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A NONSYMMETRIC ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEM

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Dedicated to Jean Leray

Introduction

I. The asymptotically homogeneous problem. In the theory of semilinear elliptic equations of the type

$$(P)\qquad \begin{cases} \Delta u + g(x,u) = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ are given functions, a very important role is played by asymptotically homogeneous equations, which are characterized by the conditions

$$\lim_{s\to-\infty} \frac{g(x,s)}{s} = \beta \quad \text{and} \quad \lim_{s\to+\infty} \frac{g(x,s)}{s} = \alpha.$$

Indeed, these equations constitute, in some sense, a connection between the (classical) case $-\infty \le \alpha < \lambda_1$ and $-\infty \le \beta < \lambda_1$, where λ_1 is the first eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the "superlinear" case $\alpha = \beta = +\infty$ and many intermediate cases, which have been object of several papers, often not related one to another.

Hence the study of asymptotically homogeneous equations can help to exhibit some components of a comprehensive theoretic framework.

Of course there are several other types of problems, for example when suitable conditions on the mutual position of α, β and $g'_s(x,0)$ are imposed (see [2]), which are, at this stage, outside at this direction of studies.

A fascinating point of view in the study of these equations is the variational one: it is well known that if g satisfies suitable assumptions, then solutions of (P) are stationary points of the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - G(x,u) + hu \right) dx,$$

where $G(x,s) = \int_0^s g(x,\sigma)\, d\sigma$ and the space $H_0^1(\Omega)$ is equipped with the usual scalar product $(u,v) = \int_\Omega \nabla u \nabla v$.

We are particularly interested in the geometric properties of the functional f.

There is an evident connection between the geometric behaviour of f and the positions of α and β with respect to the eigenvalues of the Dirichlet problem on Ω. We denote by λ_i these eigenvalues and by e_i the corresponding eigenfunctions ($\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$). We put $\mathrm{H}_i = \mathrm{span}(e_1, \ldots, e_i)$ and $\mathrm{H}_i^\perp = (\mathrm{H}_i)^\perp$.

If, for example, we consider the case $-\infty \leq \alpha < \lambda_1$ and $-\infty \leq \beta < \lambda_1$, then $\lim_{\|u\| \to +\infty} f(u) = \infty$; therefore the existence of a minimum of f can be shown by using elementary geometrical tools.

If, on the contrary, $\alpha = \beta = +\infty$, then $\lim_{t \to +\infty} f(tu) = -\infty$ for every $u \neq 0$ and the existence of critical points (assuming, for example, $g(x,0) = 0$) is due to the contrast between the asymptotic properties of f and the properties of $f''(0)$ (which has in any case some positive eigenvalues) (see [6] and [20]). The geometric behaviour of f is particularly interesting when both α and β belong to the interval $]\lambda_i, \lambda_{i+1}[$ for some i; in such a case f is a functional of the "saddle" type:

$$\lim_{\substack{u \in \mathrm{H}_i \\ \|u\| \to \infty}} f(u) = -\infty \qquad \text{and} \qquad \lim_{\substack{u \in \mathrm{H}_i^\perp \\ \|u\| \to \infty}} f(u) = +\infty.$$

Hence the functional is the object of one of the most beautiful and most expressive theorems (see [36]) concerning the existence of critical points which will be recalled below.

One important tool for proving the existence of critical points of a functional by means of some information on its sublevels is the so-called Palais-Smale condition, which we briefly recall.

Let X be Hilbert space and $F : \mathrm{X} \to \mathbb{R}$ be a C^1 function. We say that F satisfies the *Palais-Smale* (P.S.) *condition* if for any real a and b and for any sequence $(u_n)_{n\in\mathbb{N}}$ in X such that $a \leq F(u_n) \leq b$ and $\lim_n \nabla F(u_n) = 0$ in X, there exists a convergent subsequence.

THEOREM. *Let* X_1 *and* X_2 *be two closed subspaces of* X *such that* $\dim \mathrm{X}_1 < \infty$ *and* $\mathrm{X} = \mathrm{X}_1 \oplus \mathrm{X}_2$. *Assume that there exists* $R > 0$ *such that*

$$\sup_{\substack{u\in \mathrm{X}_1\\ \|u\|=R}} F(u) < \inf_{u\in \mathrm{X}_2} F(u)$$

and F *satisfies* (P.S.). *Then* F *has at least one critical point.*

II. Some known results for problems with jumping nonlinearities. The situation becomes more complex if α and β fall between some eigenvalues λ_i of the Dirichlet problem on Ω.

In the first, to our knowledge, paper in this direction [4], the authors considered the problem

$$(P_t) \qquad \begin{cases} \Delta u + g(x,u) = h_0 + te_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g \in C^2(\mathbb{R})$, $h_0 \in L^2(\Omega)$, $e_1 > 0$ in Ω and $t \in \mathbb{R}$. The following result holds.

THEOREM. *Let* Ω *be a bounded, connected and smooth open set in* $\mathbb{R}^n$. *Let* $g \in C^2(\mathbb{R},\mathbb{R})$ *be such that* $g''(s) > 0$ *for all* $s \in \mathbb{R}$, $g(0) = 0$ *and*

$$0 < \beta = \lim_{s\to-\infty} g'(s) < \lambda_1 < \alpha = \lim_{s\to+\infty} g'(s) < \lambda_2.$$

Then there exists a map $\varphi : \mathrm{H}_1^{\perp} \to \mathbb{R}$ *such that the problem* (P_t) *with* $h_0 \in \mathrm{H}_1^{\perp}$ *has exactly two solutions if* $t > \varphi(h_0)$, *exactly one solution if* $t = \varphi(h_0)$ *and no solution if* $t < \varphi(h_0)$.

The proof of the theorem does not use variational tools, but it is based on an accurate analysis of the singular points and a further argument, which gives the global result. A variational approach to improve the above result can be found in [3].

If we assume that some other eigenvalues, in addition to λ_1, fall between α and β:

$$(1,i) \qquad \beta < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_i < \alpha < \lambda_{i+1},$$

the problem (P) has been studied by several authors and it turns out to be a rather complex one.

When Ω is an interval, for example $\Omega =]0,\pi[$, it is possible to establish the exact numbers of solutions (see [24], [39] and [10]).

THEOREM. *The problem*

$$\begin{cases} u'' + g(u) = t \sin x, & x \in [0, \pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

when $g \in C^1(\mathbb{R})$ *and* $\lim_{s\to-\infty} g'(s) < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_i < \lim_{s\to+\infty} g'(s) < \lambda_{i+1}$ *(under the supplementary hypothesis:* $g_0(u) = g(u) - \alpha u^+ + \beta u^-$ *is a continuous sublinear function and* β *is positive) has at least* $2i$ *solutions if* t *is positive and large enough. Moreover, if* $g(u) = \alpha u^+ - \beta u^-$ *then for* t *positive and large enough it has exactly* $2i$ *solutions.*

When $\Omega \subset \mathbb{R}^N$ with $N > 1$, the following example is really interesting (see [14]).

THEOREM. *Assume that* Ω *is a ball in* $\mathbb{R}^N$ *with* $N > 1$ *and let us consider the problem*

$$\begin{cases} \Delta u + \alpha u^+ - \beta u^- = h_0 + t e_1 & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$$

with $\beta < \lambda_1 < \lambda_2 = \ldots = \lambda_{n+1} < \alpha < \lambda_{n+2}$ *and* $h_0 \in C^2(\Omega)$, $h_0 = 0$ *on* $\partial\Omega$, $h_0 \in \mathrm{Ker}\,(\Delta + \lambda_2 I)$, $h_0 \neq 0$. *Then for* t *positive and large enough the problem has only four solutions.*

Several other interesting papers (we mention [1], [11], [19], [21], [22], [23], [38], [41] and [42]) concern the problem (P) under assumption $(1, i)$. Under assumptions $(1, i)$ the whole picture is not yet completely clear. In this paper we do not take direct interest in this case.

Now we consider the assumption:

$$(h, i) \qquad \ldots \leq \lambda_{h-1} < \beta < \lambda_h \leq \ldots \leq \lambda_i < \alpha < \lambda_{i+1}.$$

Among some interesting known results, we wish to quote the following "alternative" theorem (see [26] and [13]), because of its generality.

We define "resonance set"

$$\Sigma_\Omega = \left\{(\alpha, \beta) \in \mathbb{R}^2 \mid \exists u \in H_0^1(\Omega),\ u \neq 0 \text{ such that } \Delta u + \alpha u^+ - \beta u^- = 0\right\}.$$

THEOREM. *Assume* $g \in C^1(\mathbb{R})$, $\lim_{s\to-\infty} g'(s) = \beta$ *and* $\lim_{s\to+\infty} g'(s) = \alpha$. *If* (h, i) *is satisfied and* $(\alpha, \beta) \notin \Sigma_\Omega$, *then the following alternative holds for the problem* (P_t): *either* (P_t) *has at least two solutions for* $t \gg 0$ *and* $t \ll 0$, *or* (P_t) *has at least one solution for* $t \gg 0$ *and three solutions for* $t \ll 0$, *or* (P_t) *has at least three solutions for* $t \gg 0$ *and one solution for* $t \ll 0$.

In [26] theorem was obtained by an analysis of the degree of solutions (under stronger assumptions on g), while in [13] the Rybakowski index was considered.

The assumption $(\alpha,\beta) \notin \Sigma_\Omega$ appears in all the papers that we know about (P_t), under condition (h,i) with $h \geq 2$, and it is automatically fullfilled if (α,β) satisfies $(1,i)$. From the variational point of view, this assumption ensures the Palais-Smale condition. However, the study of Σ_Ω turns out to be difficult, except when Ω is an interval in $\mathbb{R}$ (see [17]).

A real progress in this direction was made in the following theorem (see [15]). Let Ω_0 be an open bounded smooth subset of $\mathbb{R}^N$, $\alpha \in]0,1[$ and Q be an open neighbourhood of $\overline{\Omega}_0$. For $\varepsilon > 0$ and $\sigma \in]0,1[$ we set

$$V = \{\Phi \in C^{3,\sigma}(Q,\mathbb{R}^N) \,|\, \|\Phi\|_{C^{2,\sigma}} \leq \varepsilon\}.$$

We choose $\varepsilon > 0$ such that $I + \Phi$ is a diffeomorphism for any $\Phi \in V$.

THEOREM. *There exists a dense subset Z of V (with respect to $\| \ \|_{C^{3,\sigma}}$) such that if $\Phi \in Z$ and $\Omega = (I+\Phi)\Omega_0$ then $\overset{\circ}{\Sigma}_\Omega = \emptyset$.*

Roughly speaking, by slight perturbations of Ω and of (α,β), we get $(\alpha,\beta) \notin \Sigma_\Omega$ and so (P.S.) holds.

III. Results of this paper. As far as the Palais-Smale condition is concerned, in Section 2 we will give some sufficient conditions without requiring that $(\alpha,\beta) \notin \Sigma_\Omega$. This allows us to obtain various existence theorems, under conditions on g which are easily verifiable.

Concerning the existence theorems, we will study the case $\alpha > \lambda_1$ and $\beta > \lambda_1$, even if some results hold true also when $\alpha > \lambda_1$ and $\beta < \lambda_1$.

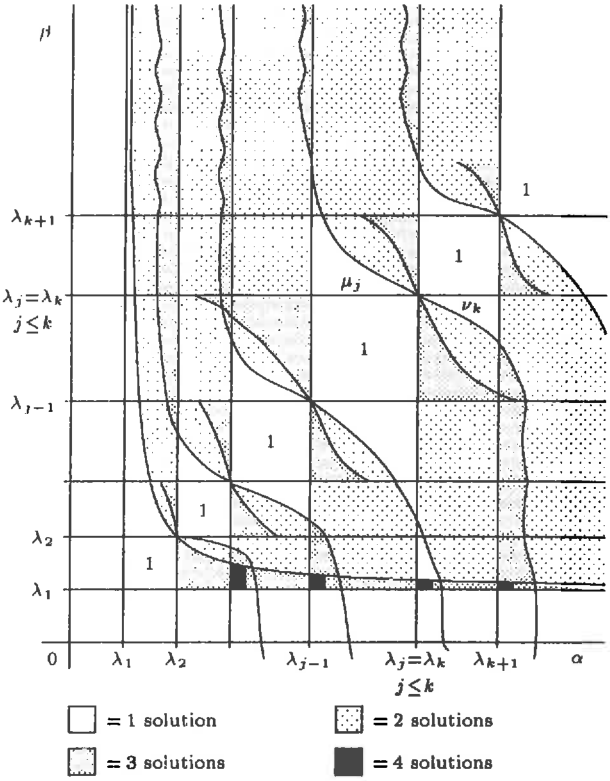
We will locate some regions in the zone $\{(\alpha,\beta) \in \mathbb{R}^2 \,|\, \alpha > \lambda_1,\ \beta > \lambda_1\}$ to which there correspond at least one, two or three solutions of (P_t) for t positive and large enough. The figure below summarizes our results. We think that our map in the (α,β) plane, which is at first stage, will be enriched and improved.

In particular a theorem on existence of one solution will be shown in Section 3. Theorems on existence of two and three solutions will be proved in Section 4 (in the case $\alpha < \beta$) and in Section 6 (in the case $\alpha > \beta$).

As we can see in the figure, the region corresponding to at least two solutions is "very large". Moreover, an infinite number of unbounded regions are contained in the set corresponding to at least three solutions.

The technical lemmas which enable us to prove the existence theorems are collected in Sections 5 and 7.

Concerning the geometric behaviour of the functional f_t, we point out that the existence of two or more solutions is connected with the fact that the functional f_t (see Definition 1.2) has more and more complex topological properties



with regard to more and more restricted regions of the (α, β) plane. From the first topological "key" (f_t separates two splitting spheres in a symmetrical way), which ensures the existence of one solution, it will be possible to develop some topological properties, progressively more complex (f_t separates a pair of linked spheres with suitable bounds and f_t separates two pairs of linked spheres in dimensional scale with suitable bounds), which ensures the existence of two and three solutions.

Abstract variational theorems which connect the topological properties of f_t with the existence and multiplicity of solutions will be shown in Section 8.

Finally, in Section 9 we will give a theorem on existence of four solutions and we will draw a "submap" which implies and describes the "alternative" theorem (without the assumption $(\alpha, \beta) \notin \Sigma_\Omega$).

1. Problem, assumptions and notation

Let Ω be an open bounded subset of $\mathbb{R}^N$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Without loss of generality we can suppose $G(x, 0) = 0$. If $G(x, s)$ is a C^1 function with respect to s, we set $g(x, s) = G'_s(x, s)$.

1.1. *The following conditions on G will be alternatively considered*

(G) $|G(x,s)| \le a(x) + b|s|^2$ *for all* $s \in \mathbb{R}$, *a.e. in* Ω, *where* $a \in L^1(\Omega)$ *and* $b \in \mathbb{R}$.

(G,α) $\lim_{s\to+\infty} 2G(x,s)/s^2 = \alpha \in \mathbb{R}$ *a.e. in* Ω.

(G,α,β) $\lim_{s\to+\infty} 2G(x,s)/s^2 = \alpha \in \mathbb{R}$ *and* $\lim_{s\to-\infty} 2G(x,s)/s^2 = \beta \in \mathbb{R}$ *a.e. in* Ω.

(g) *G is a C^1 function in s for a.e. x in Ω,*
$|g(x,s)| \le a_1(x) + b_1|s|^q$ *for all* $s \in \mathbb{R}$ *and a.e. in* Ω, *where* $a_1 \in L^p(\Omega)$ *with* $p \ge 2N/(N+2)$ (*if* $N=2$, $p>1$; *if* $N=1$, $p\ge 1$), $b_1 \in \mathbb{R}$ *and* $\frac{1}{q} \ge \frac{N-2}{N+2}$.

Set

$$G_0(x,s) = G(x,s) - \frac{1}{2}\alpha(s^+)^2 - \frac{1}{2}\beta(s^-)^2.$$

If (g) is satisfied we will consider the problem

$$(P_t)\qquad \begin{cases} \Delta y + g(x,u) = te_1 + h_0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $t \in \mathbb{R}$, $h_0 \in L^2(\Omega)$ and e_1 is a positive eigenfunction associated with the first eigenvalue of the Dirichlet problem $\Delta u + \lambda u = 0$, $u = 0$ on $\partial\Omega$. Without loss of generality we will assume $h_0 = 0$.

In general, we are interested in the variational nature of the problem (P_t). For this purpose, we make the following definition.

DEFINITION 1.2. Let $f_t : H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$f_t(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - G(x,u) + te_1 u\right) dx.$$

The Hilbert space $H_0^1(\Omega)$ is equipped with the usual norm $\|u\| = (\int_\Omega |\nabla u|^2)^{1/2}$ and the usual scalar product $(u,v) = \int_\Omega \nabla u \nabla v$.

REMARK 1.3. It is known that if (G) holds then the functional f_t is well defined. Moreover, if (g) holds, then it is a C^1 functional and its critical points are exactly the solutions of the problem (P_t).

We introduce some functionals which will be used later.

DEFINITION 1.4. Let α and β be real numbers. For $u \in H_0^1(\Omega)$ set

$$Q_\alpha(u) = \frac{1}{2}\int (|\nabla u|^2 - \alpha u^2)$$

and

$$Q_{\alpha,\beta}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \alpha(u^+)^2 - \beta(u^-)^2),$$

where $u^+(x) = u(x) \vee 0$ and $u^-(x) = -u(x) \wedge 0$.

Finally, the following classical notations will be useful.

DEFINITION 1.5. Let $(\lambda_n)_{n\geq 1}$ be the sequence of eigenvalues of the problem $\Delta u + \lambda u = 0$, $u \in H_0^1(\Omega)$. We recall that $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$ and $\lim_n \lambda_n = \infty$. Let e_n be an eigenfunction corresponding to λ_n, with $\|e_n\|_{L^2(\Omega)} = 1$. We can choose e_1 such that $e_1 > 0$ in Ω.

Moreover, set $\mathrm{H}_i = \mathrm{span}(e_1, \ldots, e_i)$ and $\mathrm{H}_i^\perp = \{w \in H_0^1(\Omega) \mid (u,w) = 0\ \forall u \in \mathrm{H}_i\}$.

2. The Palais-Smale condition

As we said in the introduction the classical Palais-Smale condition plays an essential role in studying problem (P_t). Let us recall its definition.

DEFINITION 2.1. Let X be a Hilbert space and $F \in C^1(\mathrm{X}, \mathbb{R})$. If $c \in \mathbb{R}$ we say that F satisfies the *Palais-Smale condition at level* c (i.e. $(\mathrm{P.S.})_c$ holds) if for every sequence $(u_n)_{n\in\mathbb{N}}$ in X with $\lim_n F(u_n) = c$ and $\lim_n \nabla F(u_n) = 0$, there exists a convergent subsequence.

In the following, for the sake of simplicity, we will say that F satisfies (P.S.) if $(\mathrm{P.S.})_c$ holds for any $c \in \mathbb{R}$.

It is well known that if $\lim_{s\to+\infty} g(x,s)/s = \alpha < \lambda_1$ or $\lim_{s\to-\infty} g(x,s)/s = \beta < \lambda_1$ and g satisfies the "natural" asymptotic conditions $|g(x,s)| \leq a_1(x) + b_1|s|$, a.e. in Ω, for all $s \in \mathbb{R}$, with $a_1 \in L^2(\Omega)$ and $b_1 \in \mathbb{R}$, then the Palais-Smale condition holds.

In this section we will produce two different groups of assumptions which ensure the (P.S.) condition also when both α and β are greater than λ_1.

The first, which is classical, has been recalled in the introduction. In this case we assume that (α,β) does not belong to the "resonance set" (see (2.4)). The second is stated in Theorem 2.5 and it does not need the condition that (α,β) does not belong to the "resonance set", whereas some supplementary conditions on g are required.

We recall the definition of the "resonance set".

(2.2) $$\Sigma_\Omega = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \exists u \in H_0^1(\Omega),\ u \neq 0,\ \text{with } \Delta u + \alpha u^+ - \beta u^- = 0\}.$$

The following result is classical.

PROPOSITION 2.3. *Assume the following conditions:*

(2.4)
$$\begin{cases} |g(x,s)| \leq a_1(x) + b_1|s|, \ \textit{a.e. in } \Omega,\ \forall s \in \mathbb{R},\ \textit{with } a_1 \in L^2(\Omega),\ b_1 \in \mathbb{R}; \\ \lim\limits_{s\to-\infty} \dfrac{g(x,s)}{s} = \beta \ \textit{and} \ \lim\limits_{s\to+\infty} \dfrac{g(x,s)}{s} = \alpha; \\ (\alpha,\beta) \notin \Sigma_\Omega. \end{cases}$$

Then for every t, f_t satisfies (P.S.).

From this result we can easily deduce that if $\beta < \lambda_1$ or $\alpha < \lambda_1$ then (P.S.) holds for the functional f_t.

We would like to emphasize that (2.4) ensures that every sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim_n \nabla f_t(u_n) = 0$ have a convergent subsequence in $H_0^1(\Omega)$, even if $(f_t(u_n))_{n\in\mathbb{N}}$ is unbounded.

In the introduction we have recalled an important theorem concerning Σ_Ω (see [15]), which, roughly speaking, states that by means of little perturbation of Ω we get $\overset{\circ}{\Sigma}_\Omega = \emptyset$. So this theorem makes the assumptions (2.4) more concrete.

As we stated at the beginning of this section, we can give some sufficient conditions to ensure (P.S.), without requiring that $(\alpha, \beta) \notin \Sigma_\Omega$.

Theorem 2.5 (an explicit condition which ensures (P.S.)). *Assume:*

(2.6)
- (a) $|g(x,s)| \le a_1(x) + b_1|s| \ \forall\, s \in \mathbb{R}$, *a.e. in* Ω, *where* $a_1 \in L^2(\Omega)$ *and* $b_1 \in \mathbb{R}$;
- (b) $\lim_{s\to+\infty} g(x,s)/s = \alpha \in \mathbb{R}$ *and* $\lim_{s\to-\infty} g(x,s)/s = \beta \in \mathbb{R}$ *a.e. in* Ω, *with* $(a, \beta) \ne (\lambda_i, \lambda_i)$ *if* $i \ge 2$;
- (c) $|2G(x,s) - g(x,s)s| \le a_0(x)|s| + b_0(x)$ $\forall s \in \mathbb{R}$, *a.e. in* Ω, *with* $a_0 \in L^p(\Omega)$, $p \ge 2N/N+2$ (*if* $N = 2$ *then* $p > 1$ *and if* $N = 1$ *then* $p \ge 1$) *and* $b_0 \in L^1(\Omega)$; *or, more generally: if* $\beta > \alpha$ *and* $\beta > \lambda_1$, *then* $2G(x,s) - g(x,s)s \le a_0(x)|s| + b_0(x)$, *and if* $\alpha > \beta$ *and* $\alpha > \lambda_1$, *then* $2G(x,s) - g(x,s)s \ge -a_0(x)|s| - b_0(x)$.

Then there exists $t_0 > 0$ such that if $t \ge t_0$ then the functional f_t satisfies (P.S.).

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be in $H_0^1(\Omega)$ such that:

$$\lim_n f_t(u_n) = c \quad \text{and} \quad \lim_n \nabla f_t(u_n) = 0 \quad \text{in } H_0^1(\Omega).$$

We argue by contradiction and suppose $\lim_n \|u_n\| = \infty$. By (2.6) (a), (b), using standard arguments, we obtain

$$\lim_n \frac{u_n}{\|u_n\|} = u \qquad \text{in } H_0^1(\Omega),\ \|u\| = 1 \text{ and } \Delta u + \alpha u^+ - \beta u^- = 0.$$

On the other hand, we have

$$\lim_n \left(\nabla f_t(u_n), \frac{u_n}{\|u_n\|}\right) = 0,$$

that is

$$\lim_n \frac{1}{\|u_n\|}\left(\int_\Omega |\nabla u_n|^2 - \int_\Omega g(x,u_n)u_n + t\int_\Omega e_1 u_n\right) = 0.$$

Hence

$$\lim_n \frac{1}{\|u_n\|}\left(2f_t(u_n)+\int_\Omega (2G(x,u_n)-g(x,u_n)u_n)-t\int_\Omega e_1u_n\right)=0.$$

Since $\lim_n f_t(u_n)/\|u_n\|=0$, we have

$$\lim_n \frac{1}{\|u_n\|}\int_\Omega (2G(x,u_n)-g(x,u_n)u_n)=t\int_\Omega e_1u.$$

Now if, for example, $\beta>\alpha$ and $\beta>\lambda_1$ then $\int_\Omega e_1u\geq\Lambda$, by Lemma 2.7(b). By (c), using the Hölder inequality, it is easy to see that there exist real numbers $\overline{a}$ and $\overline{b}$ so that

$$\int_\Omega (2G(x,u)-g(x,u)u)\leq \overline{a}\|u\|+\overline{b}\qquad \forall u\in H_0^1(\Omega).$$

This implies that

$$\lim_n \frac{1}{\|u_n\|}\int_\Omega (2G(x,u_n)-g(x,u_n)u_n)\leq \overline{a}.$$

Finally, if $t\Lambda>\overline{a}$ we get a contradiction. □

It is easy to prove the following lemma.

LEMMA 2.7. *Let $(\alpha,\beta)\neq(\lambda_i,\lambda_i)$ for $i\geq 2$.*

(a) *If there exists $u\in H_0^1(\Omega)$, $u\neq 0$ such that $\Delta u+\alpha u^+-\beta u^-=0$, then*

$$\beta>\alpha \text{ and } \beta>\lambda_1 \Rightarrow \int_\Omega ue_1=\frac{\beta-\alpha}{\beta-\lambda_1}\int_\Omega u^+e_1>0;$$
$$\alpha>\beta \text{ and } \alpha>\lambda_1 \Rightarrow \int_\Omega ue_1=\frac{\beta-\alpha}{\alpha-\lambda_1}\int_\Omega u^-e_1<0.$$

(b) *If $\beta>\alpha$ and $\beta>\lambda_1$ then*

$$\Lambda=\inf\left\{\int_\Omega ue_1\;\middle|\;\Delta u+\alpha u^+-\beta u^-=0,\ \|u\|=1\right\}>0;$$

If $\alpha>\beta$ and $\alpha>\lambda_1$ then

$$-\Lambda=\sup\left\{\int_\Omega ue_1\;\middle|\;\Delta u+\alpha u^+-\beta u^-=0,\ \|u\|=1\right\}<0.$$

Finally, we exhibit a class of functions g which satisfy (2.6).

EXAMPLE 2.8. If $\beta > \alpha$ and $\beta > \lambda_1$ then the function

$$g(x,s) = \alpha s^+ - \beta s^- - \gamma_1(s^+)^{1-\varepsilon_1} + \gamma_2(s^-)^{1-\varepsilon_2} + g_0(x,s),$$

where $|g_0(x,s)| \leq \gamma_0(x)$ with $\gamma_0 \in L^2(\Omega)$, $\gamma_i \in \mathbb{R}^+$ and $\varepsilon_i \in]0,1]$ for $i = 1,2$, satisfies (2.6). If $\alpha > \beta$ and $\alpha > \lambda_1$ then so does the function

$$g(x,s) = \alpha s^+ - \beta s^- + \gamma_1(s^+)^{1-\varepsilon_1} - \gamma_2(s^-)^{1-\varepsilon_2} + g_0(x,s),$$

where $|g_0(x,s)| \leq \gamma_0(x)$ with $\gamma_0 \in L^2(\Omega)$, $\gamma_i \in \mathbb{R}^+$ and $\varepsilon_i \in]0,1]$ for $i = 1,2$. (If $\gamma_i > 0$ for $i = 1,2$, it is enough to assume $|g_0(x,s)| \leq \gamma_0(x) + \delta|s|^{1-\varepsilon}$ with $1 > \varepsilon > \varepsilon_i$ for $i = 1,2$.)

REMARK 2.9. In Theorem 2.5 we can replace the vector e_1 in the functional f_t by any vector $e \in H_0^1(\Omega)$ such that $\int_\Omega eu \neq 0$ for every $u \in H_0^1(\Omega)$, $u \neq 0$, for which $\Delta u + \alpha u^+ - \beta u^- = 0$. In this case, assumption $(\alpha,\beta) \neq (\lambda_i,\lambda_i)$ has to be dropped.

3. The existence of one solution. f_t separates two splitting spheres in a symmetrical way

We first point out a feature of the behaviour of the functional f_t which is also important for the existence theorems of the following sections.

LEMMA 3.1. *Assume* (G) *and* (G,α).

(a) *There exists* $c \in \mathbb{R}$ *such that if* $u \in H_0^1(\Omega)$ *then*

$$|f_0(u) - Q_\alpha(u)| \leq c\Big(1 + \int_\Omega (u^-)^2 + \int_\Omega |G_0(x,u^+)|\Big).$$

(b) $\displaystyle\lim_{\|u\|\to\infty} \frac{1}{\|u\|^2}\int_\Omega |G_0(x,u^+)| = 0.$

We will use in the equivalent form:

$$\lim_{t\to+\infty} \frac{1}{t^2} \sup_{\|u\|\leq t} \int_\Omega |G_0(x,u^+)| = 0.$$

PROOF. (a) From (G) and (G,α), we get

$$\begin{aligned}|f_0(u) - Q_\alpha(u)| &= \Big|\int_\Omega (-G(x,-u^-) + \alpha(u^-)^2/2 - G_0(x,u^+))\Big| \\ &\leq \|a\|_{L^1(\Omega)} + (b + |\alpha|/2)\int_\Omega (u^-)^2 + \int_\Omega G_0(x,u^+).\end{aligned}$$

(b) Let $(t_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be such that $\|u_n\| \le t_n$ and $\lim_n t_n = +\infty$. We will show that $\lim_n \|G_0(x, u_n^+)\|_{L^1(\Omega)}/t_n^2 = 0$. We can suppose $\lim_n u_n^+/t_n = u \in H_0^1(\Omega)$ a.e. in Ω and in $L^2(\Omega)$ with $u \ge 0$ in Ω. Now if x is such that $u(x) > 0$, then $\lim_n G_0(x, u_n^+(x))/t_n^2 = 0$; this follows from (G, α) by observing that $\lim_n u_n^+(x) = +\infty$ and

$$\frac{G_0(x, u_n^+)}{t_n^2} = \frac{G_0(x, u_n^+)}{|u_n^+(x)|^2} \cdot \frac{|u_n^+(x)|^2}{t_n^2}.$$

Moreover, if x is such that $u(x) = 0$, we get again $\lim_n G_0(x, u_n^+(x))/t_n^2 = 0$, since

$$\left|\frac{G_0(x, u_n^+)}{t_n^2}\right| \le \frac{a(x)}{t_n^2} + \left(b + \frac{|\alpha|}{2}\right)\frac{|u_n^+(x)|^2}{t_n^2}.$$

By (G)using the Lebesgue's theorem the assertion follows.

LEMMA 3.2. *Assume $(G)$and (G, α). Let $s_t = t/(\alpha - \lambda_1)$ with $t > 0$ and $\alpha > \lambda_1$.*

(a) *There exists $c \in \mathbb{R}$ such that if $z \in H_0^1(\Omega)$*

$$f_t(s_t e_1 + z) - f_t(s_t e_1) = Q_\alpha(z) + R(t, z),$$

where

$$|R(t, z)| \le c\left(1 + \int_\Omega ((s_t e_1 + z)^-)^2 + \int_\Omega |G_0(x, s_t e_1)| + \int_\Omega |G_0(x, (s_t e_1 + z)^+)|\right).$$

(b) *If $\gamma(t) = \sup_{\|z\|\le t} \int_\Omega |G_0(x, (s_t e_1 + z)^+)|$, then $\lim_{t\to+\infty} \gamma(t)/t^2 = 0$.*

(c) *There exist $\delta, \rho > 0$ such that if $z \in H_0^1(\Omega)$*

$$\int_\Omega ((s_t e_1 + z)^-)^2 \le S^2 \|z\|^2 (\text{meas}\{x \in \Omega \mid s_t e_1(x) + z(x) \le 0\})^p,$$

(if $N \ge 3$ then S is the Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$). Moreover, for any $\varepsilon > 0$ there exists $\sigma \in [0, 1]$ such that if $\|z\| \le \sigma t$ then $(\text{meas}\{x \in \Omega \mid s_t e_1(x) + z(x) \le 0\})^p \le \varepsilon$ for any $t > 0$.

PROOF. (a) If $z \in H_0^1(\Omega)$ then we have

$$\begin{aligned} f_t(s_t e_1 + z) - f_t(s_t e_1) &= \left(Q_\alpha(s_t e_1 + z) + t\int_\Omega (s_t e_1 + z)e_1\right) - \left(Q_\alpha(s_t e_1) + t\int_\Omega s_t e_1{}^2\right) \\ &\quad + (f_0(s_t e_1 + z) - Q_\alpha(s_t e_1 + z) - (f_0(s_t e_1) - Q_\alpha(s_t e_1)) \\ &= Q_\alpha(z) + (f_0(s_t e_1 + z) - Q_\alpha(s_t e_1 + z)) - (f_0(s_t e_1) - Q_\alpha(s_t e_1)), \end{aligned}$$

since $\int_\Omega (\nabla(s_te_1)\nabla z - \alpha s_te_1 z + te_1 z) = 0$; from Lemma 3.1(a) we deduce the assertion.

(b) If $\|z\| \leq t$ then $\|(s_te_1 + z)^+\| \leq s_t\lambda_1 + t = \alpha t/(\alpha - \lambda_1)$; the assertion follows by Lemma 3.1(b).

(c) If, for example, $N \geq 3$, then by using the Holdër inequality we get

$$\begin{aligned}\int_\Omega ((s_te_1 + z)^-)^2 &\leq \int_{\{x\in\Omega \mid s_te_1(x)+z(x)\leq 0\}} (z^-)^2 \\ &\leq \|z\|^2_{L^{2N/(N-2)}(\Omega)}(\text{meas}\{x \in \Omega \mid s_te_1(x) + z(x) \leq 0\})^{2/N} \\ &\leq S^2\|z\|^2(\text{meas}\{x \in \Omega \mid s_te_1(x) + z(x) \leq 0\})^{2/N}.\end{aligned}$$

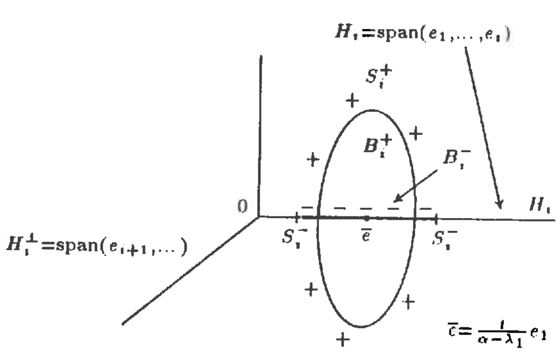
Moreover, if $t = 1$ and $(z_n)_{n\in\mathbb{N}}$ is such that $\lim_n \|z_n\| = 0$, we obtain $\lim_n z_n(x) = 0$ a.e. in Ω and hence $\lim_n(\text{meas}\{x \in \Omega \mid s_te_1(x) + z_n(x) \leq 0\}) = 0$. □

By means of the previous result, we are now able to show that, for t large enough, the functional f_t "separates two splitting spheres in a symmetrical way".

THEOREM 3.3 ("splitting spheres"). *Let $i \geq 1$. Assume (G) and (G, α) with $\lambda_i < \alpha < \lambda_{i+1}$. There exist $\sigma_0, t_0 > 0$ such that if $t \geq t_0$ then*

$$\tag{3.4} \begin{aligned}\sup_{\substack{v\in H_i\\ \|v\|=\sigma_0 t}} f_t(s_te_1 + v) &< \inf_{\substack{w\in H_i^\perp\\ \|w\|\leq\sigma_0 t}} f_t(s_te_1 + w) \\ &\leq \sup_{\substack{v\in H_i\\ \|v\|\leq\sigma_0 t}} f_t(s_te_1 + v) < \inf_{\substack{w\in H_i^\perp\\ \|w\|=\sigma_0 t}} f_t(s_te_1 + w).\end{aligned}$$

More precisely, there exists $\sigma_0 > 0$ such that for any $\sigma \in]0, \sigma_0]$ there exists $t_0 > 0$ such that if $t \geq t_0$ then (3.4) holds (where σ_0 is replaced by σ).



PROOF. If $z \in H_0^1(\Omega)$ with $\|z\| \leq t$, then from Lemma 3.2 we deduce that

$$f_t(s_t e_1 + z) - f_t(s_t e_1) = Q_\alpha(z) + R(t, z)$$

with

$$|R(t,z)| \leq c\left(1 + \int_\Omega \left((s_t e_1 + z)^-\right)^2\right) + 2\gamma(t).$$

By Lemma 3.2(c), given $\delta > 0$ there exists $\sigma_0 \in]0,1]$ such that if $\sigma \in]0, \sigma_0]$, $t > 0$ and $\|z\| \leq \sigma t$ we obtain $c \int_\Omega ((s_t e_1 + z)^-)^2 \leq \delta \|z\|^2 \leq \delta \sigma^2 t^2$. If $\sigma \in]0, \sigma_0]$ is fixed, then by Lemma 3.2(b) there exists $t_0 > 0$ such that if $t \geq t_0$ then $2\gamma(t) < \delta\sigma^2 t^2$. Finally, we see that for any $\sigma \in]0, \sigma_0]$ there exists $t_0 > 0$ such that if $t \geq t_0$ and $\|z\| \leq \sigma t$:

$$|R(t,z)| \leq c + 2\delta\sigma^2 t^2.$$

Now if $w \in \mathrm{H}_i^\perp$ and $\|w\| = \sigma t$:

$$f_t(s_t e_1 + w) - f_t(s_t e_1) \geq \frac{1}{2}\left(1 - \frac{\alpha}{\lambda_{i+1}}\right)\sigma^2 t^2 - 2\delta\sigma^2 t^2 - c.$$

On the other hand if $v \in \mathrm{H}_i$ and $\|v\| \leq \sigma t$

$$f_t(s_t e_1 + v) - f_t(s_t e_1) \leq R(t, v) \leq c + 2\delta\sigma^2 t^2.$$

In order to obtain the right inequality in (3.4), the following one has to be verified:

$$\frac{1}{2}\left(1 - \frac{\alpha}{\lambda_{i+1}}\right)\sigma^2 t^2 - 2\delta\sigma^2 t^2 - c > c + 2\delta\sigma^2 t^2,$$

that is,

$$\left(\frac{1}{2}\left(1 - \frac{\alpha}{\lambda_{i+1}}\right) - 4\delta\right)\sigma^2 t^2 > 2c,$$

which holds if $\delta > 0$ is small enough. The left inequality in (3.4) can be proved in the same way. □

Roughly speaking, the inequalities (3.4) hold because the spheres

$$S_i^- = \{s_t e_1 + v \mid v \in \mathrm{H}_i,\ \|v\| = \sigma t\}$$

and

$$S_i^+ = \{s_t e_1 + w \mid w \in \mathrm{H}_i,\ \|w\| = \sigma t\}$$

and the balls $B_i^- = \operatorname{conv} S_i^-$ and $B_i^+ = \operatorname{conv} S_i^+$ are in the "neighbourhood" of the positive functions cone and so, in this area, the functional f_t, for t positive and large enough, is, substantially, $Q_\alpha(u) + t\int_\Omega e_1 u$.

By the previous result and the variational statement 8.1, we will obtain the following existence theorem.

THEOREM 3.5 (existence of one solution). *Assume:*

(a) (g), (G), (G,α) *and* (P.S.) *for the functional* f_t *for* t *large enough;*
(b) $\alpha > \lambda_1$ *and* $\alpha \neq \lambda_i$ *if* $i \geq 1$.

Then for t *large enough the functional* f_t *has at least one critical value and hence the problem* (P_t) *has at least one solution.*

We would like to point out that assumptions (a) can be replaced by (2.4) or by the more explicit conditions (2.6).

Moreover, if $\alpha = \lambda_i$ with $i \geq 2$, we are again able to show the existence of a critical value for the functional f_t by replacing (G,α) with the (G,α,β).

THEOREM 3.6 (existence of one solution). *Assume*

(a) (g), (G), (G,α,β) *and* (P.S.) *for the functional* f_t *for* t *large enough;*
(b) $\alpha > \lambda_1$.

Then for t *large enough the functional* f_t *has at least one critical value and hence the problem* (P_t) *has at least one solution.*

PROOF. Let $\alpha = \lambda_i$. It is enough to use Lemma 4.5 (if $\beta \geq \lambda_i$) or Lemma 6.3 (if $\beta \leq \lambda_i$), which describe the behaviour of f_t, and the classical "saddle" theorem of Rabinowitz (see [36]).

Also in this case we emphasize that assumptions (a) can be replaced by the conditions (2.4) or (2.6).

4. The existence of two and three solutions if $\beta \geq \alpha$. f_t separates two linked spheres. f_t separates two pairs of linked spheres in dimensional scale

In this section we will set out some existence theorems for the problem (P_t) for t positive and large enough under assumptions (G,α,β) with $\beta \geq \alpha$. In order not to make the presentation of the existence results too heavy, we postpone the proofs of some technical lemmas and some variational abstract statements to Sections 5 and 8 respectively.

First of all, we point out a useful connection between the functionals f_0 and $Q_{\alpha,\beta}$.

PROPOSITION 4.1. *Assume* (G) *and* (G,α,β). *Then*

$$f_0(u) - Q_{\alpha,\beta}(u) = -\int_\Omega G_0(x,u) \qquad \text{for all } u \in H_0^1(\Omega)$$

and

$$\lim_{\|u\|\to\infty} \frac{\int_\Omega |G_0(x,u)|}{\|u\|^2} = 0.$$

PROOF. It is similar to the proof of Lemma 3.1(b). □

We observe that if $s_t = t/(\alpha - \lambda_1)$ then the function $s_t e_1$ is a critical point of the functional $u \to Q_{\alpha,\beta}(u) + t \int_\Omega e_1 u$.

In order to study the behaviour of the functional f_t, the increments $f_t(s_t e_1 + z) - f_t(s_t e_1)$ will be considered as in Section 3.

REMARK 4.2. (a) If $t \geq 0$, $\alpha > \lambda_1$ and $z \in H_0^1(\Omega)$ then

$$\left(Q_\alpha(s_t e_1 + z) + t \int_\Omega (s_t e_1 + z) e_1\right) - \left(Q_\alpha(s_t e_1) + t \int_\Omega s_t e_1^2\right)$$
$$= Q_\alpha(z) + \frac{1}{2}(\alpha - \beta) \int_\Omega ((s_t e_1 + z)^-)^2$$
$$= Q_{\alpha,\beta}(z) + \frac{1}{2}(\alpha - \beta) \int_\Omega (((s_t e_1 + z)^-)^2 - (z^-)^2,$$

since $s_t e_1 > 0$ and $\int_\Omega (\nabla(s_t e_1)\nabla z - \alpha s_t e_1 z + t e_1 z) = 0$.

(b) If $z \in H_0^1(\Omega)$ then

$$\int_\Omega (((e_1 + z)^-)^2 - (z^-)^2)$$
$$= \int_{\{x\in\Omega \mid z(x) \leq -e_1(x)\}} (e_1^2 + 2e_1 z) - \int_{\{x\in\Omega \mid -e_1(x) \leq z(x) \leq 0\}} z^2,$$

and

$$-2\int_\Omega e_1 z^- \leq \int_\Omega (((e_1 + z)^-)^2 - (z^-)^2) \leq 0.$$

DEFINITION 4.3. Let $i \geq 1$. Given $\alpha, \beta \in \mathbb{R}$ and $\rho > 0$, set

$$M_i(\alpha,\beta) = \sup_{v \in \mathrm{H}_i} \left\{ Q_\alpha(v) + \frac{1}{2}(\alpha - \beta) \int_\Omega ((e_1 + v)^-)^2 \right\},$$

$$m_i(\rho,\alpha,\beta) = \inf_{w \in \mathrm{H}_i^\perp, \|w\| = \rho} \left\{ Q_\alpha(w) + \frac{1}{2}(\alpha - \beta) \int_\Omega \left((e_1 + w)^-\right)^2 \right\},$$

$$E_i = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha < \beta,\ \exists \rho_i > 0 \text{ such that } M_i(\alpha,\beta) < m_i(\rho_i,\alpha,\beta)\}.$$

From Lemma 5.1, which describes the properties of M_i and m_i, we can easily deduce the following result.

LEMMA 4.4.

(a) *If $i \geq 2$ then E_i is an open set and*
$\{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha < \beta,\ \lambda_i \leq \alpha < \lambda_{i+1}\} \subset E_i \subset \{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha < \lambda_{i+1}\}$.
Moreover, if $(\alpha,\beta) \in E_i$ then $\max_{v \in \mathrm{H}_i,\, \|v\|=1} Q_{\alpha,\beta}(v) < 0$.

(b) $E_1 = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha < \beta,\ \lambda_1 \leq \alpha < \lambda_2\}$.

Now from Proposition 4.1 we can deduce a basic inequality for the functional f_t, which enables us to prove the existence of more than one solution.

LEMMA 4.5 ("saddle"). *Let $i \geq 1$. Assume (G) and (G, α, β) with $\alpha > \lambda_1$. If $(\alpha, \beta) \in E_i$ then there exist $\rho_i, t_0 > 0$ so that if $t \geq t_0$ then*

$$\sup_{v \in \mathrm{H}_i} f_t(s_t e_1 + v) < \inf_{\substack{w \in \mathrm{H}_i^{\perp} \\ \|w\| = \rho_i s_t}} f_t(s_t e_1 + w) \tag{4.6}$$

(ρ_i is in the definition of E_i).

PROOF. Given $z \in H_0^1(\Omega)$, put $u = z/s_t$. By Proposition 4.1 and Remark 4.2 we get

$$\begin{aligned} & f_t(s_t e_1 + z) - f_t(s_t e_1) \\ &= s_t^2 \left(Q_{\alpha,\beta}(u) + \frac{1}{2}(\alpha - \beta) \int_\Omega (((e_1 + u)^-)^2 - (u^-)^2) \right) + R(t, z), \end{aligned} \tag{4.7}$$

where $R(t,z) = \int_\Omega (G_0(x, s_t e_1) - G_0(x, s_t e_1 + z))$. From Proposition 4.1 we also deduce that for any $\varepsilon > 0$ there exists $d > 0$ so that

$$|R(t,z)| \leq \varepsilon(\|s_t e_1 + z\|^2 + \|s_t e_1\|^2) + d. \tag{4.8}$$

Moreover, since $M_i(\alpha, \beta) < \infty$, by Lemma 5.1($\mathrm{b}_2$) we obtain

$$-c = \max_{\substack{v \in \mathrm{H}_i \\ \|v\| = 1}} Q_{\alpha,\beta}(v) < 0.$$

Now, if $v \in \mathrm{H}_i$, then from Lemma 4.2(b) we have

$$f_t(s_t e_1 + v) - f_t(s_t e_1) \leq -c\|v\|^2 + |\beta - \alpha| \int_\Omega s_t e_1 v^- + 3\varepsilon \lambda_1 s_t^2 + 2\varepsilon \|v\|^2 + d$$

and also

$$f_t(s_t e_1 + v) - f_t(s_t e_1) \leq s_t^2 M_i(\alpha, \beta) + 3\varepsilon \lambda_1 s_t^2 + 2\varepsilon \|v\|^2 + d.$$

By using the first of the previous inequalities for $\|v\| \geq s_t \rho_\varepsilon$ with $\rho_\varepsilon = \frac{|\beta - \alpha|}{(c - 2\varepsilon)}$, we obtain

$$f_t(s_t e_1 + v) - f_t(s_t e_1) \leq 3\varepsilon \lambda_1 s_t^2 + d;$$

by using the second inequality for $\|v\| \leq s_t \rho_\varepsilon$, we obtain

$$f_t(s_t e_1 + v) - f_t(s_t e_1) \leq s_t^2 (M_i(\alpha, \beta) + 3\varepsilon \lambda_1 + 2\varepsilon \rho_\varepsilon^2) + d. \tag{4.9}$$

Therefore, since $M_i(\alpha, \beta) \geq 0$, we see that (4.9) holds for any $v \in \mathrm{H}_i$.
On the other hand if $w \in \mathrm{H}_i^{\perp}$ with $\|w\| = \rho s_t$, where $\rho > 0$, then by (4.8) we get

$$f_t(s_t e_1 + w) - f_t(s_t e_1) \geq s_t^2 (m_i(\rho, \alpha, \beta) - 3\varepsilon \lambda_1 - 2\varepsilon \rho^2) - d. \tag{4.10}$$

By assumption there exists $\rho > 0$ such that $M_i(\alpha, \beta) < m_i(\rho, \alpha, \beta)$. This allows us to choose ε so small that the coefficient of s_t^2 in (4.9) is less than the one in (4.10). Finally, by taking t positive and large enough the assertion follows. □

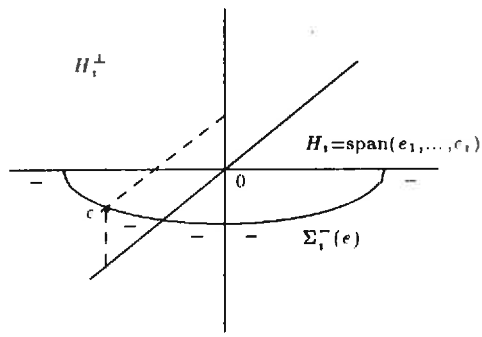
Now we observe that for particular pairs (α, β) the sublevels of the functional f_t have a sort of "topological complexity", which is connected with the "saddle" of Lemma 4.5.

DEFINITION 4.11. Let $i \geq 1$. Given $e \in \mathrm{H}_i^\perp$, $e \neq 0$. and $\alpha \in \mathbb{R}$ set:

$$\Sigma_i^-(e) = \{z = \sigma e + v \mid \|z\| = 1,\ v \in \mathrm{H}_i,\ \sigma \geq 0\},$$

$$\alpha_{i+1} = \sup\left\{ \int_\Omega |\nabla u|^2 \,\middle|\, u \in \mathrm{H}_i,\ \int_\Omega u^2 = 1,\ u \geq 0 \right\},$$

$$\mu_{i+1}(\alpha) = \inf\left\{ \beta \in \mathbb{R} \,\middle|\, \inf_{\substack{e \in \mathrm{H}_i^\perp \\ e \neq 0}} \max_{z \in \Sigma_i^-(e)} Q_{\alpha,\beta}(z) < 0 \right\}.$$



The following result can be easily deduced from Lemma 5.4 ((d) follows from Lemma 5.1(b_2)).

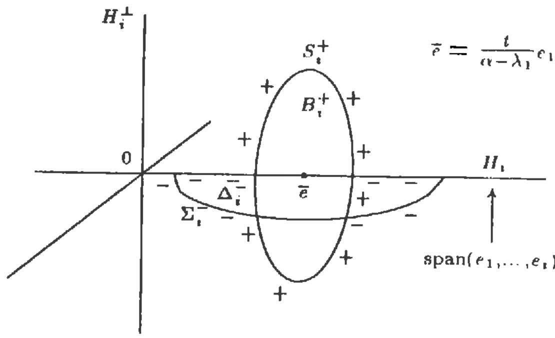
LEMMA 4.12. *Let $i \geq 1$.*

(a) $\mu_{i+1}(\alpha) \in \mathbb{R} \Leftrightarrow \alpha > \alpha_{i+1}$.
(b) $\mu_{i+1} :]\alpha_{i+1}, \infty[\;\to\;]\alpha_{i+1}, \infty[$ *is a continuous decreasing function such that:*

$$\mu_{i+1}(\lambda_{i+1}) = \lambda_{i+1},\ \mu_{i+1} \circ \mu_{i+1} = \textit{identity} \quad \textit{and} \lim_{\alpha \to \alpha_{i+1}} \mu_{i+1}(\alpha) = +\infty.$$

(c) $\alpha_2 = \lambda_1$ *and* $\lambda_1 \leq \alpha_{i+1} < \lambda_i$ *for* $i \geq 2$.
(d) *If* $i \geq 2$ *and* $(\alpha, \beta) \in E_i$ *then* $\alpha > \alpha_i$ *and* $\beta > \mu_i(\alpha)$.

By Lemmas 4.5 and 4.12 we will obtain two different results about the behaviour of f_t and also two theorems on existence of two and three solutions, with regard to suitable unbounded regions of the halfspace $\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha < \beta\}$. With this in view the regions E_i and the graphs of μ_i have been introduced.



We first show that if the pair (α, β) belongs to the region E_i above the graph of the function μ_{i+1}, then the functional f_t "separates two linked spheres" with suitable bounds.

THEOREM 4.13 ("links and bounds"). *Let $i \geq 1$. Assume (G) and (G, α, β) with $(\alpha, \beta) \in E_i$ (hence $\alpha > \alpha_{i+1}$) and $\beta > \mu_{i+1}(\alpha)$. There exist $\overline{\sigma}_i > 0$, $\rho_i > 0$ (it is in the definition of E_i), $t_0 > 0$ and $e \in \mathrm{H}_i^\perp$, $e \neq 0$, such that $\overline{\sigma}_i > \rho_i/(\alpha - \lambda_1)$ and if we set, for $\sigma_i \geq \overline{\sigma}_i$ and $t \geq t_0$,*

$$\begin{aligned}\Sigma_i^- = &\{s_t e_1 + v \mid v \in \mathrm{H}_i,\ \|v\| \leq \sigma_i t\} \\ &\cup \{s_t e_1 + v + \sigma e \mid v \in \mathrm{H}_i,\ \sigma \geq 0,\ \|v + \sigma e\| = \sigma_i t\},\end{aligned}$$

then the following inequalities hold:

(a) $\displaystyle \sup_{\Sigma_i^-} f_t < \inf_{\substack{w \in \mathrm{H}_i^\perp \\ \|w\| = \rho_i s_t}} f_t(s_t e_1 + w);$

(b) $\displaystyle \inf_{\substack{w \in \mathrm{H}_i^\perp \\ \|w\| \leq \rho_i s_t}} f_t(s_t e_1 + w) > -\infty$ *and* $\displaystyle \sup_{\substack{v \in \mathrm{H}_i, \sigma \geq 0 \\ \|v + \sigma e\| \leq \sigma_i t}} f_t(s_t e_1 + v + \sigma e) < \infty.$

PROOF. By Lemma 4.5 there exists $t_0 > 0$ such that if $t \geq t_0$ then (4.6) holds. It is enough to show that if t is large enough then there exists $e \in \mathrm{H}_i^\perp$, $e \neq 0$, and $\sigma_i > 0$ such that for t sufficiently large

$$f_t(s_t e_1 + z) - f_t(s_t e_1) \leq 0 \qquad \text{if } z = v + \sigma e,\ \sigma \geq 0,\ v \in \mathrm{H}_i \text{ and } \|z\| = \sigma_i t.$$

Since $\beta > \mu_{i+1}(\alpha)$ (see Lemma 4.12) there exists $e \in \mathrm{H}_i^\perp$, $e \neq 0$, so that $-c = \max_{\Sigma_i^-(e)} Q_{\alpha,\beta} < 0$.

By (4.7) and (4.8) we see that for any $\varepsilon > 0$ there exists $d > 0$ such that if $z = v + \sigma e$ with $\sigma \geq 0$ and $v \in \mathrm{H}_i$ then

$$\begin{aligned} &f_t(s_t e_1 + z) - f_t(s_t e_1) \\ &\qquad \leq s_t^2\left(-c\left\|\frac{z}{s_t}\right\|^2 + |\beta - \alpha| \int_\Omega e_1 \frac{z^-}{s_t} + \varepsilon \|e_1\|^2 + \varepsilon \left\|e_1 + \frac{z}{s_t}\right\|^2\right) + d. \end{aligned}$$

Now if we choose ε small enough and $\|z/s_t\| = \sigma_i$ large enough then the coefficient of s_t^2 is negative; hence for t large enough, (a) follows.

The proof of (b) is trivial. □

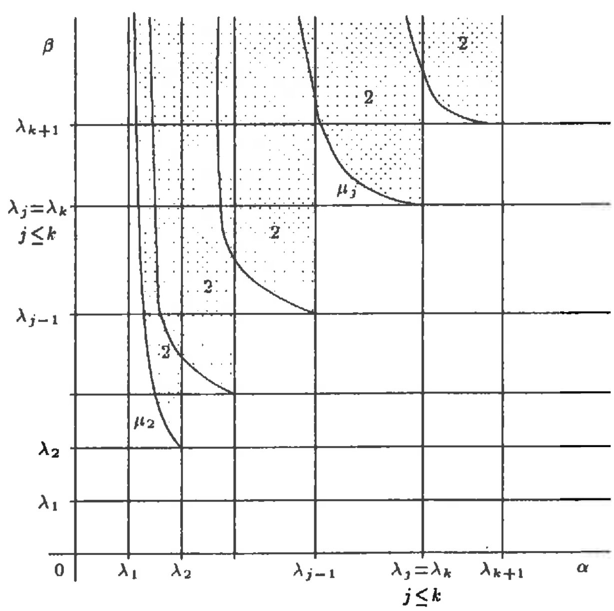
The previous lemma enables us to obtain two different existence theorems for the problem (P_t), regarding the existence of either two or three solutions. First of all by using Theorem 8.2 we easily get the following one.

THEOREM 4.14 (existence of two solutions). *Assume:*

(a) (g), (G), (G,α,β) *and* (P.S.) *for the functional* f_t *for* t *large enough;*

(b) $\alpha > \lambda_1$ *and* $\beta > \mu_j(\alpha)$, *where* λ_j *is the minimum eigenvalue strictly greater than* α.

Then, for t *large enough, the functional* f_t *has at least two critical values and hence the problem* (P_t) *has at least two solutions.*



We would like to emphasize that assumptions (a) can be replaced by (2.4) or by the more explicit (2.6).

PROOF OF THEOREM 4.14. It is enough to observe that by (b) the pair (α,β) belongs to E_{j-1} with $j \geq 2$; thus the conditions of Theorem 4.13 hold (with i replaced by $j-1$). By Theorem 8.2 the assertion follows. □

By using Theorem 4.13 for two different values of i, we obtain further information about f_t, which allows us to show the existence of three solutions of the problem (P_t) for suitable pairs (α,β).

We start by pointing out an easy consequence of Lemmas 4.4 and 4.12.

REMARK 4.15. Let λ_{j-1}, λ_j and λ_{k+1} be three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$. The set of pairs (α, β) such that $(\alpha, \beta) \in E_k \cap E_{j-1}$ (hence $\alpha > \alpha_j$ and $\beta > \mu_j(\alpha)$ by Lemma 4.12(d)) and $\beta > \mu_{k+1}(\alpha)$ (that is, (α, β) satisfies the conditions of Lemma 4.13 for both $i = k$ and $i = j-1$) is an open and non-empty set. More precisely: for every $\beta > \mu_{k+1}(\lambda_k)$ there exists $\delta > 0$ such that $]\lambda_k - \delta, \lambda_k[\times \{\beta\}$ is contained in that set.

Now we show that if (α, β) satisfies the assumptions of Remark 4.15, then the functional f_t "separates two pairs of linked spheres in dimensional scale" with a suitable lower bound.

THEOREM 4.16 ("links in scale and bounds"). *Let λ_{j-1}, λ_j and λ_{k+1} be three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$. Assume (G) and (G, α, β) with $(\alpha, \beta) \in E_k \cap E_{j-1}$ and $\beta > \mu_{k+1}(\alpha)$ (see Remark 4.15). There exist $\sigma_k, \rho_k, \sigma_{j-1}, \rho_{j-1}, t_0 > 0$ and $e \in \mathrm{H}_k^\perp$, $e \neq 0$, such that $\sigma_k > \rho_k/(\alpha - \lambda_1)$, $\sigma_{j-1} > \rho_{j-1}/(\alpha - \lambda_1)$, $\sigma_{j-1} \leq \sigma_k$ and if we set, for $t \geq t_0$,*

$$\begin{aligned}
\Sigma_{j-1}^- &= \{s_t e_1 + v \mid v \in \mathrm{H}_{j-1},\ \|v\| \leq \sigma_{j-1} t\} \\
&\quad \cup \{s_t e_1 + v + \sigma e_j \mid v \in \mathrm{H}_{j-1},\ \sigma \geq 0,\ \|v + \sigma e_j\| = \sigma_{j-1} t\}, \\
\Sigma_k^- &= \{s_t e_1 + v \mid v \in \mathrm{H}_k,\ \|v\| \leq \sigma_k t\} \\
&\quad \cup \{s_t e_1 + v + \sigma e \mid v \in \mathrm{H}_k,\ \sigma \geq 0,\ \|v + \sigma e\| = \sigma_k t\}.
\end{aligned}$$

then the following inequalities hold:

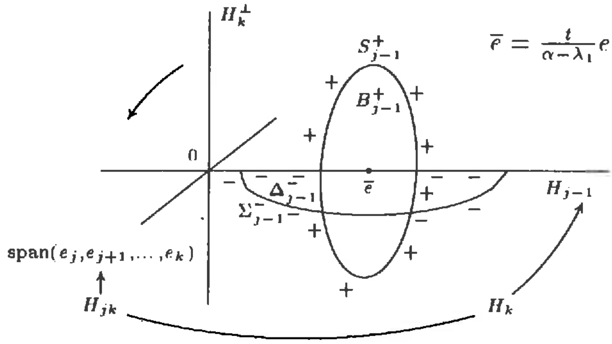
(a) $\displaystyle \sup_{\Sigma_{j-1}^-} f_t < \inf_{\substack{w \in \mathrm{H}_{j-1}^\perp \\ \|w\| = \rho_{j-1} s_t}} f_t(s_t e_1 + w) \leq \sup_{\Sigma_k^-} f_t < \inf_{\substack{w \in \mathrm{H}_k^\perp \\ \|w\| = \rho_k s_t}} f_t(s_t e_1 + w);$

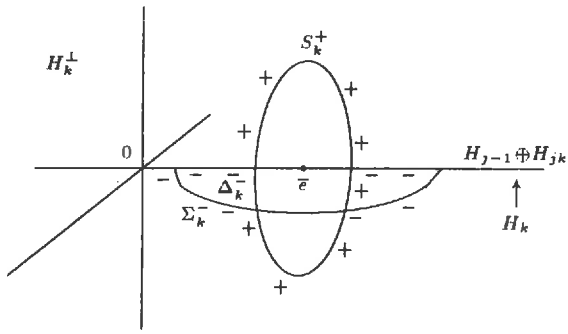
(b) $\displaystyle \inf_{\substack{w \in \mathrm{H}_{j-1}^\perp \\ \|w\| \leq \rho_{j-1} s_t}} f_t(s_t e_1 + w) > -\infty.$

PROOF. We will show the first inequality of (a). Since $(\alpha, \beta) \in E_{j-1}$ by Lemma 4.5 there exist $\rho_{j-1}, t_0 > 0$ so that if $t \geq t_0$ then

$$\sup_{v \in \mathrm{H}_{j-1}} f_t(s_t e_1 + v) < \inf_{\substack{w \in \mathrm{H}_{j-1}^\perp \\ \|w\| = \rho_{j-1} s_t}} f_t(s_t e_1 + w).$$

Since $(\alpha, \beta) \in E_k$ by Lemma 4.4 we have $\max_{v \in \mathrm{H}_k, \|v\|=1} Q_{\alpha,\beta}(v) < 0$; in particular, $\max_{\Sigma_{j-1}^-(e_j)} Q_{\alpha,\beta} < 0$.





On the other hand by using (4.7) and a similar argument to that in Theorem 4.13, we are able to show that for any σ_{j-1} and t_0 large enough, if $t \geq t_0$ then

$$f_t(s_t e_1 + z) \leq f_t(s_t e_1) \qquad \text{for } z = v + \sigma e_j,\ \sigma \geq 0,\ v \in \mathrm{H}_{j-1},\ \|z\| = \sigma_{j-1} t.$$

So we can choose $\sigma_{j-1} > \rho_{j-1}/(\alpha - \lambda_1)$ and the inequality is proved.

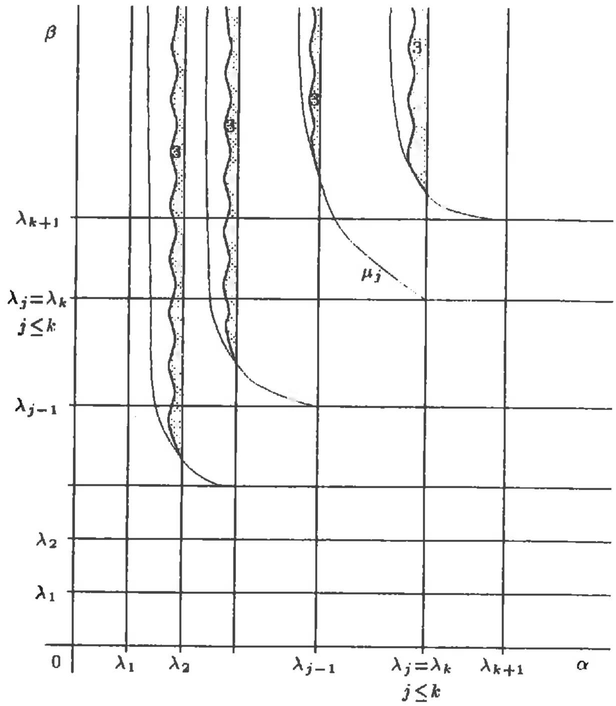
Since $(\alpha, \beta) \in E_k$ and $\beta > \mu_{k+1}(\alpha)$, by Theorem 4.13, we have the last inequality of (a) for suitable $\rho_k > 0$, $e \in \mathrm{H}_k^\perp$, $e \neq 0$, $\sigma_k > 0$ with $\sigma_k > \rho_k/\alpha - \lambda_1$ and $\sigma_{j-1} \leq \sigma_k$ and t large enough.

The proof of (b) is trivial. □

At this point by using Theorems 8.4 and 4.16 we get the following result.

THEOREM 4.17 (existence of three solutions). *Assume:*

(a) (g), (G), (G, α, β) *and* (P.S.) *for the functional* f_t *for* t *large enough;*
(b) $(\alpha, \beta) \in E_k \cap E_{j-1}$ *and* $\beta > \mu_{k+1}(\alpha)$, *where* λ_{j-1}, λ_j *and* λ_{k+1} *are three consecutive (possibly multiple) eigenvalues:* $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ *with* $k \geq j \geq 2$ (*see Remark* 4.15).



Then, for t large enough, the functional f_t has at least three critical values and hence the problem (P_t) has at least three solutions.

Also in this case assumptions (a) can be replaced by (2.4) or (2.6).

REMARK 4.18. We point out that if $\lambda_{i_1}, \ldots, \lambda_{i_h}$ and $\lambda_{i_{h+1}}$ are $h+1$ consecutive eigenvalues and if $(\alpha, \beta) \in E_{i_1} \cap \ldots \cap E_{i_h}$ and $\beta > \mu_{i_{h+1}}(\alpha)$ then the functional f_t "separates h pairs of linked spheres in dimensional scale". In this case (see Remark 8.5) if t is large enough then f_t has $h+1$ critical values and hence the problem (P_t) has $h+1$ solutions.

The following problem arises: *do there exist any pairs (α, β) with that property?*

5. Some technical lemmas for $\beta \geq \alpha$

In this section we will study the functions m_i and M_i (see Definition 4.3), in order to prove Lemma 4.4, and also the functions defined in Definition 4.11, in order to verify Lemma 4.12.

LEMMA 5.1.

(a) *If $i \geq 1$ and $\lambda_i \leq \alpha \leq \beta$ then $M_i(\alpha, \beta) = 0$.*

(b) *If* $i \geq 2$ *then:*

(b$_1$) $M_i(\alpha,\beta) = \infty$ *in a neighbourhood of the half-space* $\{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha \leq \lambda_1\}$*;*

(b$_2$) *if* $\lambda_1 < \alpha < \beta$ *then* $M_i(\alpha,\beta) < \infty \Leftrightarrow \max_{v\in H_i, \|v\|=1} Q_{\alpha,\beta}(v) < 0$*;*

(b$_3$) *if* $\lambda_1 < \alpha < \beta$ *and* $M_i(\alpha,\beta) < \infty$ *then* M_i *is a continuous function at* (α,β).

(c) *If* $i \geq 2$ *and* $\beta > \lambda_i$ *then* M_i *is a continuous function at* (λ_i,β) *(and* $M_i(\lambda_i,\beta) = 0$*).*

(d) *If* $i \geq 1$ *then* m_i *is a continuous function and*

$$\lim_{\rho\to 0} \frac{m_i(\rho,\alpha,\beta)}{\rho^2} = \frac{1}{2}\left(1 - \frac{\alpha}{\lambda_{i+1}}\right).$$

Proof. (a) This follows clearly by definition of M_i.

(b$_1$) Set $v = ae_1 + be_2$ with $a, b > 0$ and set $R = a/b$. It follows that

$$\begin{aligned} &Q_\alpha(v) + \frac{1}{2}(\alpha-\beta)\int_\Omega ((e_1+v)^-)^2 \\ &= \frac{1}{2}b^2\left(R^2(\lambda_1-\alpha) + (\lambda_2-\alpha) + (\alpha-\beta)\int_\Omega \left(\left(\frac{e_1}{b} + Re_1 + e_2\right)^-\right)^2\right). \end{aligned}$$

Fix β. If R is so large that $(Re_1+e_2)^- = 0$ and α is such that $R^2(\lambda_1-\alpha) + (\lambda_2-\alpha) > 0$, then if b tends to $+\infty$ the assertion follows.

(b$_2$) First of all it is clear that if $\max_{v\in H_i,\|v\|=1} Q_{\alpha,\beta}(v) < 0$ then $M_i(\alpha,\beta) < \infty$, because $-2\int_\Omega e_1 v^- \leq \int_\Omega(((e_1+v)^-)^2 - (v^-)^2) \leq 0$ (see Remark 4.2(b)). It is also easy to see that if $\max_{v\in H_i,\|v\|=1} Q_{\alpha,\beta}(v) > 0$ then $M_i(\alpha,\beta) = +\infty$. Finally, if there exists $v_0 \in H_i$, $\|v_0\| = 1$, such that $\max_{v\in H_i,\|v\|=1} Q_{\alpha,\beta}(v) = Q_{\alpha,\beta}(v_0) = 0$, then by Lemma 5.2 we deduce $v_0^- \neq 0$. Hence by Remark 4.2(b) we have

$$\begin{aligned} &\lim_{t\to+\infty} Q_{\alpha,\beta}(tv_0) + \frac{1}{2}(\alpha-\beta)\int_\Omega (((e_1+tv_0)^-)^2 - (tv_0^-)^2) \\ &\geq \lim_{t\to+\infty} \frac{1}{2}(\alpha-\beta)t \int_{\{x\in\Omega \mid tv_0(x)\leq -e_1(x)\}} \left(\frac{e_1^2}{t} + 2e_1 v\right) = +\infty, \end{aligned}$$

because

$$\int_{\{x\in\Omega \mid v_0(x)\leq -e_1(x)/t\}} \left(\frac{e_1^2}{t} + 2e_1 v\right) = -2\int_\Omega e_1 v_0^- < 0.$$

From Remark 4.2(a) we see that $M_i(\alpha,\beta) = +\infty$.

(b$_3$) Suppose $\lambda_1 < \alpha < \beta$ and $M_i(\alpha,\beta) < \infty$. So there exist $c, \delta > 0$ such that if $|\alpha-\alpha'| \leq \delta$ and $|\beta-\beta'| \leq \delta$ then $Q_{\alpha',\beta'}(v) \leq -c\|v\|^2$ for every $v \in H_i$. Hence the maximum points in H_i of the functions $v \to Q_{\alpha',\beta'}(v) + \frac{1}{2}(\alpha' - \beta') \int_\Omega(((e_1 + v)^-)^2 - (v^-)^2)$ are uniformly bounded. Moreover, if (α', β') tends to (α, β) these functions converge uniformly on bounded sets and the assertion follows.

(c) The statement follows from (a) and (b_3).

(d) The continuity of m_i is clear. To prove the second assertion we notice that

$$\inf_{\substack{w \in H_i^\perp \\ \|w\|=\rho}} Q_\alpha(w) = \frac{1}{2}\rho^2\left(1 - \frac{\alpha}{\lambda_{i+1}}\right).$$

So it is enough to prove that

$$\lim_{\rho \to 0} \frac{1}{\rho^2} \sup_{\substack{w \in H_i^\perp \\ \|w\|=\rho}} \int_\Omega ((e_1 + w)^-)^2 = 0.$$

But this follows

$$\begin{aligned}\int_\Omega ((e_1 + w)^-)^2 &\le \int_{\{x \in \Omega | w(x) \le -e_1(x)\}} (w^-)^2 \\ &\le S^2 \|w\|^2 (\text{meas}\{x \in \Omega \mid e_1(x) + w(x) \le 0\})^p,\end{aligned}$$

for suitable positive p and S. □

LEMMA 5.2. *Let $i \ge 1$ and $\alpha > \lambda_1$. If $v_0 \in H_i$, $\|v_0\| = 1$, and*

$$\max_{\substack{v \in H_i \\ \|v\|=1}} Q_{\alpha,\beta}(v) = Q_{\alpha,\beta}(v_0) = 0,$$

then $v_0^- \ne 0$.

PROOF. There exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega (\nabla v_0 \nabla v - \alpha v_0^+ v + \beta v_0^- v) = \lambda \int_\Omega \nabla v_0 \nabla v \qquad \forall v \in H_i.$$

Setting $v = v_0$ we obtain $\lambda = 0$. If $v_0^- = 0$ then we get $\int_\Omega (\nabla v_0 \nabla v - \alpha v_0 v) = 0$ for any $v \in H_i$ and so $\alpha = \lambda_1$. □

Now the properties stated in Lemma 4.4 follow easily from Lemma 5.1.

In order to study the curves μ_i we recall that for $i \ge 1$ we set, in Definition 4.11

$$\alpha_{i+1} = \sup\left\{ \int_\Omega |\nabla u|^2 \,\middle|\, u \in H_i,\ \int_\Omega u^2 = 1,\ u \ge 0 \right\}$$

and for $e \in H_i^\perp$, $e \ne 0$,

$$\Sigma_i^-(e) = \{z = \sigma e + v \mid \|z\| = 1,\ v \in H_i,\ \sigma \ge 0\}.$$

It is useful to introduce the functions η_i.

DEFINITION 5.3. Let $i \geq 1$. If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ we set

$$\eta_{i+1}(\alpha, \beta) = \inf_{\substack{e \in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{z \in \Sigma_i^-(e)} Q_{\alpha,\beta}(z)$$

($\Sigma_i^-(e)$ is defined in (4.11)).

LEMMA 5.4. *Let $i \geq 1$.*

(a) $\eta_{i+1}(\alpha,\beta) = \eta_{i+1}(\beta,\alpha)$ *and* $\eta_{i+1}(\lambda_{i+1}, \lambda_{i+1}) = 0$.

(b) η_{i+1} *is Lipschitz continuous.*

(c) *If* $\alpha \leq \lambda_1$ *or* $\beta \leq \lambda_1$ *then* $\eta_{i+1}(\alpha, \beta) > 0$.

(d) *In* $\{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha > 0, \beta > 0\}$ *we have* $\eta_{i+1}(\alpha,\beta) < 1$ *and moreover* η_{i+1} *is strictly indecreasing with respect to both* α *and* β.

(e) *If* $\alpha > 0$ *then* $\lim_{\beta \to +\infty} \eta_{i+1}(\alpha,\beta) = \frac{1}{2}(1 - \alpha/\alpha_{i+1})$.

(f) $\alpha_2 = \lambda_1$ *and* $\lambda_1 \leq \alpha_{i+1} < \lambda_i$ *if* $i \geq 2$.

PROOF. (a) The first equality follows from $Q_{\alpha,\beta}(u) = Q_{\beta,\alpha}(-u)$. We notice that if $\alpha = \beta$ and $e \in \mathrm{H}_i^\perp$ then $\max_{\Sigma_i^-(e)} Q_{\alpha,\beta} = Q_{\alpha,\beta}(e)$ and hence $\eta_{i+1}(\alpha,\beta) = Q_{\alpha,\beta}(e_{i+1})$, which is equal to 0 if $\alpha = \beta = \lambda_{i+1}$.

(b) If $e \in \mathrm{H}_i^\perp$, $\|e\| = 1$ and $z \in \Sigma_i^-(e)$ then

$$\begin{aligned} |Q_{\alpha,\beta}(z) - Q_{\alpha',\beta'}(z)| &\leq |\alpha - \alpha'| \int_\Omega (z^+)^2 + |\beta - \beta'| \int_\Omega (z^-)^2 \\ &\leq \frac{1}{\lambda_1}(|\alpha - \alpha'| \vee |\beta - \beta'|. \end{aligned}$$

Thus for any $e \in \mathrm{H}_i^\perp$ the function $(\alpha,\beta) \to \max_{\Sigma_i^-(e)} Q_{\alpha,\beta}$ is Lipschitz continuous with Lipschitz constant $1/\lambda_1$. The assertion follows.

(c) Indeed, $\eta_{i+1}(\alpha,\beta) \geq Q_{\alpha,\beta}(e_1/\|e_1\|) \vee Q_{\alpha,\beta}(-e_1/\|e_1\|)$. So if $\alpha < \lambda_1$ or $\beta < \lambda_1$ we obtain the assertion.

If, for example, $\alpha = \lambda_1$, from Remark 5.5 we have

$$\eta_{i+1}(\lambda_1, \beta) > Q_{\lambda_1,\beta}(e_1/\|e_1\|) = 0.$$

(d) If $\alpha > 0$ and $\beta > 0$ we have $\eta_{i+1}(\alpha,\beta) < 1$. Moreover, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathrm{H}_i^\perp$ such that $\|u_n\| = 1$ and $\lim_n \max_{\Sigma_i^-(u_n)} Q_{\alpha,\beta} = \eta_{i+1}(\alpha,\beta)$. If $\beta' > \beta$ and z_n is a maximum point for $Q_{\alpha,\beta'}$ on $\Sigma_i^-(u_n)$, since $Q_{\alpha,\beta'}(z_n) \leq \max_{\Sigma_i^-(u_n)} Q_{\alpha,\beta}$, we can assume that $\sup_n Q_{\alpha,\beta'}(z_n) < 1$. So, from Lemma 5.6(b), we get $l = \inf_n \int_\Omega (z_n^-)^2 > 0$. On the other hand,

$$\begin{aligned} \eta_{i+1}(\alpha,\beta') \leq Q_{\alpha,\beta'}(z_n) &= Q_{\alpha,\beta}(z_n) - \frac{1}{2}(\beta' - \beta) \int_\Omega (z_n^-)^2 \\ &\leq \max_{\Sigma_i^-(u_n)} Q_{\alpha,\beta} - \frac{1}{2}(\beta' - \beta) \int_\Omega (z_n^-)^2. \end{aligned}$$

Passing to the limit as $n \to \infty$ we obtain:

$$\eta_{i+1}(\alpha,\beta') \le \eta_{i+1}(\alpha,\beta) - \frac{1}{2}(\beta'-\beta)l < \eta_{i+1}(\alpha,\beta).$$

(e) First of all we prove that:

$$\lim_{\beta\to+\infty} \eta_{i+1}(\alpha,\beta) = \inf_{\substack{e\in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha,$$

where $K^+ = \{u \in H_0^1(\Omega) \,|\, u(x) \ge 0 \text{ a.e. in } \Omega\}$. Clearly,

$$\lim_{\beta\to+\infty} \eta_{i+1}(\alpha,\beta) \ge \inf_{\substack{e\in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha,$$

since for any $e \in \mathrm{H}_i^\perp$ with $\|e\| = 1$ we have $\max_{\Sigma_i^-(e)\cap K^+} Q_\alpha \le \max_{\Sigma_i^-(e)} Q_{\alpha,\beta}$.

On the other hand if $(\beta_n)_{n\in\mathbb{N}}$ satisfies $\lim_n \beta_n = +\infty$ and $(z_n)_{n\in\mathbb{N}}$ is the sequence of maximum points of Q_{α,β_n} on $\Sigma_i^-(e)$, we can assume that $\lim_n z_n = z$ in $H_0^1(\Omega)$ (because $\dim \mathrm{H}_i < \infty$), $z \in \Sigma_i^-(e)$ and $z \in K^+$ (otherwise $\lim_n Q_{\alpha,\beta_n}(z_n) = -\infty$); finally,

$$Q_{\alpha,\beta_n}(z_n) = \frac{1}{2}\int_\Omega (|\nabla z_n|^2 - \alpha(z_n^+)^2 - \beta_n(z_n^-)^2) \le \frac{1}{2}\int_\Omega (|\nabla z_n|^2 - \alpha z_n^2),$$

with

$$\lim_n \frac{1}{2}\int_\Omega (|\nabla z_n|^2 - \alpha z_n^2) = Q_\alpha(z) \le \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha.$$

Thus

$$\lim_{\beta\to+\infty} \eta_{i+1}(\alpha,\beta) \le \inf_{\substack{e\in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha$$

and so the equality is proved.

Now let us prove that

$$\inf_{\substack{e\in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha = \max\{Q_\alpha(z) \,|\, z \in \mathrm{H}_i \cap K^+,\ \|z\| = 1\} \left(= \frac{1}{2}\left(1 - \frac{\alpha}{\alpha_{i+1}}\right)\right).$$

Also in this case it is enough to prove the inequality

$$\inf_{\substack{e\in \mathrm{H}_i^\perp \\ \|e\|=1}} \max_{\Sigma_i^-(e)\cap K^+} Q_\alpha \le \max\{Q_\alpha(z) \,|\, z \in \mathrm{H}_i \cap K^+,\ \|z\| = 1\},$$

because the other one follows easily. Let us remark that there exists $u^* \in H_0^1(\Omega)$ such that $u + \varepsilon u^* \notin K^+$ for any $\varepsilon > 0$ and for any $u \in \mathrm{H}_i$ (if $n \ge 2$ it is enough to choose u^* with ess inf $u^* = -\infty$, while if $n = 1$ we take $u^* = \mathrm{dist}(x, \partial\Omega)^\gamma$ with $1/2 < \gamma < 1$).

If e^* denotes the component of u^* on $\mathrm{H}_i^\perp$ normalized in $H_0^1(\Omega)$ we have

$$\max_{\Sigma_i^-(e^*)\cap K^+} Q_\alpha = \max\{Q_\alpha(z) \mid z \in \mathrm{H}_i \cap K^+, \ \|z\| = 1\}.$$

(f) The first inequality is trivial. The second one follows since

$$\lambda_i = \sup\left\{ \int_\Omega |\nabla u|^2 \,\middle|\, u \in \mathrm{H}_i, \ \int_\Omega u^2 = 1 \right\}$$
$$> \int_\Omega |\nabla u|^2 \qquad \forall u \in \mathrm{H}_i \text{ with } \int_\Omega u^2 = 1 \text{ and } \int_\Omega e_1 u \neq 0.$$

On the other hand $\max\{\int_\Omega |\nabla u|^2 \mid u \in \mathrm{H}_i \cap K^+, \ \int_\Omega u^2 = 1\} < \lambda_i$, because the set $\{u \in \mathrm{H}_i \cap K^+ \mid \int_\Omega u^2 = 1\}$ is compact and $\int_\Omega e_1 u > 0$ for all u in it. □

REMARK 5.5. If $\alpha > 0$ (resp. $\beta > 0$) then $e_1/\|e_1\|$ (resp. $-e_1/\|e_1\|$) is a strict minimum point of $Q_{\alpha,\beta}$ on the sphere $S = \{u \in H_0^1(\Omega) \mid \|u\| = 1\}$.

Indeed

$$Q_{\alpha,\beta}(u) = Q_\alpha(u) + \frac{1}{2}(\alpha - \beta) \int_\Omega (u^-)^2.$$

Moreover, it is known that there exist $\rho, \sigma > 0$ such that

$$\int_\Omega u^2 \leq \int_\Omega \left(\frac{e_1}{\|e_1\|}\right)^2 - \sigma \left\| u - \frac{e_1}{\|e_1\|} \right\|^2 \qquad \forall u \in S \text{ with } \left\| u - \frac{e_1}{\|e_1\|} \right\| \leq \rho;$$

hence

$$Q_\alpha(u) \geq Q_\alpha\left(\frac{e_1}{\|e_1\|}\right) + \frac{\sigma}{2} \left\| u - \frac{e_1}{\|e_1\|} \right\|^2.$$

Finally, we easily have

$$\int_\Omega (u^-)^2 \leq \int_{\{x\in\Omega \mid u(x)\leq 0\}} \left(\left(u - \frac{e_1}{\|e_1\|}\right)^-\right)^2$$
$$\leq S^2 \left\| u - \frac{e_1}{\|e_1\|} \right\|^2 (\mathrm{meas}\{x \in \Omega | u(x) \leq 0\})^p,$$

with p and S suitable positive constants. So the assertion follows. □

LEMMA 5.6. *Let $i \geq 1$.*

(a) *If $e \in \mathrm{H}_i^\perp$ and $z \in \Sigma_i^-(e)$ is a maximum point for $Q_{\alpha,\beta}$ on $\Sigma_i^-(e)$, then*

$$\left(Q_{\alpha,\beta}(z) - \frac{1}{2}\left(1 - \frac{\alpha}{\lambda_1}\right)\right) \int_\Omega z^+ e_1 = \left(Q_{\alpha,\beta}(z) - \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right)\right) \int_\Omega z^- e_1.$$

(b) *If $\alpha, \beta > 0$ then for any $\varepsilon > 0$*

$$\inf\left\{ \int_\Omega (z^-)^2 \,\middle|\, Q_{\alpha,\beta}(z) = \max_{\Sigma_i^-(e)} Q_{\alpha,\beta}, \ Q_{\alpha,\beta}(z) \leq 1 - \varepsilon, \ e \in \mathrm{H}_i^\perp, \ \|e\| = 1 \right\} > 0.$$

The same inequality holds with z^- replaced by z^+.

PROOF. (a) If z is a maximum point of $Q_{\alpha,\beta}$ on $\Sigma_i^-(e)$ then there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega (\nabla z \nabla v - \alpha z^+ v + \beta z^- v) = \lambda \int_\Omega \nabla z \nabla v \qquad \forall v \in \mathrm{H}_i \oplus \operatorname{span}(z).$$

Setting $v = z$ we obtain $\lambda = 2Q_{\alpha,\beta}(z)$. Setting $v = e_1$ we have the inequality, recalling that

$$\frac{1}{2}\left(1 - \frac{\alpha}{\lambda_1}\right) = Q_{\alpha,\beta}\left(\frac{e_1}{\|e_1\|}\right) \quad \text{and} \quad \frac{1}{2}\left(1 - \frac{\beta}{\lambda_1}\right) = Q_{\alpha,\beta}\left(-\frac{e_1}{\|e_1\|}\right).$$

(b) Let $(\widetilde{e}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{H}_i^\perp$ with $\|\widetilde{e}_n\| = 1$ and let $(z_n)_{n\in\mathbb{N}}$ be the sequence of maximum points for $Q_{\alpha,\beta}$ on $\Sigma_i^-(\widetilde{e}_n)$. Suppose that $\lim_n \int_\Omega (z_n^-)^2 = 0$ and (which is always possible) that $\lim_n z_n = z$ in $L^2(\Omega)$. From (a) and since $\inf_n Q_{\alpha,\beta}(z_n) > Q_{\alpha,\beta}(e_1/\|e_1\|)$ by Remark 5.5, it follows that $\lim_n \int_\Omega z_n^+ e_1 = 0$ and so $z^+ = 0$. Hence $\lim_n \int_\Omega (z_n^+)^2 = 0$ and then $\lim_n Q_{\alpha,\beta}(z_n) = 1$. □

Finally, the functions μ_i (see Lemma 4.11) can be defined in terms of the functions η_i in the following way.

REMARK 5.7. Let $i \geq 1$. Then

$$\eta_{i+1}(\alpha,\beta) < 0 \Leftrightarrow \alpha > \alpha_{i+1} \quad \text{and} \quad \beta > \mu_{i+1}(\alpha).$$

Now the statement of Lemma 4.12 of the previous section follows immediately from Lemma 5.4.

6. The existence of two and three solutions if $\beta \leq \alpha$. f_t separates two linked spheres. f_t separates two pairs of linked spheres in dimensional scale

In this section we will set out some existence theorems for the problem (P_t) for t positive and large enough under assumptions (G, α, β) with $\beta \leq \alpha$. As in Section 4, the technical results and variational abstract statements used in this section, will be shown in Sections 7 and 8, respectively.

We would like to point out the surprising analogy between the results which were obtained in Section 4 when $\beta \geq \alpha$ and the ones which will be shown in this section when $\beta \leq \alpha$. We emphasize that t is positive and large enough in both cases.

DEFINITION 6.1. Let $i \geq 1$. Given $\alpha, \beta \in \mathbb{R}$ and $\rho > 0$, set

$$N_i(\rho, \alpha, \beta) = \sup_{\substack{v \in \mathrm{H}_i \\ \|v\| = \rho}} \left\{ Q_\alpha(v) + \frac{1}{2}(\alpha - \beta) \int_\Omega ((e_1 + v)^-)^2 \right\},$$

$$n_i(\alpha, \beta) = \inf_{w \in \mathrm{H}_i^\perp} \left\{ Q_\alpha(w) + \frac{1}{2}(\alpha - \beta) \int_\Omega ((e_1 + w)^-)^2 \right\},$$

$$F_i = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \beta < \alpha, \ \exists \rho_i > 0 \text{ with } N_i(\rho_i, \alpha, \beta) < n_i(\alpha, \beta)\}.$$

The functions N_i and n_i have properties (see Lemma 7.1) similar to the ones of the functions M_i and m_i defined in Definition 4.3. The following result can be easily deduced from Lemma 7.1.

LEMMA 6.2. *Let $i \geq 1$. Then F_i is an open set and*

$$\{(\alpha, \beta) \in \mathbb{R}^2 \mid \beta < \alpha, \ \lambda_i < \alpha \leq \lambda_{i+1}\} \subset F_i \subset \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha > \lambda_i\}.$$

Moreover, if $(\alpha, \beta) \in F_i$, then $\inf_{w \in \mathrm{H}_i^\perp, \|w\|=1} Q_{\alpha,\beta}(w) > 0$.

To the regions F_i correspond an inequality for the functional f_t which can be shown similarly to (4.6).

LEMMA 6.3 ("saddle"). *Let $i \geq 1$. Assume (G) and (G, α, β). If $(\alpha, \beta) \in F_i$ then there exist $\rho_i, t_0 > 0$ so that if $t \geq t_0$*

$$\sup_{\substack{v \in \mathrm{H}_i \\ \|v\| = \rho_i s_t}} f_t(s_t e_1 + v) < \inf_{w \in \mathrm{H}_i^\perp} f_t(s_t e_1 + w).$$

(ρ_i is in the definition of F_i.)

DEFINITION 6.4. Let $i \geq 1$. Given $e \in \mathrm{H}_i$, $e \neq 0$ and $\alpha \in \mathbb{R}$ set

$$\Sigma_i^+(e) = \{z = \sigma e + w \mid \|z\| = 1, \ w \in \mathrm{H}_i^\perp, \ \sigma \geq 0\},$$

$$\nu_i(\alpha) = \sup \left\{ \beta \in \mathbb{R} \,\middle|\, \sup_{\substack{e \in \mathrm{H}_i \\ e \neq 0}} \inf_{z \in \Sigma_i^+(e)} Q_{\alpha,\beta}(z) > 0 \right\}.$$

The functions ν_i have properties similar to the ones of the functions μ_i (see Lemma 4.12). They can be easily deduced from Lemma 7.3 ((e) follows from Lemma 7.1(b_1)).

LEMMA 6.5. *Let $i \geq 1$.*

(a) *$\nu_i(\alpha) \in \mathbb{R}$ for any $\alpha \in \mathbb{R}$.*

(b) $\nu_i : \mathbb{R} \to \mathbb{R}$ *is a continuous decreasing function in* $[\lambda_i, \infty[$ *such that* $\nu_i(\lambda_i) = \lambda_i$, $\nu_i \circ \nu_i = \mathrm{id}$ *and* $\lim_{\alpha\to+\infty} \nu_i(\alpha) = -\infty$.

(c) *If* $i \geq 2$ *then* $\nu_i(\lambda_{i+1}) > \lambda_1$.

(d) $\nu_1(\alpha) = \lambda_1$ *if* $\lambda_1 \leq \alpha \leq \lambda_2 + \varepsilon$, *for suitable* $\varepsilon > 0$.

(e) *If* $(\alpha, \beta) \in F_i$ *then* $\beta < \nu_{i+1}(\alpha)$.

Moreover, the following relationship between μ_i *and* ν_i *holds. Let* λ_{j-1}, λ_j *and* λ_{k+1} *be three consecutive (possibly multiple) eigenvalues* $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ *with* $k \geq j \geq 2$. *Then*

(f) $\nu_k(\alpha) \geq \mu_j(\alpha)$ *if* $\mu_j(\alpha) \in [\lambda_{j-1}, \lambda_{k+1}]$ *and* $\nu_k(\alpha) \in [\lambda_{j-1}, \lambda_{k+1}]$.

By using a similar argument to that used in Section 4, the curves ν_i and the regions F_i will allow us to characterize some pairs (α, β) corresponding to a more involved "topological behaviour" of f_t.

The following result can be proved by reasoning as in Theorem 4.13 and we deduce that the functional f_t "separates two linked spheres" with suitable bounds.

THEOREM 6.6 ("links and bounds"). *Let* $i \geq 1$. *Assume* (G) *and* (G, α, β) *with* $(\alpha, \beta) \in F_i$ *and* $\beta < \nu_i(\alpha)$. *Then there exist* $\overline{\sigma}_i > 0$, $\rho_i > 0$ *(see the definition of* F_i*),* $t_0 > 0$ *and* $e \in \mathrm{H}_i$, $e \neq 0$, *such that* $\overline{\sigma}_i > \rho_i/(\alpha - \lambda_1)$ *and if we set, for* $\sigma_i \geq \overline{\sigma}_i$ *and* $t \geq t_0$,

$$\begin{aligned}\Sigma_i^+ = &\{s_t e_1 + w \mid w \in \mathrm{H}_i^\perp,\ \|w\| \leq \sigma_i t\}\\ &\cup \{s_t e_1 + w + \sigma e \mid w \in \mathrm{H}_i^\perp,\ \sigma \geq 0,\ \|w + \sigma e\| = \sigma_i t\},\end{aligned}$$

then the following inequalities hold:

(a) $\displaystyle\sup_{\substack{v\in \mathrm{H}_i\\ \|v\|=\rho_i s_t}} f_t(s_t e_1 + v) < \inf_{\Sigma_i^+} f_t;$

(b) $\displaystyle\inf_{\substack{w\in \mathrm{H}_i^\perp, \sigma\geq 0\\ \|w+\sigma e\|\leq \sigma_i t}} f_t(s_t e_1 + w + \sigma e) > -\infty$ *and* $\displaystyle\sup_{\substack{v\in \mathrm{H}_i\\ \|v\|\leq \rho_i s_t}} f_t(s_t e_1 + v) < +\infty.$

Now this result will be used to obtain both a theorem on existence of two solutions and of three solutions for the problem (P_t).

First of all by using Theorems 6.6 and 8.2 and a similar argument to that in Theorem 4.14, we easily get the following result.

THEOREM 6.7 (existence of two solutions). *Assume:*

(a) (g), (G), (G, α, β) *and* (P.S.) *for the functional* f_t *for* t *large enough;*

(b) $\alpha > \lambda_1$ *and* $\beta < \nu_k(\alpha)$, *where* λ_k *is the maximum eigenvalue strictly smaller than* α.

Then, for t large enough, the functional f_t has at least two critical values and hence the problem (P_t) has at least two solutions.

We point out that assumptions (a) can be replaced by (2.4) or (2.6).

By Theorem 6.6 we can also obtain a theorem on existence of three solutions of (P_t) for suitable pairs (α, β).

We start by emphasizing an easy consequence of Lemmas 6.2 and 6.5.

Remark 6.8. Let λ_{j-1}, λ_j and λ_{k+1} be three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$.

The set of pairs (α, β) such that $(\alpha, \beta) \in F_k \cap F_{j-1}$ (hence $\beta < \nu_k(\alpha)$, by Lemma 6.5(e)) and $\beta < \nu_{j-1}(\alpha)$ (that is, (α, β) satisfies the conditions of (6.6) both for $i = k$ and $i = j - 1$) is an open and non-empty set. More precisely, for all $\beta < \nu_{j-1}(\lambda_j)$ there exists $\delta > 0$ such that $]\lambda_j, \lambda_j + \delta] \times \{\beta\}$ is contained in this set.

Now we claim that if (α, β) satisfies the conditions of Remark 6.8, then the functional f_t exhibits a very interesting behaviour from a "topological" point of view. It can be shown that, as in Theorem 4.16, the functional f_t "separates two pairs of linked spheres in dimensional scale" and satisfies a suitable lower bound.

Theorem 6.9 ("links in scale and bounds"). *Let λ_{j-1}, λ_j and λ_{k+1} be three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$. Assume (G) and (G, α, β) with $(\alpha, \beta) \in F_k \cap F_{j-1}$ and $\beta < \nu_{j-1}(\alpha)$ (see Remark 6.8). There exist $\sigma_k, \rho_k, \sigma_{j-1}, \rho_{j-1}, t_0 > 0$ and $e \in \mathrm{H}_{j-1}$, $e \neq 0$, such that $\sigma_k > \rho_k/\alpha - \lambda_1$, $\sigma_{j-1} > \rho_{j-1}/\alpha - \lambda_1$, $\sigma_k \leq \sigma_{j-1}$ and if we set, for $t \geq t_0$,*

$$\begin{aligned}
\Sigma_{j-1}^{+} = &\{s_t e_1 + w \mid w \in \mathrm{H}_{j-1}^{\perp},\ \|w\| \leq \sigma_{j-1} t\} \\
&\cup \{s_t e_1 + w + \sigma e \mid w \in \mathrm{H}_{j-1}^{\perp},\ \sigma \geq 0,\ \|w + \sigma e\| = \sigma_{j-1} t\}, \\
\Sigma_k^{+} = &\{s_t e_1 + w \mid w \in \mathrm{H}_k^{\perp},\ \|w\| \leq \sigma_k t\} \\
&\cup \{s_t e_1 + w + \sigma e_k \mid w \in \mathrm{H}_k^{\perp},\ \sigma \geq 0,\ \|w + \sigma e_k\| = \sigma_k t\},
\end{aligned}$$

then the following inequalities hold:

(a) $\displaystyle\sup_{\substack{v \in \mathrm{H}_{j-1} \\ \|v\| = \rho_{j-1} s_t}} f_t(s_t e_1 + v) < \inf_{\Sigma_{j-1}^{+}} f_t \leq \sup_{\substack{v \in \mathrm{H}_k \\ \|v\| = \rho_k s_t}} f_t(s_t e_1 + v) < \inf_{\Sigma_k^{+}} f_t$;

(b) $\displaystyle\inf_{\substack{w \in \mathrm{H}_{j-1}^{\perp} \\ \|w\| \leq \rho_{j-1} s_t}} f_t(s_t e_1 + w) > -\infty$.

The proof is similar to the one of Theorem 4.16.

Finally, by means of Theorems 8.4 and 6.9 we are able to establish the following theorem.

THEOREM 6.10 (existence of three solutions). *Assume:*

(a) (g), (G), (G, α, β) *and* (P.S.) *for the functional* f_t *for* t *large enough;*

(b) $(\alpha, \beta) \in F_k \cap F_{j-1}$ *and* $\beta < \nu_{j-1}(\alpha)$, *where* λ_{j-1}, λ_j *and* λ_{k+1} *are three consecutive (possibly multiple) eigenvalues:* $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ *with* $k \geq j \geq 2$ *(see Remark* 6.8).

Then, for t *large enough, the functional* f_t *has at least three critical values and hence the problem* (P_t) *has at least three solutions.*

Also in this case assumptions (a) can be replaced by (2.4) or (2.6).

REMARK 6.11. As in Remark 4.18, we point out that if $\lambda_{i_1}, \ldots, \lambda_{i_h}$ and λ_{i_h+1} are $h+1$ consecutive eigenvalues and if $(\alpha, \beta) \in F_{i_1} \cap \ldots \cap F_{i_h}$ and $\beta < \nu_{i_1}(\alpha)$ then the functional f_t "separates h pairs of linked spheres in dimensional scale". In this case (see Remark 8.5) if t is large enough then f_t has $h+1$ critical values and hence the problem (P_t) has $h+1$ solutions.

Also in this case the following problem arises: *do there exist any pairs* (α, β) *with that property?*

7. Some technical lemmas for $\alpha \geq \beta$

In order to prove Lemma 6.2 we need the following lemma (whose proof is similar to the one of Lemma 5.1).

LEMMA 7.1. *Let* $i \geq 1$.

(a) *If* $\beta \leq \alpha \leq \lambda_{i+1}$ *then* $n_i(\alpha, \beta) = 0$.

(b) *If* $a > \beta$ *then:*

 (b$_1$) $n_i(\alpha, \beta) > -\infty \Leftrightarrow \inf_{w \in H_i^\perp, \|w\|=1} Q_{\alpha,\beta}(w) > 0$;

 (b$_2$) *if* $n_i(\alpha, \beta) > -\infty$ *then* n_i *is a continuous function at* (α, β).

(c) *In particular, if* $\beta < \lambda_{i+1}$ *then* n_i *is continuous at* (λ_{i+1}, β) *(and* $n_i(\lambda_{i+1}, \beta) = 0$).

(d) N_i *is a continuous function and*

$$\lim_{\rho \to 0} \frac{N_i(\rho, \alpha, \beta)}{\rho^2} = \frac{1}{2}\left(1 - \frac{\alpha}{\lambda_i}\right).$$

In order to prove Lemma 6.2 we need the following definition and lemma.

DEFINITION 7.2. Let $i \geq 1$. For $\alpha, \beta \in \mathbb{R}$ we set

$$\theta_i(\alpha, \beta) = \sup_{\substack{e \in H_i \\ \|e\|=1}} \inf_{z \in \Sigma_i^+(e)} Q_{\alpha,\beta}(z).$$

(Σ_i^+ is defined in Definition 6.4.)

LEMMA 7.3. *Let $i \geq 1$.*

(a) $\theta_i(\alpha,\beta) = \theta_i(\beta,\alpha)$ *and* $\theta_i(\lambda_i,\lambda_i) = 0$.

(b) θ_i *is Lipschitz continuous.*

(c) $\lim_{\beta\to\infty}\theta_i(\alpha,\beta) = -\infty$ *and* $\lim_{\beta\to-\infty}\theta_i(\alpha,\beta) = 1$ *for any* $\alpha \in \mathbb{R}$.

(d) *If* $i \geq 2$ *then* $\theta_i(\lambda_{i+1},\lambda_1) > 0$.

(e) θ_i *is decreasing with respect to both* α *and* β; *moreover, in* $\{(\alpha,\beta) \in \mathbb{R}^2 \mid \theta_i(\alpha,\beta) = 0, \lambda_i \leq \alpha\}$ θ_i *is strictly decreasing with respect to* β.

(f) *If* $\lambda_1 \leq \alpha \leq \lambda_2 + \varepsilon$ *for suitable* $\varepsilon > 0$, *then* $\theta_1(\alpha,\lambda_1) = 0$.

PROOF. (a) and (b): These are trivial (and analogous) to (a) and (b) of Lemma 5.4.

(c) Note that $\theta_i(\alpha,\beta) \leq Q_{\alpha,\beta}(e_{i+1})$ and $e_{i+1}^+ \neq 0$; so $\lim_{\beta\to\infty}\theta_i(\alpha,\beta) = -\infty$. For the other limit, we observe that

$$\inf_{\Sigma_i^+(-e_1/\|e_1\|)} Q_{\alpha,\beta} \leq \theta_i(\alpha,\beta) \leq \inf_{\substack{w\in \mathrm{H}_i^\perp \\ \|w\|=1}} Q_{\alpha,\beta}(w).$$

We have

$$\lim_{\beta\to-\infty}\inf_{\Sigma_i^+(-e_1/\|e_1\|)} Q_{\alpha,\beta} = 1 \quad \text{and} \quad \lim_{\beta\to-\infty}\inf_{\substack{w\in \mathrm{H}_i^\perp \\ \|w\|=1}} Q_{\alpha,\beta}(w) = 1.$$

Indeed, first of all we note that $\inf_{\Sigma_i^+(-e_1/\|e_1\|)} Q_{\alpha,\beta} \leq 1$, since in $\mathrm{H}_i^\perp$ there exist functions with $H_0^1(\Omega)$-norm equal to 1 and $L^2(\Omega)$-norm as small as we want.

Now let $(\beta_n)_{n\in\mathbb{N}}$ be such that $\lim_n \beta_n = -\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma_i^+(-e_1/\|e_1\|)$. If $\liminf_n \int_\Omega (u_n^-)^2 > 0$, then $\lim_n Q_{\alpha,\beta_n}(u_n) = +\infty$. Otherwise we can suppose that $\liminf_n \int_\Omega (u_n^-)^2 = 0$, and since $\Sigma_i^+(-e_1/\|e_1\|)$ does not contain positive functions, $\lim_n u_n = 0$ in $L^2(\Omega)$ and so $\liminf_n Q_{\alpha,\beta_n}(u_n) \geq 1$. Finally,

$$\lim_{\beta\to-\infty}\inf_{\Sigma_i^+(-e_1/\|e_1\|)} Q_{\alpha,\beta} = 1.$$

The proof of the other limit is analogous.

(d) We have to verify that if $\alpha = \lambda_{i+1}$ and $\beta = \lambda_1$ then there exists $e \in \mathrm{H}_i$, $e \neq 0$, such that $\inf_{\Sigma_i^+(e)} Q_{\alpha,\beta} > 0$.

If $\sigma \geq 0$, $w \in \mathrm{H}_i^\perp$ and $e \in \mathrm{H}_i$, $e \neq 0$, then we can easily verify that for any $\varepsilon > 0$,

$$\begin{aligned} Q_{\alpha,\beta}(\sigma e + w) &\geq Q_{\alpha,\beta}(w) + \sigma^2 Q_\beta(e) - \frac{1}{2}(\alpha-\beta)\int_\Omega (2\sigma e^+ w^+ + (\sigma e^+)^2) \\ &\geq Q_{\alpha,\beta}(w) - \frac{1}{2}(\alpha-\beta)\varepsilon^2\int_\Omega w^2 \\ &\quad + \sigma^2\Big(Q_\beta(e) - \frac{1}{2}(\alpha-\beta)\Big(1+\frac{1}{\varepsilon^2}\Big)\int_\Omega (e^+)^2\Big). \end{aligned}$$

Now if ε is small enough there exists $c(\alpha,\beta)>0$ such that

$$Q_{\alpha,\beta}(w)-\frac{1}{2}(\alpha-\beta)\varepsilon^2\int_\Omega w^2\geq c(\alpha,\beta)\|w\|^2,$$

since $\alpha=\lambda_{i+1}$ and $\beta<\alpha$ (see Lemma 6.2).

On the other hand for suitable $q>0$,

$$Q_{\lambda_1}(e)\geq q\left\|e+\frac{e_1}{\|e_1\|}\right\|^2$$

(the quadratic form Q_{λ_1} has $-e_1/\|e_1\|$ as an eigenfunction corresponding to the minimum eigenvalue).

So, it is easy to verify that

$$\begin{aligned}\int_\Omega (e^+)^2 &\leq \int_{\{x\in\Omega\,|\,e(x)\geq 0\}}\left(e+\frac{e_1}{\|e_1\|}\right)^2\\ &\leq S^2\left\|e+\frac{e_1}{\|e_1\|}\right\|^2(\text{meas}\{x\in\Omega\,|\,e(x)\geq 0\})^p,\end{aligned}$$

with p and S suitable positive constants. Hence if $\|e+e_1/\|e_1\|\|$ is small enough, then for $\beta=\lambda_1$ and $\alpha>\beta$

$$d(\alpha,\beta)=Q_\beta(e)-\frac{1}{2}(\alpha-\beta)\left(1+\frac{1}{\varepsilon^2}\right)\int_\Omega(e^+)^2>0.$$

Finally, if $\alpha=\lambda_{i+1}$ and $\beta=\lambda_1$, then we still have

$$Q_{\alpha,\beta}(\sigma e+w)\geq c(\alpha,\beta)\|w\|^2+d(\alpha,\beta)\sigma^2\qquad\text{with } c(\alpha,\beta),d(\alpha,\beta)>0.$$

The assertion follows.

(d) The weak monotonicity of θ_i with respect to α and β is obvious. Now let α and β satisfy $\lambda_i\leq\alpha$ and $\theta_i(\alpha,\beta)=0$ and let $\beta'>\beta$. Since the function $e\to\inf_{\Sigma_i^+(e)}Q_{\alpha,\beta'}$ is continuous in H_i, there exists $e_0\in\mathrm{H}_i$, $\|e_0\|=1$, such that $\inf_{\Sigma_i^+(e_0)}Q_{\alpha,\beta'}=\theta_i(\alpha,\beta')$.

If $\inf_{\Sigma_i^+(e_0)}Q_{\alpha,\beta'}<0$, the statement is proved. If $\inf_{\Sigma_i^+(e_0)}Q_{\alpha,\beta'}=0$, there exists $z_0\in\Sigma_i^+(e_0)$ such that $Q_{\alpha,\beta}(z_0)=0$, since we have $\inf_{\Sigma_i^+(e_0)}Q_{\alpha,\beta}=0$. Hence there exists $\lambda\in\mathbb{R}$ such that

$$\int_\Omega(\nabla z_0\nabla w-\alpha z_0^+w+\beta z_0^-w)=\lambda\int_\Omega\nabla z_0\nabla w\qquad\forall w\in\mathrm{H}_i^\perp\oplus\text{span}(z_0).$$

Setting $w=z_0$ we obtain $\lambda=0$. If $z_0^-=0$ then we get $\int_\Omega(\nabla z_0\nabla w-\alpha z_0 w)=0$ for any $w\in\mathrm{H}_i^\perp\oplus\text{span}(z_0)$. Hence we deduce that $z_0\in\mathrm{H}_i$, and so $z_0=e_0$.

Moreover, $\alpha \le \alpha_{i+1}$ since $\int_\Omega(|\nabla z_0|^2 - \alpha z_0^2) = 0$. Finally, if $\alpha \ge \lambda_i$ then $z_0^- \ne 0$ and so

$$\inf_{\Sigma_i^+(e_0)} Q_{\alpha,\beta'} \le Q_{\alpha,\beta'}(z_0) < Q_{\alpha,\beta}(z_0) = 0.$$

If $\beta' < \beta$ then it is enough to interchange β' and β in the previous inequality.

(f) If $\beta = \lambda_1$ and $\alpha \ge \lambda_1$ then $\theta_1(\alpha, \lambda_1) \le 0$, since $Q_{\alpha,\lambda_1}(e_1/\|e_1\|) \le 0$ and $Q_{\alpha,\lambda_1}(-e_1/\|e_1\|) \le 0$.

Let us prove that for suitable $\varepsilon > 0$

$$\inf_{\Sigma_i^+(-e_1/\|e_1\|)} Q_{\alpha,\lambda_1} = 0 \qquad \text{if } \lambda_1 \le \alpha \le \lambda_2 + \varepsilon.$$

If $\sigma \ge 0$ and $w \in \mathrm{H}_i^\perp$ then

$$\begin{aligned} Q_{\alpha,\lambda_1}\Big(-\sigma\frac{e_1}{\|e_1\|} + w\Big) &= \sigma^2 Q_{\lambda_1}\Big(-\frac{e_1}{\|e_1\|}\Big) + Q_{\alpha,\lambda_1}(w) \\ &\quad + \frac{1}{2}(\alpha - \lambda_1)\int_\Omega\Big((w^+)^2 - \Big(\Big(-\sigma\frac{e_1}{\|e_1\|} + w\Big)^+\Big)^2\Big) \\ &\ge Q_{\alpha,\lambda_1}(w) \\ &= \frac{1}{2}\int_\Omega(|\nabla w|^2 - \alpha w^2) + \frac{1}{2}(\alpha - \lambda_1)\int_\Omega(w^-)^2. \end{aligned}$$

Now it is easy to verify that for $\delta, k > 0$ we get

$$c = \inf\Big\{\int_\Omega(w^-)^2 \;|\; \|w\| = 1,\ w \in \mathrm{H}_i^\perp,\ Q_{\alpha,\beta}(w) \le 1 - \delta,\ |\alpha| + |\beta| \le k\Big\} > 0,$$

because if $w \in \mathrm{H}_i^\perp$ then $w^- \ne 0$. So

$$Q_{\alpha,\lambda_1}\Big(-\sigma\frac{e_1}{\|e_1\|} + w\Big) \ge \frac{1}{2}\Big(1 - \frac{\alpha}{\lambda_2} + (\alpha - \lambda_1)c\Big)\|w\|^2.$$

This proves the assertion.

(e) The assertion follows by Lemma 7.4. □

Finally, the functions ν_i (see Lemma 6.5) can be defined in terms of the functions θ_i in the following way.

REMARK 7.4. Let $i \ge 1$. Then

$$\theta_i(\alpha, \beta) > 0 \Leftrightarrow \beta < \nu_i(\alpha).$$

Now the statement of Lemma 6.5 follows immediately from Lemma 7.3.

The following lemma is useful in order to understand the link between our curves μ_i and ν_i and the curves introduced in [16], [19] and [28]. More precisely, let λ_{j-1}, λ_j and λ_{k+1} be three eigenvalues: $\lambda_{j-1} < \lambda_j \le \ldots \le \lambda_k < \lambda_{k+1}$ with $k \ge j \ge 2$. Then in the square $[\lambda_{j-1}, \lambda_{k+1}]^2$ the curves μ_j and ν_k coincide with the ones examined in [19] and [28]. Moreover, the curve μ_2 coincides with the one considered in [16].

LEMMA 7.5. *Let λ_{j-1}, λ_j and λ_{k+1} be three eigenvalues: $\lambda_{j-1} < \lambda_j \leq \ldots \leq \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$. Assume $(\alpha, \beta) \in]\lambda_{j-1}, \lambda_{k+1}[^2$. Set*

$$\mathcal{M}(\alpha,\beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(v) = 0 \ \forall v \in \mathrm{H}_{j-1}\},$$
$$\mathcal{N}(\alpha,\beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(w) = 0 \ \forall w \in \mathrm{H}_k^\perp\},$$
$$\mathcal{Z}(\alpha,\beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(z) = 0 \ \forall z \in \mathrm{H}_{j-1} \oplus \mathrm{H}_k^\perp\}.$$

It is known that $\mathcal{M}(\alpha,\beta)$, $\mathcal{N}(\alpha,\beta)$ and $\mathcal{Z}(\alpha,\beta)$ are manifolds in $H_0^1(\Omega)$. Moreover:

(a) $\forall u \in H_0^1(\Omega) \ \exists_1 v \in \mathrm{H}_{j-1}$ *such that* $u + v \in \mathcal{M}(\alpha,\beta)$,
$u \in \mathcal{M}(\alpha,\beta) \Leftrightarrow Q_{\alpha,\beta}(u) \geq Q_{\alpha,\beta}(u+v) \ \forall \ v \in \mathrm{H}_{j-1}$;
(b) $\forall u \in H_0^1(\Omega) \ \exists_1 w \in \mathrm{H}_k^\perp$ *such that* $u + w \in \mathcal{N}(\alpha,\beta)$.
$u \in \mathcal{N}(\alpha,\beta) \Leftrightarrow Q_{\alpha,\beta}(u) \leq Q_{\alpha,\beta}(u+w) \ \forall \ w \in \mathrm{H}_k^\perp$;
(c) $\forall u \in H_0^1(\Omega) \ \exists_1 z \in \mathrm{H}_{j-1}$ *such that* $u + z \in \mathcal{Z}(\alpha,\beta)$;
(d) $u \in \mathcal{Z}(\alpha,\beta) \Rightarrow u^+ \neq 0$ *and* $u^- \neq 0$;
(e) $\beta < \mu_j(\alpha) \Leftrightarrow Q_{\alpha,\beta}(u) > 0 \ \forall u \in \mathcal{M}(\alpha,\beta) \setminus \{0\}$
$\Leftrightarrow Q_{\alpha,\beta}(u) > 0 \ \forall u \in \mathcal{Z}(\alpha,\beta) \setminus \{0\}$;
(f) $\beta > \nu_k(\alpha) \Leftrightarrow Q_{\alpha,\beta}(u) < 0 \ \forall u \in \mathcal{N}(\alpha,\beta) \setminus \{0\}$
$\Leftrightarrow Q_{\alpha,\beta}(u) < 0 \ \forall u \in \mathcal{Z}(\alpha,\beta) \setminus \{0\}$.
(g) $\beta < \mu_j(\alpha) \Rightarrow (\alpha,\beta) \notin \Sigma_\Omega$ *(see (2.2))*;
$\beta > \nu_k(\alpha) \Rightarrow (\alpha,\beta) \notin \Sigma_\Omega$ *(see (2.2))*;
(h) $(\alpha, \mu_j(\alpha)) \in \Sigma_\Omega$ *and* $(\alpha, \nu_k(\alpha)) \in \Sigma_\Omega$.

PROOF. (a), (b) and (c) follow by definition of $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{Z}$ (see for example [29] and [28]).

To prove (d) it is enough to observe that if $z \in \mathcal{Z}(\alpha,\beta)$ then $\Delta z + \alpha z^+ - \beta z^- \in \mathrm{span}(e_j, \ldots, e_k)$; so, by contradiction, if $z \geq 0$ then $z \in \mathrm{span}(e_j, \ldots, e_k)$, which is absurd.

Now we show (e). The second equivalence follows by (a) and (c). Moreover, it is easy to verify, by using the definition of μ_j and (a), that if $Q_{\alpha,\beta}(u) > 0$ for all $u \in \mathcal{M}(\alpha,\beta) \setminus \{0\}$, then $\beta < \mu_j(a)$. By the definition of μ_j and (a), it also easily follows that if $\beta < \mu_j(a)$ then $Q_{\alpha,\beta}(u) \geq 0$, for all $u \in \mathcal{M}(\alpha,\beta) \setminus \{0\}$, that is, $Q_{\alpha,\beta}(u) \geq 0$ for all $u \in \mathcal{Z}(\alpha,\beta) \setminus \{0\}$. The assertion follows from the fact that if $\beta' < \beta$ then $Q_{\alpha,\beta'}(u) > 0$, $u \neq 0$ for all $u \in \mathcal{Z}(\alpha,\beta') \setminus \{0\}$. (This is an easy consequence of the strict convexity of $Q_{\alpha,\beta}$ on $\mathrm{H}_k^\perp$ and its strict concavity on H_{j-1}.)

The proof of (f) is similar to that of (e).

We show (g). If, for example, $\beta < \mu_j(\alpha)$, then $Q_{\alpha,\beta}(u) > 0$ for all $u \in \mathcal{M}(\alpha,\beta) \setminus \{0\}$ by (e). On the other hand, if $(\alpha,\beta) \in \Sigma_\Omega$, then there exists

$u \in H_0^1(\Omega)$, $u \neq 0$, such that $\Delta u + \alpha u^+ - \beta u^- = 0$, hence $u \in \mathcal{M}(\alpha,\beta)$, and $Q_{\alpha,\beta}(u) = 0$. So we get a contradiction.

We show (h). If, for example, $\beta = \nu_k(\alpha)$, then

$$0 = \max_{\substack{z\in \mathrm{H}_k \\ \|z\|=1}} Q_{\alpha,\beta}(z + \gamma(z))$$

(we recall that $\mathcal{N}(\alpha,\beta)$ is the graph of a map $\gamma : \mathrm{H}_k \to \mathrm{H}_k^\perp$). Therefore there exists $z_0 \in \mathrm{H}_k$, $z_0 \neq 0$, such that $Q'_{\alpha,\beta}(z_0 + \gamma(z_0))(z) = \lambda \int_\Omega \nabla z_0 \nabla z$ for all $z \in \mathrm{H}_k$ and $u_0 = z_0 + \gamma(z_0) \in \mathcal{N}(\alpha,\beta) \setminus \{0\}$. Then $\Delta u_0 + \alpha u_0^+ - \beta u_0^- = 0$ and the assertion follows.

8. The variational setting

In this section we present some abstract variational theorems, from which we have obtained the existence theorems in Sections 3, 4 and 6. Theorems 8.1 and 8.2 have already been introduced in [29].

The existence Theorem 3.5 is based on the fact that the hypothesis on the functional f_t implies that f_t "separates two splitting spheres in a symmetrical way" (as is proved in Theorem 3.3) and on the following theorem.

THEOREM 8.1 ("splitting spheres"). *Let* X *be a Hilbert space which is the topological direct sum of subspaces* X_1 *and* X_2. *Let* $F \in C^1(\mathrm{X},\mathbb{R})$. *Moreover, suppose that there exist* $\rho_1, \rho_2 > 0$ *such that*

$$\sup_{\substack{u\in \mathrm{X}_1 \\ \|u\|=\rho_1}} F(u) < a = \inf_{\substack{u\in \mathrm{X}_2 \\ \|u\|\le\rho_2}} F(u) \le b = \sup_{\substack{u\in \mathrm{X}_1 \\ \|u\|\le\rho_1}} F(u) < \inf_{\substack{u\in \mathrm{X}_2 \\ \|u\|=\rho_2}} F(u).$$

If $(\mathrm{P.S.})_c$ *holds for any* $c \in [a,b]$ *and at least one of the two spaces* X_1 *and* X_2 *has finite dimension, then there exists at least one critical point* u_0 *for the functional* F *such that* $a \le F(u_0) \le b$.

PROOF. We set $B_i = \{u \in \mathrm{X}_i \mid \|u\| \le \rho_i\}$ and we denote by ∂B_i the boundary of B_i in X_i for $i = 1,2$. We assume that $\dim \mathrm{X}_1 < \infty$.

By contradiction suppose that every c with $a = \inf_{B_2} F \le c \le \sup_{B_1} F = b$ is a regular value. Then by the deformation lemma the set $F^{a-\varepsilon}$ is a strong deformation retract of F^b for ε small enough; hence there exists a deformation $\eta : [0,1] \times F^b \to F^b$ such that

$$\begin{aligned} &\eta(0,u) = u && \forall u \in F^b, \\ &\eta(t,u) = u && \forall t \in [0,1],\ \forall u \in F^{a-\varepsilon}, \\ &\eta(1,u) \in F^{a-\varepsilon} && \forall u \in F^b. \end{aligned}$$

Since $B_1 \subset F^b$, we have $\eta(1, B_1) \subset F^{a-\varepsilon}$. On the other hand, from the assumptions, $B_2 \cap F^{a-\varepsilon} = \emptyset$, and so $\eta(1, B_1) \cap B_2 = \emptyset$. Now we will get the contradiction by showing $\eta(1, B_1) \cap B_2 \neq \emptyset$. We consider the continuous map $\phi : [0,1] \times B_1 \to \mathbb{R}^+ \times \mathrm{X}_1$ defined by $\phi(t,u) = (\|P_2\eta(t,u)\|/\rho_2, P_1\eta(t,u))$, where P_1 and P_2 are the orthogonal projections of X onto X_1 and X_2 respectively.

By the assumptions and the properties of η we easily see that

$$\begin{aligned} \phi(0,u) &= (0,u) && \forall u \in B_1, \\ \phi(t,u) &= (0,u) && \forall t \in [0,1],\ \forall u \in \partial B_1, \\ (1,0) &\notin \mathrm{Im}(\phi), && \text{because } \partial B_2 \cap F^b = \emptyset. \end{aligned}$$

Thus there exist $\sigma \in [0,1]$ and $u \in B_1$ such that $\phi(1,u) = (\sigma, 0)$. In fact, by the properties of ϕ, the homotopy $\Phi : [0,1] \times [-1,1] \times B_1 \to \mathbb{R} \times \mathrm{X}_1$ defined by $\Phi(t,s,u) = \phi(t,u) + (s,0)$ is such that $\Phi(0,\cdot)$ is the identity and $(0,0) \notin \Phi([0,1] \times \partial([-1,1] \times B_1))$; hence, by homotopy invariance, we obtain $\deg(\Phi(1,\cdot), [-1,1] \times B_1, (0,0)) = 1$ and thus there exist $\sigma \in [0,1]$ and $u \in B_1$ such that $\phi(1,u) = (\sigma, 0)$.

Finally, this fact implies that there exists $u \in B_1$ such that $\eta(1,u) \in B_2$ and this concludes the proof. □

We remark that, as usual, the hypothesis of finite dimension for one subspace can be replaced by a suitable hypothesis on ∇F.

Theorem 4.14 in the case $\alpha < \beta$ and Theorem 6.7 in the case $\beta < \alpha$ provide the existence of at least two critical points for the functional f_t. These theorems are based on the fact that f_t "separates two linked spheres with suitable bounds" (see Theorems 4.13 and 6.6) and on the following theorem.

THEOREM 8.2 ("links and bounds"). *Let X be a Hilbert space which is the topological direct sum of subspaces X_1 and X_2. Let $F \in C^1(\mathrm{X}, \mathbb{R})$. Moreover, suppose that there exist $e \in \mathrm{X}_1$, $e \neq 0$, and $\rho_1, \rho_2 > 0$ such that*

(a) $|\rho_2 - \rho_1| < \|e\| < \rho_2 + \rho_1$,
(b) $\sup_{\partial B_1} F < \inf_{\partial B_2} F$,
(c) $-\infty < a = \inf_{B_2} F$ *and* $b = \sup_{B_1} F < \infty$,

where B_1 denotes the ball in X_1 centered at 0 with radius ρ_1, ∂B_1 is its boundary in X_1 and B_2 denotes the ball in $\mathrm{X}_2 \oplus \mathrm{span}(e)$ centered at e with radius ρ_2 and ∂B_2 is its boundary in $\mathrm{X}_2 \oplus \mathrm{span}(e)$. If $(\mathrm{P.S.})_c$ holds for any $c \in [a,b]$ and at least one of the two spaces X_1 and X_2 has finite dimension, then there exist at least two critical levels c_1 and c_2 for the functional F such that

$$\inf_{B_2} F \leq c_2 \leq \sup_{\partial B_1} F < \inf_{\partial B_2} F \leq c_1 \leq \sup_{B_1} F.$$

PROOF. We assume that $\dim X_1 < \infty$. The existence of c_1 follows by the classical linking theorem (see [36]).

We are going to show the existence of the critical value c_2. By contradiction suppose that every c between $a' = \sup_{\partial B_1} F$ and $a = \inf_{B_2} F$ is a regular value. Then $F^{a-\varepsilon}$ is a strong deformation retract of $F^{a'}$ for ε small enough by means of the deformation $\eta : [0,1] \times F^{a'} \to F^{a'}$ such that

$$\begin{aligned} \eta(0,u) &= u && \forall u \in F^{a'}, \\ \eta(t,u) &= u && \forall t \in [0,1],\ \forall u \in F^{a-\varepsilon}, \\ \eta(1,u) &\in F^{a-\varepsilon} && \forall u \in F^{a'}. \end{aligned}$$

Since $\partial B_1 \subset F^{a'}$ and $B_2 \cap F^{a-\varepsilon} = \emptyset$, we have $\eta(1, \partial B_1) \cap B_2 = \emptyset$. We will get the contradiction by showing $\eta(1, \partial B_1) \cap B_2 \neq \emptyset$.

Let $P : \mathrm{X} \to \mathrm{span}(e) \oplus \mathrm{X}_2$ and $Q : \mathrm{X} \to \mathrm{X}^*$ be the orthogonal projections, where X^* is such that $\mathrm{X}_1 = \mathrm{X}^* \oplus \mathrm{span}(e)$. Let $\phi : [0,1] \times \partial B_1 \to \mathrm{H}_1$ be a continuous map defined by $\phi(t,u) = Q\eta(t,u) + \|P\eta(t,u) - e\| e/\|e\|$. First of all $\rho_2 e/\|e\| \notin \mathrm{Im}(\phi)$. Otherwise we will find $\eta(t,u) \in \partial B_2$ with $u \in \partial B_1$ and $t \in [0,1]$, while (b) implies $\partial B_2 \cap F^{a'} = \emptyset$.

Consider now the continuous map $\Phi : [0,1] \times \partial B_1 \to S$ defined by $\Phi(t,u) = \pi\phi(t,u)$, where $S = \partial B(\rho_2 e/\|e\|, 1) \cap \mathrm{H}_1$ and $\pi : \mathrm{H}_1 \setminus \{\rho_2 e/\|e\|\} \to S$ is the radial projection. By well known properties of degree, we get $\deg(\Phi(0,\cdot)) \neq 0$, since if $w \in \mathrm{span}(e) \cap S$, by (a) there exists a unique $u \in \partial B_1$ such that $\Phi(0,u) = w$. By the homotopy invariance property, if $\deg(\Phi(0,\cdot)) \neq 0$ then $\deg(\Phi(1,\cdot)) \neq 0$ and therefore $\Phi(1, \partial B_1) = S$. Thus there exists $u \in \partial B_1$ such that $\phi(1,u) = \sigma e/\|e\|$ with $\sigma < \rho_2$ and hence $\eta(1,u) \in B_2$. This completes the proof. □

In order to compare this theorem and the classical result of Rabinowitz (see [36]), suppose (to fix ideas) that $\dim \mathrm{X}_1 < \infty$. Then the classical linking theorem gives the existence of the critical level c_1, but not of c_2.

Also in the case of Theorem 8.2 we can remark that the assumption that X_1 or X_2 has finite dimension can be replaced by a suitable assumption on ∇F.

Finally, let us remark that in Theorem 4.13 the sphere ∂B_1 is replaced by Σ_i^- and in Theorem 6.6 the sphere ∂B_2 is replaced by Σ_i^+. It is obvious that this modification does not change the assertion of Theorem 8.2.

The theorems on existence of three solutions (4.7) and Theorem 6.10 are based on Theorems 4.16 and 6.9 and on the following abstract variational theorem, where a functional that "separates pairs of spheres in dimensional scale" with suitable bounds is considered.

It is useful to make the following definition.

DEFINITION 8.3. Let X be a Hilbert space, Y a subspace, $\rho > 0$ end $e \in X\backslash Y$, $e \neq 0$. Set

$$\begin{aligned}
B_\rho(Y) &= \{x \in Y \mid \|x\|_X \leq \rho\},\\
S_\rho(Y) &= \{x \in Y \mid \|x\|_X = \rho\},\\
\Delta_\rho(e, Y) &= \{x = \sigma e + v \mid \sigma \geq 0,\ v \in Y,\ \|\sigma e + v\|_X \leq \rho\},\\
\Sigma_\rho(e, Y) &= \{x = \sigma e + v \mid \sigma \geq 0,\ v \in Y,\ \|\sigma e + v\|_X = \rho\}\\
&\qquad \cup \{v \mid v \in Y,\ \|v\|_X \leq \rho\}.
\end{aligned}$$

THEOREM 8.4 ("links in scale and bounds"). *Let X be a Hilbert space which is the topological direct sum of four subspaces X_0, X_1, X_2 and X_3. Let $F \in C^1(X, \mathbb{R})$. Moreover, assume that*

(a) $\dim X_i < \infty$ *for* $i = 0, 1, 2$;

(b) *there exist* $\rho, R > 0$ *and* $e \in X_2$, $e \neq 0$, *such that*

$$\rho < R \quad \text{and} \quad \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F \qquad \text{(first link)};$$

(c) *there exist* $\rho', R' > 0$ *and* $e' \in X_1$, $e' \neq 0$, *such that*

$$\rho' < R' \quad \text{and} \quad \sup_{S'_\rho(X_0 \oplus X_1)} F < \inf_{\Sigma'_R(e', X_2 \oplus X_3)} F \qquad \text{(second link)};$$

(d) $R \leq R'$ (*hence* $\Rightarrow$ $\Delta_R(e, X_3) \subset \Sigma'_R(e', X_2 \oplus X_3)$);

(e) $-\infty < a = \inf_{\Delta'_R(e, X_2 \oplus X_3)} F$ (*lower bound*);

(f) (P.S.)$_c$ *holds for any* $c \in [a, b]$, *where* $b = \sup_{B_\rho(X_0 \oplus X_1 \oplus X_2)} F$.

Then there exist three critical levels c_1, c_2 and c_3 for the functional F such that

$$\begin{aligned}
a \leq c_3 \leq \sup_{S'_\rho(X_0 \oplus X_1)} F < \inf_{\Sigma'_R(e', X_2 \oplus X_3)} F \leq \inf_{\Delta_R(e, X_3)} F\\
\leq c_2 \leq \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F \leq c_1 \leq b.
\end{aligned}$$

PROOF. From (a), (b), (e) and (f) using Theorem 8.2, substituting $S_\rho(X_0 \oplus X_1 \oplus X_2)$ for the sphere ∂B_1 and $\Sigma_R(e, X_3)$ for the sphere ∂B_2 we obtain the existence of critical values c_1 and c_2 of F such that

$$\inf_{\Delta_R(e, X_3)} F \leq c_2 \leq \sup_{S_\rho(X_0 \oplus X_1 \oplus X_2)} F < \inf_{\Sigma_R(e, X_3)} F \leq c_1 \leq \sup_{B_\rho(X_0 \oplus X_1 \oplus X_2)} F.$$

From (a), (c), (e) and (f) using Theorem 8.2, substituting $S'_\rho(X_0 \oplus X_1)$ for ∂B_1 and $\Sigma'_R(e', X_2 \oplus X_3)$ for ∂B_2 we obtain the existence of critical values $\bar{c}_1$ and $\bar{c}_2$ of F such that

$$\inf_{\Delta'_R(e', X_2 \oplus X_3)} F \leq \bar{c}_1 \leq \sup_{S'_\rho(X_0 \oplus X_1)} F < \inf_{\Sigma'_R(e', X_2 \oplus X_3)} F \leq \bar{c}_2 \leq \sup_{B'_\rho(X_2 \oplus X_3)} F.$$

By hypothesis (d) it follows that

$$\inf_{\Sigma'_R(e',X_2\oplus X_3)} F \leq \inf_{\Delta_R(e,X_3)} F,$$

and the assertion follows. □

REMARK 8.5. Of course we can consider a dimensional scale of h pairs of linked spheres S_i and Σ_i separated by F (that is, $\sup_{S_i} F < \inf_{\Sigma_i} F$) with $\dim S_{i+1} < \dim S_i < \infty$ for $i = 1, \dots, h$, with $\Delta_i \subset \Sigma_{i+1}$ (where Δ_i is the convex generated by Σ_i) and with $-\infty < \inf_{\Delta_h} F$. If F satisfies the Palais-Smale condition, then F has at least $h+1$ critical values.

Finally, we point out that in Theorem 4.17 the notations S_i and Σ_i are interchanged.

9. The existence of four solutions and some other results

We are able to draw a "submap" of the map of the Introduction, which also implies an "alternative theorem".

REMARK 9.1. We recall that in [23], [24] and [8] some regions of the (α, β) plane, to which at least three solutions of (P_t) for t large enough correspond, are shown. In substance they show that if λ_{j-1}, λ_j and λ_{k+1} are three consecutive (possibly multiple) eigenvalues $\lambda_{j-1} < \lambda_j = \dots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$ and either $(\alpha, \beta) \in]\lambda_k, \lambda_{k+1}[\times]\lambda_{j-1}, \lambda_j[$ with $\beta < \mu_j(\alpha)$ or $(\alpha, \beta) \in]\lambda_{j-1}, \lambda_j[\times]\lambda_k, \lambda_{k+1}[$ with $\beta > \nu_k(\alpha)$, under suitable conditions on g, then the problem (P_t) has at least three solutions if t is large enough. (In these cases the nonlinearity g does not depend on x and a bound of $(g(s_1) - g(s_2))/(s_1 - s_2)$ with respect to the eigenvalues λ_{j-1} and λ_{k+1} is required.)

A different statement was proved in [29].

THEOREM 9.2. *Let $j \geq 2$ and $\lambda_{j-1} < \lambda_j$. Assume (g), (G) and:*

(a) *there exists $k > 0$ such that*

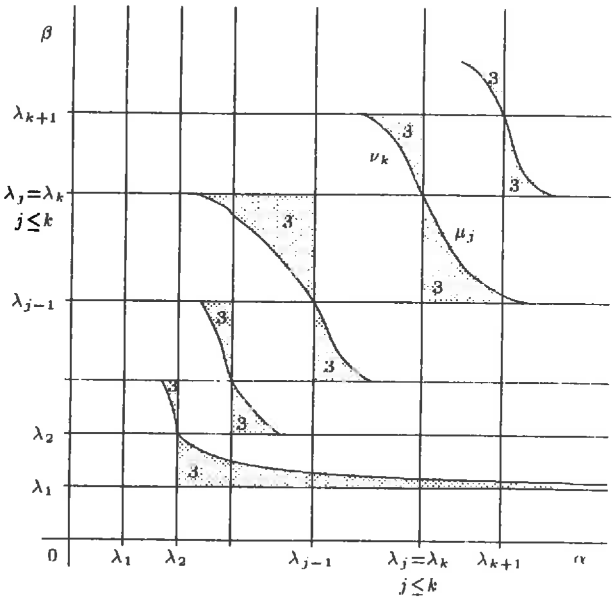
$$G(x,s) = \frac{\alpha}{2}(s^+)^2 + \frac{\beta}{2}(s^-)^2 + G_0(x,s)$$

a.e. in Ω and for $|s| \geq k$, where $|G_0(x,s)| \leq c_0(x)$ with $c_0 \in L^1(\Omega)$; moreover, $\alpha > \lambda_j$ and $\lambda_{j-1} < \beta < \mu_j(\alpha)$;

(b) *there exist $\overline{\beta}$ and γ such that*

$$\lambda_{j-1} < \overline{\beta} \leq \frac{g(x,s_1) - g(x,s_2)}{s_1 - s_2} \leq \gamma$$

a.e. in Ω, and for any s_1, s_2 with $s_1 \neq s_2$.



Then, for t large enough, the functional f_t has at least two critical values and at least three critical points, so the problem (P_t) has at least three solutions.

PROOF. By (b) the set

$$\mathcal{M}(f_t) = \{u \in H_0^1(\Omega) \mid f_t'(u)(v) = 0 \ \forall v \in \mathrm{H}_j\}$$

is the graph of a Lipschitz function $\gamma_t : \mathrm{H}_j^\perp \to \mathrm{H}_j$.

By Lemma 7.5 we see that $Q_{\alpha,\beta}(u) > 0$ for $u \in \mathcal{M}(\alpha,\beta) \setminus \{0\}$, hence we obtain easily

$$\lim_{\substack{u \in \mathcal{M}(f_t) \\ \|u\| \to \infty}} f_t(u) = \infty.$$

Thus $\inf_{\mathcal{M}(f_t)} f_t > -\infty$.

Now let $i \geq j$ be such that $\lambda_i < \alpha \leq \lambda_{i+1}$. By Lemma 6.3 we can easily deduce that for t positive and large enough,

$$\sup_S f_t < \inf_{\mathcal{M}^+(f_t)} f_t,$$

where

$$S = \{u + \gamma_t(u) \mid u \in \operatorname{span}(e_j, \dots, e_i), \ \|u\| = \rho\}$$

and

$$\mathcal{M}^+(f_t) = \mathcal{M}(f_t) \cap (\mathrm{H}_j \oplus \mathrm{H}_i^\perp).$$

Finally, it is obvious that f_t restricted to the manifold $\mathcal{M}(f_t)$ satisfies (P.S.) (see Lemma 7.5).

Thus we get the existence of two critical points, which are at levels less than $\sup_S f_t$, since S is not contractible in $\mathcal{M}(f_t) \setminus \mathcal{M}^+(f_t)$. Moreover, there is also a critical level greater than $\sup_S f_t$, since the functional f_t restricted to $\mathcal{M}(f_t)$ has a "saddle" point (see Lemma 6.3). □

A similar tool was used in [32].

The previous theorem ensures the existence of three solutions when $\alpha > \beta$. Of course a similar result holds when $\alpha < \beta$.

REMARK 9.3. We point out that in the three solutions region $\{(\alpha,\beta) \mid \lambda_j < \alpha, \lambda_{j-1} < \beta < \mu_j(\alpha)\}$ it is possible to find pairs (α,β) such that many eigenvalues fall between α and β. This is true, for example, for the unbounded region $\{(\alpha,\beta) \mid \lambda_1 < \alpha,\ \lambda_1 < \beta < \mu_2(\alpha)\}$. An analogous statement holds for the function ν_k if $\lambda_k < \lambda_{k+1}$. This type of result was obtained in [34], where weaker assumptions than (b) of Theorem 9.2 are required.

Theorem 9.2 and the theorems on existence of one solution (see Theorem 3.6) and of two solutions (see Theorems 4.14 and 6.7) allow us to give the following version of the "alternative theorem", recalled in the introduction, without the assumption $(\alpha,\beta) \notin \Sigma_\Omega$.

To this end we introduce an exceptional one-dimensional subset of the (α,β) plane. For any eigenvalue λ_k such that $\lambda_k < \lambda_{k+1}$, with $k \geq 2$, we set

$$N_k = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \beta = \nu_k(\alpha) < \lambda_{k+1},\ \alpha \leq \lambda_{k+1}\}$$

and for any eigenvalue λ_j such that $\lambda_{j-1} < \lambda_j$, with $j \geq 2$

$$M_j = \{(\alpha,\beta) \in \mathbb{R}^2 \mid \lambda_{j-1} \leq \beta = \mu_j(\alpha),\ \alpha \geq \lambda_{j-1}\}.$$

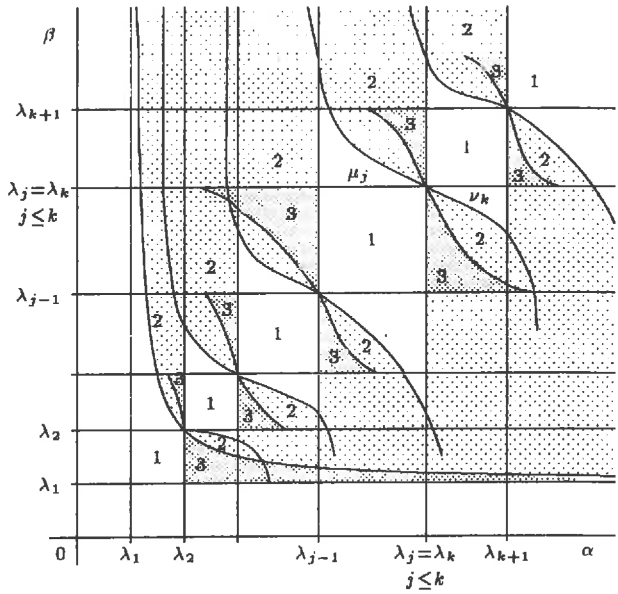
Finally, we set

$$T_\Omega = \Big(\bigcup_{j\geq 2} M_j\Big) \cup \Big(\bigcup_{k \geq 2} N_k\Big).$$

COROLLARY 9.4. *Assume* (2.6) *with* $\alpha,\beta > \lambda_1$ *and* $(\alpha,\beta) \notin T_\Omega$. *Moreover, assume that there exist* $\overline{\alpha}$ *and* $\overline{\beta}$ *such that a.e. in* Ω *and for any* s_1, s_2 *with* $s_1 \neq s_2$

$$\lambda_{j-1} < \overline{\beta} \leq \frac{g(x,s_1) - g(x,s_2)}{s_1 - s_2} \leq \overline{\alpha} < \lambda_{k+1}, \tag{9.5}$$

where $\lambda_{j-1} = \max\{\lambda_i \mid \alpha,\beta > \lambda_i\}$ *and* $\lambda_{k+1} = \min\{\lambda_i \mid \alpha,\beta < \lambda_i\}$. *Then the following alternative holds for the equation* (P_t): *either* (P_t) *has at least two*



The "alternative map"

solutions for t large positive and t small negative, or (P_t) has at least one solution for t large positive and three solutions for t small negative, or (P_t) has at least three solutions for t large positive and one solution for t small negative.

PROOF. The assertion follows from Theorem 9.2 by replacing the function $g(\cdot, s)$, when s is negative and small enough, with the function $-g(\cdot, -s)$. □

We emphasize that assumption (9.5) is stronger than necessary and it can be dropped or reduced with regard to the pair (α, β). More precisely, if, for example, $\lambda_k < \alpha < \lambda_{k+1}$ and $\nu_k(\alpha) < \beta < \lambda_k$ then it is enough to assume the inequality

$$\frac{g(x, s_1) - g(x, s_2)}{s_1 - s_2} \leq \overline{\alpha} < \lambda_{k+1}.$$

Moreover, if the pair (α, β) does not belong to any "curvilinear triangle", like those considered in Theorem 9.2, then Theorem 9.5 can be removed.

Now we will give a theorem on existence of four critical points for the functional f_t. In order to simplify the notation, we make the following definition.

DEFINITION 9.6. If λ_{j-1}, λ_j and λ_{k+1} are three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$ then we set (see Theorem 6.10)

$$R_{j,k} = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \lambda_1 < \beta < \mu_2(\alpha),\ (\alpha, \beta) \in F_{j-1} \cap F_k,\ \beta < \nu_{j-1}(\alpha)\}.$$

THEOREM 9.7 (existence of four solutions). *Let λ_{j-1}, λ_j and λ_{k+1} be three consecutive (possibly multiple) eigenvalues: $\lambda_{j-1} < \lambda_j = \ldots = \lambda_k < \lambda_{k+1}$ with $k \geq j \geq 2$. Assume:*

(a) *(g) and (G, α, β) with $(\alpha, \beta) \in R_{j,k}$;*

(b) *there exist $\overline{\beta}$ and γ such that a.e. in Ω and for any s_1, s_2 with $s_1 \neq s_2$*

$$\lambda_1 < \overline{\beta} \leq \frac{g(x, s_1) - g(x, s_2)}{s_1 - s_2} \leq \gamma.$$

Then there exists an open bounded subset $\widetilde{R}_{j,k}$ contained in $R_{j,k}$ such that if $(\alpha, \beta) \in \widetilde{R}_{j,k}$ and if t is large enough, then the functional f_t has at least three critical values and at least four critical points, hence the problem (P_t) has at least four solutions.

More precisely, we will show that the region $\widetilde{R}_{j,k}$ is like the dark one sketched in the figure in the Introduction. Indeed, we will observe that

$$(9.8) \qquad \begin{cases} \text{for each } \beta \in]\lambda_1, \mu_2(\lambda_k)[\text{ there exists } \delta > 0 \text{ such that} \\ \text{if } \alpha \in]\lambda_k, \lambda_k + \delta[\text{ then } (\alpha, \beta) \in \widetilde{R}_{j,k}. \end{cases}$$

PROOF. By (b) the set

$$\mathcal{M}(f_t) = \{u \in H_0^1(\Omega) \mid f_t'(u)(v) = 0 \ \forall v \in \mathrm{H}_1\}$$

is the graph of a Lipschitz function $\gamma_t : \mathrm{H}_1^\perp \to \mathrm{H}_1$. By Lemma 7.5 we find that $Q_{\alpha,\beta}(u) > 0$ for $u \in \mathcal{M}(\alpha, \beta) \setminus \{0\}$, hence by Lemma 9.12 we have

$$\lim_{\substack{u \in \mathcal{M}(f_t) \\ \|u\| \to \infty}} f_t(u) = +\infty.$$

Thus $\inf_{\mathcal{M}(f_t)} f_t > -\infty$. Hence f_t has a minimum point on $\mathcal{M}(f_t)$. On the other hand, by Theorems 6.10 and 8.4, if t is large enough f_t has three critical values c_1, c_2 and c_3; more precisely, there exist $\rho_{j-1}, \rho_k > 0$, Σ_{j-1}^+ and Σ_k^+ (see notation in Theorem 6.10) such that

$$\begin{aligned} c_3 &\leq \sup_{\substack{z \in \mathrm{H}_{j-1} \\ \|z\| = \rho_{j-1} s_t}} f_t(s_t e_1 + z) < \inf_{\Sigma_{j-1}^+} f_t \leq \inf_{\Delta_k^+} f_t \leq c_2 \\ &\leq \sup_{\substack{v \in \mathrm{H}_k \\ \|v\| = \rho_k s_t}} f_t(s_t e_1 + v) < \inf_{\Sigma_k^+} f_t \leq c_1 \\ &\leq \sup_{\substack{v \in \mathrm{H}_k \\ \|v\| \leq \rho_k s_t}} f_t(s_t e_1 + v), \end{aligned}$$

where

$$\Delta_k^+ = \{\sigma e + w \mid w \in \mathrm{H}_k^\perp,\ \sigma \geq 0,\ \|\sigma e + w\| \leq \sigma_k t\}$$

(Σ_k^+ is its boundary). Moreover, it is trivial that f_t restricted to the manifold $\mathcal{M}(f_t)$ satisfies (P.S.) (see Lemma 7.5). Finally, by Lemma 9.9, if (α, β) belongs to a suitable set $\widetilde{R}_{j,k}$ ($\widetilde{R}_{j,k} = A_{j-1,k} \cap R_{j,k}$, see (9.9) and (9.6)) then there exists $\rho > 0$ such that

$$\sup_{S(\rho)} f_t < \inf_{w \in \mathrm{H}_{j-1}^\perp} f_t(s_t e_1 + w) < \inf_{\mathcal{M}(f_t) \cap (\mathrm{H}_1 \oplus \mathrm{H}_{j-1}^\perp)} f_t,$$

where $S(\rho) = \{z + \gamma_t(z) \mid z \in \mathrm{H}_1^\perp,\ \|z\| = \rho\}$.

Now by using Lemma 6.6 and taking into account $(\alpha, \beta) \in F_{j-1}$ and $\beta < \nu_j(\alpha)$ we obtain inequalities

$$\sup_{S(\rho)} f_t < \inf_{\Sigma_{j-1}^+} f_t \leq c_2.$$

Finally, there exist two critical points which are at level less than $\sup_{S(\rho)} f_t$, since $S(\rho)$ is not contractible in $\mathcal{M}(f_t) \setminus (\mathrm{H}_1 \oplus \mathrm{H}_{j-1}^\perp)$. □

LEMMA 9.9. *There exists an open subset $A_{j-1,k}$ contained in $F_{j-1} \cap F_k$ such that if $(\alpha, \beta) \in A_{j-1,k}$ then there exists $\rho > 0$ so that*

$$\sup_{S(\rho)} f_t < \inf_{w \in \mathrm{H}_{j-1}^\perp} f_t(s_t e_1 + w),$$

where

$$S(\rho) = \{z + \gamma_t(z) \mid z \in \mathrm{H}_1^\perp,\ \|z\| = \rho\}.$$

PROOF. By definition of γ_t we get

$$f_t(w + \gamma_t(w)) \geq f_t(w + s_t e_1) \geq \inf_{w \in \mathrm{H}_{j-1}^\perp} f_t(s_t e_1 + w).$$

Moreover, by the definition of n_{j-1} (see Definition 6.1), arguing as in Lemmas 4.5 and 6.3, we see that for any $\varepsilon > 0$ there exist d_ε and ρ_ε such that

$$\inf_{w \in \mathrm{H}_{j-1}^\perp} f_t(s_t e_1 + w) \geq f_t(s_t e_1) + s_t^2(n_{j-1}(\alpha, \beta) - 3\varepsilon\lambda_1 - 2\varepsilon\rho_\varepsilon^2) - d_\varepsilon. \tag{9.10}$$

Moreover, since the Lipschitz constant L of γ_t does not depend on t and

$$\|\gamma_t(0) - s_t e_1\| \leq \rho_{j-1} s_t$$

(ρ_{j-1} is defined for F_{j-1}, see Definition 6.1), if $\|u\| = \rho_{j-1}s_t$ then

$$\rho_{j-1}^2 s_t^2 \leq \|u + \gamma_t(u) - s_t e_1\|^2 \leq (2L^2+3)\rho_{j-1}^2 s_t^2.$$

Now if $v = u + \gamma_t(u) - s_t e_1$, then arguing as in Lemmas 4.5 and 6.3, we see that for any $\varepsilon > 0$ there exists d_ε such that

$$(9.11)\quad f_t(s_t e_1 + v) - f_t(s_t e_1) \leq s_t^2\left(\widetilde{N}_{j-1}(\rho_{j-1},\alpha,\beta) + 3\varepsilon\lambda_1 + 2\varepsilon\left\|\frac{v}{s_t}\right\|^2\right) + d_\varepsilon,$$

where

$$\widetilde{N}_{j-1}(\rho,\alpha,\beta) = \sup_{\substack{v\in \mathrm{H}_{j-1}\\ \rho\leq\|v\|\leq\sqrt{2L^2+3}\rho}} \left\{Q_\alpha(v) + \frac{1}{2}(\alpha-\beta)\int_\Omega ((e_1+v)^-)^2\right\}.$$

Now we define the set

$$A_{j-1,k} = \{(\alpha,\beta)\in\mathbb{R}^2 \mid \beta<\alpha,\ \exists\rho>0 \text{ with } \widetilde{N}_{j-1}(\rho,\alpha,\beta) < n_{j-1}(\alpha,\beta)\}.$$

By the properties of n_{j-1} (see Lemma 7.1) and by the fact that

$$\lim_{\rho\to 0}\frac{\widetilde{N}_{j-1}(\rho,\lambda_j,\beta)}{\rho^2} = \frac{1}{2}\left(1-\frac{\lambda_j}{\lambda_{j-1}}\right) < 0,$$

arguing as in Lemma 6.2 we find that $A_{j-1,k}$ is an open set with the property (9.8).

Finally, if $(\alpha,\beta)\in A_{j-1,k}$ we can choose ε small enough so that the coefficient of s_t^2 in (9.11) is less than the one in (9.10). Therefore if t is large enough, the assertion follows. □

LEMMA 9.12. *If $\alpha > \lambda_1$ and $\lambda_1 < \beta < \mu_2(\alpha)$ then*

$$\lim_{\substack{u\in\mathcal{M}(f_t)\\ \|u\|\to\infty}} f_t(u) = +\infty,$$

where

$$\mathcal{M}(f_t) = \{u \in H_0^1(\Omega) \mid f_t'(u)(v) = 0\ \forall v\in \mathrm{H}_1\}.$$

PROOF. Let us recall that the manifold (see Lemma 7.5)

$$\mathcal{M}((\alpha,\beta)) = \{u\in H_0^1(\Omega)\mid Q'_{\alpha,\beta}(u)(v) = 0\ \forall v\in\mathrm{H}_1\},$$

is the graph of a Lipschitz function $\gamma : \mathrm{H}_1^\perp \to \mathrm{H}_1$. By definition of γ_t we get

$$f_t(z+\gamma_t(z)) \geq f_t(z+\gamma(z)) = Q_{\alpha,\beta}(z+\gamma(z)) - \int_\Omega G_0(x,u) + t\int_\Omega e_1\gamma(z).$$

By Lemma 4.1, since γ is Lipschitz, it is enough to show that

$$\lim_{\substack{z \in H_1^\perp \\ \|z\| \to \infty}} \frac{Q_{\alpha,\beta}(z + \gamma(z))}{\|z\|^2} > 0.$$

Arguing by contradiction and taking into account that (see Lemma 7.5)

$$Q_{\alpha,\beta}(u) > 0 \qquad \text{for any } u \in \mathcal{M}(\alpha, \beta) \setminus \{0\},$$

we can easily obtain the assertion. □

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