX + lY) + X^2(mX + nY) + pX^3Y, \\ \dot{Y} &= Y(kX + lY) + Y^2(qX + rY) + sXY^3, \end{aligned}$$

which after the change $X \to 1/X + C_1$, $Y \to 1/Y + C_2$ becomes

$$\dot{X} = aX + bY + (lX + kY)X, \qquad \dot{Y} = cX + dY + (lX + kY)Y.$$

If $|Z| > 0$ then we can apply Lemma 2(h) (here are included also the cases with $|T| > 2$, $|\phi_1| + |\psi_1| > 0$). But if $|Z| = 0 < |Z|_{\max}$ then in the variables $X, U = Y/X$ we have

$$\dot{X} = k + lU + X(m + nU), \qquad \dot{Y} = U[rU^2 + (q-n)U - m],$$

a system with a DSC integral (not Darboux or DHE).

Therefore we have $|Z| = 0$, $p = s = 0$, the case CR_{15} with $X = T/x$, $Y = T/y$, $T = ax^2 + bxy + cy^2 + dx + ey + f$ and

$$CR_{15}: \quad \begin{aligned} \dot{x} = &- kxy - lx^2 - myT - nxT \\ &+ (bx + 2cy + e)[-rx^2 + (n-q)xy + my^2] \\ \dot{y} = &- kxy - ly^2 - qyT - rxT \\ &+ (2ax + by + d)[rx^2 + (q-n)xy - my^2]. \end{aligned}$$

We can assume that the constant term f is not zero because otherwise we have the situation as in Proposition 64(b). So, we put $f = 1$. □

PROOF OF PROPOSITION 69(e). We omit the proof that $X = x^2/y$, $Y = x^3/\eta$, $|\eta| = 2$,

$$\dot{X} = 2X(kY + mX^2 + nXY), \qquad \dot{Y} = 3Y^2(k + rX)$$

with $|Z| = 1$. Let us check when we get $|Z| = 1$.

Let $\eta = \alpha x + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \zeta$. After simple calculations we get $\det(d\Phi) = x^4 y^{-2}\eta^{-2}(3\eta - x\eta_x - 2y\eta_y) = x^4 y^{-2}\eta^{-2}(\alpha x^2 - \gamma y^2 + 2\delta x + \varepsilon y + 3\zeta)$. Because the latter monomial is reducible and Γ_Φ has a real component, we have $\alpha\gamma > 0$. We can assume that $\alpha = \gamma = 1$. Then $Z = x - y + \mu$, $W = x + y - \nu$, where $\Gamma = \{W = 0\}$ and $Z = 0$ is invariant for V and $Z' = 0$ (the image of $Z = 0$) is invariant for V'.

Let us introduce the variables $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y$. Then

$$\dot{\widetilde{X}} = 2(k\widetilde{X}^2 + m\widetilde{Y} + n\widetilde{X}), \qquad \dot{\widetilde{Y}} = 3\widetilde{Y}(k\widetilde{X} + r),$$

a system from Lemma 6. Applying Lemma 6 we deduce that $\{\widetilde{Z} = 0\} = \widetilde{\Phi}(\{Z = 0\})$, $\widetilde{\Phi} = (\widetilde{X}, \widetilde{Y})$ is cubic (rational elliptic) with a point of double tangency with the line $\widetilde{Y} = 0$.

We have $\widetilde{Y} = [x^2 + \beta xy - y^2 + (\mu - \nu)x/2 + (\mu + \nu)y - \mu\nu/3]x^{-3}$,

$$\widetilde{X}|_{Z=0} = \tau + \mu\tau^2,$$
$$\widetilde{Y}|_{Z=0} = \beta\tau + \left(\beta + \frac{1}{2}(\mu - \nu)\right)\tau + \frac{2}{3}\mu\nu\tau^3,$$

with $\tau = 1/x$, $\mu\nu \neq 0$. We need the property $\widetilde{X} = \mathrm{const}(\tau - \tau_0) + \dots$, $\widetilde{T} = \mathrm{const}\,(\tau - \tau_0)^3$. Because τ_0 is a unique zero of $\widetilde{Y}$, we have $\tau_0 = 0$, $\beta = 0$, $\mu = \nu$. After changing x, y we can assume that $\mu = \nu = 1$,

$$\widetilde{X} = \tau + \tau^2, \qquad \widetilde{Y} = \frac{2}{3}\tau^3.$$

This gives

$$\widetilde{Z} = 4\widetilde{X}^3 - 9\widetilde{Y}^2 - 18\widetilde{X}\widetilde{Y} - 6\widetilde{Y}$$

and the condition $\dot{\widetilde{Z}} = g\widetilde{Z}$ leads to $k = -4m/3$, $r = 4m = n/2$, or after normalization,

$$\dot{\widetilde{X}} = \widetilde{X}^2 - 3\widetilde{Y} + 2\widetilde{X}, \qquad \dot{\widetilde{Y}} = 6\widetilde{Y}(\widetilde{X} + 1).$$

It remains to show the non-existence of a center for V. We have to find Γ' and prove that V' is not tangent to Γ' from the outside of its local image. We shall check it for $\widetilde{\Gamma} = \widetilde{\Phi}(\Gamma)$.

We have $\Gamma = \{y = 1 - x\}$, $\widetilde{X}|_\Gamma = \tau^2 - \tau$, $\widetilde{Y}|_\Gamma = \frac{2}{3}\tau^3$. This gives $\widetilde{\Gamma} = \{\widetilde{W} = 0\}$, $\widetilde{W} = 4\widetilde{X}^3 - 9\widetilde{Y}^2 + 18\widetilde{X}\widetilde{Y} + 6\widetilde{Y}$; $\widetilde{\Gamma}$ is the reflection of $\widetilde{Z} = 0$ with respect to the axis $\widetilde{Y} = 0$. Calculations give

$$\dot{\widetilde{W}} = (12\widetilde{X} + 6)\widetilde{W} - 18\widetilde{Y}(4\widetilde{X}^2 + 7\widetilde{Y}).$$

But $4\widetilde{X}^2 + 7\widetilde{Y} = \frac{1}{3}\tau^2(12\tau^2 - 10\tau + 12) > 0$. So, there are no tangency points outside the line $\widetilde{Y} = 0$ which is invariant. $\square$

PROOF OF PROPOSITION 70(a). As in the proof of Proposition 69(e) we start with $X = x^3/y$, $Y = x^2/\eta$, $\eta = \alpha x + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \zeta$ and

$$\dot{X} = 3X^2(l + nY), \qquad \dot{Y} = 2Y(lX + rXY + sY^2 + tXY^2).$$

It is useful to pass to the variables $\widetilde{X} = 1/X$, $\widetilde{Y} = 1/Y$. Then

$$\dot{\widetilde{X}} = 3\widetilde{X}(l\widetilde{Y} + n), \qquad \dot{\widetilde{Y}} = 2(l\widetilde{Y}^2 + r\widetilde{Y} + s\widetilde{X} + t).$$

After applying a translation of $\widetilde{Y}$ (or $Y \to Y/(1 + cY)$) we can put $t = 0$ and we get the system (18) from Lemma 6.

We have $\det(d\Phi) = x^4y^{-2}\eta^{-2}(2\eta - x\eta_x - 3y\eta_y) = x^4y^{-2}\eta^{-2}(-2\beta xy - \gamma y^2 - \delta x - \varepsilon y + \zeta)$. Assuming that the latter polynomial is reducible we need that $\beta \neq 0$. Assume that $\beta = 1$, $-2WZ = (y-\mu)(x - \nu y + \rho)$ and $\widetilde{Y} = [(\alpha x^2 + xy - 2\nu y^2 + 2\mu x + 2(\rho + \mu\nu)y + \mu\rho]x^{-2}$.

By Lemma 6 we are interested in the case when the curve $\widetilde{Z} = 0$ is cubic and doubly tangent to the axis $\widetilde{X} = 0$. We have the following possibilities for $Z = 0$.

(i) $Z = y$, $(\mu = 0)$. Then $\widetilde{Y} - \alpha = y(x - 2\nu y + 2\rho)x^{-2}$ and we can apply the change $(X', Y') = (X, (\widetilde{Y} - \alpha)X)$ giving a simpler case.
(ii) $Z = y - \mu$, $\mu \neq 0$.
(iii) $Z = x$ $(\nu = \rho = 0)$. Then $Y = x/(\text{linear})$.
(iv) $Z = x + \rho$. Then $\widetilde{X} = Ay$, $\widetilde{Y} = By + C$, $\widetilde{Z} = 0$ a line.
(v) $Z = x - \nu y$. Then $\widetilde{X} = \tau^2/\nu$, $\widetilde{Y}(\tau)$ quadratic, $\tau = 1/x$, and $\widetilde{Z} = 0$ conic.
(vi) $Z = x - \nu y + \rho$. Then $\widetilde{X} = \frac{1}{\nu}\tau^2 + \frac{\rho}{\nu}\tau^3$, $\widetilde{Y}$ quadratic and due to Lemma 6 we need $1/\nu = 0$.

Therefore there remains the case (ii), where we can put $\mu = 1$. We have

$$\widetilde{X}|_{Z=0} = \tau^3, \qquad \widetilde{Y}|_{Z=0} = \alpha + 3\tau + 3\rho\tau^2.$$

Because $\widetilde{Z} = 0$ is tangent to $\widetilde{X} = 0$, we have $\alpha = 0$. We also need $\rho \neq 0$ because if $\rho = 0$ then $\widetilde{Z} = 0$ is different from the one given in Lemma 6. So, changing x if necessary we can assume that $\rho = 1$, $\widetilde{X} = \tau^3$, $\widetilde{Y} = 3(\tau + \tau^2)$. From the proof of Proposition 69 we get

$$\widetilde{Z} = \widetilde{Y}^3 - 27\widetilde{X}^2 - 27\widetilde{X}\widetilde{Y} - 27\widetilde{X}$$

and

$$\widetilde{V}: \ \dot{\widetilde{X}} = 3k\widetilde{X}(\widetilde{Y} + 3), \qquad \dot{\widetilde{Y}} = k(2\widetilde{Y}^2 - 9\widetilde{X} + 3\widetilde{Y}).$$

Now we have to check the existence of a center. We have $y = (x+1)/\nu$ on Γ and

$$\widetilde{X}|_\Gamma = \frac{1}{\nu}(\tau^2 + \tau^3), \qquad \widetilde{Y}|_\Gamma = \frac{1}{\nu}[-1 + (4\nu - 1)\tau + 3\nu\tau^2].$$

The phase portrait of $\widetilde{V}$ is fixed but the image $\widetilde{\Gamma} = \widetilde{\Gamma}(\nu)$ of the fold curve is varying with ν. The right kind of tangency of $\widetilde{V}$ to $\widetilde{\Gamma}$ can be seen either by drawing a picture or by calculation of $\dot{\widetilde{W}}$ which is a polynomial of fifth degree in τ.

Finally, we present the formula for the initial system:

$$CR_{17}: \quad \begin{aligned} \dot{x} &= k[-9xy - 3\eta y\eta_y + 3x^2\eta_y + 2\eta^2 + 9x^2\eta y]/(y-1), \\ \dot{y} &= ky[-27y + 9x\eta + 3\eta\eta_x + 9x^2\eta_x - 18x\eta]/(y-1). \end{aligned}$$

□

PROOF OF PROPOSITION 78(a). We have $X = TX_1$, $Y = TY_1$, $|X_1| = -|Y_1| = 1$, $T = 0$ not invariant,

$$\dot{X} = X(k + lY + mY^2) + nY^2, \qquad \dot{Y} = Y(k + pY + qY^2 + rY^3)$$

with a DSC integral (not Darboux or DHE). If $|T| > 1$ or $X_1 \neq x$ or $Y_1 \neq 1/y$ then $|Z| > 0$ and we apply Lemma 2(f).

So $X = Tx$, $Y = T/y$, $|T| = 1$, i.e.

$$CR_6: \quad \begin{aligned} \dot{x} &= -n + ny + (l-k-p)xy^2 + (m-l-q)xyT - (m+r)xT^2, \\ \dot{y} &= -ny + (p-l)xy^2 + ky^3 + (q-m)xyT \\ &\quad + py^2T + rxT^2 + qyT^2 + rT^3. \end{aligned}$$

□

PROOF OF PROPOSITION 81(d). We have $X = TX_1$, $Y = TY_1$, $|X_1| = -|Y_1| = 1$,

$$\dot{X} = X(k + lX), \qquad \dot{Y} = Y(k + mX + nY + pXY)$$

with DSC integral

$$H = Y^{-1}X^{-1}(k + lX)^{(l-m)/l} - \int X^{-2}(k + lX)^{-m/l}(n + pX)\, dX$$

(not Darboux or DHE). So $|T| = 1$, $X = Tx$, $Y = T/y$, i.e.

$$CR_7: \quad \begin{aligned} \dot{x} &= x[-(n+k) + (l-m)xy + (l-p)xT], \\ \dot{y} &= nx + ky + nT + (m-l)x^2y + px^2T + nxyT + pxT^2. \end{aligned}$$

□

PROOF OF PROPOSITION 81(e). We have $X = TX_1$, $Y = T^2Y_1$, $|X_1| = -|Y_1| = 1$,

$$\dot{X} = X(k + lX), \qquad \dot{Y} = 2Y(k + mX + nY)$$

with DSC integral

$$H = Y^{-1}X^{-2}(k + lX)^{2(l-m)/l} - n\int X^{-3}(k + lX)^{(l-2m)/l} dX$$

(not Darboux or DHE). Therefore $X = Tx$, $Y = T^2/y$, $|T| = 1$, i.e.

$$CR_8: \quad \begin{aligned} \dot{x} &= x[-k - 2nT + 2(l-m)xy - lxT], \\ \dot{y} &= 2[ky + nxT + nT^2 + (m-l)x^2y + mxyT]. \end{aligned}$$

$\square$

Proof of Proposition 81(f). We have $X = TX_1$, $Y = T^3Y_1$, $|X_1| = -|Y_1| = 1$,

$$\dot{X} = X(k + lX), \qquad \dot{Y} = 3Y(k + mX + nY)$$

with DSC integral

$$H = Y^{-1}X^{-3}(k + lX)^{3(l-m)/l} - n\int X^{-4}(k + lX)^{(2l-3m)/l}\,dX$$

(not Darboux or DHE). Therefore $X = Tx$, $Y = T^3/y$, $|T| = 1$, i.e.

$$CR_{10}: \quad \begin{aligned} \dot{x} &= x[-k + 3(l-m)xy - lxT - 3nT^2], \\ \dot{y} &= 3[ky + nxT^2 + (m-l)x^2y + nT^3]. \end{aligned}$$

$\square$

Proof of Proposition 82(b). We have $X = TX_1$, $Y = T^2Y_1$, $|X_1| = -|Y_1| = 1$,

$$\dot{X} = X(kX + lY), \qquad \dot{Y} = 2Y(kX + mY + nY^2).$$

If $|\omega| > 0$ or $|\psi_1| > 0|$ then $l = m = n = 0$ (Lemma 2(a)). If $|T| > 1$ then $|Y| > 0$ and $n = 0$ (Lemma 2(c)). So $X = Tx$, $Y = T^2/y$, $|T| = 1$, i.e.

$$CR_9: \quad \begin{aligned} \dot{x} &= x[-lT + 2(l-m)y - kxy - 2nT^2], \\ \dot{y} &= 2[(m-l)xy + myT + kxy^2 + nxT^2 + nT^3]. \end{aligned}$$

To prove the non-integrability we make the change $X \to 1/X$, $Y \to 1/Y$ and get

$$\dot{X} = X(kY + lX), \qquad \dot{Y} = 2[X + (kY + nX)Y].$$

Next we follow the proof of Proposition 58(b) (irrational $\lambda(p_i)$ and the perturbation $\dot{x} = -2xy$, $\dot{y} = ax + bx^2 + cy^2 + \varepsilon xy$, $H = x^c(y^2 + \alpha x^2 + \beta x)$). $\square$

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