Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 4, 1994, 31–57

MULTIBUMP PERIODIC SOLUTIONS OF A FAMILY OF HAMILTONIAN SYSTEMS

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Dedicated to Jean Leray

1. Introduction

Several recent papers have used global variational methods to establish the establish the existence of multibump solutions of families of superquadratic Hamiltonian systems. Such solutions are homoclinic solutions of the equations. See e.g. Séré [10–11], Chang and Liu [3], Bessi [2], Alama and Li [1] and Coti Zelati and Rabinowitz [4–5]. This paper shows how to modify the methods of [4–5] to obtain what we call multibump periodic solutions for the setting of [4]. To describe these solutions, the setting of [4] will be recalled. Thus consider the second order Hamiltonian system

$$\text{(HS)} \qquad \ddot{q} - L(t)q + V_q(t,q) = 0$$

where L and V satisfy

(L) L is a symmetric $n \times n$ matrix, continuous and T-periodic in t, and uniformly positive definite for $t \in [0,T]$,

(V_1) $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and is T-periodic in t,

(V_2) $V_{qq}(t,0) = 0$, $t \in \mathbb{R}$.

(V_3) There is a $\mu > 0$ such that for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$,

1991 *Mathematics Subject Classification.* Primary 34C25; Secondary 34C37.

[1]This research was sponsored in part by the National Science Foundation under grant #MCS–8110556 and the U.S. Army Research Office under Contract #DAAL03–87–K–0043.

$$0 < \mu V(t,q) \le q \cdot V_q(t,q).$$

Homoclinic solutions of (HS) were obtained as critical points of the functional

$$I(q) = \int_{\mathbb{R}} \left[\tfrac{1}{2}(|\dot q|^2 + L(t)q\cdot q) - V(t,q)\right] dt \tag{1.1}$$

on $E \equiv W^{1,2}(\mathbb{R},\mathbb{R}^n)$. Indeed any critical point q of I on E satisfies (HS) and $q(t)$, $\dot q(t) \to 0$ as $|t| \to \infty$. Thus q is homoclinic to 0.

Some further notation is needed. For $a, b \in \mathbb{R}$, let

$$I^a = \{q \in E \mid I(q) \le a\}, \quad I_b = \{q \in E \mid I(q) \ge b\}$$

and $I_b^a = I^a \cap I_b$. Let $\mathcal{K}$ denote the set of critical points of I on E, i.e.

$$\mathcal{K} = \{q \in E \mid I'(q) = 0\}.$$

Set

$$\mathcal{K}^a = \mathcal{K} \cap I^a; \qquad \mathcal{K}_b = \mathcal{K} \cap I_b; \qquad \mathcal{K}_b^a = \mathcal{K} \cap I_b^a$$

and $\mathcal{K}(a) = \mathcal{K} \cap I_a^a$. Let

$$\Gamma = \{g \in C([0,1],E) \mid g(0) = 0 \text{ and } g(1) \in I^0\backslash\{0\}\} \tag{1.2}$$

and

$$c = \inf_{g\in\Gamma} \max_{\theta\in[0,1]} I(g(\theta)), \tag{1.3}$$

i.e. c is the mountain pass minimax value associated with I. Note that by (L) and (V_1), the functional I possesses a natural $\mathbb{Z}$ symmetry. Namely if $j \in \mathbb{Z}$, $q \in E$, and

$$\tau_j q(t) = q(t - jT), \tag{1.4}$$

then

$$I(\tau_j q) = I(q) \tag{1.5}$$

for all $q \in E$. Suppose that

($*$) there is an $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}/\mathbb{Z}$ is finite.

Under the hypotheses (L), (V_1)–(V_3), and ($*$), it was shown in [4] that c is a critical value of I and moreover for each $k \in \mathbb{N}\backslash\{1\}$, $\mathcal{K}_{kc-\alpha}^{kc+\alpha}$ is infinite. This latter fact is a consequence of a more precise result: there is a finite set $A \subset \mathcal{K}(c)$ such that for any sufficiently small $r > 0$ and for any $k \in \mathbb{N}\backslash\{1\}$, there is an $\ell_0 = \ell_0(r,k)$ having the property that whenever $\ell = (\ell_1,\dots,\ell_k) \in \mathbb{Z}^k$ with $\ell_{i+1} - \ell_i \ge \ell_0$, $1 \le i \le k-1$, then

$$N_r(\mathcal{A}(\ell)) \cap \mathcal{K} \ne \emptyset. \tag{1.6}$$

In (1.6),

$$\mathcal{A}(\ell) = \left\{ \sum_{i=1}^{k} \tau_{\ell_i} v_i \,\middle|\, v_i \in A \right\}$$

and

$$N_r(S) = \{x \in E \mid \|x - S\| < r\}.$$

The number ℓ_0 is such that for $v \in A$, most of the "mass" of $v(t)$ lies in a region whose diameter is small compared to ℓ_0, i.e. $v(t)$ is near 0 for e.g. $|t| \geq \ell_0/4$. Hence if $v_1, \ldots, v_k \in A$, the bulk of the mass of $\sum_{i=1}^{k} \tau_{\ell_i} v_i$ lies in k disjoint intervals and the function has k-"bumps". Thus for r small, any $q \in B_r(\sum_{i=1}^{k} \tau_{\ell_i} v_i) \cap \mathcal{K}$ is a k-bump solution of (HS).

Note that ℓ_0 above depends on k. In a more refined result in his setting, Séré [11] obtains (1.6) with ℓ_0 independent of k but B_r replaced by a larger neighborhood. See also Bessi [2] for a k-independent ℓ_0.

In this paper, the existence of multibump periodic solutions of (HS) will be obtained. In particular, it will be shown there is a $j_0 \in \mathbb{N}$ such that for $j \in \mathbb{N}$, $j \geq j_0$, there is a jT, periodic solution q of (HS) which is "near" some element of A in a sense that will be made precise later. Such a q will be called a *1-bump periodic solution* of (HS). Similarly, there is a jT-periodic solution of (HS) near $\mathcal{A}(\ell)$ provided that $\ell_{i+1} - \ell_i \geq \ell_0(r)$ is appropriately large and $j \geq k\ell_0$. Such a q will be called a *k-bump periodic solution* of (HS).

There is a classical dynamical systems approach to homoclinics (see e.g. [6]) where if a certain tranversality condition is satisfied, there is a symbolic dynamics that associates with each bi-infinite sequence of 0's and 1's a solution of (HS). The transversality condition can often be satisfied for perturbation problems or somewhat more generally when $n = 1$. In the global variational setting, it is conjectured that condition $(*)$ or similar hypotheses are weaker than the classical transversality condition. However, this has only been shown for one special case by Bessi [2]. In any event, the multibump solutions of [1–5, 10–11] and the multibump periodic solutions obtained here correspond to subsets of the symbolic dynamics of solutions obtained classically.

The existence of subharmonic solutions (that is, jT-periodic solutions) for systems like (HS) has been proved in various settings (see for example [9]). They differ from the solutions found here in two ways. Our solutions are multi-bumps and they are bounded in L^∞ independently of j. In [9], however, the solutions go to zero or to ∞ as j tends to ∞ depending on the behavior of V at ∞.

The existence of 1-bump periodics will be proved in §2 and the general k-bump case will be treated in §3. Lastly, infinite bump solutions will be obtained in §4.

2. One-bump solutions

The existence of 1-bump periodic solutions can be obtained as a special case of the k-bump result. However, we prefer to prove the 1-bump case first since it enables us to introduce several of the ideas needed in a considerably simpler setting and it allows us to be sketchy in parts of the treatment of the k-bump case. Several results from [4] will be required and the notation of [4] will mainly be followed.

To begin, for simplicity we take $L(t) =$ id and also set $T = 1$.

By $(*)$, the critical points of I on E in $\mathcal{K}^{c+\alpha}$ are isolated. Moreover, by (V_1), $v \in \mathcal{K}^{c+\alpha}$ implies $\tau_j v \in \mathcal{K}^{c+\alpha}$ for all $j \in \mathbb{Z}$. Therefore if $v \in \mathcal{K}(c)$ is normalized to make it unique by requiring e.g. that $\|v\|_{L^\infty} = |v(\bar{t})|$ for some $\bar{t} \in [0,1)$ and $|v(t)| < \|v\|_{L^\infty}$ for $t < 0$, there are only finitely many normalized $v \in \mathcal{K}(c)$. Let $\mathcal{F}$ denote the set of normalized $v \in \mathcal{K}(c)$ and set

$$\mathcal{T}_\ell(\mathcal{F}) = \left\{ \sum_{i=1}^{j} \tau_{k_i} v_i \;\middle|\; 1 \leq j \leq \ell,\ v_i \in \mathcal{F},\ k_i \in \mathbb{Z} \right\}.$$

Then (see e.g. Proposition 1.55 of [4])

$$\mu(\mathcal{T}_\ell(\mathcal{F})) \equiv \inf\{\|x - y\| \mid x \neq y \in \mathcal{T}_\ell(\mathcal{F})\}$$

is positive. It was also shown in [4] that there is a $\underline{c} > 0$ such that $I(q) \geq \underline{c}$ for all $q \in \mathcal{K}\setminus\{0\}$. Let $\bar{\ell}$ be the largest integer not exceeding $(c+\alpha)\underline{c}^{-1}$ and set $\mu = \mu_{\bar{\ell}}(\mathcal{F})$.

The fact that points in $\mathcal{K}^{c+\alpha}$ are isolated leads to a uniform lower bound on $\|I'\|$ in associated annular regions that is crucial for what follows. In the sequel, even if not explicitly stated, it is always assumed that (L), (V_1)–(V_3), and $(*)$ hold.

PROPOSITION 2.1. *Let $0 < s \leq r < \frac{1}{24}\mu$. Then there is a $\delta = \delta(r,s) > 0$ such that*

$$\|I'(x)\| \geq 4\delta \tag{2.2}$$

for

$$x \in \overline{N}_{8r}(\mathcal{K}(c)) \setminus N_{s/32}(\mathcal{K}(c)).$$

PROOF. Since $\|I'(z)\| = \|I'(\tau_j z)\|$ for all $j \in \mathbb{Z}$, it suffices to prove (2.2) for all $z \in \overline{B}_{8r}(v)\setminus B_{s/32}(v)$ for all normalized $v \in \mathcal{K}(c)$. If (2.2) fails, then for some such v, there is a sequence $(z_m) \subset \overline{B}_{8r}(v)\setminus B_{s/32}(v)$ such that

$$I'(z_m) \to 0. \tag{2.3}$$

Moreover,

$$(2.4)\qquad \begin{aligned} I(z_m) &= c + \int_0^1 \frac{d}{d\theta} I(v + \theta(z_m - v))\, d\theta \\ &= c + \int_0^1 I'(v + \theta(z_m - v))(z_m - v)\, d\theta. \end{aligned}$$

The form of I' shows that it is bounded on bounded sets. Consequently, (z_m) is a Palais-Smale sequence, i.e. $I(z_m)$ is bounded and (2.3) holds. By Proposition 1.24 of [4], there are $j \in \mathbb{N}$, $w_1, \dots, w_j \in \mathcal{K}$ and $(k_m^i) \subset \mathbb{Z}$, $1 \le i \le j$, such that as $m \to \infty$, along a subsequence of z_m,

$$(2.5)\qquad \left\| z_m - \sum_{i=1}^{j} \tau_{k_m^i} w_i \right\| \to 0,$$

$$(2.6)\qquad |k_m^i - k_m^p| \to \infty \qquad \text{if } i \neq p$$

and

$$(2.7)\qquad \sum_{i=1}^{j} I(w_i) = c.$$

By (2.5), (2.7) and the fact that $I(w) > 0$ for $w \in \mathcal{K}(c)\backslash\{0\}$ (see [4]) $z_m \to \mathcal{T}_{\bar{\ell}}(\mathcal{F})$. But this is impossible since $(z_m) \subset \overline{B}_{8r}(v)\backslash B_{s/32}(v)$.

To continue, let

$$(2.8)\qquad \alpha_1 = \sup\{\beta < \alpha \mid \mathcal{K}_{c-\beta}^{c+\beta} = \mathcal{K}(c)\}.$$

Then $\alpha_1 > 0$ by (∗). The following result from [4] provides us with the set A mentioned in the introduction and which is needed to formulate the existence theorems here.

PROPOSITION 2.9. *There is a finite set $A \subset \mathcal{K}(c)$ with the property that whenever $\bar{\varepsilon}_1 \in (0, \alpha_1)$, $r_1 \in (0, \mu/12)$, and $p \in \mathbb{N}$, then there is a constant $\varepsilon_1 \in (0, \bar{\varepsilon}_1)$ and $g_1 \in \Gamma$ satisfying*

1° $\max_{\theta \in [0,1]} g_1(\theta) \le c + \varepsilon_1/p$ *and*
2° $I(g_1(\theta)) > c - 2\varepsilon_1$ *implies* $g_1(\theta) \in N_{r_1/16}(A)$.

Now let $r < \mu/24$, set $s = r$ and take $\delta = \delta(r, r)$ as given by Proposition 2.1. Then for each $v \in A$, a $v^* \in E$ can be chosen such that v^* has compact support and

$$(2.10)\qquad \|v - v^*\| \le r/32.$$

Let $A^* = \{v^* \mid v \in A\}$. By Proposition 2.1 and (2.10),

$$(2.11)\qquad \|I'(x)\| \ge 4\delta, \qquad x \in \overline{N}_{7r}(A^*)\backslash N_{r/16}(A^*).$$

By Proposition 2.9 with

$$\overline{\varepsilon}_1 = \min(\alpha_1/2, r\delta, (r), 2, c/2), \tag{2.12}$$

$r_1 = r$, and $p = 8$, there exist $\varepsilon \in (0, \overline{\varepsilon}_1)$ and $g_1 \in \Gamma$ satisfying 1°–2° of the proposition. Then $g \in \Gamma$ can be chosen so that g has compact support (uniformly in $\theta \in [0,1]$) and satisfies

$$\max_{\theta \in [0,1]} I(g(\theta)) \le c + \varepsilon/4 \tag{2.13}$$

and

$$I(g(\theta)) > c - \varepsilon \quad \text{implies} \quad g(\theta) \in N_{r/8}(A). \tag{2.14}$$

Consequently, by (2.10),

$$I(g(\theta)) > c - \varepsilon \quad \text{implies} \quad g(\theta) \in N_{r/4}(A^*). \tag{2.15}$$

Since the functions $v^* \in A^*$ all have compact support as do $g(\theta)$, $\theta \in [0,1]$, there is an $R \in \mathbb{N}$ such that

$$\operatorname{supp} v^*, \operatorname{supp} g(\theta) \subset [-R, R] \tag{2.16}$$

for all $v^* \in A^*$ and $\theta \in [0,1]$. Note that $R = R(r)$.

Let $j \in \mathbb{N}$ and define

$$E_j = \{q \in W^{1,2}([-j,j], \mathbb{R}^n) \mid q \text{ is } 2j\text{-periodic in } t\}. \tag{2.17}$$

Then E_j is a Hilbert space under the associated norm

$$\|q\|_j^2 = \int_{-j}^{j} (|\dot{q}|^2 + |q|^2)\, dt.$$

Note that if $j > R$, then $v^*|_{[-j,j]}$ extends in a natural way to an element of E_j which will be denoted by $\widehat{v}$. Define $\widehat{A} = \{\widehat{v} \mid v^* \in A^*\}$. Let

$$I_j(q) = \int_{-j}^{j} \mathcal{L}(q)\, dt$$

where

$$\mathcal{L}(q) = \tfrac{1}{2}(|\dot{q}|^2 + |q|^2) - V(t,q).$$

Any critical point of I_j in E_j is a $2j$-periodic solution of (HS). Let $\mathcal{K}_j$ denote the set of critical points of I_j on E_j. Let $B_r^j(z)$ denote an open ball of radius r in E_j about z and $N_r^j(S)$ the analogue of $N_r(S)$ for E_j.

Now the basic existence theorem for 1-bump periodic solutions of (HS) can be stated.

THEOREM 2.18. *Let* (L), (V_1)–(V_3), *and* (*) *hold. Then for r sufficiently small, there is a $j_0(r) > R$ such that for each $j \in \mathbb{N}$ with $j \geq j_0$,*

$$N_r^j(\widehat{A}) \cap \mathcal{K}_j \neq \emptyset.$$

The idea of the proof of Theorem 2.18 is the following. Suppose

$$N_r^j(\widehat{A}) \cap \mathcal{K}_j = \emptyset. \tag{2.19}$$

Consider $g \in \Gamma$ obtained earlier satisfying (2.13)–(2.14). Since supp $g \subset [-R, R]$, g extends naturally to $\widehat{g} \in C([0,1], E_j)$ and by (2.13),

$$\max_{\theta\in[0,1]} I_j(\widehat{g}(\theta)) = \max_{\theta\in[0,1]} I(g(\theta)) \leq c + \varepsilon/4. \tag{2.20}$$

Using (2.19) and Proposition 2.1, $\widehat{g}$ will be deformed to $\widehat{\eta} \in C([0,1], E_j)$ such that

$$\|\widehat{\eta}(\theta) - \widehat{g}(\theta)\|_j \leq 2r \tag{2.21}$$

for all $\theta \in [0,1]$ and

$$\max_{\theta\in[0,1]} I_j(\widehat{\eta}(\theta)) \leq c - \varepsilon/2. \tag{2.22}$$

Using (2.21), $\widehat{\eta}$ will be approximated by a nearby $\widehat{h} \in C([0,1], E_j)$ such that $\widehat{h}$ vanishes near $|t| = j$ and

$$\max_{\theta\in[0,1]} I_j(\widehat{h}(\theta)) \leq c - \varepsilon/4. \tag{2.23}$$

Thus defining

$$h(\theta)(t) = \begin{cases} \widehat{h}(\theta)(t), & |t| \leq j, \\ 0, & |t| > j, \end{cases}$$

produces $h \in \Gamma$ satisfying

$$\max_{\theta\in[0,1]} I(h(\theta)) \leq c - \varepsilon/4. \tag{2.24}$$

But (2.24) contradicts to (1.3).

To carry out the details of this sketch, first the behavior of I_j in $N_r^j(\widehat{A})$ will be studied.

PROPOSITION 2.25. *There is a $j_0(r) > R$ such that whenever $j \geq j_0$ and $x \in \overline{N_r^j}(\widehat{A}) \backslash N_{r/8}^j(\widehat{A})$, there exists a $\varphi(x) \in E_j$ with $\|\varphi(x)\|_j = 1$ and*

$$I_j'(x)\varphi(x) \geq 2\delta(r) \tag{2.26}$$

where $\delta(r) \equiv \delta(r,r)$ is given by Proposition 2.1.

PROOF. Let $\widehat{w} \in \widehat{A}$ and $\zeta \in E_j$ and $r/8 \leq \|\zeta\|_j \leq r$. Set $j_0(r) = \ell_0(r) \equiv \gamma(r) + R$ where $\gamma(r) \in \mathbb{N}$ and is free for now. Since E_j consists of $2j$-periodic functions, we can identify $t = j$ with $t = -j$. Then, by the definition of R, $\widehat{w}$

vanishes on an interval S_1 of length 2γ. Therefore there is an interval $S \subset S_1$ of length 2 such that

$$\|\zeta\|_{W^{1,2}(S)} \leq r\gamma^{-1/2}. \tag{2.27}$$

Let

$$M_1 = \max_{v \in A} \|v\| + 1.$$

Then $\widehat{v}$ can be chosen so that

$$M_1 \geq \max_{\widehat{v} \in \widehat{A}} \|\widehat{v}\|_{L^\infty} + 1. \tag{2.28}$$

Set

$$M = \max_{t \in \mathbb{R}, |x| \leq M_1} |V_{qq}(t,x)| \tag{2.29}$$

where $|V_{qq}|$ denotes the sum of the components of V_{qq}. Choose $\gamma(r)$ so large that if $\widehat{z} \in E_j$ is defined via

$$\widehat{z}(t) = \begin{cases} \zeta(t), & t \notin S, \\ 0, & t \text{ in a unit interval about the center of } S, \\ \text{linearly interpolated otherwise,} \end{cases} \tag{2.30}$$

then

$$\|\zeta - \widehat{z}\|_j \leq \frac{\delta}{4(1+M)}. \tag{2.31}$$

It can be assumed that δ satisfies

$$\frac{\delta}{4(1+M)} \leq \frac{r}{16}. \tag{2.32}$$

Therefore

$$\frac{r}{16} \leq \|\zeta\|_j - \|\zeta - \widehat{z}\|_j \leq \|\widehat{z}\|_j \leq \|\zeta\|_j + \|\zeta - \widehat{z}\|_j \leq \frac{3r}{2}. \tag{2.33}$$

Moreover, if $\overline{\varphi} \in E_j$ and $\|\overline{\varphi}\|_j \leq 1$, then

$$\begin{aligned} |I_j'(\widehat{w} + \zeta)\overline{\varphi} - I_j'(\widehat{w} + \widehat{z})\overline{\varphi}| &\leq \|\zeta - \widehat{z}\|_j \\ &+ \left| \int_S (V_q(t, \widehat{w} + \zeta) - V_q(t, \widehat{w} + \widehat{z})) \cdot \overline{\varphi}\, dt \right| \\ &\leq (1+M)\|\zeta - \widehat{z}\|_j \leq \frac{\delta}{4}. \end{aligned} \tag{2.34}$$

Suppose $j \in S$. Then defining $z(t) = \widehat{z}(t)$ for $t \in [-j, j]$, and $z(t) = 0$ for $|t| > j$, gives

$$I_j'(\widehat{w} + \widehat{z})\overline{\varphi} = I'(w + z)\overline{\varphi}. \tag{2.35}$$

Since $w = w^* \in A^*$ and $\|z\| = \|\widehat{z}\|_j$, it follows from (2.33) and (2.11) that there is a $\widetilde{\varphi} \in E$ such that

$$I'(w+z)\widetilde{\varphi} \geq 4\delta.$$

Since $w+z$ has compact support in $[-\ell_0, \ell_0]$, it is possible to find $\overline{\varphi}$ with supp $\overline{\varphi} \subset [-\ell_0, \ell_0]$, $\|\overline{\varphi}\| = 1$, and

$$I'(w+z)\overline{\varphi} \geq 3\delta \,. \tag{2.36}$$

Redefining $\overline{\varphi}$ outside $[-\ell_0, \ell_0]$ to get a $2j$-periodic function φ with $\|\varphi\|_j = 1$, (2.36) yields

$$I_j'(\widehat{w}+\widehat{z})\varphi \geq 3\delta \,. \tag{2.37}$$

Combining (2.37) and (2.34) gives (2.26) and Proposition 2.25. If $j \notin S$, replacing φ by $\tau_{-k}\varphi$ for an appropriate k reduces the problem to the case just treated.

With the aid of Proposition 2.25 and (2.19) a locally Lipschitz continuous function $\mathcal{V}$ on E_j can be constructed so that $\|\mathcal{V}(x)\|_j \leq 1$ for all $x \in E_j$,

$$I_j'(x)\mathcal{V}(x) \geq 2\delta, \qquad x \in \overline{N}_r^j(\widehat{A}) \backslash N_{r/8}^j(\widehat{A}), \tag{2.38}$$

and for some $\delta_j > 0$,

$$I_j'(x)\mathcal{V}(x) \geq \delta_j. \tag{2.39}$$

Indeed, by (2.19) for each $x \in \overline{N}_{r/8}(\widehat{A})$, there is a $\varphi(x)$ such that $\|\varphi(x)\|_j \leq 1$ and

$$I_j'(x)\varphi(x) \geq \tfrac{1}{2}\|I_j'(x)\| > 0. \tag{2.40}$$

Let

$$\delta_j = \inf_{x \in N_{r/8}^j(\widehat{A})} \|I_j'(x)\| \,. \tag{2.41}$$

Then $\delta_j > 0$ for otherwise there are $\widehat{w} \in \widehat{A}$ and $(z_m) \subset B_{r/8}^j(\widehat{w})$ such that $I_j'(\widehat{w}+z_m) \to 0$. But I_j satisfies the Palais-Smale condition (see e.g. [8]) and therefore $z_m \to z \in \overline{B}_{r/8}^j(\widehat{w})$ such that $I_j'(\widehat{w}+z) = 0$, contrary to (2.19).

Now by a standard argument (see e.g. [8] or Proposition 3.50) the function $\mathcal{V}$ can be constructed on $\overline{N}_r^j(\widehat{A})$ from the vectors $\varphi(x)$ and in fact $\mathcal{V}$ can easily be extended to satisfy $\|\mathcal{V}(x)\|_j \leq 1$ all of E_j.

Set

$$f(x) = \frac{\|x - (I_j)^{c-\varepsilon}\|_j}{\|x - (I_j)^{c-\varepsilon}\|_j + \|x - (I_j)_{c-\varepsilon/2}^{c+\varepsilon/2}\|_j} \tag{2.42}$$

where $(I_j)^s$, $(I_j)_\sigma^s$ have the obvious meaning. Then f is locally Lipschitz continuous on E_j. Finally, set

$$\mathcal{W}(x) = f(x)\mathcal{V}(x), \tag{2.43}$$

a bounded locally Lipschitz continuous function on E_j. Consider the ordinary differential equation

$$\frac{d\eta}{ds} = -\mathcal{W}(\eta) \tag{2.44}$$

on E_j, with the initial conditions

$$\eta(0,x) = x \tag{2.45}$$

for $x = \widehat{g}(\theta)$. Since $\mathcal{W}$ is bounded, a solution of (2.44)–(2.45) exists for all $s \in \mathbb{R}$. By (2.20),

$$I_j(x) \le c + \varepsilon/4. \tag{2.46}$$

If $I_j(x) \le c - \varepsilon/2$, set $\sigma(x) = 0$. If

$$I_j(x) > c - \varepsilon/2, \tag{2.47}$$

since $I_j(x) = I(x)$, by (2.15), $x \in N^j_{r/4}(\widehat{A})$. The behavior of the orbit $\eta(s,x)$ will be analyzed. Suppose $\eta(s,x)$ leaves $N^j_r(\widehat{A})$. Then there are numbers $0 < s_1 < s_2$ such that $\eta(s,x) \in \overline{N^j_r}(\widehat{A}) \backslash N^j_{r/4}(\widehat{A})$ for $s \in [s_1,s_2]$, $\eta(s_1,x) \in \partial N^j_{r/4}(\widehat{A})$, $\eta(s_2,x) \in \partial N^j_r(\widehat{A})$ and

$$\begin{aligned} \frac{3r}{4} &\le \|\eta(s_2,x) - \eta(s_1,x)\|_j = \left\| \int_{s_1}^{s_2} \frac{d\eta}{ds}\, ds \right\|_j \\ &\le \int_{s_1}^{s_2} \left\| \frac{d\eta}{ds} \right\|_j ds \le \int_{s_1}^{s_2} f(\eta(s,x)) \|\mathcal{V}(\eta(s,x))\|_j \, ds \\ &\le \int_{s_1}^{s_2} f(\eta(s,x))\, ds \end{aligned} \tag{2.48}$$

since $\|\mathcal{V}\|_j \le 1$. Note that

$$\frac{d}{ds} I_j(\eta(s,x)) = -f(\eta(s,x)) I_j'(\eta(s,x)) \mathcal{V}(\eta(s,x)) \le 0 \tag{2.49}$$

for $x \in [0,s_2]$, i.e. $I_j(\eta(s,x))$ is a nonincreasing function for this s interval. Therefore by (2.49) and the form of (2.44),

$$\begin{aligned} \frac{3}{4}\varepsilon &\ge I_j(\eta(s_1,x)) - I_j(\eta(s_2,x)) \\ &= \int_{s_2}^{s_1} \frac{dI_j(\eta(s,x))}{ds}\, ds = \int_{s_1}^{s_2} I_j'(\eta(s,x)) f(\eta(s,x)) \mathcal{V}(\eta(s,x))\, ds \\ &\ge 2\delta \int_{s_1}^{s_2} f(\eta(s,x))\, ds. \end{aligned} \tag{2.50}$$

Combining (2.48) and (2.50) yields

$$\varepsilon \ge 2\delta r, \tag{2.51}$$

contrary to (2.12). Hence $\eta(s,x)$ remains in $N_r^j(\widehat{A})$ for all $s > 0$. Consequently there is a unique $s = \sigma(x) > 0$ such that $I_j(\eta(\sigma(x),x)) = c - \varepsilon/2$ for otherwise,

$$I_j(\eta(s,x)) > c - \varepsilon/2 \tag{2.52}$$

for all $s > 0$. Then by (2.42), $f(\eta(s,x)) = 1$ and as in (2.50), for any $s > 0$,

$$c + \varepsilon/2 - I_j(\eta(s,x)) \geq \int_0^s \delta_j\,ds = s\delta_j. \tag{2.53}$$

Choosing $s = \delta_j^{-1}$ gives

$$c + \varepsilon/2 - 1 \geq I_j(\eta(\delta_j^{-1},x)) > c - \varepsilon/2 \tag{2.54}$$

which is impossible. Thus for each $\theta \in [0,1]$, there is a unique $s = \sigma(\widehat{g}(\theta))$ such that

$$I_j(\eta(\sigma(\widehat{g}(\theta)), \widehat{g}(\theta)) \leq c - \varepsilon/2. \tag{2.55}$$

It is easily seen that $\sigma(\widehat{g}(\theta))$ is continuous in θ. Moreover, since we have $\widehat{\eta}(\theta) = \eta(\sigma(\widehat{g}(\theta)), \widehat{g}(\theta)) \in N_r^j(\widehat{A})$,

$$\|\widehat{\eta}(\theta) - \widehat{g}(\theta)\|_j \leq 2r\,. \tag{2.56}$$

It remains to use (2.55)–(2.56) to pass from $\widehat{\eta}$ to an approximation $\widehat{h}$ and corresponding $h \in \Gamma$ and obtain the contradiction (2.24). A comparison argument from [4] will be employed to construct $\widehat{h}$. Observe first that

$$\widehat{\eta}(0) = \eta(\sigma(\widehat{g}(0)), \widehat{g}(0)) = \eta(\sigma(0),0) = 0 \tag{2.57}$$

since $I_j(\widehat{g}(0)) = I(0) = 0 < c - \varepsilon/2$ (via (2.12)). Similarly, $I_j(\widehat{g}(1)) = I(g(1)) \leq 0$ so $\sigma(g(1)) = 0$ and

$$\widehat{\eta}(1) = \widehat{g}(1). \tag{2.58}$$

By the properties of $\widehat{g}$ and (2.56), there is an interval Y in $[-j,j]$ (with end points identified) of length at least 2γ such that for all $\theta \in [0,1]$,

$$\|\widehat{\eta}(\theta)\|_{W^{1,2}(Y)} \leq 2r\,. \tag{2.59}$$

For each $\theta \in [0,1]$, let

$$\widehat{E}_\theta = \left\{x \in W^{1,2}(Y,\mathbb{R}^n) \;\middle|\; x|_{\partial Y} = \widehat{\eta}(\theta)|_{\partial Y} \quad \text{and} \quad \|x\|_{W^{1,2}(Y)} \leq 8r\right\}. \tag{2.60}$$

For $x \in \widehat{E}_\theta$, define

$$\Psi(x) = \int_Y \mathcal{L}(x)\,dt. \tag{2.61}$$

By Proposition 5.7 of [5] or Proposition 4.26 of [4], for r sufficiently small, there is a unique $\widehat{x}_\theta \in \widehat{E}_\theta$ such that

$$\Psi(\widehat{x}_\theta) = \inf_{x \in \widehat{E}_\theta} \Psi(x). \tag{2.62}$$

Moreover, $\widehat{x}_\theta$ depends continuously on θ.

For each $\theta \in [0,1]$, set

$$U_\theta(t) = \begin{cases} \widehat{x}_\theta(t), & t \in Y \\ \widehat{\eta}(\theta)(t), & t \in [-j,j)\backslash Y \end{cases} \tag{2.63}$$

Then $U_\theta \in E_j$ for all $\theta \in [0,1]$ and by (2.62) and (2.55),

$$I_j(U_\theta) \leq I_j(\widehat{\eta}(\theta)) \leq c - \varepsilon/2. \tag{2.64}$$

By (2.57)–(2.58) and the construction of U_θ,

$$U_0 = 0 \quad \text{and} \quad U_1 = \widehat{g}(1). \tag{2.65}$$

Let W be the solution of the boundary value problem

$$\begin{aligned} \overline{L}W &\equiv -\frac{d^2}{dt^2}W + 2W = 0, \qquad t \in Y, \\ W &= a \equiv \max_{\theta \in [0,1],\, t \in \partial Y} |U_\theta(t)|. \end{aligned} \tag{2.66}$$

It was shown in [4] that

$$|\widehat{x}_\theta(t)|^2 \leq W(t), \qquad t \in Y. \tag{2.67}$$

Indeed, W can be written down explicitly and is exponentially small near the center of Y. E.g. if $Y = [-j, j+\gamma] \cup [j-\gamma, j]$, then

$$|\widehat{x}_\theta(\pm j + s)| \leq \sqrt{2a}\, e^{\sqrt{2}(1-\gamma)/2} \tag{2.68}$$

for $|s| \leq 1$. Define $\widehat{h}_\theta \in E_j$ by

$$\widehat{h}_\theta(t) = \begin{cases} U_\theta(t), & |t| < j - \gamma, \\ 0, & |t-j|,\, |t+j| \leq 1/2, \end{cases} \tag{2.69}$$

and linearly interpolated for $1/2 \leq |t-j|,\ |t+j| \leq 1$.

Thus for γ sufficiently large (see e.g. [4])

$$\max_{\theta \in [0,1]} I_j(\widehat{h}_\theta) \leq c - \varepsilon/4. \tag{2.70}$$

Associated with $\widehat{h}_\theta$ is $h(\theta) \in C([0,1] \times E, E)$ with $h \in \Gamma$ via (2.65). Moreover,

$$I_j(\widehat{h}_\theta) = I(h(\theta)) \leq c - \varepsilon/4. \tag{2.71}$$

Thus (2.24) has been verified. If Y is not the above interval, a simple translation argument yields the same conclusion and Theorem 2.18 is proved.

3. Multibump periodic solutions

The existence of multibump periodic solutions of (HS) will be established in this section. This case has several ideas in common with §2 but there are also new features and considerable technical complication. Some preliminaries are needed before the main results can be stated.

For $k \in \mathbb{N}\backslash\{1\}$, $1 \le i \le k$, $\theta \in [0,1]^k$, and $\varphi \in \{0,1\}$, let

$$\varphi_i = (\theta_1, \dots, \theta_{i-1}, \varphi, \theta_{i+1}, \dots, \theta_k). \tag{3.1}$$

A family of maps, Γ_k, related to Γ and which play an important part in the existence proof is defined as follows:

$$\Gamma_k = \left\{ G = \sum_{i=1}^{k} g_i \;\middle|\; g_i \text{ satisfies } (\mathrm{g}_1)\text{–}(\mathrm{g}_3) \right\} \tag{3.2}$$

where for $1 \le i \le k$,

(g_1) $g_i \in C([0,1]^k, E)$,

(g_2) $g_i(0_i) = 0$, $g_i(1_i) \in I^0\backslash\{0\}$,

(g_3) there are real numbers $p_1 < \dots < p_{k-1}$ independent of $\theta \in [0,1]^k$ such that if $p_0 = -\infty$ and $p_k = \infty$, then

$$\operatorname{supp} g_i(\theta) \subset (p_{i-1}, p_i).$$

In (g_3), $\operatorname{supp} f$ denotes the support of f (as an element of E).

Associated with each set Γ_k is a minimax value (b_k) defined via

$$b_k = \inf_{G \in \Gamma_k} \max_{\theta \in [0,1]^k} I(G(\theta)). \tag{3.3}$$

It was shown in [4] that $b_k = kc$. This fact is not needed here but the key step in its proof is essential for our existence argument. A new and simpler proof of this step will be given next.

PROPOSITION 3.4. *Let g_i satisfy (g_1)–(g_2), $1 \le i \le k$. Then there is a $\overline{\theta} \in [0,1]^k$ such that $I(g_i(\overline{\theta})) \ge c$, $1 \le i \le k$.*

PROOF. Let F_{i0} denote the face of $[0,1]^k$ containing (0_i) and F_{i1} the face containing 1_i. If γ is a curve joining F_{i0} to F_{i1}, then by (g_2), $g_i(\gamma) \in \Gamma$. Therefore by the definition of c,

$$c \in I(g_i(\gamma([0,1]))). \tag{3.5}$$

Since this is true for any such γ, $(I(g_i))^{-1}(c)$ separates F_{i0} and F_{i1} (in $[0,1]^k$).

Let $\varepsilon > 0$ and set

$$A_i \equiv \{\theta \in [0,1]^k \mid I(g_i(\theta)) \ge c - \varepsilon\}.$$

Then for any $\delta = \delta(\varepsilon)$ sufficiently small, A_i contains a uniform, δ neighborhood of $(I(g_i))^{-1}(c)$. Since $[0,1]^k$ is compact, this neighborhood contains only a finite number of components. The same is true for $B_i = [0,1]^k \backslash A_i$. Since A_i separates F_{i0} and F_{i1}, there is a component C_i of B_i containing F_{i1} but not F_{i0}.

Define

$$\sigma_i(\theta) = \begin{cases} |\theta - A_i|, & \theta \in [0,1]^k \backslash C_i, \\ -|\theta - A_i|, & \theta \in C_i. \end{cases} \tag{3.6}$$

Then $\sigma_i \in C([0,1]^k, [-1,1])$, $1 \le i \le k$,

$$\sigma_i(0_i) > 0, \qquad \sigma_i(1_i) < 0, \tag{3.7}$$

and

$$\sigma_i(\theta) = 0 \quad \text{if and only if} \quad I(g_i(\theta)) \ge c - \varepsilon. \tag{3.8}$$

We claim there is a $\overline{\theta}_\varepsilon \in [0,1]^k$ such that

$$I(g_i(\overline{\theta}_\varepsilon)) \ge c - \varepsilon, \qquad 1 \le i \le k. \tag{3.9}$$

By (3.8), this is equivalent to finding a zero for $\sigma(\theta) = (\sigma_1(\theta), \dots, \sigma_k(\theta))$. Consider the Brouwer degree of σ with respect to $(0,1)^k$ and 0. Denote it by $d(\sigma, (0,1)^k, 0)$. By (3.7), the degree is defined. If it is nonzero, then there exists $\overline{\theta}_\varepsilon$ as desired. To verify this, consider the homotopy

$$h(\lambda, \theta) = (1-\lambda)\sigma(\theta) + \lambda\eta(\theta), \qquad \lambda \in [0,1],$$

where $\eta(\theta) = (\eta_1(\theta), \dots, \eta_k(\theta))$ and $\eta_i(\theta) = -2\theta_i + 1$. If $h(\lambda,\theta) = 0$ for some $\lambda \in [0,1]$ and $\theta \in \partial(0,1)^k$, then $\theta = 0_i$ or 1_i for some i. If $\theta = 0_i$, then by (3.7),

$$0 = (1-\lambda)\sigma_i(0_i) + \lambda(-2 \cdot 0 + 1) > 0, \tag{3.10}$$

while if $\theta = 1_i$, than again by (3.7),

$$0 = (1-\lambda)\sigma_i(1_i) + \lambda(-2 \cdot 1 + 1) < 0. \tag{3.11}$$

Consequently, $h(\lambda, \theta) \neq 0$ for $\lambda \in [0,1]$ and $\theta \in \partial(0,1)^k$. Hence by the properties of degree,

$$d(\sigma, (0,1)^k, 0) = d(\eta, (0,1)^k, 0) = (-1)^k \neq 0. \tag{3.12}$$

Thus there is a $\overline{\theta}_\varepsilon$ satisfying (3.9). Letting $\varepsilon \to 0$ then yields Proposition 3.4.

Another proof of Proposition 3.4 can be given using a theorem of Miranda [6]. In [4], the fact that $b_k = kc$ was used together with a construction to show that for each $k \in \mathbb{N}$ and r sufficiently small, there is an $\ell_0 = \ell_0(r,k)$ such that if $\ell = (\ell_1, \dots, \ell_k) \in \mathbb{Z}^k$ with $\ell_{i+1} - \ell_0 \ge \ell_0$ and $\mathcal{A}(\ell)$ is as in §1, then

$$N_r(\mathcal{A}(\ell)) \cap \mathcal{K} \neq \emptyset.$$

In other words, near the collection of sums of translates associated with ℓ of members of A, there is an actual solution of (HS) provided that ℓ_0 is sufficiently large. The main result here is that a similar theorem prevails with A replaced by $\widehat{A}$ and $\mathcal{K}$ by $\mathcal{K}_j$. In fact, by working with a different norm as in Séré [11], ℓ_0 can be chosen to be independent of k. This observation can also be used to improve the results of [4].

To state our results more precisely, a more careful choice of parameters and sets must be made than in §2. Let $0 < r < \mu/24$ and $s = r$ in Proposition 2.1. Then with A as in §2, there is a $\delta = \delta(r) > 0$ such that

$$\|I'(x)\| \geq 4\delta, \qquad x \in \overline{N}_{8r}(A)\backslash N_{r/32}(A). \tag{3.14}$$

Let

$$0 < \varepsilon_1 < \min(r\delta(r)/24, r, 1). \tag{3.15}$$

Choose $\rho = \rho(r) < 1$ so that

$$0 < \rho < \varepsilon_1/32 \tag{3.16}$$

and

$$I(x) \leq c + \varepsilon_1/32, \qquad x \in N_{4\rho}(A). \tag{3.17}$$

For each $v \in A$, choose $v^* \in E$ having compact support and such that

$$\|v - v^*\| \leq \rho/2. \tag{3.18}$$

Therefore by (3.14) and (3.18),

$$\|I'(x)\| \geq 4\delta, \qquad x \in \overline{N}_{2r}(A^*)\backslash N_{r/16}(A^*), \tag{3.19}$$

where $A^* = \{v^* \mid v \in A\}$. Choose $g \in \Gamma$ with compact support such that (2.13) and (2.15) hold, i.e.

$$\max_{\theta\in[0,1]} I(g(\theta)) = \max_{\theta\in[0,1]} I_j(g(\theta)) \leq c + \varepsilon/4 \tag{3.20}$$

and

$$I(g(\theta)) > c - \varepsilon \quad \text{implies} \quad g(\theta) \in N_{r/4}(A^*). \tag{3.21}$$

In (3.20)–(3.21), $\varepsilon \in (0, \widehat{\varepsilon}_1)$ is given by Proposition 2.9 where $\widehat{\varepsilon}_1 = \min(\varepsilon_1, \overline{\varepsilon}_1)$ and $\overline{\varepsilon}_1$ was defined in (2.12). Finally, choose R so that

$$\operatorname{supp} A^*, \operatorname{supp} g(\theta) \subset [-R, R]. \tag{3.22}$$

Now let $\ell \in \mathbb{Z}^k$ with

$$\ell_{i+1} - \ell_i \geq \ell_0 = 2(R + \gamma), \tag{3.23}$$

where $\gamma = \gamma(r)$ will be chosen to satisfy several conditions later. Choose $j \geq j_0 = \ell_k - \ell_1 + \ell_0 \geq k\ell_0$. Then for any choice of $v_1^*, \ldots, v_k^* \in A^*$ we have

$$\operatorname{supp} \tau_{\ell_i} v_i^* \subset [m_{i-1} + \gamma, m_i - \gamma], \quad 1 \leq i \leq k, \tag{3.24}$$

where $m_0 = \ell_1 - R - \gamma(R)$, $m_k = \ell_k + R + \gamma(R)$, and $m_i = (\ell_i + \ell_{i+1})/2$, $1 \leq i \leq k-1$. After a change of variables, it can be assumed that $m_0 = 0$ and $m_k = 2j$. As in §2, to each $v^* \in A^*$, there corresponds a $\widehat{v} \in E_j$ such that $v^*|_{[-j,j]} = \widehat{v}$. Let $\widehat{A} = \{\widehat{v} \mid v^* \in A^*\}$.

The space E_j will be renormed via

$$|||x|||_j = \max_{1 \leq i \leq k} \|x\|_{W^{1,2}([m_{i-1}, m_i], \mathbb{R}^n)}. \tag{3.25}$$

The norm depends on ℓ which is fixed for what follows. A ball of radius r about x under $|||\cdot|||_j$ will be denoted by $\mathcal{B}_r(x)$. For $S \subset E_j$, set $\mathcal{N}_r(S) = \{x \in E_j \mid |||x - S|||_j < r\}$. Let

$$\widehat{\mathcal{A}}(\ell) = \left\{ \sum_{i=1}^{k} \tau_{\ell_i} \widehat{v}_i \;\middle|\; \widehat{v}_i \in \widehat{A} \right\}.$$

Now our main result can be stated.

THEOREM 3.26. *Let* (L), (V$_1$)–(V$_3$), *and* (*) *hold. Then for any r sufficiently small, there is an $\ell_0(r)$ such that for any $k \in \mathbb{N}$ and $\ell = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k$ with $\ell_{i+1} - \ell_i \geq \ell_0(r)$ and $j \geq j_0(r) = \ell_k - \ell_1 + \ell_0 \geq k\ell_0(r)$,*

$$\mathcal{A}_j \equiv \mathcal{N}_r(\widehat{\mathcal{A}}(\ell)) \cap \mathcal{K}_j \neq \emptyset. \tag{3.27}$$

In brief, the strategy of the proof of Theorem 3.26 is similar to that of Theorem 2.18. If $\mathcal{A}_j = \emptyset$, a $G \in \Gamma_k$ will be chosen so that

$$\max_{\theta \in [0,1]^k} I(G(\theta)) = \max_{\theta \in [0,1]^k} I_j(\widehat{G}(\theta)) \leq k(c + \varepsilon) \tag{3.28}$$

for ε as in (3.20)–(3.21). After a deformation and modification process in the spirit of §2 but more complicated, $H = \sum_{i=1}^k h_i \in \Gamma_k$ will be constructed from G such that

$$\max_{\theta \in [0,1]^k} I(h_i(\theta)) \leq c - \varepsilon/4, \tag{3.29}$$

for some i. But (3.29) contradicts Proposition 3.4.

To carry out the details of this argument, we begin with a refinement of Proposition 2.1.

PROPOSITION 3.30. *Let $0 < r < \frac{\mu}{12}$ and $\rho(r)$ satisfy* (3.16)–(3.17). *Then there are $\widetilde{\delta} = \widetilde{\delta}(r) > 0$ and $\gamma = \gamma(r) > 0$ such that whenever $k \in \mathbb{N}$, $\ell = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k$, $\ell_{i+1} - \ell_i \geq \ell_0(r) = 2(\gamma(r) + R)$, and $x \in \overline{\mathcal{N}}_{4r}(\widehat{\mathcal{A}}(\ell)) \setminus \mathcal{N}_\rho(\widehat{\mathcal{A}}(\ell))$, there exists $\varphi_x \in E_j$ with $\|\varphi_x\|_j = 1$ and*

$$I_j'(x)\varphi_x \geq 4\widetilde{\delta}.$$

PROOF. Suppose first that $x = \sum \tau_{\ell_i}\widehat{v}_i + z$ where $z \in \overline{\mathcal{B}}_{4r}(0)\backslash\mathcal{B}_{r/4}(0)$. Set $z_i = z|_{[m_{i-1},m_i]}$. Then

$$\|z_i\|_{W^{1,2}[m_{i-1},m_i]} \leq 4r, \qquad 1 \leq i \leq k, \tag{3.31}$$

and for some p in $[1,k]$,

$$\|z_p\|_{W^{1,2}[m_{p-1},m_p]} \geq r/4. \tag{3.32}$$

By (3.31), there is a unit interval $U_i^+ \subset [m_i - \gamma, m_i]$ with integer endpoints such that

$$\|z_i\|_{W^{1,2}(U_i^+)} \leq 4r\gamma^{-1/2}. \tag{3.33}$$

Similarly, there is a unit interval $U_i^- \subset [m_{i-1}, m_{i-1}+\gamma]$ such that

$$\|z_i\|_{W^{1,2}(U_i^-)} \leq 4r\gamma^{-1/2}. \tag{3.34}$$

Choosing $i = p$, let $z^*(t)$ be a function such that

$$z^*(t) = \begin{cases} 0, & t \in \text{center half of } U_{p-1}^+, U_p^\pm, U_{p+1}^-, \\ 0, & t \text{ to the left of the center half of } U_{p-1}^+ \\ & \text{and to the right of the center half of } U_{p+1}^-, \\ z(t), & t \text{ between } U_{p-1}^+ \text{ and } U_p^-, \\ & U_p^- \text{ and } U_p^+,\ U_p^+ \text{ and } U_{p+1}^-, \\ \multicolumn{2}{l}{\text{linear combination of 0 and } z \text{ in the}} \\ \multicolumn{2}{l}{\text{remaining subintervals of } U_{p-1}^+, U_p^\pm, \text{ and } U_{p+1}^-.} \end{cases} \tag{3.35}$$

Note that in any of the intervals $U = U_{p-1}^+$ etc.,

$$\|z - z^*\|_{W^{1,2}(U)} \leq 8r\gamma^{-1/2}. \tag{3.36}$$

Let $\varphi(t)$ have support in the interval X_p bounded on the left by U_{p-1}^+ and on the right by U_{p+1}^- and suppose $\|\varphi\|_j = 1$. Then since z and z^* differ on X_p only on the U intervals,

$$\begin{aligned} I_j'\left(\sum_{i=1}^{k} \tau_{\ell_r} v_r^* + z\right)\varphi &= I_j'(\tau_{\ell_p} v_p^* + z)\varphi \\ &= I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi + (I_j'(\tau_{\ell_p} v_p^* + z) - I_j'(\tau_{\ell_p} v_p^* + z^*))\varphi \\ &\geq I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi - \delta/2 \end{aligned} \tag{3.37}$$

provided that γ is sufficiently large. Moreover, again for γ sufficiently large,

$$8r \geq \|z_p^*\|_{W^{1,2}[m_{p-1},m_p]} \geq r/8. \tag{3.38}$$

Now two cases are considered. Suppose Y_p is the interval bounded by U_p^- on the left and U_p^+ on the right and

$$\|z_p^*\|_{W^{1,2}(Y_p)} \geq r/16. \tag{3.39}$$

Define $Z_p \in E_j$ via

$$Z_p(t) = \begin{cases} z_p^*(t), & t \in Y_p, \\ 0, & t \in [-j,j] \backslash Y_p. \end{cases}$$

Considering Z_p and $\tau_{\ell_p} v_p^*$ extended by zero outside Y_p as elements of E, by Proposition 2.1, there is a $\varphi \in E$ with support in Y_p and $\|\varphi\| = 1$ such that

$$I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi = I'(\tau_{\ell_p} v_p^* + Z_p)\varphi \geq 4\delta(r). \tag{3.40}$$

Next suppose that

$$\|z_p^*\|_{W^{1,2}(Y_p)} < r/16. \tag{3.41}$$

Then

$$\|z_p^* - Z_p\|_{W^{1,2}[m_{p-1},m_p]} \geq r/16. \tag{3.42}$$

Take $\varphi = \beta^{-1}(z^* - Z_p)$ where β is free for the moment. Note that supp $\varphi \subset X_p \backslash Y_p$ and

$$I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi = \int_{X_p \backslash Y_p} (\beta^{-1}(|\dot{z}^*|^2 + |z^*|^2) - V_q(t, z^*) \cdot \varphi)\, dt. \tag{3.43}$$

On $X_p \backslash Y_p$, for r sufficiently small,

$$|V_q(t, z^*)| \leq \frac{1}{10}|z^*|. \tag{3.44}$$

Therefore

$$\begin{aligned} I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi &\geq \int_{X_p \backslash Y_p} \left(\beta^{-1}(|\dot{z}^*|^2 + |z^*|^2) - \frac{\beta^{-1}}{10}|z^*|^2\right) dt \\ &\geq \frac{9}{10}\beta^{-1}\|z^*\|^2_{W^{1,2}(X_p \backslash Y_p)} = \frac{9}{10}\beta^{-1}\|z^* - Z_p\|^2_{W^{1,2}(X_p)} \\ &\geq \frac{9}{10}\beta^{-1}\|z^* - Z_p\|^2_{W^{1,2}([m_{p-1},m_p])} \geq \frac{9}{10}\beta^{-1}\left(\frac{r}{16}\right)^2. \end{aligned} \tag{3.45}$$

Choose β so that $\|\varphi\|_j = 1$. Since

$$\frac{r}{16} \leq \|z^* - Z_p\|_j \leq 8r, \tag{3.46}$$

this gives k-independent bounds for β. Now since it can be assumed that δ is small compared to r, (3.45)–(3.46) show

$$I_j'(\tau_{\ell_p} v_p^* + z^*)\varphi \geq 4\delta(r). \tag{3.47}$$

Combining (3.40) and (3.47) and taking $\varphi = \varphi_x$ gives the lower bound for $I_j'(x)\varphi_x$ for these cases and above choices of φ_x. If $x = \sum \tau_{\ell_p} v_p^* + z$ with

$z \in \overline{\mathcal{B}}_{r/4}(0)\backslash\mathcal{B}_\rho(0)$, arguing exactly as above with $\gamma(r)$ still larger and $\delta = \delta(r,\rho(r))$ from Proposition 2.1 gives the desired lower bound. Finally, taking $\widetilde{\delta}(r) = \min(\delta(r), \delta(r,\rho(r)))$ yields Proposition 3.30.

REMARK 3.48. The above construction yields φ_x such that $\|\varphi_x\|_j = |||\varphi_x|||_j = 1$. If there are several values of p such that (3.32) holds, a more refined choice of φ_x is needed. Namely, take φ_x to be the sum of the corresponding φ_{x_p} as obtained above. Then by (3.25) and the way in which the supports of the φ_{x_p} are situated,

$$|||\varphi_x|||_j \leq 3. \tag{3.49}$$

The next result pieces together the vectors φ_x to form a vector field on E_j which will be used as in §2 for the deformation process.

PROPOSITION 3.50. *If $\mathcal{A}_j = \emptyset$, there exists a locally Lipschitz continuous function $\mathcal{V}(x)$ on E_j and $\delta_j(r) \leq 2\widetilde{\delta}(r)$ such that*

$$|||\mathcal{V}(x)|||_j \leq 3, \qquad \text{for all } x \in E_j, \tag{3.51}$$

$$I_j'(x)\mathcal{V}(x) \geq 2\delta(r), \qquad x \in \overline{\mathcal{N}}_r(\widehat{\mathcal{A}}(\ell))\backslash\mathcal{N}_{r/4}(\widehat{\mathcal{A}}(\ell)), \tag{3.52}$$

$$I_j'(x)\mathcal{V}(x) \geq 2\widetilde{\delta}(r), \qquad x \in \overline{\mathcal{N}}_{r/4}(\widehat{\mathcal{A}}(\ell))\backslash\mathcal{N}_\rho(\widehat{\mathcal{A}}(\ell)), \tag{3.53}$$

$$I_j'(x)\mathcal{V}(x) \geq \delta_j(r) > 0, \qquad x \in \overline{\mathcal{N}}_\rho(\widehat{\mathcal{A}}(\ell)). \tag{3.54}$$

Moreover, if

$$\Phi_{ji}(x) \equiv \int_{m_i-1}^{m_i} \mathcal{L}(x)\,dt, \qquad 1 \leq i \leq k \tag{3.55}$$

and $x = y + z$ with $y \in \widehat{\mathcal{A}}(\ell)$ then

$$\Phi_{ji}'(x)\mathcal{V}(x) \geq 2\delta(r), \quad r/4 \leq \|z_i\|_{W^{1,2}[m_{i-1},m_i]} \leq r \tag{3.56}$$

and

$$\Phi_{ji}'(x)\mathcal{V}(x) \geq 2\widetilde{\delta}(r), \quad \rho \leq \|z_i\|_{W^{1,2}[m_{i-1},m_i]} \leq r/4. \tag{3.57}$$

PROOF. There is a standard argument to construct $\mathcal{V}(x)$ from φ_x. It involves taking convex combinations of the vectors φ_x and using appropriate cut-off functions. See e.g. [7, Lemma A-2]. Thus (3.49) leads to (3.51), (3.47) to (3.52), and likewise going from $\delta(r)$ to $\widetilde{\delta}(r)$ yields (3.53). Property (3.54) follows as in (2.41) since I_j satisfies the Palais-Smale condition. To obtain (3.56)–(3.57), note that they hold with $\mathcal{V}(x)$ replaced by φ_x and hence for $\mathcal{V}$ since $\mathcal{V}$ is obtained as a convex combination of φ_x's (see [8]). Finally, note that without loss of generality $\delta_j < 2\widetilde{\delta}$.

To continue, set

$$G(\theta) = \sum_{i=1}^{k} \tau_{\ell_i} g(\theta_i) \tag{3.58}$$

where $g \in \Gamma$ satisfying (3.20)–(3.21) was obtained from Proposition 2.9. Then $G \in \Gamma_k$ and its periodic extension $\widehat{G}$ satisfies

$$I_j(\widehat{G}(\theta)) = I(G(\theta)) \le k(c + \varepsilon/4). \tag{3.59}$$

The function $\widehat{G}$ will be deformed and modified to obtain $H \in \Gamma$ satisfying (3.29) as indicated earlier. To do so, a flow will be employed. Let ε_1 satisfy (3.15). Define locally Lipschitz continuous functions as follows: For $1 \le i \le k$,

$$\psi_i(x) \begin{cases} = 0 & \text{if } \Phi_{ji}(x) \ge c + 2\varepsilon_1, \\ = 1 & \text{if } \Phi_{ji}(x) \le c + \varepsilon_1, \\ \in (0,1) & \text{if } \Phi_{ji}(x) \in (c + \varepsilon_1, c + 2\varepsilon_1), \end{cases} \tag{3.60}$$

$$\chi_i(x) \begin{cases} = 0 & \text{if } \Phi_{ji}(x) \le c - 2\varepsilon, \\ = 1 & \text{if } \Phi_{ji}(x) \ge c - \varepsilon, \\ \in (0,1) & \text{if } \Phi_{ji}(x) \in (c - 2\varepsilon, c - \varepsilon). \end{cases} \tag{3.61}$$

Set

$$\psi(x) = \prod_{i=1}^{k} \psi_i(x) \tag{3.62}$$

and

$$\chi(x) = \prod_{i=1}^{k} \chi_i(x). \tag{3.63}$$

Consider the ordinary differential equation

$$\frac{d\eta}{ds} = -\psi(\eta)\chi(\eta)\mathcal{V}(\eta) \tag{3.64}$$

in E_j, with the initial condition

$$\eta(0,x) = x. \tag{3.65}$$

We are only interested in $x = G(\theta)$ for $\theta \in [0,1]^k$. Since the right hand side of (3.64) is bounded, $\eta(s,x)$ exists for all $s \in \mathbb{R}$.

As in §2, set $\sigma(x) = 0$ if $\Phi_{ji}(x) \le c - \varepsilon/2$ for some i with $1 \le i \le k$. Otherwise choose $\sigma(x)$ to be the smallest positive value of s such that $\Phi_{ji}(\eta(s,x)) = c - \varepsilon/2$ for some i. That such a $\sigma(x)$ exists and is continuous in θ is a consequence of the arguments that follow.

Thus consider any $x = G(\theta)$ such that for each i with $1 \le i \le k$,

$$c - \varepsilon/2 < \Phi_{ji}(x). \tag{3.66}$$

Since for $1 \le i \le k$,

$$\Phi_{ji}(x) = I(\tau_{\ell_i} g(\theta_i)) = I(g(\theta_i)) \le c + \varepsilon/4, \tag{3.67}$$

by (3.66)–(3.67) and (3.20)–(3.21) we have

$$g(\theta_i) \in N_{r/4}(A^*). \tag{3.68}$$

Therefore

$$x = G(\theta) \in \mathcal{N}_{r/4}(\widehat{\mathcal{A}}(\ell)) \tag{3.69}$$

and $x \in \mathcal{B}_{r/4}(y)$ for some $y \in \widehat{\mathcal{A}}(\ell)$. Suppose that $\eta(s,x)$ crosses from $\mathcal{B}_{r/4}(y)$ to $\partial\mathcal{B}_{r/2}(y)$. Then for some $s_2 > s_1 > 0$, $\eta(s,x) \in \overline{\mathcal{B}}_{r/2}(y)\backslash\mathcal{B}_{r/4}(y)$ for all $s \in [s_1, s_2]$ and for some p,

$$\begin{aligned} &\|\eta(s_1,x) - y\|_{W^{1,2}[m_{p-1},m_p]} = r/4 \\ &\le \|\eta(s,x) - y\|_{W^{1,2}[m_{p-1},m_p]} \le \|\eta(s_2,x) - y\|_{W^{1,2}[m_{p-1},m_p]} = r/2. \end{aligned} \tag{3.70}$$

Hence

$$\begin{aligned} r/4 &\le \|\eta(s_2,x) - y\|_{W^{1,2}[m_{p-1},m_p]} - \|\eta(s_1,x) - y\|_{W^{1,2}[m_{p-1},m_p]} \\ &\le \|\eta(s_2,x) - \eta(s_1,x)\|_{W^{1,2}[m_{p-1},m_p]} \\ &= \left\| \int_{s_1}^{s_2} \frac{d\eta}{ds}\, ds \right\|_{W^{1,2}[m_{p-1},m_p]} \\ &\le \int_{s_1}^{s_2} \psi(\eta(s,x))\chi(\eta(s,x))\|\mathcal{V}(\eta(s,x))\|_{W^{1,2}[m_{p-1},m_p]}\, ds \\ &\le 3\int_{s_1}^{s_2} \psi(\eta(s,x))\chi(\eta(s,x))\, ds, \end{aligned} \tag{3.71}$$

the last inequality following from (3.51). On the other hand, by (3.56),

$$\begin{aligned} &\Phi_{jp}(\eta(s_1,x)) - \Phi_{jp}(\eta(s_2,x)) \\ &\quad = \int_{s_2}^{s_1} \frac{d}{ds}\Phi_{jp}(\eta(s,x))\, ds = \int_{s_2}^{s_1} \Phi'_{jp}(\eta(s,x))\frac{d\eta}{ds}\, ds \\ &\quad = \int_{s_1}^{s_2} \psi(\eta(s,x))\chi(\eta(s,x))\Phi'_{jp}(\eta(s,x))\mathcal{V}(\eta(s,x))\, ds \\ &\quad \ge 2\delta(r)\int_{s_1}^{s_2} \psi(\eta(s,x))\chi(\eta(s,x))\, ds. \end{aligned} \tag{3.72}$$

Combining (3.71)–(3.72) and (3.15) shows

$$\Phi_{jp}(\eta(s_1,x)) - \Phi_{jp}(\eta(s_2,x)) > 4\varepsilon_1. \tag{3.73}$$

But the form of the equation (3.64), in particular the choice of the cut-off functions ψ and χ implies $\Phi_{jp}(\eta(s,x)) \in (c - 2\varepsilon, c + 2\varepsilon_1)$. Hence (3.73) is not possible and for $s \ge 0$, $\eta(s,x) \in \mathcal{B}_{r/2}(y)$.

Let

$$T_j = 2k\varepsilon\delta_j^{-1}. \tag{3.74}$$

We claim that for each x satisfying (3.66), there is some i with $1 \le i \le k$ and $s \in (0, T_j)$ such that

$$\Phi_{ji}(\eta(s,x)) = c - \varepsilon/2. \tag{3.75}$$

Otherwise for all $s \in (0, T_j]$ and $1 \le i \le k$,

$$\Phi_{ji}(\eta(s,x)) > c - \varepsilon/2. \tag{3.76}$$

Consequently,

$$\chi(\eta(s,x)) = 1 \tag{3.77}$$

and

$$I_j(\eta(s,x)) > k(c - \varepsilon/2) \tag{3.78}$$

for $s \in (0, T_j]$. By (3.77),

$$\begin{aligned} I_j(\eta(s,x)) &= I_j(x) + \int_0^s \frac{dI_j}{ds}(\eta(s,x))\,ds \\ &= I_j(x) - \int_0^s \psi(\eta(s,x))I_j'(\eta(s,x))\mathcal{V}(\eta(s,x))\,ds. \end{aligned} \tag{3.79}$$

Since $\eta(s,x) \in \mathcal{B}_r(y)$, by Proposition 3.50,

$$I_j(x) - I_j(\eta((T,x)) \ge \delta_j \int_0^{T_j} \psi(\eta(s,x))\,ds. \tag{3.80}$$

Hence by (3.78) and (3.59),

$$\tfrac{3}{4}k\varepsilon = k\,(c + \varepsilon/4) - k\,(c - \varepsilon/2) \ge \delta_j \textstyle\int_0^{T_j} \psi(\eta(s,x))\,ds. \tag{3.81}$$

It remains to analyze $\psi(\eta(s,x))$. When $s = 0$, $\eta(s,x) = x$ satisfying (3.67). Therefore $\psi_i(x) = 1$ for $1 \le i \le k$. By its definition, $\psi_i(\eta(s,x)) = 1$ whenever $\Phi_{ji}(\eta(s,x)) \le c + \varepsilon_1/2$. Thus suppose there is a smallest $\overline{s} \in (0, T_j]$ such that

$$\Phi_{ji}(\eta(\overline{s},x)) = c + \varepsilon_1/2 \tag{3.82}$$

for some i. Let $\eta_i(\overline{s},x) = \eta(\overline{s},x)|_{[m_{i+1},m_i]} \equiv \tau_{\ell_i}v_i^* + z_i$. If

$$\rho \le \|z_i\|_{W^{1,2}[m_{i-1},m_i]} \le r, \tag{3.83}$$

then by Proposition 3.50,

$$\frac{d\Phi_{ji}(\eta(\overline{s},x))}{ds} = -\Phi_{ji}'(\eta(\overline{s},x))\mathcal{V}(\eta(\overline{s},x)) < 0, \tag{3.84}$$

i.e. $\Phi_{ji}(\eta(\overline{s},x))$ is decreasing at $\overline{s}$, contrary to the choice of $\overline{s}$. Hence

$$\|z_i\|_{W^{1,2}[m_{i-1},m_i]} < \rho. \tag{3.85}$$

Define a function $\zeta \in E$ via

$$\zeta(t) = \begin{cases} 0, & t \notin [m_{i-1}, m_i], \\ z_i(t), & t \in [m_{i-1}+1, m_i - 1], \\ \text{convex combination of 0 and } z_i \\ \text{in the remaining intervals.} \end{cases}$$

Then $\|\zeta\| < 3\rho$ and

(3.86) $$|\Phi_{ji}(\tau_{\ell_i} v_i^* + \zeta) - \Phi_{ji}(\eta(\bar{s}, x))| \leq 3\rho^2 < 3\rho.$$

Therefore by (3.82), (3.86), and (3.16),

(3.87) $$\Phi_{ji}(\tau_{\ell_i} v_i^* + \zeta) \geq c + \varepsilon_1/4.$$

But

(3.88) $$\Phi_{ji}(\tau_{\ell_i} v_i^* + \zeta) = I(\tau_{\ell_i} v_i^* + \zeta) = I(v_i^* + \tau_{-\ell_i}\zeta)$$

and by (3.16) and (3.18), $v_i^* + \tau_{-\ell_i}\zeta \in N_{4\rho}(A)$. Hence by (3.17),

(3.89) $$I(v_i^* + \tau_{-\ell_i}\zeta) \leq c + \varepsilon_1/32,$$

contrary to (3.87)–(3.88). Thus there is no $\bar{s}$ satisfying (3.82) and $\psi_i(\eta(s,x)) = 1$ for all $s \in [0, T_j]$. Now by (3.81),

(3.90) $$\tfrac{3}{4}k\varepsilon \geq T_j \delta_j.$$

But (3.90) violates (3.74).

Thus we have shown that for each $\theta \in [0,1]^k$, there is a unique $\sigma(G(\theta)) \in [0, T_j]$ such that for some i with $1 \leq i \leq k$,

(3.91) $$\Phi_{ji}(\eta(\sigma(G(\theta)), G(\theta)) \leq c - \varepsilon/2.$$

As in [4], $\sigma(G(\theta))$ is continuous in θ.

Next, by modifying $\eta(\sigma(G(\theta)), G(\theta)) = \widehat{\eta}(\theta)$, an $h \in \Gamma_k$ will be obtained satisfying (3.29) and thereby completing the proof of Theorem 3.26. The modification procedure is similar to that of §2 and [4] so we will be brief.

Note that by construction,

(3.92) $$|||\widehat{\eta}(\theta) - G(\theta)|||_j \leq r$$

and $G(\theta)(t) = 0$ for $t \in [m_i - \gamma, m_i \pm \gamma]$, $1 \leq i \leq k$. Therefore

(3.93) $$\|\widehat{\eta}(\theta)\|_{W^{1,2}[m_i - \gamma, m_i + \gamma]} \leq 2r$$

for $1 \leq i \leq k$. Define $Y_i^- = [m_{i-1}, m_{i-1} + \gamma]$ and

(3.94) $$\Psi_i^-(x) = \int_{Y_i^-} \mathcal{L}(x)\, dt$$

for $x \in \widehat{E}_i^-(\theta)$ where

$$\widehat{E}_i^-(\theta) = \{x \in W^{1,2}(Y_i^-) \mid x|_{\partial Y_i^-} = \widehat{\eta}(\theta)|_{\partial Y_i^-} \text{ and } \|x\|_{W^{1,2}(Y_i^-)} \le 8r\}.$$

Similarly, define $Y_i^+ = [m_i - \gamma, m_i]$ and

$$\Psi_i^+(x) = \int_{Y_i^+} \mathcal{L}(x)\, dt \tag{3.95}$$

for $x \in \widehat{E}_i^+(\theta)$ where

$$\widehat{E}_i^+(\theta) = \{x \in W^{1,2}(Y_i^+) \mid x|_{\partial Y_i^+} = \widehat{\eta}(\theta)|_{\partial Y_i^+} \text{ and } \|x\|_{W^{1,2}(Y_i^+)} \le 8r\}.$$

By Proposition 4.26 of [4], for r sufficiently small there is a unique $x_i^\pm(\theta)$ minimizing $\Psi_i^\pm$ over $\widehat{E}_i^\pm(\theta)$ for $1 \le i \le k$. Define $U_\theta(t) \in C([0,1]^h, E_j)$ via

$$U_\theta(t) = \begin{cases} x_i^-(\theta)(t), & t \in Y_i^-,\ 1 \le i \le k, \\ x_i^+(\theta)(t), & t \in Y_i^+,\ 1 \le i \le k, \\ \widehat{\eta}(\theta)(t) & \text{otherwise.} \end{cases} \tag{3.96}$$

By the construction of the functions $x_i^\pm(\theta)$,

$$\Phi_{ji}(U_\theta) \le \Phi_{ji}(\widehat{\eta}(\theta)), \qquad 1 \le i \le k. \tag{3.97}$$

In particular, for each $\theta \in [0,1]^k$ there is an $i(\theta)$ such that

$$\Phi_{ji(\theta)}(U_\theta) \le c - \varepsilon/2. \tag{3.98}$$

The comparison argument indicated in §2 (see also [4]) shows $x_i(\theta)(t) \to 0$ in C^1 like $e^{-\sqrt{\gamma}}$ as $\gamma \to +\infty$ for t in the unit interval S_i^- centered about $m_{i-1} + \gamma/2$. The same is true for x_i^+ in S_i^+ centered about $m_i - \gamma/2$. Therefore U_θ can be modified in $S_i^\pm$ for $1 \le i \le k$ to obtain a function $W(\theta)$ such that W vanishes near the center of $S_i^\pm$ for $1 \le i \le k$ and satisfies

$$\Phi_{ji}(W(\theta)) \le \Phi_{ji}(U_\theta) + \varepsilon/4. \tag{3.99}$$

In particular,

$$\Phi_{ji(\theta)}(W(\theta)) \le c - \varepsilon/4. \tag{3.100}$$

Finally, note that in the interval X_i between the centers of S_i^+ and S_{i+1}^-, by (3.93), the $W^{1,2}$ norms of $\widehat{\eta}(\theta)$, U_θ, and $W(\theta)$ over X_i are all bounded by $a_1 r$ where a_1 is independent of r. Hence for r sufficiently small,

$$\int_{X_i} \mathcal{L}(W(\theta))\, dt \ge \frac{1}{4} \int_{X_i} (|\dot{W}(\theta)|^2 + |W(\theta)|^2)\, dt. \tag{3.101}$$

Set

$$H(\theta)(t) = \begin{cases} W(\theta)(t), & t \notin \bigcup_{r=1}^k X_r, \\ 0, & t \in \bigcup_{r=1}^k X_r. \end{cases}$$

Then

$$\Phi_{ji}(H(\theta)) \le \Phi_{ji}(W(\theta)), \qquad 1 \le i \le k, \tag{3.102}$$

and in particular,

$$\Phi_{ji(\theta)}(W(\theta)) \le c - \varepsilon/4. \tag{3.103}$$

As in §2, H can be identified with an element of $C([0,1]^k, E)$ satisfying (g_1) and (g_3). If H also satisfies (g_2) then (3.103) provides us with the desired contradiction. Thus it remains to show $H_i(0_i) = 0$ and $H_i(1_i) \in I^0\backslash\{0\}$ for $1 \le i \le k$. Recall that $\sigma(G(\theta)) = 0$ if $\Phi_{ji}(G(\theta)) \le c - \varepsilon/2$. But this is the case if $\theta = 0_i$ or 1_i. E.g.

$$G(1_i) = \tau_{\ell_1} g(\theta_1) + \ldots + \tau_{\ell_i} g(1) + \ldots$$

and

$$\begin{aligned}\Phi_{ji}(G(1_i)) &= \Phi_{ji}(\tau_{\ell_i} g(1)) = I(\tau_{\ell_i}(g(1))) \\ &= I(g(1)) \le 0.\end{aligned} \tag{3.104}$$

The definition of $x_i^{\pm}(\theta)$ then implies that $x_i^{\pm}(\theta)(t) = 0$ in $Y_i^{\pm}$. Thus

$$U_{0_i}(t) = W(0_i)(t) = H(0_i)(t), \qquad t \in Y_i^- \cup Y_i^+.$$

Therefore $H_i(0_i) = 0$ and similarly $H_i(1_i) = \tau_{\ell_i} g(1) \in I^0\backslash\{0\}$. Consequently, $H \in \Gamma_k$ and the proof of Theorem 3.26 is complete.

REMARK 3.105. Since $I_j' \ne 0$ in $\mathcal{N}_r(\widehat{\mathcal{A}}(\ell))\backslash\mathcal{N}_\rho(\widehat{\mathcal{A}}(\ell))$, in fact $\mathcal{K}_j \cap \mathcal{N}_\rho(\widehat{\mathcal{A}}(\ell)) \ne \emptyset$.

4. Infinite bump solutions

As a simple application of Theorem 3.26, we can obtain infinite bump solutions of (HS). Of course k-bump periodic solutions can be considered to be infinite bump solutions but the class of infinite bump solutions that will be constructed is of a more general nature.

Let $(k_i) \subset \mathbb{Z}$ be a doubly infinite sequence with

$$k_{i+1} - k_i \ge \ell_0(r) \tag{4.1}$$

for all $i \in \mathbb{Z}$ where r, $\ell_0(r)$ are as in Proposition 3.30. For each $p \in \mathbb{N}$, set $z_p = (k_{-p}, \ldots, k_p) \in \mathbb{Z}^{2p+1}$. Let $j_p \in \mathbb{N}$ such that $j_p \ge k_p - k_{-p} + \ell_0(r)$. As in §§2–3, let

$$\widehat{A} = \widehat{A}_p = \{\widehat{v} \in E_{j_p} \mid \widehat{v} = v^*|_{[-j_p, j_p]} \text{ for some } v^* \in A^*\}.$$

Set

$$\widehat{\mathcal{A}}(z_p) = \left\{ \sum_{i=-p}^{p} \tau_{k_i} \widehat{v}_i \,\middle|\, \widehat{v}_i \in \widehat{A}_p \right\}.$$

Then by Theorem 3.26, there is a solution $Q_p \in E_{j_p}$ of (HS) with $Q_p \in \mathcal{N}_r(\widehat{\mathcal{A}}(z_p))$. Therefore for each i with $-p \le i \le p$, there is a v_i^* (depending on p) $\in A^*$ such that

$$\|Q_p - \tau_{k_i} v_i^*\|_{W^{1,2}[m_{i-1},m_i]} \le r. \tag{4.2}$$

Note that for any $i \in \mathbb{Z}$ and $p > |i|$, $m_i = \frac{1}{2}(k_i + k_{i+1})$ and is independent of p. Since A^* is a finite set, by passing to a subsequence if necessary, it can be assumed that v_i^* is independent of p. The local bounds on Q_p provided by (4.2) and the fact that Q_p are solutions of (HS) imply Q_p are bounded in $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. Therefore, via (HS), Q_p converges along a subsequence to a solution Q of (HS) satisfying

$$\|Q - \tau_{k_i} v_i^*\|_{W^{1,2}[m_{i-1},m_i]} \le r \tag{4.3}$$

for all $i \in \mathbb{Z}$.

By (4.3), Q is an "infinite bump" solution of (HS). Recalling from (3.18) that for each $v^* \in A^*$, there is a $v \in A$ such that $\|v - v^*\| \le \rho/2 < r$, we have proved:

THEOREM 4.4. *Let V satisfy* (V_1)–(V_3) *and* ($*$) *holds. Then for any r sufficiently small, there is an $\ell_0(r) > 0$ such that if (k_i) is a doubly infinite sequence of integers satisfying* (4.1) *for all $i \in \mathbb{Z}$, then there is a finite set $A \subset \mathcal{K}(c)$ and a solution Q of* (HS) *satisfying*

$$\|Q - \tau_{k_i} v_i\|_{W^{1,2}[(k_{i-1}+k_i)/2,(k_i+k_{i+1})/2]} \le 2r$$

for all $i \in \mathbb{Z}$ and some $v_i \in A$.

REMARK 4.5. Since A is bounded in $C(\mathbb{R}, \mathbb{R}^n)$ so is Q and in fact it is then bounded in $C^2(\mathbb{R}, \mathbb{R}^n)$ via (HS).

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Manuscript received April 25, 1994

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