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MORSE THEORY FOR C^1-FUNCTIONALS AND CONLEY BLOCKS

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Dedicated to Jean Leray

Introduction

Let Ω be an open subset of a Hilbert manifold Λ of class C^2, f a functional of class C^2 in a neighborhood of $\overline{\Omega}$ and $K_f = K_f(\overline{\Omega})$ the set of critical points of f in $\overline{\Omega}$.

If f is a Morse function, i.e. its critical points are nondegenerate (cf. Definition 1.5) and $m(x, f)$ denotes the Morse index of x as critical point of f (cf. Definition 1.4), the Morse polynomial of f is given by

$$m_\lambda(K_f, f) = \sum_{x \in K_f} \lambda^{m(x,f)}.$$

If $\overline{\Omega} = \{x \in \Lambda \mid a \leq f(x) \leq b\} \equiv f_a^b$, the classical Morse theory states that

$$m_\lambda(K_f, f) = \mathcal{P}_\lambda(f^b, f^a) + (1 + \lambda)\mathcal{Q}(\lambda), \tag{1}$$

where $f^c = \{x \in \Lambda : f(x) \leq c\}$, $\mathcal{P}_\lambda(f^b, f^a)$ is the Poincaré polynomial of (f^b, f^a) (cf. Section 2) and $\mathcal{Q} \in S$, the set of formal series with natural coefficients (possibly ∞).

In this paper we introduce an index i_λ generalizing the Morse polynomial m_λ to the following cases:

(a) the critical points of f are degenerate;
(b) f and/or Λ are of class C^1.

Moreover, we define a class Σ_F of closed sets (called Conley blocks) relative to a vector field F such that $\langle \operatorname{grad} f, F\rangle \leq 0$ (more precisely, satisfying the assumptions (i)–(iv) of Lemma 4.1) and an index $I_\lambda(\cdot, F) : \Sigma_F \to S$. This index coincides with $\mathcal{P}_\lambda(f^b, f^a)$ whenever $\overline{\Omega} = f_a^b$. It allows us to write the Morse relations even if we are under the conditions (a), (b) and

(c) *$\overline{\Omega}$ is not the strip f_a^b.*

Under the generalizations above, the Morse relations (1) become

$$i_\lambda(K_f, f) = I_\lambda(\overline{\Omega}, F) + (1+\lambda)Q(\lambda). \tag{2}$$

We notice that i_λ is a differential invariant, while I_λ is a topological invariant. In [2] and [6] a formula similar to the formula (2) has been obtained, but in those cases, i_λ is replaced by a topological invariant. Such a topological invariant assumes values which are different from i_λ and depend on the field of coefficients chosen for a homology theory.

The utility of (a) is obvious to get existence results for critical points by Morse theory. Indeed, we cannot a priori assume that they are nondegenerate. The generalization (b) is needed e.g. in partial differential equations where the functionals used are often of class C^1 and not C^2 (cf. Section 6). The generalization (c) allows one to apply Morse theory in situations where, usually, the topological degree is used. Indeed, the a priori bounds which make it possible to use the topological degree on a ball B_R of a Hilbert space, also allow one to establish that $B_R \in \Sigma_F$ and to evaluate the index $I_\lambda(B_R, F)$.

The definition of I_λ takes its inspiration from the Conley index (cf. [7]) and a generalization given by the first author (cf. [2]).

The class Σ_F is sufficiently large to include also the closures of submanifolds of Λ (cf. Theorem 3.8(ix)).

In Section 6 we have a typical application of this theory.

The definition of the generalized Morse index was announced in [5] together with some applications.

1. The Morse index

The main part of Morse theory consists in the "Morse relations", i.e. in the relations between two polynomials, the Morse polynomial and the Poincaré (or Betti) polynomial. The Morse polynomial depends on the critical points of a function and its definition will be recalled in this section. The Poincaré polynomial is a topological invariant and its definition will be recalled in Section 2.

Throughout this section Λ will be a Hilbert manifold of class C^1, Ω an open subset of Λ and f a functional of class C^1 in a neighborhood of $\overline{\Omega}$.

DEFINITION 1.1. A point $x \in \Lambda$ is called a *critical point* of f if $df(x) = 0$. If x is not a critical point, it is called *regular point*. The set of critical points of f will be denoted by K_f.

$a \in \mathbb{R}$ is called a *critical value* of f if there exists $x \in K_f$ such that $f(x) = a$; $a \in \mathbb{R}$ is called a *regular value* for f if it is not a critical value.

DEFINITION 1.2. If x is a critical point of f, and f is of class C^2 in a neighborhood of x, we define the*Hessian form* of f at x,

$$H^f(x) : T_x\Lambda \times T_x\Lambda \to \mathbb{R},$$

as follows:

$$H^f(x)[v,w] = \frac{\partial^2}{\partial t \partial s} f(\gamma_{v,w}(t,s))\bigg|_{t=0,s=0},$$

where $T_x\Lambda$ is the tangent space of Λ at x and $\gamma_{v,w}(t,s)$ is a smooth function such that

$$\gamma_{v,w}(0,0) = x, \qquad \partial_t\gamma_{v,w}(0,0) = v, \qquad \partial_s\gamma_{v,w}(0,0) = w.$$

REMARK 1.3. Since x is a critical point of f, it is not difficult to see that the bilinear form $H^f(x)$ is well defined (i.e. it depends only on v and w, but not on $\gamma_{v,w}$).

DEFINITION 1.4. Let $x \in K_f$ be a critical point such that $H^f(x)$ is defined. The (restricted) *Morse index* of x is the maximal dimension of a subspace of $T_x\Lambda$ on which $H^f(x)$ is negative definite; it is denoted by $m(x, f)$ (or simply by $m(x)$).

The *nullity* of x is the dimension of the kernel of $H^f(x)$ (i.e. the subspace consisting of all v such that $H^f(x)[v,w] = 0$ for all $w \in T_x\Lambda$).

The *large Morse index* is the sum of the restricted Morse index and the nullity and it will be denoted by $m^*(x, f)$.

Notice that if f is defined in an infinite-dimensional manifold it is possible that $m(x, f) = \infty$.

DEFINITION 1.5. Let $x_0 \in K_f$ be a critical point such that $H^f(x_0)$ is defined. Then x_0 is called *nondegenerate* if there exist a splitting $H^+ \oplus H^-$ of $T_{x_0}\Lambda$ and a constant $\nu > 0$ such that

(i) $H^f(x_0)[v,v] \geq \nu|v|^2$ for all $v \in H^+$,
(ii) $H^f(x_0)[v,v] \leq -\nu|v|^2$ for all $v \in H^-$.

Thus, if L is a selfadjoint operator such that

$$H^f(x_0)[v,v] = \langle Lv, v\rangle,$$

then x_0 is nondegenerate if and only if L is invertible.

In applications, particularly to partial differential equations, it may be necessary to consider functionals of class C^1 or/and functionals defined on manifolds of class C^1. In this section we show that sometimes it is possible to define the Morse index also in this case. In Section 5 the definition of Morse index will even be extended to more general situations.

First we need a lemma.

LEMMA 1.6. *Let Λ be of class C^1, f and Ω as above and $x \in K_f(\overline{\Omega}) \equiv K_f \cap \overline{\Omega}$. Assume that there exists a C^1-chart (U,ϕ) (with $x \in U$) such that $\phi(x)$ is a nondegenerate critical point of $f \circ \phi^{-1}$ according to Definition 1.5. Then, for any C^1-chart (V,ψ) (with $x \in V$), $f \circ \psi^{-1}$ has a second derivative at $\psi(x)$, $\psi(x)$ is a nondegenerate critical point of $f \circ \psi^{-1}$ and*

$$m(\psi(x), f \circ \psi^{-1}) = m(\phi(x), f \circ \phi^{-1}).$$

PROOF. Let $\widetilde{U}, \widetilde{V}$ be open subsets of the Hilbert space upon which Λ is modeled, such that $\phi : U \cap V \to \widetilde{U}$ and $\psi : U \cap V \to \widetilde{V}$ are homeomorphisms. Then, there exists a diffeomorphism $\theta : \widetilde{V} \to \widetilde{U}$ of class C^1.

Let $\alpha = \phi(x)$ and $\beta = \psi(x)$. Denoting by L the Hessian at $\phi^{-1}(x) = \alpha$ for $f \circ \phi^{-1}$, we have, for any $y \in \widetilde{U}$,

$$f \circ \phi^{-1}(y) = f \circ \phi^{-1}(\alpha) + \frac{1}{2}\langle L(y-\alpha), y-\alpha\rangle + O(|y-\alpha|^2).$$

Therefore, since $\phi^{-1} \circ \theta = \psi^{-1}$ and $\theta(\beta) = \alpha$, if $z = \theta^{-1}(y) \in \widetilde{V}$, then

$$\begin{aligned} f \circ \psi^{-1}(z) &= f \circ \phi^{-1}(\theta(z)) \\ &= f \circ \phi^{-1}(\alpha) + \frac{1}{2}\langle L(\theta(z)-\theta(\beta)), \theta(z)-\theta(\beta)\rangle + O(|\theta(z)-\theta(\beta)|^2). \end{aligned}$$

Since θ is differentiable at β,

$$\theta(z) - \theta(\beta) = d\theta(\beta)(z-\beta) + O(|z-\beta|).$$

Thus, since $d\theta(\beta)$ is a linear isomorphism,

$$f \circ \psi^{-1}(z) = f \circ \psi^{-1}(\beta) = \frac{1}{2}\langle L \circ d\theta(\beta)[z-\beta], d\theta(\beta)[z-\beta]\rangle + O(|z-\beta|^2).$$

Therefore $f \circ \psi^{-1}$ is twice differentiable at β and its Hessian is given by $[d\theta(\beta)]^* \circ L \circ d\theta(\beta)$. Since $d\theta(\beta)$ is a linear isomorphism the assertion of Lemma 1.6 follows immediately. □

By virtue of the above lemma, we can give the following definition:

DEFINITION 1.7. Let Λ be of class C^1, f and Ω as above and $x \in K_f(\overline{\Omega})$. We say that x is a *nondegenerate critical point* if there exists a C^1-chart (U, ϕ) (with $x \in U$) such that $\phi(x)$ is a nondegenerate critical point of $f \circ \phi^{-1}$ according to Definition 1.5. In this case, we set

$$m(x, f) = m(\phi(x), f \circ \phi^{-1}).$$

A function $f \in C^2$ is called a *Morse function* if its critical points are all nondegenerate (and, consequently, isolated).

DEFINITION 1.8. Let f be a Morse function. The *Morse polynomial* of a set $K \subset K_f$ is defined as follows:

$$m_\lambda(K) = \sum_{x \in K} \lambda^{m(x)},$$

with the convention that $\lambda^\infty = 0$.

Thus $m_\lambda(K)$ is a polynomial $\sum_k a_k \lambda^k$ whose coefficients a_k are integers representing the number of critical points in K_f having Morse index k.

REMARK 1.9. By virtue of Lemma 1.6, the Morse polynomial makes sense also for functions of class C^1 whose critical points are non degenerate in the sense of Definition 1.7. Clearly it is possible that K is an infinite set; in this case, the Morse polynomial becomes a formal series.

The above remark makes it useful to define the family S of formal series in one variable λ with coefficients in $\mathbb{N} \cup \{\infty\}$.

On S the sum and product are defined in the usual way:

$$\sum a_k \lambda^k + \sum b_k \lambda^k = \sum (a_k + b_k) \lambda^k$$

and

$$\sum a_k \lambda^k \cdot \sum b_k \lambda^k = \sum_k \left(\sum_{j=0}^{k} a_{k-j} b_j \right) \lambda^k$$

(and we set, as usual, $0 \cdot \infty = 0$).

If $\mathcal{P} \in S$ we set

$$c_k(\mathcal{P}) = a_k \Leftrightarrow \mathcal{P}(\lambda) = \sum_k a_k \lambda^k.$$

For the further development of the theory, it is necessary to impose on S an order structure. We define a relation of total order as follows:

$$\text{(1.1)} \qquad \sum a_k \lambda^k < \sum b_k \lambda^k \Leftrightarrow \exists n \in \mathbb{N} : a_k = b_k \text{for } k \leq n-1, \text{ and } a_n < b_n.$$

We define the notion of limit in S in the following way:

$$\mathcal{R} = \lim_{n\to\infty} \mathcal{P}_n \;\Leftrightarrow\; c_k(\mathcal{P}_n) \xrightarrow{n} c_k(\mathcal{R}) \qquad \text{for any } k \in \mathbb{N}. \tag{1.2}$$

If we identify the formal series $\sum a_k\lambda^k$ with the sequence $\{a_k\}$, then the topology introduced by (1.2) is equivalent to the product topology on $\prod_{i=0}^{\infty} X_i$, where $X_i = \mathbb{N} \cup \{\infty\}$; hence by Tikhonov's theorem, S is compact. If $A \subseteq S$, we denote by $\overline{A}$ the closure of A, i.e.

$$\overline{A} = \{\mathcal{P} \in S \mid \exists\{\mathcal{P}_n\} \subseteq A : \mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n\}.$$

Now, it makes sense to define the infimum and supremum as follows:

DEFINITION 1.10. If $A \subseteq S$, we put

$$\mathcal{R} = \inf A \quad \text{if } \mathcal{R} = \min \overline{A} \qquad \text{and} \qquad \mathcal{R} = \sup A \quad \text{if } \mathcal{R} = \max \overline{A}.$$

We have the following result:

THEOREM 1.11. *For any set $A \subseteq S$, $\inf A$ and $\sup A$ exist and are unique.*

PROOF. The uniqueness is trivial.

Existence of the infimum. We set

$$b_0 = \min\{c_0(\mathcal{P}) \mid \mathcal{P} \in \overline{A}\}, \qquad \mathbb{B}_0 = \{\mathcal{P} \in \overline{A} \mid c_0(\mathcal{P}) = b_0\},$$

$$b_n = \min\{c_n(\mathcal{P}) \mid \mathcal{P} \in \mathbb{B}_{n-1}\}, \qquad \mathbb{B}_n = \{\mathcal{P} \in \mathbb{B}_{n-1} \mid c_n(\mathcal{P}) = b_n\}.$$

Since the $\mathbb{B}_n$'s are compact and $\mathbb{B}_{n-1} \subseteq \mathbb{B}_n$ for every n, their intersection is not empty; $\mathcal{R} \in \bigcap_{n=0}^{\infty} \mathbb{B}_n$ is $\min \overline{A}$.

Existence of the supremum: we argue in the same way. □

REMARK 1.12. Notice that the topology induced by this notion of convergence, is not the topology induced by the order relation. Thus, we might have

$$\sup A \neq \inf\{\mathcal{P} \in S \mid \forall \mathcal{Q} \in S : \mathcal{P} \geq \mathcal{Q}\}.$$

For example, take

$$A = \{n\lambda^0 \in S \mid n \in \mathbb{N}\} \cup \{n\lambda^0 + \lambda^1 \in S \mid n \in \mathbb{N}\}.$$

In this case, we have $\sup A = \infty\lambda^0 + \lambda^1$, but

$$\inf\{\mathcal{P} \in S \mid \forall \mathcal{Q} \in S : \mathcal{P} \geq \mathcal{Q}\} = \infty\lambda^0.$$

However, the inequalities "pass to the limit": for example

$$\mathcal{P}_n \geq \mathcal{Q} \text{ for any } n \Rightarrow \lim_{n\to\infty} \mathcal{P}_n \geq \mathcal{Q},$$

and similarly for inf and sup.

REMARK 1.13. If $\{\mathcal{P}_k\}$ is a non-decreasing sequence and $\mathcal{R} = \sup\{\mathcal{P}_k\}$ is a polynomial, then, for k large, $\{\mathcal{P}_k\}$ is constantly equal to $\mathcal{R}$. In fact, by the definition (1.2), the values $c_n(\mathcal{R})$ are achieved by $c_n(\mathcal{P}_k)$ for k sufficiently large.

2. The Poincaré polynomial

The Poincaré polynomial of a topological pair (X, A) is a topological invariant which carries the information on the homology of (X, A).

DEFINITION 2.1. Given a homology theory $H_*(\cdot, \cdot, Z_2)$ and a topological pair (X, A), we set

$$\mathcal{P}_\lambda(X, A) = \sum_{q \in \mathbb{N}} \dim [H_q(X, A, Z_2)] \cdot \lambda^q.$$

Moreover, we set $\mathcal{P}_\lambda(X) = \mathcal{P}_\lambda(X, \emptyset)$.

The natural numbers $\dim [H_q(X, A, Z_2)]$ are the *called Betti numbers* (in fact, sometimes, the Poincaré polynomial is called the *Betti polynomial*). Notice that in general the Poincaré polynomial is not a "polynomial" but a formal series in S (cf. Remark 1.9).

We have chosen the field of coefficients to be Z_2 in order to avoid orientation problems. However, the theory which we will develop also works with any other field K of coefficients. The only difference arises actually in the computation of $\mathcal{P}_\lambda(X, A)$.

In the following, we will encounter the situations where the choice of coefficients does make a difference.

Many properties of homology can be transferred to the Poincaré polynomial and the operations and relations in S have a topological interpretation.

THEOREM 2.2. *Let (X, A) and (Y, B) be pairs of topological spaces.*

(i) *If (X, A) and (Y, B) are homotopically equivalent, then $\mathcal{P}_\lambda(X, A) = \mathcal{P}_\lambda(Y, B)$.*

(ii) *If $X \cap Y = \emptyset$, then $\mathcal{P}_\lambda(X \cup Y, A \cup B) = \mathcal{P}_\lambda(X, A) + \mathcal{P}_\lambda(Y, B)$.*

(iii) $\mathcal{P}_\lambda(X \times Y) = \mathcal{P}_\lambda(X) \cdot \mathcal{P}_\lambda(Y)$.

(iv) *If (X, A, B) is a topological triple and B is a weak deformation retract of A then*

$$\mathcal{P}_\lambda(X, A) = \mathcal{P}_\lambda(X, B);$$

if A is a weak deformation retract of X, then

$$\mathcal{P}_\lambda(X, B) = \mathcal{P}_\lambda(A, B).$$

(v) *If (X, A, B) is a topological triple then there exists a $\mathcal{Q}_\lambda = \mathcal{Q}_\lambda(X, A, B) \in S$ such that*

$$\mathcal{P}_\lambda(X, A) + \mathcal{P}_\lambda(A, B) = \mathcal{P}_\lambda(X, B) + \mathcal{Q}_\lambda.$$

(vi) (Excision property) *Let (X, A) be a topological pair; if $\overline{C} \subset \operatorname{int} A$, then*

$$\mathcal{P}_\lambda(X, A) = \mathcal{P}_\lambda(X \backslash C, A \backslash C).$$

(vii) *Let $\varphi_1 : (X, A) \to (Y, B)$ and $\varphi_2 : (Y, B) \to (Z, C)$ be two maps such that $(\varphi_2 \circ \varphi_1)_*$ is an isomorphism; then there exists $\mathcal{Z}_\lambda \in S$ such that*

$$\mathcal{P}_\lambda(Y, B) = \mathcal{P}_\lambda(X, A) + \mathcal{Z}_\lambda$$

(in particular, this happens e.g. if $(X, A) = (Z, C)$ and $\varphi_2 \circ \varphi_1$ is homotopically equivalent to the identity).

(viii) *Let $\mathcal{M}$ be a manifold and let $\mathcal{N} \subset \mathcal{M}$ be a closed (in $\mathcal{M}$) submanifold of codimension n. If W is a subset of $\mathcal{N}$ closed in $\mathcal{N}$, then*

$$\mathcal{P}_\lambda(\mathcal{M}, \mathcal{M} \backslash W) = \lambda^n \mathcal{P}_\lambda(\mathcal{N}, \mathcal{N} \backslash W)$$

(in this case, if the coefficient field K is not Z_2, we need to assume $\mathcal{N}$ and $\mathcal{M}$ to be orientable).

(ix) *If x_0 is a single point, then $\mathcal{P}_\lambda(\{x_0\}) = 1$.*

(x) *If B_n is an n-dimensional ball, then*

$$\mathcal{P}_\lambda(B_n, \partial B_n) = \lambda^n.$$

PROOF. (i)–(iv), (vi)–(vii) and (ix)–(x) are standard results in algebraic topology (cf. e.g. [9]). A proof of (v) can be found e.g. in [2]. The proof of (viii) follows from Corollary 8.11.20 of [8] (cf. also the remark below the Corollary) because the manifold $\mathcal{N}$ is an A. N. R. (cf. [11]). Moreover, notice that the Thom Theorem, in this form, holds even if the dimension of the manifold $\mathcal{M}$ is infinite. □

For any $a < b \in \mathbb{R}$ set

$$f^b = \{x \in \Lambda \,|\, f(x) \leq b\}, \qquad f_a^b = \{x \in \Lambda \,|\, a \leq f(x) \leq b\}.$$

Now we can state the Morse relations in their simplest form:

THEOREM 2.3. *Let Λ be a smooth manifold, let $f \in C^2(\Lambda, \mathbb{R})$ be a Morse function and let $a < b$ be two regular values for f. Then if f_a^b is compact, we have*

$$\sum_{x \in K(f_a^b)} \lambda^{m(x)} = \mathcal{P}_\lambda(f^b, f^a) + (1 + \lambda)\mathcal{Q}_\lambda$$

where $K(f_a^b) = K_f \cap f_a^b$ *and* $Q(t)$ *is a polynomial with integer nonnegative coefficients.*

If f_a^b is not compact, Theorem 2.3 is non longer valid as shown by simple examples. However, there are assumptions on the pair (Λ, f) which guarantee the validity of the Morse relations. The most famous (but not the most general) condition is the condition of Palais and Smale:

DEFINITION 2.4. We say that f satisfies the *Palais-Smale condition* in a set $\Omega \subseteq \Lambda$ (P.S. in Ω) if any sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Lambda$ such that

$$f(x_n) \xrightarrow{n} c \in \mathbb{R} \qquad \text{and} \qquad f'(x_n) \xrightarrow{n} 0$$

has a subsequence which converges to some $x \in \Omega$.

Clearly, if f_a^b is compact, then f satisfies P.S. in f_a^b; however, it is possible for f to satisfy P.S. even if f_a^b is not locally compact.

Moreover, as we have seen in Remark 1.9, the Morse polynomial also makes sense for C^1-functions. These remarks make natural the following definition:

DEFINITION 2.5. Let Ω be an open set in Λ. A function $f \in C^1(\Omega)$ is called a *generalized Morse function* if

(i) the critical points of f are nodegenerate in the sense of Definition 1.5;
(ii) f satisfies P.S. in Ω;
(iii) f can be extended to a functions of class C^1 in a neighborhood of $\overline{\Omega}$.

The set of generalized Morse function will be denoted by $\mathcal{M}(\overline{\Omega})$.

REMARK 2.6. If $f \in \mathcal{M}(\overline{\Omega})$ and it is bounded in Ω, then it has a finite number of critical points.

Now we can generalize Theorem 2.3:

THEOREM 2.7. *Let* Λ *be a complete manifold of class* C^1 *and let* $f \in \mathcal{M}(\mathrm{int}\, f_a^b)$. *Then*

$$\sum_{x \in K(f_a^b)} \lambda^{m(x)} = \mathcal{P}_\lambda(f^b, f^a) + (1+\lambda)\mathcal{Q}_\lambda,$$

where $K(f_a^b) = K_f \cap f_a^b$ *and* $\mathcal{Q}_\lambda$ *is a polynomial with integer nonnegative coefficients.*

Clearly Theorem 2.3 is an immediate consequence of Theorem 2.7. The proof of Theorem 2.7 is quite involved. We will prove it in Section 4 (in a more general form), using the notion of "Conley block" which will be introduced in the next section.

3. The Conley blocks

In this section we will introduce the notion of index pair and Conley block. They will allow us to prove the Morse relations in a very general context.

Let Λ be a Hilbert manifold and $T\Lambda$ its tangent bundle. Consider a vector field $F : \Lambda \to T\Lambda$ and denote by $\eta(t,x)$ the solution of the Cauchy problem

$$(3.1) \qquad \begin{cases} d\eta/dt = F(\eta), \\ \eta(0,x) = x. \end{cases}$$

Assume that (3.1) is well posed and

$$(3.2) \qquad \text{for any } x \in \Lambda,\ \eta(t,x) \equiv x \cdot t \text{ is defined for any } t \in \mathbb{R}.$$

For any set $A \subset \Lambda$ put

$$W_+(A) \equiv W_+(A,F) = \bigcap_{t \le 0} \eta(t,A) = \{x \in A, \mid \eta(t,x) \in A \text{ for any } t \ge 0\},$$

$$W_-(A) \equiv W_-(A,F) = \bigcap_{t \ge 0} \eta(t,A) = \{x \in A, \mid \eta(t,x) \in A \text{ for any } t \le 0\},$$

$$G(A) \equiv G(A,F) = W_+(A,F) \cap W_-(A,F) = \{x \in A \mid \eta(t,x) \in A \text{ for any } t \in \mathbb{R}\}.$$

The set $W_+(A)$ is usually called the *positively maximal invariant set* relatively to A (with respect to the flow η) and $W_-(A)$ the *negatively maximal invariant set* relative to A. $G(A)$ is called the *maximal invariant set* in A. (We recall that a set $E \subseteq A$ is called *positively invariant* relative to A if

$$x \in E \ \text{ and } \ \eta([0,t],x) \cap \Lambda \backslash E \ne \emptyset \Rightarrow \exists t_* \in [0,t] : \eta(t_*,x) \notin A).$$

Moreover for any $A \subset \Lambda$ and $T \ge 0$ put

$$W_+^T(A) \equiv W_+^T(A,F) = \bigcap_{t \in [-T,0]} \eta(t,A),$$

$$W_-^T(A) \equiv W_-^T(A,F) = \bigcap_{t \in [0,T]} \eta(t,A),$$

$$G^T(A) \equiv G^T(A,F) = W_+^T(A) \cap W_-^T(A).$$

REMARK 3.1. It is easy to verify that:

(i) If A is closed, then $W_+(A)$, $W_-(A)$, $G(A)$, $W_+^T(A)$, $W_-^T(A)$ and $G^T(A)$ are closed.

(ii) $W_+(W_+^T(A)) = W_+(A)$, for any $T \ge 0$ and for any A.

(iii) $G(W_+^T(A)) = G(A)$, for any $T \ge 0$ and for any A.

(iv) $\eta_T(W_+^T(A)) = W_-^T(A)$.

(v) $\eta_T(W_+^{2T}(A)) = G^T(A)$.

(vi) $G^T(A) \backslash W_+(G^T(A)) = G^T(A) \backslash W_+(A)$.

(vii) $G^{T_1}(G^{T_2}(A)) = G^{T_1+T_2}(A)$.

For any closed set $A \subset \Lambda$, define the following set which we will call the *exit set* (relative to A):

$$\Gamma(A) = \{x \in \partial A \mid \forall \varepsilon_0 \geq 0,\ \exists \varepsilon \in (0, \varepsilon_0) : x \cdot \varepsilon \notin A\};$$

the points in $\Gamma(A)$ are called *exit points* with respect to A. Even if A is closed, $\Gamma(A)$ need not be closed. Thus, it makes sense to define

$$\Sigma = \Sigma_F = \{N \subseteq \Lambda \mid \Gamma(N) \text{ is closed}\}.$$

A set $A \in \Sigma$ is called a *Conley block.*

Notice that $\Gamma(\eta(t, N)) = \eta(t, \Gamma(N))$ for any $t \in \mathbb{R}$.

DEFINITION 3.2. If $N \in \Sigma$, we define the *index* of N as follows:

$$I_\lambda(N) = I_\lambda(N, F) = \mathcal{P}_\lambda(N, \Gamma(N)).$$

DEFINITION 3.3. A couple (N, E) of closed subsets of Λ (with $E \subset N$) is called an *index pair* (with respect to F) if

(i) E is positively invariant relative to N, i.e.

$$x \in E \quad \text{and} \quad x \cdot [0, t] \cap \Lambda \backslash E \neq \emptyset \Rightarrow \exists t_* \in [0, t] : x \cdot t_* \notin N.$$

(ii) E is an exit set for N, i.e.

$$x \in N \quad \text{and} \quad x \cdot [0, t] \cap \Lambda \backslash E \neq \emptyset \Rightarrow \exists t_* \in [0, t] : x \cdot t_* \in E.$$

Clearly, if $\Gamma(N)$ is closed, the topological pair $(N, \Gamma(N))$ is an index pair of a particular type. Clearly the Conley blocks are index pairs.

The next theorem gives a simple but very useful method to construct sets in Σ.

THEOREM 3.4. *Let g_1 and g_2 be two differentiable functions defined on Λ. Set*

$$N = \{x \in \Lambda \mid g_j(x) \leq 0,\ j = 1, 2\}, \qquad \Gamma = \{x \in \partial N \mid g_1(x) = 0\}$$

and suppose that

$$x \in \partial N \text{ and } g_1(x) = 0 \ \Rightarrow\ \langle \nabla g_1(x), F(x) \rangle > 0,$$
$$x \in \partial N \text{ and } g_2(x) = 0 \ \Rightarrow\ \langle \nabla g_2(x), F(x) \rangle < 0.$$

Then (N, Γ) is a Conley block, $N \in \Sigma_F$ and $I_\lambda(N, F) = \mathcal{P}_\lambda(N, \Gamma)$.

PROOF. It is immediate to verify that $\Gamma(N) = \Gamma$ and that Γ is closed. □

REMARK 3.5. The condition on g_2 can be weakened as follows:

$$x \in \partial N \text{ and } g_2(x) = 0 \ \Rightarrow\ \frac{d}{dt}(g_2 \circ \eta(t, x))\bigg|_{t=0} \leq 0.$$

Theorem 3.4 can be easily generalized.

THEOREM 3.6. *Let g_i $(i = 1, \ldots, k)$ be functions of class C^1 on Λ. Set*

$$N = \bigcap_{i=1}^{k} \{x \in \Lambda \mid g_i(x) \le 0\}.$$

Suppose that for any $x \in \partial A$, there exists g_i satisfying

$$g_i(x) = 0 \Rightarrow \langle \nabla g_i(x), F(x) \rangle \ne 0.$$

Then $N \in \Sigma_F$ and $I_\lambda(N) = \mathcal{P}_\lambda(N, \Gamma(N))$.

An interesting application of Theorem 3.4 is the following:

THEOREM 3.7. *Let Λ be an Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and let L be a continuous linear operator satisfying the following assumptions:*

- (L_1) *there exist closed subspaces H^+ and H^- of Λ such that $L(H^\pm) \subset H^\pm$ and $H^- \oplus H^+ = \Lambda$;*
- (L_2) *there exists $\nu > 0$ such that $\langle Lx, x \rangle \le -\nu |x|^2$ for any $x \in H^-$ and $\langle Ly, y \rangle \ge \nu |y|^2$ for any $y \in H^+$, where $|\cdot|$ is the norm induced by the scalar product $\langle \cdot, \cdot \rangle$.*

Let

$$F(z) = -Lz + K(z),$$

where $z = x + y$, $x \in H^-$, $y \in H^+$ and $K : \Lambda \to \Lambda$ is a map of class $C^{0,1}_{\mathrm{loc}}$ such that there exists $\rho > 0$ satisfying

$$|K(x, y)| \le \frac{\nu}{2}|x| \qquad \text{if } |x| = \rho \text{ and } |y| \le \rho, \tag{3.3}$$

$$|K(x, y)| \le \frac{\nu}{2}|y| \qquad \text{if } |y| = \rho \text{ and } |x| \le \rho. \tag{3.4}$$

Then, the set $Q_\rho = \{x + y \in \Lambda \mid |x|, |y| \le \rho\}$ is in Σ_F and $I_\lambda(Q_\rho) = \lambda^m$, where $m = \dim H^-$.

PROOF. Set

$$g_1(x + y) = \frac{1}{2}|x|^2 - \frac{1}{2}\rho^2 \qquad \text{and} \qquad g_2(x + y) = \frac{1}{2}|y|^2 - \frac{1}{2}\rho^2.$$

Then by (3.3) and (3.4), we have

$$x \in \partial Q_\rho \text{ and } g_1(x) = 0$$
$$\Rightarrow \langle F(x, y), \nabla g_1(x) \rangle = \langle -Lx, x \rangle + \langle K(x, y), x \rangle \ge \frac{\nu}{2}|x|^2 > 0$$

and

$$x \in \partial Q_\rho \text{ and } g_1(x) = 0$$
$$\Rightarrow \langle F(x,y), \nabla g_2(x)\rangle = \langle -Ly, y\rangle + \langle K(x,y), y\rangle \leq -\frac{\nu}{2}|y|^2 < 0.$$

Set $\Gamma = \{x + y \in Q_\rho \mid |x| = \rho\}$. Then, by Theorem 3.4, (Q_ρ, Γ) is a Conley block and

$$I_\lambda(Q_\rho) = \mathcal{P}_\lambda(Q_\rho, \Gamma) = \mathcal{P}_\lambda(B_\rho \cap H^-, \partial B_\rho \cap H^-) = \lambda^m$$

(cf. Theorem 2.2(x)). □

The next theorem describes the main properties of the index.

THEOREM 3.8. *Let $N \in \Sigma$. Then:*

(i) $I_\lambda(N) = \mathcal{P}_\lambda(N, N\backslash W_+(N))$.

(ii) $\eta(t,N) \in \Sigma$ *and* $I_\lambda(\eta(t,N)) = I_\lambda(N)$.

(iii) $W_+^T(N) \in \Sigma$ *and* $I_\lambda(W_+^T(N)) = I_\lambda(N)$.

(iv) $W_-^T(N) \in \Sigma$ *and* $I_\lambda(W_-^T(N)) = I_\lambda(N)$.

(v) $G^T(N) \in \Sigma$ *and* $I_\lambda(G^T(N)) = I_\lambda(N)$.

(vi) *If $N_1, N_2 \in \Sigma$, and for some $T > 0$, $G^T(N_1) \subseteq N_2$ and $G^T(N_2) \subseteq N_1$ then*

$$I_\lambda(N_1) = I_\lambda(N_2).$$

(vii) *If $N_1, N_2 \in \Sigma$ are disjoint sets, then $N_1 \cup N_2 \in \Sigma$ and*

$$I_\lambda(N_1 \cup N_1) = I_\lambda(N_2) + I_\lambda(N_2).$$

(viii) *Let F_1 and F_2 be two vector fields which satisfy* (3.1) *and* (3.2); *if $N \in \Sigma_{F_1}$ and $F_2 = F_1$ on a neighborhood of ∂N, then $N \in \Sigma_{F_2}$ and*

$$I_\lambda(N, F_1) = I_\lambda(N, F_2).$$

(ix) *If for some $T > 0$, $G^T(N) \subset \operatorname{int} N$, then*

$$I_\lambda(N) = \mathcal{P}_\lambda(\operatorname{int} N, (\operatorname{int} N)\backslash W_+(N)),$$

where $\operatorname{int} N$ *denotes the interior on N whenever N is the closure of an open subset of Λ, while if $N = \operatorname{cl} M$ and M is a submanifold of Λ, then* $\operatorname{int} N$ *denotes the manifold M itself.*

(x) *Let $\mathcal{M} \in \Sigma$ and $\mathcal{N} \subset \mathcal{M}$ be two manifolds in Λ such that*

(1) $\overline{\mathcal{N}}$ *is positively invariant relative to* $\overline{\mathcal{M}}$,

(2) $G^T(\overline{\mathcal{M}}) \subset \mathcal{M}$ *for some* $T > 0$,

(3) $\partial\mathcal{N} \equiv \overline{\mathcal{N}}\backslash\mathcal{N} \subset \partial\mathcal{M} \equiv \overline{\mathcal{M}}\backslash\mathcal{M}$,

(4) $W_+(\overline{\mathcal{M}}) \subset \overline{\mathcal{N}}$.

Then $\overline{\mathcal{N}} \in \Sigma$. Moreover, if $\mathcal{N}$ has codimension d in $\mathcal{M}$, then

$$I_\lambda(\overline{\mathcal{M}}) = \lambda^d I_\lambda(\overline{\mathcal{N}})$$

(in this case, if the coefficient field K is not Z_2, we need to assume $\mathcal{N}$ and $\mathcal{M}$ to be orientable).

PROOF. (i) By the definition of $\Gamma(N)$ it is evident that $\Gamma(N) \subset N\backslash W_+(N)$. Then, by Theorem 2.2 (iv) applied to the topological triple $(N, N\backslash W_+(N), \Gamma(N))$, it is sufficient to prove that $\Gamma(N)$ is a deformation retract of $N\backslash W_+(N)$.

In order to prove this, consider $x \in N\backslash W_+(N)$. By the definition of $W_+(N)$ there exists $t > 0$ such that $\eta(t,x) \notin N$. Let

$$\tau(x) = \inf\{t \in \mathbb{R}^+ \mid \eta(t,x) \notin N\}.$$

We want to prove that τ is a continuous function. Let $x \in N\backslash W_+(N)$ and let x_k be a sequence in $N\backslash W_+(N)$ such that $x_k \to x$. We will prove that $\tau(x_k) \to \tau(x)$.

Choose any $\varepsilon_0 > 0$; then there exists $\varepsilon \in (0,\varepsilon_0)$ such that $\eta(\tau(x)+\varepsilon, x) \notin N$. By the continuity of η, for k large enough, we have $\eta(\tau(x)+\varepsilon, x_k) \notin N$ and hence $\tau(x_k) \le \tau(x) + \varepsilon$. Since ε can be choosen arbitrarily small, this proves that

$$\tau(x) \ge \limsup_{k\to\infty} \tau(x_k).$$

Now let $\tau = \liminf_{k\to\infty} \tau(x_k)$. If $\tau = \infty$, the proof is finished. Now, suppose $\tau < \infty$ and take a subsequence $\tau(x_{k_m})$ such that

$$\tau = \lim_{m\to\infty} \tau(x_{k_m}).$$

Since $\eta(\tau(x_{k_m}), x_{k_m}) \in \Gamma$ and Γ is closed, we have $\eta(\tau, x) \in \Gamma$, and therefore, by the definitions of Γ and $\tau(x)$,

$$\tau(x) \le \tau = \liminf_{k\to\infty} \tau(x_k).$$

Thus $\tau(x)$ is continuous, and the map $x \mapsto \eta(\tau(x), x)$ is a strong retraction of $N\backslash W_+(N)$ on $\Gamma(N)$.

(ii) Since $\eta(t,\cdot)$ is a homeomorphism,

$$\begin{aligned} I_\lambda(N) &= \mathcal{P}_\lambda(N, \Gamma(N)) = \mathcal{P}_\lambda(\eta(t,N), \eta(t,\Gamma(N))) \\ &= \mathcal{P}_\lambda(\eta(t,N), \Gamma(\eta(t,\Gamma(N)))) = I_\lambda(\eta(t,N)). \end{aligned}$$

(iii) Put $C = N\backslash W_+^T(N)$. Then

$$(W_+^T(N),\ W_+^T(N)\backslash W_+(N)) = (N\backslash C, (N\backslash W_+(N))\backslash C). \tag{3.5}$$

Since $\mathrm{cl}_N C \subset \mathrm{int}_N N\backslash W_+(N)$, by Theorem 2.2(vi) we have

$$\mathcal{P}_\lambda(N, N\backslash W_+(N)) = \mathcal{P}_\lambda(N\backslash C, (N\backslash W_+(N))\backslash C)$$

and, by (3.5),

$$\mathcal{P}_\lambda(N, N\backslash W_+(N)) = \mathcal{P}_\lambda(W_+^T(N), W_+^T(N)\backslash W_+(N)). \tag{3.6}$$

Now it is not difficult to prove that $W_+^T(N) \in \Sigma$. Then using Theorem 3.8(i), (3.6) and Remark 3.1(ii), we have

$$I_\lambda(N) = \mathcal{P}_\lambda(N, N\backslash W_+(N)) = \mathcal{P}_\lambda(W_+^T(N), W_+^T(N)\backslash W_+(N)) = I_\lambda(W_+^T(N)).$$

(iv) By (iii), (ii) and Remark 3.1(iv), we have

$$I_\lambda(N) = I_\lambda(W_+^T(N)) = I_\lambda(\eta(T, W_+^T(N))) = I_\lambda(W_-^T(N)).$$

(v) By (iii), (ii) and Remark 3.1(v), we have

$$I_\lambda(N) = I_\lambda(W_+^{2T}(N)) = I_\lambda(\eta(T, W_+^{2T}(N))) = I_\lambda(G^T(N)).$$

Notice that, in order to prove the above equality, we have only used the homeomorphism $\eta(-T,\cdot)$ and the excision map. Thus, since $G^T(N)\backslash W_+(G^T(N)) = G^T(N)\backslash W_+(N)$, denoting by $i : W_+^{2T}(N) \to N$ the inclusion map, we find that the map

$$(i \circ \eta(-T,\cdot))_* : H_*(G^T(N), G^T(N)\backslash W_+(N)) \to H_*(N, N\backslash W_+(N))$$

is an isomorphism.

Moreover, using the homotopy $H(\sigma, x) = \eta(-\sigma T, x)$ shows that the inclusion

$$j : (G^T(N), G^T(N)\backslash W_+(N)) \to (N, N\backslash W_+(N))$$

is homotopically equivalent to the map $i \circ \eta(-T,\cdot)$. Therefore

$$j_* : H_*(G^T(N), G^T(N)\backslash W_+(N)) \to H_*(N, N\backslash W_+(N)) \tag{3.7}$$

is an isomorphism.

(vi) By our assumptions,

$$G^{2T}(N_2) \subset G^T(N_1) \subset N_2.$$

Thus, we have the embeddings

$$i_1 : (G^{2T}(N_2), G^{2T}(N_2)\backslash W_+(N_2)) \to (G^T(N_1), G^T(N_1)\backslash W_+(N_2))$$

and

$$i_2 : (G^T(N_1), G^T(N_1)\backslash W_+(N_2)) \to (N_2, N_2\backslash W_+(N_2)).$$

By (3.7), $(i_2 \circ i_1)_*$ is an isomorphism. Therefore, by Theorem 2.2(vii), there exists $\mathcal{Z}_\lambda \in S$ such that

$$\mathcal{P}_\lambda(G^T(N_1), G^T(N_1)\backslash W_+(N_2)) = \mathcal{P}_\lambda(N_2, N_2\backslash W_+(N_2)) + \mathcal{Z}_\lambda.$$

Since $G^T(N_1)\backslash W_+(N_2) = G^T(N_1)\backslash W_+(G^T(N_1))$, using (i) and (v), we have

$$\begin{aligned} I_\lambda(N_1) = I_\lambda(G^T(N_1)) &= \mathcal{P}_\lambda(G^T(N_1), G^T(N_1)\backslash W_+(G^T(N_1))) \\ &= \mathcal{P}_\lambda(N_2, N_2\backslash W_+(N_2)) + \mathcal{Z}_\lambda = I_\lambda(N_2) + \mathcal{Z}_\lambda. \end{aligned}$$

Arguing in the same way, we also have the existence of $\widetilde{\mathcal{Z}}_\lambda \in S$ such that $I_\lambda(N_1) + \widetilde{\mathcal{Z}}_\lambda = I_\lambda(N_2)$; hence $I_\lambda(N_1) = I_\lambda(N_2)$.

(vii) and (viii) are immediate consequences of the definition of I_λ.

(ix) We define the homotopy

$$H(t,x) = \begin{cases} \eta(t,x) & \text{if } t \le \tau(x), \\ \eta(\tau(x),x) & \text{if } t \ge \tau(x), \end{cases}$$

where $\tau(x)$ is defined in the proof of (i). If $H(2T,x) \notin \Gamma(N)$, then $H(T,x) \in G^T(N) \subset \operatorname{int} N$, hence $H(2T,x) \in \operatorname{int} N$. Therefore $K(\sigma,x) = H(2\sigma T, x)$ is a weak deformation of N onto $\operatorname{int} N \cup \Gamma(N)$ and also a weak deformation of $N\backslash W_+(N)$ onto $(\operatorname{int} N\backslash W_+(N)) \cup \Gamma(N)$. Thus, by Theorem 2.2(iv) we have

$$I_\lambda(N) = \mathcal{P}_\lambda(N, N\backslash W_+(N)) = \mathcal{P}_\lambda(\operatorname{int} N \cup \Gamma(N), (\operatorname{int} N\backslash W_+(N)) \cup \Gamma(N)).$$

The conclusion follows by the excision property (Theorem 2.2(vi)) on taking $C = \Gamma(N)$.

(x) First of all we have to prove that $\Gamma(\overline{\mathcal{N}})$ is closed. Let x_k be a converging sequence of exit points with respect to $\overline{\mathcal{N}}$; by assumption (1) they are also exit points with respect to $\overline{\mathcal{M}}$; thus they converge to a point x_0 which is an exit point with respect to $\overline{\mathcal{M}}$ and which belongs to $\overline{\mathcal{N}}$ (by (3)). Thus $x_0 \in \Gamma(\overline{\mathcal{N}})$. Now, by (3), $\overline{\mathcal{N}} \subset \overline{\mathcal{M}}$, therefore $G^T(\overline{\mathcal{N}}) \subset G^T(\overline{\mathcal{M}})$ and, by (2) and (3), $G^T(\overline{\mathcal{N}}) \subset \mathcal{N}$. Then by (ix),

$$I_\lambda(\overline{\mathcal{M}}) = \mathcal{P}_\lambda(\mathcal{M}, \mathcal{M}\backslash W_+(\overline{\mathcal{M}})) \qquad I_\lambda(\overline{\mathcal{N}}) = \mathcal{P}_\lambda(\mathcal{N}, \mathcal{N}\backslash W_+(\overline{\mathcal{N}})).$$

Moreover, by (4), $W_+(\overline{\mathcal{N}}) = W_+(\overline{\mathcal{M}})$ and hence $I_\lambda(\overline{\mathcal{N}}) = \mathcal{P}_\lambda(\mathcal{N}, \mathcal{N}\backslash W_+(\overline{\mathcal{M}}))$.

Now, by (3), $\mathcal{N}$ is closed in $\mathcal{M}$, and also $W_+(\overline{\mathcal{M}}) \cap \mathcal{N}$ is closed in $\mathcal{N}$; thus we can apply Theorem 2.2(viii) with $W = W_+(\overline{\mathcal{M}})$ to obtain

$$\mathcal{P}_\lambda(\mathcal{M}, \mathcal{M}\backslash W_+(\overline{\mathcal{M}})) = \lambda^d \mathcal{P}_\lambda(\mathcal{N}, \mathcal{N}\backslash W_+(\overline{\mathcal{M}})),$$

and the conclusion follows. □

REMARK 3.9. Let $S = G(A)$ with $A \in \Sigma$. Then S is an invariant maximal set in A. If the following condition (introduced in [2]) is satisfied

(C) for every neighborhood B of S, $G^T(A) \subset B$ for some $T > 0$,

then, by Theorem 3.8(i)), $I_\lambda(B)$ (for $B \in \Sigma$) is independent of B.

In particular, if Λ is compact, we have

$$I_\lambda(B) = \sum_{k \geq 0} \dim H_k(\operatorname{Con} S, K) \cdot \lambda^k,$$

where $\operatorname{Con} S$ is the Conley index of S (cf. [7]).

4. The Morse relations

The theory developed in the previous section concerns general flows. Now we can apply it to the study of the critical points of a C^1-functional f and to Morse theory. To do this it is necessary to construct a vector field F such that (3.1) and (3.2) hold and

$$\forall x \in \Lambda\backslash K_f, \qquad df(x)[F(x)] < 0.$$

This relation implies that f is decreasing along the orbit flow η, i.e.

$$\frac{d}{dt}(f \circ \eta(t,x)) < 0 \qquad \text{for any } x \notin K_f \text{ and } t \in \mathbb{R}.$$

If Λ has a Riemannian structure, $f \in C^{1,1}_{\mathrm{loc}}$ and ∇f is bounded, then the vector field F can be obtained by taking

$$F = -\nabla f.$$

If ∇f is not bounded, then F can be obtained as follows:

$$F = \frac{-\nabla f}{1 + |\nabla f|}.$$

However, in applications, particularly to P.D.E., we are interested in functionals which are merely of class C^1. In this case the construction of F is more delicate.

As essentially proved in [12] the following lemma holds:

LEMMA 4.1. *Given $f \in C^1(\Lambda)$, there exists a vector field $-F$ (called a pseudo-gradient vector field for f) such that*

(i) $F \in C^{0,1}_{\mathrm{loc}}(\Lambda \backslash K_f)$, *for any* $x \notin K_f$,

(ii) $|F(x)| \leq M d(x, K_f)$, *where M is a constant,*

(iii) $\langle \nabla f(x), F(x) \rangle < 0$ *if* $x \notin K_f$.

Moreover, if f satisfies P.S. and it is bounded on an open set Ω, then (iii) *strengthened to*

(iv) *for any neighborhood U of $K_f(\Omega)$, there exists $\nu = \nu(\Omega, U) > 0$ such that*

$$\forall x \in \Omega \backslash U, \qquad \langle \nabla f(x), F(x) \rangle \leq -\nu.$$

Moreover, we can construct F near the critical points by the following lemma:

LEMMA 4.2. *Let $f \in C^1(\Lambda)$ and let x_0 be a nondegenerate critical point of f (in the sense of Definition 1.7). Then there exists a neighborhood U of x_0 and a vector field $F \in C^{0,1}(U)$ such that*

$$\forall x \in U, \qquad \langle \nabla f(x), F(x) \rangle \leq -\nu |\nabla f(x)|^2,$$

where $\nu > 0$ is a constant.

PROOF. Let (U, ϕ) be a chart as in Definition 1.7 and let L be the Hessian of $f \circ \phi^{-1}$ at the point $\phi(x_0)$ as in the proof of Lemma 1.6. Then we can define F by the following formula:

$$F(x) = (d\phi^{-1}(x))^* \circ L[\phi(x) - \phi(x_0)],$$

i.e. F is L pulled back by ϕ^{-1}. It is not difficult to check that F has the required properties. □

As in Section 2 set

$$f_a^b = \{x \in \Lambda \mid a \leq f(x) \leq b\},$$

and, for $c \in \mathbb{R}$,

$$f^c = \{x \in \Lambda \mid f(x) \leq c\}.$$

THEOREM 4.3 (deformation theorem). *Let $f \in \mathcal{M}(\Lambda)$. There exists a pseudo-gradient vector field $-F$ such that the Cauchy problem*

$$\begin{cases} \dfrac{d\eta}{dt} = F(\eta), \\ \eta(0, x) = x, \end{cases}$$

is well posed, $\eta(t,x)$ is defined for any $x \in \Lambda$ and $t \in \mathbb{R}$ and

$$\text{(i)} \qquad \frac{d}{dt}(f \circ \eta(t,x)) < 0 \qquad \text{for any } x \notin K_f \text{ and } t \in \mathbb{R}.$$

Moreover, for any neighborhood U of K_f,

(ii) *if f is bounded on an open set Ω, there exists $\nu = \nu(U,\Omega) > 0$, such that*

$$\forall x \in \Omega \backslash U, \qquad \frac{d}{dt}(f \circ \eta(0,x)) = \langle \nabla f(x), F(x) \rangle \leq -\nu,$$

(iii) *if c is the only critical value of f in (a,b), then there exists $T = T(U,a,b) > 0$ such that*

$$G^T(f_a^b) \subseteq U.$$

PROOF. Let F_1 be the vector field given by Lemma 4.1 and let F_2 be a vector field which coincides with the vector field given by Lemma 4.2 in an open neighborhood U of the critical points. Since, by our assumptions, these points are isolated, we can assume that U is given by the union of local charts. Now let ϕ be a Lipschitz continuous function which is 0 in $\Lambda \backslash U$ and which is 1 in an open neighborhood V ($\overline{V} \subset U$) of the critical points.

Then the vector field F given in every local chart by

$$F(x) = \phi(x)F_1(x) + (1 - \phi(x))F_2(x)$$

has the properties (i) and (ii). Let us prove (iii).

Let U_0 be a neighborhood of the critical points, with closure included in U. Using a standard argument (cf. e.g. [13]) we see that

$$\begin{aligned} &\exists T_1 > 0 : x \in U_0 \Rightarrow \eta_t(x) \in U \ \forall t \in [-T_1, T_1], \\ &\exists \varepsilon > 0 : G^{T_1}(f_{c-\varepsilon}^{c+\varepsilon}) \subset U, \\ &\exists T_2 > 0 : G^{T_2}(f_a^b) \subset f_{c-\varepsilon}^{c+\varepsilon}. \end{aligned}$$

Then the assertion follows by Remark 3.1(vii). □

REMARK 4.4. If $f \in \mathcal{M}(\overline{\Omega})$ where Ω is an open set in Λ, then the same result of Theorem 4.3 holds, except that $\eta(t,x)$ is defined only for $x \in \overline{\Omega}$ and $t \leq \tau(x)$, the exit time from $\overline{\Omega}$.

In the variational case, the strip f_a^b is the simplest set to which we can apply the theory of Section 1.3.

THEOREM 4.5. *Let f and F be as above, let $a, b \in \mathbb{R}$ with $a < b$ and suppose that a is a regular value of f. Then $f_a^b \in \Sigma$ and*

$$I_\lambda(f_a^b, F) = \mathcal{P}_\lambda(f^b, f^a),$$

PROOF. Take $g_1(x) = -f(x) + a$ and $g_2(x) = f(x) - b$. Then the conclusion follows by Theorem 3.4 and Remark 3.5. □

Notice that in this theorem it is not necessary that F satisfies P.S. Moreover, in this case $I_\lambda(f_a^b, F)$ does not depend on F, but only on f: any vector field F which satisfies (i) of Theorem 4.3 gives the same index.

The next theorem relates the Morse index of a nondegenerate critical point to the index I_λ:

THEOREM 4.6. *Let $f \in C^1(\Lambda)$, let x_0 be a nondegenerate critical point of f (in the sense of Definition 1.3) and let F be as in Lemma 4.2. Then there exists a neighborhood U_0 of x_0 such that $U_0 \in \Sigma$ and*

$$I_\lambda(U_0) = \lambda^{m(x_0)}.$$

PROOF. Choosing a local chart V for x_0 we can assume to work in a Hilbert space. Let $Q_\rho \subset V$ be the set of Theorem 3.7 "centered" at x_0. If we take ρ sufficiently small, the assumptions of Theorem 3.7 are satisfied. Thus the statement of Theorem 4.6 follows. □

In the variational case, we can produce sets in Σ by intersecting of sets in Σ.

LEMMA 4.7. *Let f and F be as in Theorem 4.3, let $N \in \Sigma$ and let b be a regular value of f. Then $N \cap f^b, N \cap f_b \in \Sigma$ and there exists $\mathcal{Q}_\lambda = \mathcal{Q}_\lambda(F) \in S$ such that*

$$I_\lambda(N \cap f^b) + I_\lambda(N \cap f_b) = I_\lambda(N) + (1+\lambda)\mathcal{Q}_\lambda(F).$$

PROOF. Set

$$A = N \cap f_b \quad \text{and} \quad B = N \cap f^b.$$

Consider the triple $(\Gamma(N), B \cup \Gamma(N), N)$. By Theorem 2.2(v), there exists $\mathcal{Q} \in S$ such that

$$\mathcal{P}_\lambda(N, B \cup \Gamma(N)) + \mathcal{P}_\lambda(B \cup \Gamma(N), \Gamma(N)) = \mathcal{P}_\lambda(N, \Gamma(N)) + (1+\lambda)\mathcal{Q}_\lambda.$$

Moreover, since b is a regular value for f, it is not difficult to check that $\Gamma(B) = \Gamma(N) \cap B$, and $\Gamma(A) = (\Gamma(N) \cap A) \cup f^{-1}(b)$, so that $\Gamma(B)$ and $\Gamma(A)$ are closed.

Now set

$$C_\varepsilon = \{\eta_t(x) \mid t \in [0,\varepsilon], x \in f^{-1}(b)\}.$$

Since b is a regular value, if ε is sufficiently small, by Theorem 2.2(iv), we have

$$\mathcal{P}_\lambda(A, \Gamma(A)) = \mathcal{P}_\lambda(A \cap C_\varepsilon, \Gamma(A) \cap C_\varepsilon).$$

Moreover, following the proof of Theorem 3.8(i) and using Theorem 2.2(iv) and 2.2(vi) shows that (if ε is sufficiently small)

$$\mathcal{P}_\lambda(N, B \cap \Gamma(N)) = \mathcal{P}_\lambda(A \cap C_\varepsilon, \Gamma(A) \cup C_\varepsilon),$$

therefore,

$$\mathcal{P}_\lambda(N, B \cup \Gamma(N)) = \mathcal{P}_\lambda(A, \Gamma(A)) = I_\lambda(A, F).$$

Moreover, $B \cup \Gamma(N) \backslash (\Gamma(N) \backslash \Gamma(B)) = B$ and, since b is a regular value,

$$\mathrm{cl}_{B \cup \Gamma(N)}(\Gamma(N) \backslash \Gamma(B)) \subset \mathrm{int}_{B \cup \Gamma(N)}(\Gamma(N).$$

Therefore, by the excision property,

$$\mathcal{P}_\lambda(B \cup \Gamma(N), \Gamma(N)) = \mathcal{P}_\lambda(B, \Gamma(B)) = I_\lambda(B, F).$$

Since $I_\lambda(N) = \mathcal{P}_\lambda(N, \Gamma(N))$, we get the conclusion. □

COROLLARY 4.8. *Let f and F be as in Theorem 4.3, let $N \in \Sigma$ and let b_k $(k = 1, \ldots, n)$ be a sequence of regular values of f; moreover, set $b_0 = -\infty$ and $b_{n+1} = \infty$. Then $N \cap f_{b_k}^{b_{k+1}} \in \Sigma$ and there exists $\mathcal{Q}_\lambda = \mathcal{Q}_\lambda(F) \in S$ such that*

$$\sum_{k=0}^{n} I_\lambda(N \cap f_{b_k}^{b_{k+1}}) = I_\lambda(N) + (1+\lambda)\mathcal{Q}_\lambda(F).$$

PROOF. It is an easy consequence of Lemma 4.7. □

Now we are ready to prove Theorem 2.7 in a more general form:

THEOREM 4.9. *Let Ω be an open set in Λ and let $f \in \mathcal{M}(\overline{\Omega})$ with $\overline{\Omega} \in \Sigma$. If f is bounded, then*

$$\sum_{x \in K(\Omega)} \lambda^{m(x)} = I_\lambda(\overline{\Omega}, F) + (1+\lambda)\mathcal{Q}_\lambda.$$

where $K(\Omega) = K_f \cap \Omega$, $\mathcal{Q}_\lambda$ is a polynomial with integer nonnegative coefficients and F is given by Theorem 4.3.

(Note that if $f \in \mathcal{M}(\overline{\Omega})$, then $K_f(\overline{\Omega}) = K_f(\Omega)$.)

Theorem 2.7 follows from the above theorem and Theorem 4.5 if we take $\overline{\Omega} = f_a^b$.

First we need the following lemma.

LEMMA 4.10. *Let $\overline{\Omega} \in \Sigma$ and let $f \in \mathcal{M}(\overline{\Omega})$. If f is bounded and has only the critical value $c \in \mathbb{R}$, then*

$$I_\lambda(N) = \lambda^{m(x_1)} + \cdots + \lambda^{m(x_k)},$$

where $\{x_1, \ldots, x_k\}$ are the critical points of f and $m(x_j)$ is the Morse index of x_j (notice that we have only a finite number of critical points by Remark 2.6).

PROOF. Let $U_1, \ldots, U_k$ be disjoint neighborhoods of our critical points taken according to Theorem 4.6, and let $U = \bigcup_{h=1}^k U_h$. By Theorems 3.8(vii) and 4.6, we have

$$I_\lambda(U) = \sum_h I_\lambda(U_h) = \lambda^{m(x_1)} + \cdots + \lambda^{m(x_k)}.$$

Moreover, by Theorem 4.3(iii), there exists T so large that $G^T(\overline{\Omega}) \subseteq U$. Then, by Theorem 3.8.(vi), $I_\lambda(\overline{\Omega}) = I_\lambda(U)$, from which the conclusion follows.□

PROOF OF THEOREM 4.9. Since $f \in \mathcal{M}(\overline{\Omega})$ and it is bounded, by Remark 2.6, it has only a finite number of critical values $\{c_1, \ldots, c_n\}$. Now let $\{b_0, \ldots, b_{n+1}\}$ be a sequence of numbers such that

$$-\infty = b_0 < c_1 < b_1 < \ldots < b_n < c_n < b_{n+1} = \infty.$$

Then, by Corollary 4.8,

$$\sum_{k=0}^{n} I_\lambda(\overline{\Omega} \cap f_{b_k}^{b_{k+1}}) = I_\lambda(\overline{\Omega}) + (1+\lambda)Q_\lambda$$

and the conclusion follows from Lemma 4.10 with $\overline{\Omega}$ replaced by $\overline{\Omega} \cap f_{b_k}^{b_{k+1}}$. □

In Theorem 4.9, we have assumed that f is bounded, but this assumption can be partially removed if we allow both the Morse and Poincaré polynomials be formal series (in the space S).

THEOREM 4.11. *Let Ω be an open set in Λ and let $f \in \mathcal{M}(\overline{\Omega})$ with $\overline{\Omega} \in \Sigma$; if f is bounded from below on Ω, then*

$$\sum_{x \in K(\Omega)} \lambda^{m(x)} = I_\lambda(\overline{\Omega}, F) + (1+\lambda)Q_\lambda, \tag{4.1}$$

where $K(\Omega) = K_f \cap \Omega$ is a countable set, Q_λ is a formal series in S and F is given by Theorem 4.3.

PROOF. Since every critical point of f is nondegenerate and f satisfies P.S. on Ω and is bounded from below, there exist $\{b_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ and $\{c_h\}_{h\in\mathbb{N}} \subset \mathbb{R}$ such that

- every b_n is a regular value for f,

- $b_0 < \inf_{\overline{\Omega}} f < b_1 < \ldots < b_h < b_{h+1} < \ldots$,
- $\lim_{h\to\infty} b_h = \infty$,
- for any $h \in \mathbb{N}$, $f_{b_h}^{b_{h+1}} \cap K_f(\Omega) = f^{-1}(c_h) \cap K_f(\Omega)$,
- for any $h \in \mathbb{N}$, $f^{-1}(c_h) \cap K_f(\Omega)$ is finite.

Then, by Lemma 4.7 and Corollary 4.8, arguing by induction on h gives, for any $h \in \mathbb{N}$, the existence of $\mathcal{Q}_h \in S$ such that

$$m_\lambda(f^{b_h} \cap K_f(\overline{\Omega}), f) + I_\lambda(f_{b_h}, F) = I_\lambda(\overline{\Omega}, F) + (1+\lambda)\mathcal{Q}_h(\lambda), \tag{4.2}$$

where $f^b = \{x \in \overline{\Omega} \mid f(x) \leq b\}$ and $f_b = \{x \in \overline{\Omega} \mid f(x) \geq b\}$.

Fix $k \in \mathbb{N}$. Our goal is to prove (4.1), using (4.2) and considering the coefficients of any fixed degree of the formal series in (4.1).

If the set M_k of points of $K_f(\Omega)$ having Morse index k is infinite, taking the limit in (4.2) as $h \to \infty$, gives immediately the proof of (4.1) in degree k, since the coefficient of degree k of $m_\lambda(f^{b_h} \cap K_f(\overline{\Omega}), f)$ is nondecreasing with respect to h and tends to ∞.

Now suppose that M_k is finite and let b a regular value such that

$$b > \max_{M_k} f. \tag{4.3}$$

By (4.2), to prove (4.1) (degree k) it is sufficient to prove that

(4.4) the coefficient of degree k of $I_\lambda(f_b, F)$ is zero.

Now let $c > b$ be a regular value for f. Then, using the flow η and the excision property as in proving Lemma 4.7 shows that

$$\mathcal{P}_\lambda(\Gamma(\overline{\Omega}) \cup f^c, \Gamma(\overline{\Omega}) \cup f^b) = \mathcal{P}_\lambda(f_b^c, (\Gamma(\overline{\Omega}) \cap f_b^c) \cup f^{-1}(b)). \tag{4.5}$$

Since $(\Gamma(\overline{\Omega}) \cap f_b^c) \cup f^{-1}(b) = \Gamma(f_b^c)$, by (4.5) we have

$$\mathcal{P}_\lambda(\Gamma(\overline{\Omega}) \cup f^c, \Gamma(\overline{\Omega}) \cup f^b) = \mathcal{P}_\lambda(f_b^c, \Gamma(f_b^c)). \tag{4.6}$$

Then, by Corollary 4.8 with $N = f_b^c$, combining (4.3) and (4.6) gives

$$H_k(\Gamma(\overline{\Omega}) \cup f^c, \Gamma(\overline{\Omega}) \cup f^b) = 0. \tag{4.7}$$

Since c is a regular value using the flow and the excision property we also get

$$I_\lambda(f_b, F) = \mathcal{P}_\lambda(f_b, (\Gamma(\overline{\Omega}) \cap f_b) \cup f^{-1}(b)) = \mathcal{P}_\lambda(\overline{\Omega}, \Gamma(\overline{\Omega}) \cup f^b). \tag{4.8}$$

Now consider the exact homology sequence

(4.9)
$$\to H_k(\Gamma(\overline{\Omega}) \cup f^c, \Gamma(\overline{\Omega}) \cup f^b)) \xrightarrow{i_k^*} H_k(\overline{\Omega}, \Gamma(\overline{\Omega}) \cup f^b) \xrightarrow{j_k^*} H_k(\overline{\Omega}, \Gamma(\overline{\Omega}) \cup f^c) \to$$

where i_k^* and j_k^* are the maps induced by the inclusion maps.

If, by contradiction, (4.4) does not hold, then by (4.8) there exists

$$\alpha \in H_k(\overline{\Omega}, \Gamma(\overline{\Omega}) \cup f^c), \qquad \alpha \neq 0.$$

Now, denoting by Δ the support of α and choosing $c > \max_\Delta f$, gives $j_k^*(\alpha) = 0$. Then, by the exactness of the homology sequence (4.9), there exists $\beta \in H_k(\Gamma(\overline{\Omega}) \cup f^c, \Gamma(\overline{\Omega}) \cup f^b))$ such that $i_k^*(\beta) = \alpha$, contrary to (4.7). Thus (4.4) is proved and the proof of Theorem 4.11 is complete. □

5. Morse theory for degenerate critical points

In this section we introduce an index i_λ generalizing the Morse polynomial m_λ when the critical points of f are degenerate. More generally, we define a Morse polynomial also for an isolated set K of critical points of f.

First of all we shall describe a class $\mathcal{F}(\overline{\Omega})$ of C^1-functionals where the generalized Morse index will be defined.

For any $\varepsilon > 0$ and $A \subset \Lambda$, we put

$$N_\varepsilon(A) = \{x \in \Lambda \mid d(x, A) < \varepsilon\},$$

where d is the distance induced by the Hilbert structure of Λ. Now consider the class

$$\mathcal{M}_f^\varepsilon(\overline{\Omega}) = \{g \in \mathcal{M}(\overline{\Omega}) \mid g(x) = f(x) \ \text{ for } \ x \notin N_\varepsilon(K_f(\overline{\Omega}))\},$$

where $\mathcal{M}(\overline{\Omega})$ is defined in Definition 2.5.

Notice that

$$\varepsilon_1 < \varepsilon_2 \Rightarrow \mathcal{M}_f^{\varepsilon_1}(\overline{\Omega}) \subset \mathcal{M}_f^{\varepsilon_2}(\overline{\Omega}).$$

We define

$$\mathcal{F}(\overline{\Omega}) = \{f \in C^1(\overline{\Omega}) \,|\, \mathcal{M}_f^\varepsilon(\overline{\Omega}) \neq \emptyset \ \text{ for any } \varepsilon > 0\}.$$

Clearly, if $\Omega \subseteq \Lambda$ is bounded and Λ is a finite-dimensional manifold, then $\mathcal{F}(\overline{\Omega}) = C^1(\overline{\Omega})$. In general, we do not know how general this class is; however, we can prove that many interesting functionals belong to it.

EXAMPLE 5.1. Let f be of class C^2 in a neighborhood of $\overline{\Omega}$ and satisfy P.S. in Ω. Suppose that, for any degenerate critical point $x \in K_f(\overline{\Omega})$, the linear operator associated with $H^f(x)$ is a Fredholm operator (of index 0). Then $f \in \mathcal{F}(\overline{\Omega})$ (cf. [10]).

EXAMPLE 5.2. Let Λ be a separable Hilbert space and let f_0 satisfy the assumptions of Example 5.1 (for instance $f_0(x) = \frac{1}{2}\langle Lx, x\rangle$, where $L : \Lambda \to \Lambda$ is a bounded selfadjoint strictly positive operator). Let $\psi \in C^1(\Lambda)$ be a functional whose gradient is completely continuous, i.e.

(5.1) if x_k weakly converges to x, then $\psi'(x_k)$ strongly converges to $\psi'(x)$.

Assume that $f(x) = f_0(x) + \psi(x)$ is bounded in $\overline{\Omega}$ and satisfies P.S. in Ω. Then $f \in \mathcal{F}(\overline{\Omega})$.

PROOF. Since Λ is separable, there exists a sequence $\{E_n\}_{n\in\mathbb{N}'}$ of linear subspaces of Λ such that $E_n = \operatorname{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system for Λ. Consider the orthogonal projection P_n of Λ on E_n and put

$$\psi_n(x) = \psi(P_n(x)).$$

The definition of P_n and the assumption (5.1) easily give

$$\sup_{x\in B} |\psi_n'(x) - \psi'(x)| \xrightarrow{n} 0 \qquad \text{for any bounded subset } B,$$

which implies that

$$\sup_{x\in B} |\psi_n(x) - \psi(x)| \xrightarrow{n} 0 \qquad \text{for any bounded subset } B.$$

Since $\psi_n|_{E_n} : E_n \to \mathbb{R}$, and E_n is finite-dimensional, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of C^∞-maps defined on E_n such that

$$|\psi_n|_{E_n} - \varphi_n|_{C^1(E_n)} \leq 1/n.$$

Now, for any $x \in \Lambda$, set

$$\widetilde{\psi}_n(x) = \varphi_n(P_n(x)).$$

Then for any bounded subset B,

$$\widetilde{\psi}_n(x) \xrightarrow{n} \psi(x) \qquad \text{in } C^1(B). \tag{5.2}$$

Now, set $K = K_f(\Omega)$, and for any $\varepsilon > 0$, let $\phi_\varepsilon \in C^2(\Lambda, [0,1])$ be a function such that

- $\phi_\varepsilon = 0$ on $\Lambda \setminus N_\varepsilon(K)$,
- $\phi_\varepsilon = 1$ on $N_{\varepsilon/2}(K)$,
- $|\phi_\varepsilon'|$ is bounded in Λ.

Now we define

$$g_{\varepsilon,n}(x) := f_0(x) + \psi(x) + \phi_\varepsilon(x)(\psi_n(x) - \psi(x)) \tag{5.3}$$
$$= f(x) + \phi_\varepsilon(x)(\psi_n(x) - \psi(x)).$$

Since f satisfies P.S. and it is bounded on Ω, there exists $\nu(\varepsilon) > 0$ such that $|f'(x)| \geq \nu(\varepsilon)$ for any x in $\Omega \backslash N_{\varepsilon/2}(K)$, so

$$|g'_{\varepsilon,n}(x)| \geq |f'(x)| - |\phi'_\varepsilon(x)|\,|\psi_n(x) - \psi(x)| - |\phi_\varepsilon(x)|\,|\psi'_n(x) - \psi'(x)|.$$

Since f satisfies P.S. and is bounded in Ω, K is compact and hence $N_\varepsilon(K)$ is bounded. Therefore, by (5.2), if n is sufficiently large, there are no critical points of $g_{\varepsilon,n}$ in $N_\varepsilon(K)\backslash N_{\varepsilon/2}(K)$ and hence in $\Omega\backslash N_{\varepsilon/2}(K)$. Moreover, on $N_{\varepsilon/2}(K)$,

$$g_{\varepsilon,n} = f_0 + \widetilde{\psi}_n.$$

Since $g_{\varepsilon,n}$ satisfies the assumptions of Example 5.1 on $N_{\varepsilon/2}(K)$ and it does not have critical points on $\partial N_{\varepsilon/2}(K_f)$, using the perturbation methods of Marino and Prodi (cf. [10]), $g_{\varepsilon,n}$ can be modified to a function $\widetilde{g}_{\varepsilon,n}$ such that

- $\widetilde{g}_{\varepsilon,n}|_{N_{\varepsilon/2}(K)}$ is a Morse function,
- $\widetilde{g}_{\varepsilon,n}$ does not have critical points in $N_\varepsilon(K)\backslash N_{\varepsilon/2}(K)$,
- $\widetilde{g}_{\varepsilon,n} = f$ on $\Omega\backslash N_\varepsilon(K)$.

Then, for n large, $\widetilde{g}_{\varepsilon,n} \in \mathcal{M}^\varepsilon_f(\overline{\Omega})$ and hence $f \in \mathcal{F}(\overline{\Omega})$. □

REMARK 5.3. Note that $f \in \mathcal{F}(\overline{\Omega})$ might not satisfy P.S. on $\overline{\Omega}$. This happens if $K_f(\overline{\Omega})$ is not compact. For example, let Λ be an infinite-dimensional Hilbert space and $f(x) = \varphi(|x|^2)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and

$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq 1, \\ (s-1)^2 & \text{if } s \geq 1. \end{cases}$$

Let $\varphi_\varepsilon \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ be a function such that $\varphi''_\varepsilon(0) \neq 0$ and $\varphi'_\varepsilon(s) = 0$ if and only if $s = 0$ and $\varphi_\varepsilon(s) = \varphi(s)$ for any $s \geq 1 + \varepsilon$, and set $g_\varepsilon(x) = \varphi_\varepsilon(|x|^2)$; then $g_\varepsilon \in \mathcal{M}^\varepsilon_f(\overline{\Omega}) \neq \emptyset$.

Another example is the map $\sin(\cdot) \in \mathcal{F}(\mathbb{R})$.

Let $f \in C^1(\Omega)$. A compact set $K \subset K_f$ is called an *isolated critical* set if there exists an open set ω such that $K = K_f \cap \omega$. The open set ω will be called an *isolating set* (for K).

Finally, we define the Morse index for an isolated critical set $K \subset K_f$, which is the main point of this section. Let ω be an isolating set for K.

DEFINITION 5.4. We set

$$i_\lambda(K,f) \equiv i_\lambda(K,f,\omega) = \sup_{\varepsilon>0} \inf_{g\in\mathcal{M}_f^\varepsilon(\overline{\omega})} m_\lambda(K_g(\overline{\omega}),g), \tag{5.4}$$

where m_λ, inf and sup are given by Definitions 1.8 and 1.10 respectively.

The formal series $i_\lambda(K,f)$ is called the (generalized) *Morse index* of K.

REMARK 5.5. It is easy to see that the index (5.4) of an isolated critical set does not depend on the isolating set ω.

DEFINITION 5.6. If $x \in K_f(\overline{\Omega})$ is an isolated critical point, we call the integer $i_1(\{x\},f)$ the *multiplicity* of x. Analogously it is possible to define the multiplicity of an isolated critical set.

Note that a critical set K contains at least $i_1(K,f)$ critical points if counted with multiplicities.

REMARK 5.7. If a critical point x has multiplicity $h \in \mathbb{N}$, then every perturbation of f producing nondegenerate critical points has at least h critical points near x. In fact, by Remark 1.13, for ε sufficiently small,

$$\inf_{g\in\mathcal{M}_f^\varepsilon(\overline{\omega})} m_\lambda(K_g(\overline{\omega}),g)$$

is constant.

The next theorem describes the basic properties of the index i_λ and shows that it is a generalization of the Morse polynomial.

THEOREM 5.8. *Let $f \in \mathcal{F}(\overline{\Omega})$. Then*

(i) *If x_0 is a nondegenerate critical point of f, then*

$$i_\lambda(\{x\},f) = \lambda^{m(x,f)}.$$

(ii) *If $K_1, K_2 \subset K$ are isolated compact sets, and $K_1 \cap K_2 = \emptyset$, then*

$$i_\lambda(K_1 \cup K_2, f) = i_\lambda(K_1,f) + i_\lambda(K_2,f).$$

(iii) *If K is a discrete set, then*

$$i_\lambda(K,f) = \sum_{x\in K} i_\lambda(x,f).$$

(iv) *If $f \in \mathcal{M}(\overline{\Omega})$, then*

$$i_\lambda(K_f,f) = m_\lambda(K_f,f) = \sum_{x\in K_f} \lambda^{m(x,f)}.$$

PROOF. Let (U_x, ϕ_x) be a local chart for the nondegenerate critical point x and $\alpha_x = \phi_x(x)$. Putting $L = d\phi_x^{-1}(\alpha_x) \circ (H^{f \circ \phi_x}(\alpha_x))$ in Theorem 3.7 gives the existence of $\rho > 0$ such that $Q_\rho \subset U_x$, $Q_\rho \in \Sigma_F$ and

$$I_\lambda(Q_\rho, F) = \lambda^{m(x)}. \tag{5.5}$$

Moreover, ρ can be chosen so small that f satisfies P.S. on Q_ρ, and

$$K_f(Q_\rho) = \{x\}. \tag{5.6}$$

Then, for any $\varepsilon > 0$, $f \in \mathcal{M}_f^\varepsilon(Q_\rho)$ and, by Definition 5.4,

$$i_\lambda(\{x\}, f) \leq \lambda^{m(x)}. \tag{5.7}$$

Moreover, by Theorem 4.9 and (5.5), for any $g \in \mathcal{M}_f^\varepsilon(Q_\rho)$, there exists $\mathcal{Q}_{\lambda,g} \in S$ such that

$$m_\lambda(\{x\}, g) = \lambda^{m(x)} + (1+\lambda)\mathcal{Q}_{\lambda,g}.$$

Therefore, by Definition 5.4,

$$i_\lambda(\{x\}, f) \geq \lambda^{m(x)}, \tag{5.8}$$

and combining (5.7) and (5.8) gives the proof of (i).

(ii) and (iii) are simple consequences of Definition 5.4; (iv) follows from (iii) and (i). □

By Theorem 5.8 it turns out that the multiplicity of a nondegenerate critical point is one.

Notice that it is possible for a critical point to have multiplicity zero (consider for example $i_1(\{0\}, f)$ with $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$).

Of course the index i_λ has been constructed in such a way that the Morse relations are still valid. In fact, we have the following theorem.

THEOREM 5.9. *Assume f is bounded from below on Ω and $f \in \mathcal{M}(\overline{\Omega})$. If F is a vector field as in Theorem 4.3, and $\overline{\Omega} \in \Sigma$, then*

$$i_\lambda(K_f, f) = I_\lambda(\overline{\Omega}, F) + (1+\lambda)\mathcal{Q}_\lambda,$$

where $\mathcal{Q}_\lambda \in S$.

PROOF. By Definitions 5.4 and 1.10, there are sequences $\varepsilon_k \to 0^+$ and g_k such that

$$g_k \in \mathcal{M}_f^{\varepsilon_k}(\overline{\omega}) \quad \text{and} \quad \lim_{k\to\infty} m_\lambda(K_{g_k}, g_k) = i_\lambda(K_f, f).$$

By Theorem 4.11, for any $k \in \mathbb{N}$, there exists $Q_\lambda^k \in S$ such that

$$m_\lambda(K_{g_k}, g_k) = I_\lambda(\overline{\Omega}, F) + (1+\lambda)Q_\lambda^k. \tag{5.9}$$

Taking the limit of the coefficients of the above equation, we conclude the proof. □

If some of the coefficients of $i_\lambda(K_f, f)$ are ∞, then, by taking the limit in (5.9), some of the coefficients of Q_λ might not be uniquely determined; nevertheless the Morse relations hold upon applying the usual algebra to ∞.

The next theorem states some information given by I_λ for the degenerate points of C^2-functionals.

THEOREM 5.10. *Let f satisfy the assumptions of Example 5.1 and suppose that $K_f(\overline{\Omega})$ is compact. Then*

(i) *If $I_\lambda(K, f) = \sum_{k\geq 0} a_k\lambda^k$ with $a_k \neq 0$, then there exists $x \in K$ such that*

$$m(x) \leq k \leq m^*(x).$$

(ii) *If we set $m(K) = \inf_{x\in K} m(x)$ and $m^*(K) = \sup_{x\in K} m^*(x)$, then*

$$I_\lambda(K, f) = \sum_{k=m(K)}^{m^*(K)} a_k\lambda^k.$$

PROOF. (i) follows by the perturbation methods of Marino and Prodi (cf. [10]), and (since the Hessian is a Fredholm map) by the lower semicontinuity of the strict Morse index and the upper semicontinuity of the large Morse index. (ii) is a simple consequence of (i). □

We end this section with a theorem which will be useful when we apply the Morse theory to P.D.E.

THEOREM 5.11. *Let f be a function as in Example 5.2, and let x_0 be an isolated critical point of f. Let $E_n = \operatorname{span}\{e_1, \ldots, e_n\}$ be a sequence of linear subspaces of Λ, where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system for Λ, and set*

$$V_n = x_0 + E_n.$$

Suppose that

(i) *$f|_{V_n}$ is twice differentiable at x_0,*

(ii) *for n sufficiently large, x_0 is a nondegenerate critical point of $f|_{V_n}$ and*

$$m(x_0, f|_{V_n}) = k.$$

Then $i(x_0) = \lambda^k$.

PROOF. It is not restrictive to suppose that $x_0 = 0$. Let N be a Conley block such that $K_f \cap N = \{x_0\}$. Such a block can be constructed in the following way: take a neighborhood $\mathcal{V}$ of x_0 such that $K_f \cap \mathcal{V} = \{x_0\}$; if $\varepsilon > 0$ is sufficiently small and $c = f(x_0)$, take

$$N = \mathcal{V} \cap G^1(f_{c-\varepsilon}^{c+\varepsilon})$$

($G^T(A)$ is defined in Section 3). Now, let $g_{\varepsilon,n}$ be the function defined by (5.3) in the proof of Example 5.2 with ε small enough such that $N_\varepsilon(x_0) \subseteq N$ and n large enough such that (ii) holds. Then, by the construction of $g_{\varepsilon,n}$, we see that x_0 is the only critical point of $g_{\varepsilon,n}$ in N and

$$m(x_0, g_{\varepsilon,n}) = k.$$

Since N is a Conley block for $g_{\varepsilon,n}$, by the Morse relations for $g_{\varepsilon,n}$, we get $\lambda^k = I_\lambda(N)$; then, by the Morse relations for f,

$$i(x_0) = I_\lambda(N) = \lambda^k.$$

□

REMARK 5.12. Let Λ be a Hilbert space. Suppose that $f \in \mathcal{F}(\overline{\Omega})$, f is bounded on $\overline{\Omega}$, F is as in Theorem 4.3 and

$$F = \mathrm{id} + \psi,$$

where id is the identity on Λ and $\psi : \overline{\Omega} \to \Lambda$ is a compact operator.

Assume $\overline{\Omega} \in \Sigma$ and $F(x) \neq 0$ for any $x \in \partial\Omega$. Then by Theorem 4.9 and the definitions of i_λ, I_λ and the topological degree we see that

$$I_{-1}(\overline{\Omega}, F) = \deg(\overline{\Omega}, -F, 0), \tag{5.10}$$

and it is also possible to show that (5.10) can be generalized to situations where F is not a pseudo-gradient vector field (nonvariational case) by the approach to Morse inequalities for the nonvariational case.

6. A typical application

In this section we give a simple application of the theory developed in the previous sections. Such an application has been chosen since it is simple and, at the same time, some of the results are new and it is not easy to obtain them using different methods.

Consider the following problem:

$$\begin{cases} -\Delta u - \mu u + g(u) = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{6.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with sufficiently smooth boundary, $\mu \in \mathbb{R}$, and $g \in C^0(\mathbb{R}, \mathbb{R})$ satisfies the following assumption:

$$|g(s)| \le c(|s|^{\vartheta} + 1), \qquad c \in \mathbb{R},\ \vartheta \in [0, 1[. \tag{6.2}$$

Let

$$G(s) = \int_0^s g(\sigma)\, d\sigma$$

and $f : H_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional of class C^1 given by

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \frac{\mu}{2}\int_\Omega u^2(x)\, dx + \int_\Omega G(u(x))\, dx, \tag{6.3}$$

whose critical points are solutions of (6.1).

Notice that the functional f is of class C^1 even if G is of class C^2. In fact, f is of class C^2 provided that

$$|g'(s)| \le c(|s|^{p^*-2} + 1), \tag{6.4}$$

where $p^* = 2N/(N-2)$ and $c \in \mathbb{R}$.

This assumption is too restrictive. In fact, using topological degree or the Lusternik and Schnirelmann theory, it is possible to get existence results without it. So the classical Morse theory is not adequate to study problem (6.1). However, using the theory developed in this paper, not only is it possible to use Morse theory, but also it provides better estimates on the number of solutions and sometimes on their qualitative properties.

THEOREM 6.1. *Let $\mu \in]\mu_k, \mu_{k+1}[$, where μ_j is the j-th eigenvalue of $-\Delta$. Then*

(a) *The problem (6.1) has at least one solution $\overline{u}$ such that*

$$i_\lambda(\overline{u}) = \lambda^k + \text{other terms.}$$

(b) *If g is of class C^1 and (6.4) holds, then*

$$m(\overline{u}, f) \le k \le m^*(\overline{u}, f),$$

where $m^*(\overline{u}, f)$ is the large Morse index for $\overline{u}$.

REMARK 6.2. The result in (a) can be easily obtained by using topological degree, but without the information on the Morse index as in (b). On the other hand, the "classical" Morse theory cannot be used for two reasons. The first one is that the functional f is, in general, only of class C^1. The second one is that it is not trivial to evaluate the Poincaré polynomial of the "strips" $\{a \leq f(x) \leq b\}$, while the evaluation of the Poincaré polynomial of the block is not difficult (under the assumptions of Theorem 6.1).

The information about the Morse index can be used e.g. to evaluate the number of the nodal regions of the solutions of (6.1). (We recall that a *nodal region* of u is a connected component of $\Omega \backslash u^{-1}(0)$).

Suppose that

$$g'(s)s^2 < g(s)s \qquad \text{for any } s \in \mathbb{R}\backslash\{0\}$$

(for example $g(s) = \arctan s$). By Theorem 6.1(b), the same argument used in [4] (where the superlinear case $-g(s) = s|s|^{p-2}$, $2 < p \leq 2N/(N-2)$, has been studied) shows that the number of nodal regions of $\overline{u}$ is less than or equal to k.

PROOF OF THEOREM 6.1. By standard arguments it is possible to prove that there exists a compact operator K such that $F = -L + K$ is a pseudo-gradient vector field for f, which satisfies the assumptions of Theorem 3.7 (cf. e.g. [3] for details). Then (a) follows by Theorem 5.9. (b) follows by Theorem 5.10 and Example 5.2. □

The following theorem concerns the concept of multiplicity introduced in Section 5.

THEOREM 6.3. *Let $\mu \in \,]\mu_k, \mu_{k+1}[$, where μ_j is the j-th eigenvalue of $-\Delta$. Suppose that $g(0) = 0$. Then*

(a) *If $g'(0)$ exists and $g'(0) \in \,]\mu_h, \mu_{h+1}[$ with $h \neq k$, then the problem* (6.1) *has at least* 3 *solutions if counted with multiplicities.*

(b) *Moreover, if g is of class C^1 and* (6.4) *holds, then there are at least two solutions u_1, u_2 of* (6.1) (*or one solution with multiplicity* 2) *such that*

$$m(u_1, f) \leq k \leq m^*(u_1, f)$$

and

$$m(u_2, f) - 1 \leq k \leq m^*(u_2, f) + 1.$$

PROOF. If 0 is not an isolated critical point the conclusion follows, so we can assume that 0 is isolated. The functional f satisfies the assumptions of Theorem

5.11 for the critical point $u \equiv 0$. Let $h = m(x_0, f|_{V_n})$ for n sufficiently large. Then, denoting by K the set of critical points of f, we obtain

$$i_\lambda(K) = \lambda^h + Z(\lambda).$$

We have to show that $Z(1) \geq 2$. By Theorem 5.9 and Theorem 3.7,

$$\lambda^h + Z(\lambda) = \lambda^k + (1+\lambda)\mathcal{Q}(\lambda). \tag{6.5}$$

Since $h \neq k$, by (6.5) we get (a) since $Z(\lambda)$ must have two terms at least. Moreover, if $g'(0) \in]\mu_h, \mu_{h+1}[$, then 0 is a nondegenerate critical point of f and $m(0, f) = h$. Therefore (b) easily follows from (6.5) and Theorem 5.10. □

REMARK 6.4. Theorem 6.3 illustrates the concept of multiplicity and generalizes the Theorem of 3 solutions of Amann (cf. [1]). Notice that in Theorem 6.3, it is possible to have only one nontrivial solution of multiplicity 2. Thus, if we make a minimal assumption to have a multiplicity result, we have to consider the notion of multiplicity.

If, in the proof of Theorem 6.3 we use the topological degree instead of the generalized Morse index, we need an assumption assuring that the topological degree at 0 is different from the topological degree at infinity, i.e. $(-1)^h \neq (-1)^k$.

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