Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 4, 1994, 59–78

SUPERLINEAR INDEFINITE ELLIPTIC PROBLEMS AND NONLINEAR LIOUVILLE THEOREMS

H. BERESTYCKI — I. CAPUZZO DOLCETTA — L. NIRENBERG

A Jean Leray, en témoignage de profonde admiration

1. Introduction

We study the following elliptic boundary value problem in a bounded domain Ω in $\mathbb{R}^N$, with smooth boundary:

$$\begin{aligned} u > 0, \quad Lu + a(x)g(u) &= 0 \qquad \text{in } \Omega, \\ Bu &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here, L is a linear elliptic operator—we use summation convention—

$$L = a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

with $a_{ij} \in C^2(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty$. We assume uniform ellipticity

$$c_0|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le C_0|\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \ \forall x \in \Omega,$$

with $c_0, C_0 > 0$. The boundary operator B is one of the following:

$$Bu := u, \tag{1.2a}$$

$$Bu := \nu_j a_{jk} u_{x_k} + \alpha(x)u, \tag{1.2b}$$

The first and third authors were partially supported under grant ARO-DAAL-03-92-G-0143.

The third author was also supported by NSF DMS 9114456; he wishes in addition to thank the Forschungs Institut ETH for its kind hospitality.

The second author received partial support from MURST.

where $\nu_i = (\nu_1, \ldots, \nu_N)$ denotes the exterior unit normal on $\partial\Omega$; α is a given continuous nonnegative function on $\partial\Omega$.

We are interested in the case that the function a changes sign. The problem is then called one with indefinite nonlinearity. In case

$$L = \Delta - m(x), \qquad m \in L^\infty,$$

several recent papers treat the problem for special functions g, by variational methods (see [4] and [1], when $g = u^p$, $1 < p < (N+2)/(N-2)$, [8] treats the problem on a compact manifold using bifurcation analysis). [1] also treats a wider class of functions g for the Dirichlet problem, i.e., for B given by (1.2a). In case g is odd they also obtain multiple solutions.

For general L, problem (1.1) does not admit a variational approach. We always assume that g is a C^1 function on $\mathbb{R}^+$ with

$$g(0) = g'(0) = 0 \quad \text{and} \quad g(s) > 0 \qquad \text{for } s > s_1 > 0. \tag{1.3}$$

Our main result, which is proved with the aid of degree theory, is concerned with functions g which have precise power-like growth at infinity:

$$\lim_{s\to\infty} \frac{g(s)}{s^p} = l > 0, \qquad \text{for some } p > 1. \tag{1.4}$$

Concerning the function a, we assume it belongs to $C^2(\overline{\Omega})$, that

$$\Omega^+ := \{x \in \Omega : \ a(x) > 0\} \qquad \text{and} \qquad \Omega^- := \{x \in \Omega : \ a(x) < 0\}$$

are nonempty, and that

$$\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega, \qquad \text{with } \nabla a(x) \neq 0 \ \forall x \in \Gamma. \tag{1.5}$$

Our results depend on the sign of the principal eigenvalue $\lambda_1 = \lambda_1(-L)$ of the operator $-L$ in Ω under the boundary conditions $Bu = 0$. The eigenvalue λ_1 is such that there is a function φ satisfying

$$\begin{aligned} \varphi > 0, \quad (L + \lambda_1)\varphi &= 0 \qquad \text{in } \Omega, \\ B\varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

In case B is as in (1.2b), it follows easily with the aid of the Hopf lemma that

$$\varphi > 0 \qquad \text{in } \overline{\Omega}.$$

Our main existence result is

THEOREM 1. *Assume* (1.3), (1.4) *and* (1.5) *and assume that* $\lambda_1(-L) > 0$. *Then problem* (1.1) *has a solution provided*

$$1 < p < \frac{N+2}{N-1}. \tag{1.6}$$

The upper bound in (1.6) is less than the usual critical exponent $(N+2)/(N-2)$. The limitation (1.6) is due to that in our nonlinear Liouville Theorem 4 below. We believe that Theorem 1 should hold if $1 < p$ for $N \leq 2$, and if $N \geq 3$, also for $p < (N+2)/(N-2)$.

The next existence results refer to the problem involving a parameter $\tau \leq 0$:

$$\begin{aligned} u > 0, \quad (L-\tau)u + a(x)g(u) = 0 \qquad & \text{in } \Omega, \\ Bu = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

in case $\lambda_1(-L) = 0$.

THEOREM 2. *Assume* (1.3), (1.5) *and that* $\lambda_1(-L) = 0$. *Assume further that*

$$\lim_{s\to 0} \frac{g(s)}{s^q} = \alpha \neq 0, \qquad \textit{for some } q > 1, \tag{1.8}$$

and that

$$\alpha \int_\Omega a(x)\varphi^q\psi < 0, \tag{1.9}$$

where $\psi > 0$ *is the principal eigenfunction of the adjoint operator* $L^* = \partial_{ij}a_{ij} - \partial_i b_i + c$, *i.e.* $L^*\psi = 0$ *in* Ω *under the adjoint boundary condition* $B^*\psi = 0$, *with*

$$\begin{aligned} B^*\psi &= \psi && \textit{in case } (1.2a), \\ B^*\psi &= \nu_i(a_{ij}\psi)_j + (b_i\nu_i - \alpha)\psi && \textit{in case } (1.2b). \end{aligned}$$

Then there exists τ^*, $0 > \tau^* > -\infty$, *such that for* $0 > \tau > \tau^*$, *problem* (1.7) *has a solution, but for* $\tau < \tau^*$ *it has no solution.*

THEOREM 3. *Assume the conditions of Theorem* 2 *and, in addition, that* (1.4) *holds with* $p < (N+2)/(N-1)$. *Then* (1.7) *has a solution when* $\tau = 0$.

The proofs of the theorems involve several ingredients. Theorem 1 relies on Leray-Schauder degree theory, and for this purpose it is necessary to establish a priori estimates. These are derived in Section 3. There we obtain a bound for the L^∞ norm of solutions of a one-parameter family of problems

$$\begin{aligned} \rho \geq 0, \quad Lu + a(x)g(u^+) + \rho = 0 \qquad & \text{in } \Omega, \\ Bu = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

The bound

$$u \leq \overline{C} \tag{1.10}$$

is established, with $\overline{C}$ independent of ρ, but depending on L, the function a, and on Ω.

The derivation of (1.10) forms the heart of paper. The proof follows a line of argument similar to that used in Gidas and Spruck in [6]. It is indirect: we assume there is no such bound and then use blow up arguments to obtain a contradiction. A number of cases must be treated. To derive the contradiction we use two nonlinear forms of Liouville's theorem, Theorems 1.2 and 1.3 in [6]. In addition, we establish some new nonlinear Liouville theorems. In proving (1.10) we use the following one. (It is a consequence of a more general result of Liouville type, Theorem 2.1 in Section 2, which we present in the belief that it will prove useful in other problems.)

THEOREM 4. *In the half space*

$$\Sigma = \{x \in \mathbb{R}^N : \ x_N > 0\}$$

let u be a nonnegative function in $C^2(\Sigma)$ which is also bounded near the origin and satisfies

$$\Delta u + x_N u^p \leq 0 \qquad \text{in } \Sigma. \tag{1.11}$$

Then

$$u \equiv 0 \qquad \text{provided } p < \frac{N+2}{N-1}. \tag{1.12}$$

REMARKS. Note that no condition is assumed about the behaviour of u near infinity or near $\{x_N = 0\}$, except at the origin. If, in place of (1.11),

$$\Delta u + x_N u^p \equiv 0 \qquad \text{in } \Sigma \tag{1.13}$$

holds, then we believe that the conclusion $u \equiv 0$ should be true for a larger range of p than given in (1.12).

For simplicity we carry out the proofs of Theorems 1–3 for L of the form

$$L = \Delta - m(x);$$

the arguments work also for our general L which is not self-adjoint.

It is clear from the arguments that more general functions a could be treated. For example, one might permit Γ to intersect $\partial\Omega$ transversally or Γ to have clean self-intersections. The more general Liouville theorem of Section 2 would then be called upon. We leave this for further consideration.

In Section 4, using the a priori estimate (1.10) we prove Theorem 1. Theorem 3 is then proved in Section 5, using Theorem 1 and a bifurcation analysis. Similar bifurcation analysis then yields small solutions of (1.7) in case $0 < -\tau$

is small. Finally, in Section 6, we use sub- and supersolutions to complete the proof of Theorem 2.

In the proof of Theorem 2 we make use of the following simple

LEMMA 1. *Suppose* Ω^+ *is nonempty and that* g *satisfies* (1.3). *If* u *is a solution of* (1.7) *for some* $\tau < 0$, *then*

$$-\tau \le \widetilde{C}, \tag{1.14}$$

where $\widetilde{C}$ *depends only on* L, *on* a, *and on a constant* C *which is such that*

$$|g(s)| \le Cs \qquad \text{on } [0, s_1].$$

PROOF. Let B be an open ball in Ω^+ and let μ be the principal eigenvalue of $-L$ in B under Dirichlet boundary conditions, i.e. there is a function $\widetilde{\varphi}$ in B satisfying

$$\begin{aligned} \widetilde{\varphi} > 0, \quad (L+\mu)\widetilde{\varphi} &= 0 \qquad \text{in } B, \\ \widetilde{\varphi} &= 0 \qquad \text{on } \partial B. \end{aligned}$$

Since $a > 0$ in B, and $g(u) > 0$ for $u > s_1$, we see from (1.7) that in B,

$$(L-\tau)u = -a(x)g(u) \le C||a||_{L^\infty} u.$$

Hence,

$$u > 0, \quad Lu - (\tau + C||a||_{L^\infty})u \le 0 \qquad \text{in } B.$$

It follows (see [5]) that $-(\tau + C||a||_{L^\infty}) \le \mu$, which yields (1.14).

The proof in Section 6 of Theorem 2, using sub- and supersolutions, is the same as one in [4].

It would be interesting to obtain some information about τ^*—even for the operator $\Delta - m(x)$—and to determine if (1.7) has a solution when $\tau = \tau^*$.

A FINAL REMARK. The derivation in Section 3 of the a priori estimate (1.10) involves blow up arguments. When dealing with $L = \Delta - m(x)$, these arguments lead to equations of the form

$$v > 0, \qquad \Delta v + v^p = 0 \tag{1.15}$$

in all of $\mathbb{R}^N$ or in a half space, with then

$$v = 0 \quad \text{or} \quad \nu.\nabla v = 0 \qquad \text{on the boundary.} \tag{1.16}$$

Or else, the blow up leads to

$$v > 0, \quad \Delta v + x_N v^p = 0 \qquad \text{in } \mathbb{R}^N. \tag{1.17}$$

For general L, these equations would, instead, take the form

$$v > 0, \qquad a_{jk} v_{x_j x_k} + v^p = 0$$

in $\mathbb{R}^N$ or in a half space, with then

$$v = 0 \quad \text{or} \quad \nu_j a_{jk} v_{x_k} = 0 \qquad \text{on the boundary.}$$

Or else we would find

$$v > 0, \quad a_{jk} v_{x_j x_k} + a_j x_j v^p = 0 \qquad \text{in } \mathbb{R}^N$$

where the a_{jk} and a_j are constants, $\sum |a_j| > 0$, with $a_{jk}\partial_{jk}$ elliptic of course. After a suitable linear transformation of independent variables, and multiplication of v by a factor, we are easily reduced to the cases (1.15), (1.16) or (1.17).

Some of the results of this paper have been announced in [3].

2. Liouville theorems in cones

Let Σ be an open connected cone in $\mathbb{R}^N$, $N \geq 2$, with vertex at the origin and with $\overline{\Sigma} \neq \mathbb{R}^N$. Let u be a nonnegative function in $C^2(\Sigma)$, bounded near the origin, satisfying

$$\Delta u + h(x) u^p \leq 0, \tag{2.1}$$

where $0 \leq h \in C(\Sigma)$, bounded near the origin, and

$$h(x) = a|x|^\gamma \qquad \text{for } |x| \text{ large}, \ \gamma > -2, \ a > 0. \tag{2.2}$$

THEOREM 2.1. *Let λ_1 be the principal eigenvalue for the Dirichlet problem on $\Sigma \cap S^{N-1}$ of $-\Delta_S$, the Laplacian on S^{N-1}. Define $\alpha > 0$ by the identity*

$$\lambda_1 = \alpha(N + \alpha - 2). \tag{2.3}$$

Assume that p satisfies

$$1 < p \leq \frac{N + \alpha + \gamma}{N + \alpha - 2} := \sigma. \tag{2.4}$$

Let $u \geq 0$ satisfy (2.1)*, with h as above. Then $u \equiv 0$.*

In the theorem no regularity of $\partial\Sigma$ is assumed and we always take $a = 1$.

COROLLARY 2.1. *Let Σ and u be as above. Assume that u satisfies* (2.1)*, where h is a positive continuous function in Σ which is homogeneous of degree $\gamma > -2$ for $|x|$ large. Then $u \equiv 0$ if*

$$1 < p < \frac{N + \alpha + \gamma}{N + \alpha - 2}, \tag{2.5}$$

where α is defined as in Theorem 2.1.

PROOF OF COROLLARY 2.1. Observe that $h(x)|x|^{-\gamma}$, for large $|x|$, may vanish on $\partial\Sigma$. Then we simply apply Theorem 2.1 in a slightly smaller cone $\widetilde{\Sigma}$. Corresponding to $\widetilde{\Sigma}$ we may have $\widetilde{\lambda}_1 > \lambda_1$, with $\widetilde{\lambda}_1 - \lambda_1$ as small as we like (see

[5]). For the corresponding $\widetilde{\alpha}$ in (2.3), we have $0 < \widetilde{\alpha} - \alpha$ small, hence $0 < \sigma - \widetilde{\sigma}$ small. By the theorem, $u \equiv 0$ in $\widetilde{\Sigma}$ and hence in Σ.

PROOF OF THEOREM 2.1. Set $\Omega = \Sigma \cap S^{N-1}$ and let $\{\Omega_j\}$ be an increasing sequence of domains on S^{N-1}, with smooth boundaries, such that

$$\Omega_j \subset \overline{\Omega}_j \subset \Omega_{j+1} \subset \ldots \subset \Omega, \qquad \bigcup \Omega_j = \Omega.$$

Let φ_j be the principal eigenfunction for the Dirichlet problem of $-\Delta_S$ on Ω_j with principal eigenvalue λ_j, i.e.

$$\begin{aligned} \varphi_j > 0 \quad (\Delta_S + \lambda_j)\varphi_j = 0, &\qquad \text{in } \Omega_j, \\ \varphi_j = 0 &\qquad \text{on } \partial\Omega_j. \end{aligned}$$

We normalize the φ_j by requiring them to equal 1 at some fixed point $x_0 \in \Omega$. The functions φ_j are then uniformly bounded by some fixed constant C_1 (see Theorem 2.1 and its proof in [5]).

Let Σ_j be the cone generated by Ω_j, with vertex at 0. With $\alpha_j > 0$ chosen as in (2.3), the functions

$$g_j = |x|^{\alpha_j} \varphi_j(x/|x|)$$

are positive and harmonic in Σ_j and vanish on $\partial\Sigma_j$.

Let ζ be a C^∞ function defined on $[0, \infty)$ with $0 \le \zeta \le 1$, $\zeta(t) \equiv 1$ on $[0, 1/2]$, $\zeta(t) \equiv 0$ for $t \ge 1$. For $R > 0$, let $\zeta_R(x) = \zeta(|x|/R)$. With ε, R respectively small and large positive numbers and with $q = p/(p-1)$, set

$$I_{j,\varepsilon} = \int_{\Sigma_j} \zeta_R^q (1 - \zeta_\varepsilon) g_j h u^p.$$

By (2.1),

$$I_{j,\varepsilon} \le - \int_{\Sigma_j} \zeta_R^q (1 - \zeta_\varepsilon) g_j \Delta u$$

so that

$$I_{j,\varepsilon} \le \int_{\partial\Sigma_j} u \zeta_R^q (1 - \zeta_\varepsilon) \frac{\partial}{\partial\nu} g_j - \int_{\Sigma_j} u \Delta(\zeta_R^q (1 - \zeta_\varepsilon) g_j),$$

where ν is the exterior normal to $\partial\Sigma_j$. Since $\frac{\partial}{\partial\nu} g_j \le 0$ on $\partial\Sigma_j$ we find

$$I_{j,\varepsilon} \le - \int_{\Sigma_j} u \Delta(\zeta_R^q (1 - \zeta_\varepsilon) g_j).$$

We now let ε go zero. The only term which requires some care is

$$\int u \zeta_R^q \Delta(\zeta_\varepsilon g_j).$$

This is integrated over $x \in \Sigma_j$, $|x| < \varepsilon$. Since u is bounded near the origin this term is

$$O(\varepsilon^{\alpha_j} \varepsilon^{N-2}).$$

Here is the only place where we use $\alpha > 0$, in case $N = 2$. Consequently,

$$I_j := \int_{\Sigma_j} \zeta_R^q g_j h u^p \leq - \int_{\Sigma_j} u \Delta(\zeta_R^q g_j);$$

hence

$$I_j \leq - \int_{\Sigma_j} u g_j \Delta \zeta_R^q - 2 \int_{\Sigma_j} u \frac{\partial}{\partial r} \zeta_R^q \frac{\partial}{\partial r} g_j,$$

where $\partial/\partial r$ represents radial differentiation.

Using C to denote various constants independent of u, R and γ, we have

$$-\frac{C}{R} \leq \frac{\partial}{\partial r} \zeta_R^q \leq 0, \quad |\Delta \zeta_R| \leq \frac{C}{R^2} \quad \text{and} \quad -\Delta(\zeta_R^q) \leq -q \zeta_R^{q-1} \Delta \zeta.$$

Hence, since $\frac{\partial}{\partial r} g_j = \frac{\alpha_j}{r} g_j$,

$$\begin{aligned} I_j &\leq - \int_{\Sigma_j} u g_j q \zeta_R^{q-1} \Delta \zeta + \frac{C}{R^2} \int_{\Sigma_j} u \zeta_R^{q-1} g_j \\ &\leq \frac{C}{R^2} \int_{\Sigma_{j,R}} u \zeta_R^{q-1} g_j = \frac{C}{R^2} \int_{\Sigma_{j,R}} u \zeta_R^{q-1} h^{1/p} |x|^{-\gamma/p} g_j, \qquad \text{for } R \text{ large}, \end{aligned}$$

where $\Sigma_{j,R} = \Sigma_j \cap \{x \in \mathbb{R}^N : R/2 < |x| < R\}$.

Thus, by Hölder's inequality,

$$\begin{aligned} I_j &\leq \frac{C}{R^2} \left[\int_{\Sigma_{j,R}} u^p \zeta_R^q h g_j \right]^{1/p} \left[\int_{\Sigma_{j,R}} |x|^{-\gamma/(p-1)} g_j \right]^{(p-1)/p} \\ &\leq \frac{C}{R^2} \left[\int_{\Sigma_{j,R}} u^p \zeta_R^q h g_j \right]^{1/p} R^{[N+\alpha_j-\gamma/(p-1)](p-1)/p}, \end{aligned}$$

with a different C. It follows that

$$I_j^{1-1/p} \leq C R^{[N+\alpha_j-\gamma/(p-1)](p-1)/p-2}.$$

Now let $j \to \infty$. The functions φ_j tend to φ, the principal eigenfunction of $-\Delta_S$ in Ω (see Section 4 in [5]). With $g = |x|^\alpha \varphi(x/|x|)$, we find that

$$I := \int_\Sigma u^p \zeta_R^q h g < \infty$$

and, furthermore

$$I_j \leq \frac{C}{R^2} \left[\int_{\Sigma_R} u^p \zeta_R^q h g \right]^{1/p} R^{[N+\alpha-\gamma/(p-1)](p-1)/p}, \tag{2.6}$$

where $\Sigma_R := \Sigma \cap \{x \in \mathbb{R}^n : R/2 < |x| < R\}$. Consequently,

$$I^{1-1/p} \leq C R^{[N+\alpha-\gamma/(p-1)](p-1)/p-2}.$$

Condition (2.4) means that $\tau := [N + \alpha - \gamma/(p-1)]/(p-1)/p - 2 \leq 0$.

If $\tau < 0$, let $R \to \infty$. Then we conclude that

$$J := \int_\Sigma h g u^p = 0.$$

Thus, $u = 0$ for $|x|$ large. By the Maximum Principle it follows that $u \equiv 0$.

If $\tau = 0$ and we let $R \to \infty$, we conclude that $J < \infty$. But then, returning to (2.6) and letting $R \to \infty$, we find that the right-hand side in (2.6) tends to zero. Hence, $J = 0$ also in this case and we conclude as before that $u \equiv 0$.

PROOF OF THEOREM 4 OF INTRODUCTION. It is a special case of Corollary 2.1 when Σ is the half space $\{x_N > 0\}$ and $u \geq 0$ satisfies

$$\Delta u + x_N u^p \leq 0 \qquad \text{in } \Sigma$$

and u is bounded near the origin. In this case, $g = x_N$ and so $\alpha = 1$, while $\gamma = 1$. From Corollary 2.1 it then follows that $u \equiv 0$ if $1 < p < (N+2)/(N-1)$.

The proof of Theorem 2.1 yields also the following simpler result:

THEOREM 2.2. *Let u be a nonnegative function in $\Sigma = \mathbb{R}^N - \{0\}$, $N \geq 3$, bounded near 0, and satisfying*

$$\Delta u + h(x)u^p \leq 0 \qquad \textit{in } \Sigma.$$

Here, $h \in C(\Sigma)$, $h \geq 0$ and $h(x) = a|x|^\gamma$, $a > 0$, $\gamma > -2$, for $|x|$ large. If

$$p \leq \frac{N+\gamma}{N-2},$$

then $u \equiv 0$.

3. A priori estimates

In this section, for $\rho \geq 0$ we derive a priori estimates, independent of ρ, for the L^∞ norm of positive solutions of the problem

$$\begin{aligned} \rho \geq 0, \quad (\Delta - m(x))u + a(x)g(u) + \rho = 0 \qquad & \text{in } \Omega, \\ Bu = 0 \qquad & \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and B is one of the following boundary operators:

$$Bu := u, \tag{3.2a}$$

$$Bu := \partial_\nu u + \alpha(x)u, \tag{3.2b}$$

where ν is the outer normal to $\partial\Omega$ and $\alpha \geq 0$ is a given continuous function on $\partial\Omega$. We recall the assumption (1.4):

$$\lim_{s\to\infty} \frac{g(s)}{s^p} = l > 0 \qquad \text{for some } p,\ 1 < p < \frac{N+2}{N-1}. \tag{3.3}$$

By scaling we may suppose $l = 1$. Recall also (1.5):

$$\begin{aligned} &a \in C^2(\overline{\Omega}), \quad \Omega^+ := \{x \in \Omega : a(x) > 0\} \neq \emptyset, \\ &\Omega^- := \Omega \setminus \overline{\Omega^+} \neq \emptyset, \\ &\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega \quad \text{and} \quad \nabla a \neq 0 \text{ on } \Gamma. \end{aligned} \tag{3.4}$$

Thus, a neighborhood in Ω of $\partial\Omega$ belongs either to Ω^+ or to Ω^-. We assume $m \in L^\infty$. In case of (3.2b) it follows with the aid of the Hopf lemma that $u > 0$ in $\overline{\Omega}$.

This section is devoted to the proof of

THEOREM 3.1. *Let $u \in W^{2,q}(\Omega)$, for all $q < \infty$, be a positive solution of* (3.1). *Suppose that* (3.3) *and* (3.4) *hold. Then, if $\|m\|_{L^\infty} \leq m_0$,*

$$0 \leq u(x) \leq \overline{C}, \qquad \forall x \in \overline{\Omega}, \tag{3.5}$$

where $\overline{C}$ is a constant depending only on m_0, the function a, and Ω, and it is independent of $\rho \geq 0$.

From (3.5), with the aid of standard elliptic estimates we obtain the a priori estimate

$$\|u\|_{W^{2,q}} \leq \widetilde{C} \qquad \text{for } \ q < \infty, \tag{3.6}$$

where $\widetilde{C}$ depends only on q, m_0, Ω, and the function a.

We need the following

LEMMA 3.1. *If u is a positive solution of* (3.1) *with $\|m\|_{L^\infty} \leq m_0$, then*

$$\rho \leq C \max u,$$

where C depends only on m_0 and Ω^+.

PROOF. Since $g(u) > 0$ for u large, it follows from (3.1) that for some constant C_1 independent of ρ,

$$\Delta u - m_0 u \leq C_1 - \rho \leq -\rho/2 \qquad \text{in } \Omega^+$$

if $\rho > 2C_1$. Let σ be the solution of

$$\begin{aligned} (\Delta - m_0)\sigma = -1 \qquad &\text{in } \Omega^+, \\ \sigma = 0 \qquad &\text{on } \partial\Omega^+. \end{aligned}$$

Clearly, $\sigma > 0$ and for $\rho > 2C_1$, by the maximum principle,

$$w := u - \frac{\rho}{2}\sigma \geq 0 \qquad \text{in } \Omega^+.$$

Since $C_2 := \max \sigma$ depends only on Ω^+ and on m_0, it follows that $\rho \leq (2/C_2)u$ at a maximum point of σ and this yields the lemma.

PROOF OF THEOREM 3.1. We follow the approach in [6]. The proof is by contradiction and makes use of a blow up argument to reduce the problem of establishing the a priori estimate (3.5) to results of Liouville type.

Step 1. Suppose that the conclusion of Theorem 3.1 is false. Then there exist sequences m_j with $||m_j||_{L^\infty} \leq m_0$, $\rho_j \geq 0$ and a sequence $u_j \in W^{2,q}(\Omega)$ such that

$$\begin{aligned} u_j > 0, \quad (\Delta - m_j(x))u_j + a(x)g(u_j) + \rho_j = 0 \qquad &\text{in } \Omega, \\ Bu_j = 0 \qquad &\text{on } \partial\Omega \end{aligned} \tag{3.7}$$

and

$$M_j := \max_{\overline{\Omega}} u_j \to \infty \qquad \text{as } j \to \infty. \tag{3.8}$$

We may assume that $M_j = u_j(x_j)$ for some $x_j \in \overline{\Omega}$, and that, for some $x_0 \in \overline{\Omega}$, $x_j \to x_0$ as $j \to \infty$. Let us introduce new scaled coordinates by setting

$$y = \frac{x - x_j}{\lambda_j}.$$

The positive scale factors λ_j will be chosen later with $\lambda_j \to 0$ as $j \to \infty$. Accordingly, we define a blow up function v_j by

$$v_j(y) = \frac{1}{M_j} u_j(x) = \frac{1}{M_j} u_j(\lambda_j y + x_j). \tag{3.9}$$

The function v_j is well defined for y in a suitable domain and

$$\max v_j = v_j(0) = 1, \qquad j = 1, 2, \ldots, \tag{3.10}$$

On the other hand, a direct computation shows that v_j satisfies

$$L_y v_j + \lambda_j^2 M_j^{p-1}\left(a(\lambda_j y + x_j)\frac{g(M_j v_j)}{M_j^p} + \frac{\rho_j}{M_j^p}\right) = 0, \tag{3.11}$$

where

$$L_y = \Delta_y - \lambda_j^2 m_j(\lambda_j y + x_j).$$

By Lemma 3.1, in every case

$$\rho_j / M_j^p \to 0 \qquad \text{as } j \to \infty. \tag{3.12}$$

Observe also that for $\lambda_j \to 0$, $\lambda_j^2 m_j(\lambda_j y + x_j) \to 0$.

To proceed with the proof we consider several cases according to the location of the limit point x_0, namely:

(1) Case A: $x_0 \in \Gamma$,
(2) Case B: $x_0 \in \Omega^+ \cup \Omega^-$,
(3) Case C: $x_0 \in \partial\Omega$.

Step 2. Let us deal first with Case A. Set

$$\delta_j := \operatorname{dist}(x_j, \Gamma) = |x_j - z_j|, \qquad z_j \in \Gamma.$$

Since $\nabla a \neq 0$ on Γ, it follows that δ_j, which tends to zero, is given by

$$\delta_j = \pm \frac{\nabla a(z_j)}{|\nabla a(z_j)|}(x_j - z_j) \qquad \text{for } j \text{ large},$$

where the plus and the minus occur according as $x_j \in \Omega^+$ or $x_j \in \Omega^-$. Since $a(z_j) = 0$ and $a \in C^2(\Omega)$, we find by Taylor expansion,

$$a(\lambda_j y + x_j) = \pm |\nabla a(z_j)| \delta_j + \lambda_j \nabla a(z_j) \cdot y + O(\lambda_j^2 |y|^2 + \delta_j^2).$$

Substituting this expression into equation (3.11) we find

$$\text{(3.13)} \quad L_y v_j + \lambda_j^3 M_j^{p-1} \Big[\nabla a(z_j) \cdot y \pm \frac{\delta_j}{\lambda_j} |\nabla a(z_j)| + O\Big(\lambda_j^2 |y|^2 + \frac{\delta_j^2}{\lambda_j} \Big) \Big] \cdot \frac{g(M_j v_j)}{M_j^p} + \frac{\lambda_j^2 \rho_j}{M_j} = 0$$

We now choose

$$\text{(3.14)} \qquad \lambda_j = M_j^{(1-p)/3}.$$

Observe that equation (3.13) holds, for large j, in the ball

$$|y| < \frac{1}{2\lambda_j} \operatorname{dist}(x_0, \partial\Omega).$$

We must now consider several cases:

Suppose

(i) $\qquad \delta_j / \lambda_j \to \infty \qquad$ possibly for a subsequence.

Then set

$$\alpha_j := (\lambda_j / \delta_j)^{1/2} \qquad \text{and} \qquad \eta := y / \alpha_j.$$

Under this change of variables, equation (3.13) with λ_j specified by (3.14) becomes

$$\text{(3.15)} \quad L_\eta v_j + \alpha_j^2 \Big[\alpha_j \nabla a(z_j) \cdot \eta \pm \frac{|\nabla a(z_j)|}{\alpha_j^2} + O\Big(\lambda_j \alpha_j^2 |\eta|^2 + \frac{\delta_j}{\alpha_j} \Big) \Big] \frac{g(M_j v_j)}{M_j^p} + \alpha_j^2 \lambda_j^2 \frac{\rho_j}{M_j} = 0,$$

where

$$L_\eta = \Delta_\eta - \alpha_j^2 \lambda_j^2 m_j(\lambda_j \alpha_j \eta + x_j).$$

By standard $W^{2,q}$ theory one may obtain estimates on the $v_j \leq 1$ ensuring that, for a subsequence, $v_j \to v$ locally uniformly, with v defined in all of $\mathbb{R}^N$

and $v \in W^{2,q}_{\text{loc}}$, for all $q < \infty$. Letting $j \to \infty$ in (3.15) we obtain, since $z_j \to x_0$, $\lambda_j, \delta_j, \alpha_j \to 0$ and using Lemma 3.1 and (3.3), that

$$\Delta_\eta v \pm |\nabla a(x_0)| v^p = 0 \qquad \text{in } \mathbb{R}^N. \tag{3.16}$$

This should be explained in more detail. We have only to verify that

$$\sigma_j := \frac{g(M_j v_j)}{M_j^p} \to v^p. \tag{3.17}$$

At points x where $v(x) > 0$ this is clear. Suppose that $v(x) = 0$. Since the v_j are locally uniformly continuous, the set A where $v_j \to 0$ is closed and one easily verifies that $\sigma_j \to 0$ on A. Indeed, if $M_j v_j$ is bounded, then $\sigma_j \to 0$, while if, for a subsequence, $M_j v_j \to \infty$, then $\sigma_j \to v^p(x) = 0$. Hence, from (3.15) it follows that

$$\int_{\mathbb{R}^N} v\Delta\zeta \pm |\nabla a(x_0)| v^p \zeta = 0, \qquad \forall \zeta \in C_0^\infty(\mathbb{R}^N),$$

which implies (3.16).

From (3.16) we find that $v \in C^2$. Furthermore, from (3.10),

$$v \geq 0, \qquad \max v = v(0) = 1.$$

Because of the maximum principle, the minus sign cannot occur in (3.16), for $\nabla a(x_0) \neq 0$. Therefore, v satisfies

$$v \geq 0, \quad \Delta_\eta v + |\nabla a(x_0)| v^p = 0 \qquad \text{in } \mathbb{R}^N.$$

By Theorem 1.2 in [6], this implies $v \equiv 0$, a contradiction with $v(0) = 1$.

Consider next the case

(ii) $\qquad \delta_j/\lambda_j \to 0 \qquad$ for a subsequence.

Now let $j \to \infty$ in (3.13). As before, for a subsequence, $v_j \to v$. The limit equation is now

$$v \geq 0, \quad \Delta_y v + (\nabla a(x_0) \cdot y) v^p = 0 \qquad \text{in } \mathbb{R}^N.$$

After a suitable rotation and rescaling, the equation reads

$$v \geq 0, \quad \Delta_\zeta v + \zeta_N v^p = 0 \qquad \text{in } \mathbb{R}^N.$$

We now apply Theorem 4 and conclude that $v \equiv 0$, again a contradiction with $v(0) = 1$.

Finally, the case

(iii) $\qquad \delta_j/\lambda_j \to \delta_0 > 0 \qquad$ for some subsequence.

The limit equation is now

$$v \geq 0, \quad \Delta_y v + (\pm\delta_0|\nabla a(x_0)| + \nabla a(x_0) \cdot y) v^p = 0 \qquad \text{in } \mathbb{R}^N. \tag{3.18}$$

The minus sign is impossible since at the maximum point $y = 0$, (3.18) would imply

$$\Delta_y v = \delta_0 |\nabla a(x_0)| > 0.$$

After a suitable change of variables and rescaling of v, (3.18) can be written as

$$v \geq 0, \quad \Delta_\zeta v + \zeta_N v^p = 0 \qquad \text{in } \mathbb{R}^N,$$

with $v(0, \dots, 0, h) = 1$ for some $h \geq 0$.

Applying once more Theorem 4 we are led again to a contradiction.

Step 3. Consider now Case B, i.e. $x_0 \in \Omega^+ \cup \Omega^-$. This time choose

$$\lambda_j = M_j^{(1-p)/2}$$

in (3.11). Since $x_0 \in \Omega$, the same argument—Lemma 3.1 ensures that $\lambda_j^2 \rho_j / M_j \to 0$—employed in the previous case leads to the limit equation

$$0 \leq v \leq 1, \quad \Delta_y v + a(x_0) v^p = 0 \qquad \text{in all of } \mathbb{R}^N.$$

Since $v(0) = 1 = \max v$, the above and the maximum principle imply $a(x_0) > 0$.

Once more, by Theorem 1.2 in [6], $v \equiv 0$, contradicting $v(0) = 1$.

Step 4. We pass then to Case C, i.e. $x_0 \in \partial\Omega$. Here

$$d_j := \operatorname{dist}(x_j, \partial\Omega) \to 0.$$

We need to consider two subcases:

(a) $d_j M_j^{(p-1)/2} \to \infty$ for a subsequence,

(b) $d_j M_j^{(p-1)/2} \to \delta_0 \geq 0$ for a subsequence.

Without loss of generality, we may assume that x_0 is the origin and that the exterior normal there is $-e_N = -(0, \dots, 0, 1)$. For all cases, we choose

$$\lambda_j = M_j^{(1-p)/2}.$$

In case (a), $d_j / \lambda_j \to \infty$. As before, letting $j \to \infty$, through a subsequence, in (3.11), we obtain a limit function v defined, then, in all of $\mathbb{R}^N$, and it satisfies

$$0 \leq v \leq 1 = v(0), \quad \Delta_y v + a(x_0) v^p = 0 \quad \text{in } \mathbb{R}^N. \tag{3.19}$$

Since $a(x_0) \neq 0$, the desired contradiction follows as in Case B.

In case (b), for a subsequence, $d_j / \lambda_j \to \delta_0 \geq 0$. Going to the limit in (3.11) we nowin fd that (3.19) holds in $\{y \in \mathbb{R}^N : y_N > -\delta_0\}$, and

$$v = 0 \tag{3.20a}$$

or

$$v_N = 0, \tag{3.20b}$$

and, since now $\psi = \varphi$,

$$\alpha \int_\Omega a(x)\varphi^{q+1} < 0. \tag{5.3}$$

We have to prove that (5.1) has a solution when $\tau = 0$. For $\tau > 0$, (5.1) has a solution u_τ and $u_\tau \le \overline{C}$. We have only to show that, as $\tau \to 0$, any positive solution stays away from the origin. Then, for a sequence $\tau_i \to 0$, the u_{τ_i} converge to a nonzero solution of (5.1) with $\tau = 0$. To show that u_τ stays away from zero we carry out a standard bifurcation analysis, using Lyapunov-Schmidt decomposition.

We show that for $0 < \tau$ small, there is no solution of (5.1) with small L^∞ norm. Suppose there were such a solution u; decompose it as a sum

$$u = t\varphi + v \qquad \text{with } \int_\Omega v\varphi = 0. \tag{5.4}$$

Then

$$(-\Delta + m)v = -\tau(t\varphi + v) + a(x)g(t\varphi + v) \tag{5.5}$$

and, necessarily, the right-hand side of (5.5) is L^2-orthogonal to φ.

For the general operator L we would take v and the right-hand side of (5.5) orthogonal to ψ.

In the space of functions orthogonal to φ, the operator $-\Delta+m$ has a bounded inverse from L^p to $W^{2,p}$ for any p in $(1,\infty)$. It follows that

$$||v||_{W^{2,p}} \le C\tau(|t| + ||v||_{L^p}) + C||g(t\varphi + v)||_{L^\infty}. \tag{5.6}$$

If the L^∞ norm of u is small, so are $|t|$ and the L^∞ norm of v. Consequently, for τ small, by (5.6),

$$||v||_{W^{2,p}} \le C\tau|t| + C(|t|^q + ||v||^q_{L^\infty}).$$

Since $q > 1$ and $||v||_{L^\infty} \le C||v||_{W^{2,p}}$, it follows that

$$||v||_{L^\infty} = |t|O(\tau + |t|^{q-1}). \tag{5.7}$$

From $u > 0$ it follows that $t \ne 0$, in fact $t > 0$. Next, we use the condition that the right-hand side of (5.5) is L^2-orthogonal to φ; we may now suppose $\int_\Omega \varphi^2 = 1$. Setting $v/t = w$ we find, on multiplying (5.5) by $\psi = \varphi$ and integrating,

$$\begin{aligned} t\tau &= \int_\Omega a(x)g(t\varphi + tw)\varphi \\ &= \alpha t^q \int_\Omega a(x)(\varphi + w)^q\varphi + o(t^q) \qquad \text{(by (5.2))} \\ &= \alpha t^q \int_\Omega a(x)\varphi^{q+1} + O(\tau t^q) + o(t^q) \quad \text{(by (5.7))}. \end{aligned} \tag{5.8}$$

Since $\tau, t > 0$, we see from (5.3) that this is impossible.

6. Proof of Theorem 2

Step 1. Using the same bifurcation analysis as above, we show first that for $0 < -\tau$ small, (5.1) has a solution $u = u_\tau$ which is close to 0 (we may assume $g(s) = 0$ for $s < 0$).

This follows from

LEMMA 6.1. *Under the conditions of Theorem* 2, *i.e.* (1.3) *and* (5.2), (5.3), *there is an interval* $I = [0, t_0)$ *and a continuous function* $\delta(t)$ *on* I, *with (recall* $\varphi = \psi$)

$$\delta(0) = \delta_0 := \alpha \int_\Omega a\varphi^{q+1} < 0,$$

and a C^1 *function* $u(t,x)$ *for* $t \in I$, $x \in \overline{\Omega}$, *which for* $0 < t < t_0$ *is a solution of* (5.1) *with* $\tau = t^{q-1}\delta(t)$.

PROOF. In (5.1) write u as in (5.4). We carry out the standard Lyapunov-Schmidt procedure. Consider the Banach spaces $X = \{u \in W^{2,p}(\Omega) : Bu = 0\}$ and $Y = L^p(\Omega)$, for some fixed $p > N$. Let P be the L^2-orthogonal projection onto the subspace Y_1 of Y consisting of functions orthogonal to φ. We decompose (5.1) into two pieces which are to be solved for $v \in X$ and $\tau \leq 0$, depending on t:

$$(-\Delta + m)v = -\tau v + P[ag(t\varphi + v)] \tag{6.1}$$

$$t\tau = \int_\Omega ag(t\varphi + v)\varphi. \tag{6.2}$$

The right-hand side of (6.1) lies in $Y_1 = PY$ and it is of class C^1 in v, τ and t. With the aid of the Implicit Function Theorem we may solve (6.1) uniquely for $v = v(t, \tau)$ with $v(0,0) = 0$. For t in some small interval I, and $|\tau|$ small, v belongs to C^1. Furthermore, (5.6) holds and the derivatives v_t and v_τ vanish at $(0,0)$. In fact, we find from (6.1), after taking the τ derivative,

$$(-\Delta + m)v_\tau = -v - \tau v_\tau + P[ag'(t\varphi + v)v_\tau].$$

Using (5.6) it follows that

$$||v_\tau||_{W^{2,p}} \leq C(t|\tau| + t^q). \tag{6.3}$$

We now substitute $v(t,\tau)$ in (6.2) and obtain

$$t\tau = \int_\Omega ag(t\varphi + v(t,\tau))\varphi. \tag{6.3}$$

This we solve for τ in the form $\tau = \delta t^{q-1}$, i.e. we solve for δ,

$$\delta = \int_\Omega a\varphi \frac{g(t\varphi + v(t, \delta t^{q-1}))}{t^q} =: G(t, \delta). \tag{6.4}$$

As a function of $t \in I$ and of δ, $G(0,\delta) = \delta_0$. Furthermore, G is continuous in t and of class C^1 in δ. Indeed,

$$\frac{\partial}{\partial \delta} \frac{g(t\varphi + v(t, \delta t^{q-1}))}{t^q} = \frac{g'(t\varphi + v(t, \delta t^{q-1}))v_\tau}{t}.$$

By (6.3) we see that $G_\delta(t,\delta) = O(t^{q-1})$. We may therefore use the Implicit Function Theorem again and solve (6.4) for $\delta(t)$ with $\delta(0) = \delta_0$. The function $\delta(t)$ is continuous on I (possibly shortened). Then $u = t\varphi + v(t, t^{q-1}\delta(t))$ is a solution of (5.1) with $\tau = t^{q-1}\delta(t)$. The set of such τ covers a small interval $(\tau_0, 0)$.

Step 2. Completion of the proof of Theorem 2.

In Lemma 1 we showed that for τ large negative, problem (5.1) has no solution. The last assertion of Theorem 2 then follows from Lemma 6.1 and the following:

LEMMA 6.2. *If* (5.1) *has a solution* u_{τ_1} *for some* $\tau_1 < 0$, *then it has a solution for every* τ *in* $(\tau_1, 0)$.

PROOF. For every given τ in $(\tau_1, 0)$ consider problem (5.1). The function $\overline{u} = u_{\tau_1}$ is then a supersolution for (5.1). On the other hand, for $0 < \varepsilon$ small, the function $\underline{u} = \varepsilon\varphi$ is a subsolution. Furthermore, for ε small, $\underline{u} < \overline{u}$. This is clear for the Dirichlet problem, with the aid of the Hopf lemma. For B given by (1.2b), it holds because $\overline{u}$ and φ are positive in $\overline{\Omega}$.

By the well known theory of sub- and supersolutions (see for example [2]), there is a solution $u = u_\tau$ of (5.1) with $\underline{u} < u < \overline{u}$.

Theorem 2 is proved.

REFERENCES

[1] S. ALAMA AND G. TARANTELLO, *On semilinear elliptic equations with indefinite nonlinearities*, Calculus of Variations and PDE **1** (1993), 439–475.

[2] H. AMANN, *On the existence of positive solutions of nonlinear elliptic boundary value problems*, Indiana Univ. Math. J. **21** (1971), 125–146.

[3] H. BERESTYCKI, I. CAPUZZO DOLCETTA ET L. NIRENBERG, *Problèmes elliptiques indéfinis et théorèmes de Liouville nonlinéaires*, C. R. Acad. Sci. Paris Sér I **317** (1993), 945–950.

[4] ———, *Variational methods for indefinite superlinear homogeneous elliptic problems* (to appear).

[5] H. BERESTYCKI, L. NIRENBERG AND S. R. S. VARADHAN, *The principal eigenvalue and maximum principle for second order elliptic operators in general domains*, Comm. Pure Appl. Math. **47** (1994), 47–93.

[6] B. GIDAS AND J. SPRUCK, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **8** (1981), 883–901.

[7] J. LERAY ET J. SCHAUDER, *Topologie et équations fonctionnelles*, Ann. Sci. Ecole Norm. Sup. **51** (1934), 45–75.

[8] T. OUYANG, *On positive solutions of semilinear equations* $\Delta u+\lambda u+hu^p = 0$ *on compact manifolds,* Part II, Indiana Univ. Math. J. **40** (1991), 1083–1141.

Manuscript received May 10, 1994

H. BERESTYCKI
Analyse Numérique
Université Pierre et Marie Curie
Tour 55-65, 5ème étage, 4 Place Jussieu
75252 Paris Cedex 05, FRANCE

I. CAPUZZO DOLCETTA
Università di Roma La Sapienza
Dipartimento di Matematica, Instituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2, I-00185 Roma, ITALY

L. NIRENBERG
Courant Institute, NYU
251 Mercer St.
New York, NY 10012, USA

E-mail address: nirenl@cims5.cims.nyu.edu