Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 4, 1994, 253–287

GEOSTROPHIC ASYMPTOTICS OF THE PRIMITIVE EQUATIONS OF THE ATMOSPHERE

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Dedicated to Jean Leray

Introduction

The various components of the motion and state of the atmosphere are governed by the general thermodynamic and hydrodynamic equations of a compressible fluid. Thanks to the fact that the vertical scale of the atmosphere is much smaller than the horizontal one, the vertical momentum equation of the atmosphere can be well approximated by the hydrostatic equation. The resulting system is called the *primitive equations* of the atmosphere (*the PEs*). By retaining in the vertical momemtum equation the viscosity terms, one can also obtain the *primitive equations with vertical viscosity* (the PEV^{2}s) of the atmosphere. These systems of equations are now considered to be the fundamental equations of the atmosphere and serve as starting points of dynamic meteorology and climatology. The mathematical and theoretical numerical analysis of these equations have been conducted in our previous articles (see J.-L. Lions, R. Temam and S. Wang [24] and S. Wang [39]).

Another important characteristic of the atmosphere is the rotating effect of the earth. The rotation of the earth introduces an apparent force into the equations of the atmosphere, i.e. the Coriolis force. The Coriolis force or acceleration

This work was partially supported by the National Science Foundation under Grant NSF-DMS9400615, by the Department of Energy under grant DOE-DE-FG02-92ER25120, and by the Research Fund of Indiana University.

is important for motion whose period is not very short as compared to the reference temporal scale of the order of one day. By scale analysis, for the motion with large horizontal spatial scale and with temporal scale comparable with the reference temporal scale, we can see that the dominant terms in the horizontal momentum equations are generally the pressure gradient and the Coriolis terms. Reducing the general hydrodynamic momentum equations to include only these terms and the hydrostatic equation mentioned before, we obtain the following:

$$2\Omega \cos\theta\, k \times v \; + \frac{1}{\rho}\nabla p = 0, \tag{0.1}$$

where v is the horizontal velocity of the air, p the pressure, ρ the density, Ω the angular velocity of the earth, θ the colalitude, z is the height above the sea level, and ∇ is the horizontal gradient. Equations (0.1) are termed the ***geostrophic wind relations***, the geostrophic wind being the hypothetical wind resulting from a perfect balance between the Coriolis and pressure gradient forces.

The emphasis on the importance of the rotation effects of the earth and their study should be traced back to the work of Laplace [17] in the eighteenth century. He recognized the importance of rotation in his theory of the tides and developed the appropriate equations, describing the rotation effects. However, the question of how a fluid adjusts in a uniformly rotating system was not completely discussed until the time of Rossby (1938), when Rossby considered the process of adjustment to the geostrophic equilibrium. This process is now referred to as the *Rossby adjustment.* Roughly speaking the Rossby adjustment process explains why the atmosphere and ocean are always close to geostrophic equilibrium, for if any force tries to upset such an equilibrium, the gravitational restoring force quickly restores a near geostrophic equilibrium.

The main objective of this article is to justify from the mathematical standpoint the Rossby adjustment, and to obtain a systematic asymptotic analysis related to the Rossby adjustment process, leading to the quasi-geostrophic equations. The quasi-geostrophic equations (abbreviated as QGs) were first introduced by J. Charney [4].

In this article, we intend to derive and study the Geostrophic and Quasi-geostrophic equations. The derivation is made by asymptotic analysis from the PEs equations, first in a formal way in Section 2, then in a more rigorous, albeit not complete way in Section 4.

We start in Section 1 by recalling the general atmospheric equations (namely the primitive equations, PEs), based on the hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g. \tag{0.2}$$

This equation allows the utilization of the p-coordinate system, where the vertical coordinate is the pressure p, whereas the horizontal coordinates are the colatitude θ and the eastward longitude φ. The resulting equations, namely the PEs, are given by (1.1)–(1.4) in Section 1 of this article.

Then we nondimensionalize the PEs equations in a way appropriate for the asymptotic analysis we have in view, which we call here the *Rossby asymptotics.* The following two points are important here:

(i) Noticing that the Coriolis parameter $f = 2\cos\theta$ is close to zero near the equator, we restrict ourselves to the mid-latitude regions, where the Coriolis parameter is bounded away from zero, and the Coriolis force is the dominant force. With this in mind, we form a nondimensional parameter called the *Rossby number:*

$$\varepsilon = \mathrm{Ro} = \frac{U}{L\widetilde{f}^0}, \qquad \widetilde{f}^0 = 2\Omega\cos\theta_0, \tag{0.3}$$

where U is the typical horizontal velocity and L is the typical horizontal length scale, namely the east-west size of the domain under consideration, and θ_0 is the colatitude value of a latitude in the specified mid-latitude region. The Rossby number represents essentially the ratio of the characteristic scales of the acceleration U^2/L to the characteristic Coriolis force $2\Omega\cos\theta U$.

(ii) Another important character needed for the scaling concerns the Coriolis parameter $f = 2\Omega\cos\theta$. Restricting ourselves to the mid-latitude regions and using the Taylor expansion, we have

$$\frac{f}{\Omega} = 2\cos\theta = 2\cos\theta_0\left[1 - \frac{\sin\theta_0}{\cos\theta_0}(\theta-\theta_0) + \dots\right] \tag{0.4}$$

θ_0 being a specific (fixed) colatitude in the mid-latitude region. The essential assumption for the Rossby asymptotics is to consider the mid-latitude region such that

$$|\theta - \theta_0| \leq \frac{L}{a} \approx \varepsilon = \mathrm{Ro} = \frac{U}{L\Omega}. \tag{0.5}$$

Therefore we can expand f/Ω as

$$f/\Omega = f^0 + f^1\varepsilon + f^2\varepsilon^2 + \dots \tag{0.6}$$

Taking the above discussions into consideration, we obtain a systematic (formal) Rossby asymptotics in Section 2 of this article. The zeroth order approximation of the PEs provides the geostrophic equations (cf. (2.3) in Section 2);

while the first order approximation establishes the quasi-geostrophic equations. Let us point out here that the derivation of the geostrophic and quasi-geostrophic equations is essentially classical (see e.g. [4], [13], [31], [33] and [34]). Namely we introduce a small parameter ε (= Ro here) and we "filter" the equations accordingly; then we (try to) develop an *asymptotic analysis* based on this formulation. Here we depart from the classical approach since we start from the *primitive equations* and proceed in a systematic way. As we know the viscosity of the atmosphere has to be taken into consideration if the prediction period extends to three or four days (cf. [5] and the discussion about the PEV^{2}s below). The geostrophic asymptotics for local strong solutions for the 3D Navier-Stokes equations of an incompressible fluid were studied from the mathematical point of view in [3].

We also study in Section 2 (and then in Section 4) the Rossby asymptotics for the primitive equations with vertical viscosity (PEV^{2}s). Indeed, emphasizing the viscosity term in the vertical momentum equations, i.e. replacing the hydrostatic equation by the hydrostatic equation with vertical viscosity, we also studied in [24] the primitive equations with vertical viscosity (the PEV^{2}s). The Rossby asymptotics of the PEV^{2}s leads to the *same* geostrophic and quasi-geostrophic equations as the PEs. Roughly speaking, from the physical point of view, the geostrophic and the quasi-geostrophic equations filter out the high frequency gravitational waves, and provide models with vanishing vertical velocity. Therefore it is very natural to obtain the same geostrophic and quasi-geostrophic equations from both the PEs and the PEV^{2}s.

Whereas the previous asymptotic expansion were performed in a formal way, directly from the equations, we aim at performing in Sections 3 and 4 a rigorous asymptotic expansion in the appropriate function spaces. Some results are rigourosly proven, others lead to open questions (and reasonable conjectures).

In Section 3 we recall the functional formulation of the PEs and PEV^{2}s and the corresponding results of existence of solutions; these results are essentially borrowed from [24]. Then we introduce the functional framework for the geostrophic and quasi-geostrophic equations which appeared in Section 2; unusual and interesting function spaces appear, raising new functional analysis questions. In particular, we show how the kernel and the range of certain linear operators are related to certain auxiliary unknown functions appearing as Lagrange multipliers in the equations. We also obtain a decomposition of the function space for the PEs into the function space of geostrophic motions and that of nongeostrophic motions. This decomposition shows how the general motion and state of the atmosphere are related to the geostrophic and non-geostrophic motions and states. Finally, we establish the existence of solutions of the quasi-geostrophic equations, obtained in a standard manner by Galerkin approximation.

In Section 4 we write the asymptotic expansion of the PEs and the PEV^{2}s, in the function space, starting this time from the weak (functional) formulation of the equations. We show in a rigorous way that the solutions of the PEs and the PEV^{2}s converge, as $\varepsilon \to 0$, to functions satisfying the geostrophic relations. We are not able to show the similar result for the QGs (next terms in the asymptotic expansions); this difficulty which is related to the appearance of an antisymmetric penalization operator (see Remark 4.3) may be caused perhaps by persistent oscillations. We prove nevertheless some partial results, namely the corresponding convergences for the linearized PEs (and PEV^{2}s) equations and for the stationary nonlinear equations.

A huge number of interesting papers are devoted to the questions encountered in the present paper. We confine ourselves to add to the Bibliography mentioned in the text, the papers by A. F. Bennett and P. E. Kloeden [1], [2], P. Constantin, A. Majda and G. E. Tabak [9], [10] and P. Constantin [8] and the references therein.

1. The primitive equations of the atmosphere

1.1. The equations. In this article we study the atmospheric equations in mid-latitude regions, using the so-called β-plane approximation. The horizontal coordinates are then the Cartesian coordinates x and y, directed eastward and northward, and the vertical coordinate is the pressure, increasing downward. We begin by recalling two systems of equations, called the *primitive equations* (PEs) and the *primitive equations with vertical viscosity* (PEV^{2}s). These systems of equations are fundamental equations of the atmosphere, consisting of the horizontal momentum equations, the hydrostatic equation, the mass continuity equation and the thermodynamic equation. We recall from [24] and [26] the PEs (without the underlined terms) and the PEV^{2}s (including the underlined terms), in which the horizontal coordinates are replaced by the Cartesian coordinates x and y:

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \omega\frac{\partial v}{\partial p} + \tilde{f}k \times v + \nabla\Phi + \tilde{L}_1 v = 0, \tag{1.1}$$

$$\frac{\partial \phi}{\partial p} + \frac{RT}{p} + \underline{\tilde{L}_4\omega} = 0, \tag{1.2}$$

$$\operatorname{div} v + \frac{\partial \omega}{\partial p} = 0, \tag{1.3}$$

$$\frac{R^2}{c^2}\left\{\frac{\partial T}{\partial t} + v \cdot \nabla T + \omega \frac{\partial T}{\partial p}\right\} - \frac{R\omega}{p} + \widetilde{L}_2 T = \widetilde{Q}. \tag{1.4}$$

Here the vertical variable is p, and $\Phi = gz$. Hence all functions above, v, ω, Φ and T, depend on t, x, y and p. As explained in the Introduction, and in order to avoid at this time the difficulties related to geometry of the sphere, the horizontal spatial domain is a rectangle (with periodic boundary conditions), instead of the sphere.

The notations used in (1.1)–(1.4) are as follows:

1. The pseudo-spatial domain is given by

$$\widetilde{M} = (0, 2\pi L) \times (0, 2\pi L) \times (p_0, P), \tag{1.5}$$

where $L > 0$ is the horizontal scale. Since we only consider the mid-latitude atmosphere, we have $L < a$, a being the radius of the earth. The constants p_0 $(0 < p_0 < P)$ and P are the pressure of the air on the top and bottom of the atmosphere, respectively. Namely, we study the motion of the atmosphere between the (very high) isobar $p = p_0$, and the isobar $p = P$, where P is slightly smaller than the pressure of the surface of the earth and the ocean so that the isobar $p = P$ is above the surface of the earth.

2. The unknown functions of the PEs are the (two-component) horizontal velocity field v, the vertical component of the velocity ω (in the p-variable), the temperature function T, and the geopotential Φ. As indicated before, they depend on t, x, y and p.

3. The linear operators related to viscous effects $\widetilde{L}_1, \widetilde{L}_2$ and $\widetilde{L}_4$ are defined (see [24] and [26]) by

$$\begin{cases} \widetilde{L}_1 v = -\mu_1 \Delta v - \nu_1 \dfrac{\partial}{\partial p}\left(\left(\dfrac{gp}{R\overline{T}}\right)^2 \dfrac{\partial v}{\partial p}\right), \\ \widetilde{L}_2 T = -\mu_2 \Delta T - \nu_2 \dfrac{\partial}{\partial p}\left(\left(\dfrac{gp}{R\overline{T}}\right)^2 \dfrac{\partial T}{\partial p}\right), \\ \widetilde{L}_4 \omega = -\mu_1^* \Delta \omega - \nu_1^* \dfrac{\partial}{\partial p}\left(\left(\dfrac{gp}{R\overline{T}}\right)^2 \dfrac{\partial \omega}{\partial p}\right). \end{cases}$$

In (1.1)–(1.4) and (1.6), the differential operators Δ, ∇ and div are the horizontal (two-dimensional) Laplacian, gradient and divergence operators.

4. The constants μ_i, ν_i $(i = 1, 2)$, μ_1^* and ν_1^* are viscosity coefficients respectively in horizontal and vertical directions; R and c^2 in (1.4) are gas constants. The (given) function $\overline{T} = \overline{T}(p)$:

$$\overline{T} \in C^\infty([p_0, P])$$

is determined by

$$c^2 = R\left(\frac{R\overline{T}}{c_p} - p\frac{\partial \overline{T}}{\partial p}\right) = \text{const}, \tag{1.7}$$

the positive constant c_p being the heat capacity of the air. We refer to [24], [26] and [41] for more detailed explanations.

5. Let θ be the colatitude of the earth; then the Coriolis parameter $\widetilde{f}$ is given by

$$\widetilde{f} = 2\Omega \cos\theta. \tag{1.8}$$

Here $\Omega = 7.29 \times 10^{-5}\,\text{rad}\cdot\text{s}^{-1}$ is the angular velocity of the earth. For the study of the mid-latitude atmosphere, we assume that θ is close to some mid-latitude value θ_0 (so that $\cos\theta_0$ and $\sin\theta_0$ are not too small in magnitude); then $\widetilde{f}$ is close to $\widetilde{f}^0 = 2\Omega\cos\theta_0$,

$$\widetilde{f} = 2\Omega\cos\theta = 2\Omega\cos\theta_0\left[1 - \frac{\sin\theta_0}{\cos\theta_0}(\theta - \theta_0) + \dots\right]. \tag{1.9}$$

We then consider the corresponding Rossby number $(U/L\widetilde{f})$ whose value at colatitude $\theta = \theta_0$ is

$$\varepsilon = \text{Ro} = \frac{U}{L\widetilde{f}^0}; \tag{1.10}$$

here $\widetilde{L}$ is the horizontal length scale ($\widetilde{L} = 2\pi L$, see (1.5)); typically $\widetilde{L} = 10^6$ m and U is the horizontal velocity scale, typically $10\,\text{ms}^{-1}$; since $\Omega = 7.3\times 10^{-5}\,\text{s}^{-1}$, we see that $\varepsilon \simeq 0.1$ for e.g. $\cos\theta_0 = 0.68$.

We now rewrite (1.9) as

$$\begin{aligned} &\widetilde{f} = 2\Omega\cos\theta = \widetilde{f}^0 + \varepsilon\widetilde{f}^1 + \varepsilon^2\widetilde{f}^2 + \dots, \\ &\widetilde{f}^0 = 2\Omega\cos\theta_0, \qquad \widetilde{f}^1 = 2\Omega\sin\theta_0\frac{\theta - \theta_0}{\varepsilon}, \end{aligned} \tag{1.11}$$

and this is consistent with our choice of $\widetilde{L}$ provided $|\theta - \theta_0| < \varepsilon$; indeed,

$$|\theta - \theta_0| \le \frac{\widetilde{L}}{2a} = \frac{500}{6{,}400} = 0.078 < \varepsilon.$$

Let us now recall the boundary conditions:

$$\begin{cases} \omega = 0, \quad \dfrac{\partial T}{\partial p} = \alpha_s(\widetilde{T}_s - T), \quad \text{at } p = P, \\ \dfrac{\partial v}{\partial p} = 0, \quad \omega = 0, \quad \dfrac{\partial T}{\partial p} = 0, \quad \text{at } p = p_0, \\ v, \omega, T \text{ and } \Phi \text{ are periodic with respect to } x \text{ and } y \\ \qquad \text{with period } 2\pi L \text{ in each direction,} \end{cases} \tag{1.12}$$

and either

$$v = 0 \qquad \text{at } p = P, \tag{1.13a}$$

or

$$\frac{\partial v}{\partial p} = \widetilde{\gamma}_s(\widetilde{v}_s - v) \qquad \text{at } p = P. \tag{1.13b}$$

Condition (1.13b) is physically more realistic while condition (1.13a) is sometimes conveniently simpler. Here α_s and γ_s are given positive functions of x and y, $\widetilde{v}_s$ vanishes on the land and is equal to the velocity of water on the oceans; $\widetilde{T}_s$ is the apparent temperature of the atmosphere on the surface of the earth.[1]

1.2. Scaling of the primitive equations. To study the quasi-geostrophic theory, we need to renormalize the PEs, i.e. to write them in a nondimensional form. With the previous choice of horizontal length scale L and horizontal velocity scale U, we set

$$\begin{cases} v = v'U, \quad \omega = \dfrac{(P-p_0)U}{L}\omega', \quad T = \dfrac{LU\Omega}{R}T', \\ \Phi = 2\cos\theta_0 LU\Omega\Phi', \quad x = Lx', \quad y = Ly', \\ t = \dfrac{L}{U}t', \quad p = P - (P-p_0)\eta, \end{cases} \tag{1.14}$$

$$\begin{cases} f = \dfrac{\widetilde{f}}{f^0} = \dfrac{\cos\theta}{\cos\theta_0}, \quad \varepsilon = \mathrm{Ro} = \dfrac{U}{L\Omega}, \\ \alpha = \dfrac{L^2\Omega^2}{c^2}, \quad Q = \dfrac{\widetilde{Q}L^2\Omega}{U^2R}, \\ \dfrac{1}{Re_1} = \dfrac{\mu_1}{LU}, \quad \dfrac{1}{Re_2} = \dfrac{P^2\nu_1 Lg^2}{(P-p_0)^2UR^2\overline{T}_0^2}, \\ \dfrac{1}{Rt_1} = \dfrac{\mu_2L^2\Omega^2}{LUR^2}, \quad \dfrac{1}{Rt_2} = \left(\dfrac{P}{P-p_0}\right)^2\dfrac{L^3\Omega^2\nu_2g^2}{\overline{T}_0^2R^2U,}, \\ K_2 = 2\Omega\cos\theta_0\left[\dfrac{P}{P-p_0} - \eta\right], \quad K_1(\eta) = \left(\dfrac{p\overline{T}_0}{P\overline{T}}\right)^2, \\ \overline{T}_0 = \dfrac{LU\Omega}{R}, \\ \dfrac{1}{Rv_1} = \dfrac{\mu_1^*(P-p_0)^2}{L^3U^2}, \quad \dfrac{1}{Rv_2} = \dfrac{\nu_1^*g^2p^2}{R^2(\overline{T}_0)^2LU^2}. \end{cases} \tag{1.15}$$

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[1]In general $\widetilde{T}_s \neq \overline{T}(P)$, for $\widetilde{T}_s$ is the apparent temperature of the air on the surface depending on the location. Of course $\overline{T}(P)$ is related to the spatial average of $\widetilde{T}_s$ on the surface.

Substituting (1.14) and (1.15) into (1.1)–(1.4) and dropping all primes in the resulting equations, we obtain the following nondimensional PEs (without the underlined terms) and the PEV²s (including the underlined terms):

$$\varepsilon\left[\frac{\partial v}{\partial t} + v\cdot\nabla v - \omega\frac{\partial v}{\partial \eta}\right] + fk\times v + \nabla\Phi + \varepsilon L_1 v = 0, \tag{1.16}$$

$$\varepsilon\alpha\left[\frac{\partial T}{\partial t} + v\cdot\nabla T - \omega\frac{\partial T}{\partial \eta}\right] - \frac{\omega}{K_2} + \varepsilon L_2 T = \varepsilon Q, \tag{1.17}$$

$$\operatorname{div} v - \frac{\partial \omega}{\partial \eta} = 0, \tag{1.18}$$

$$\frac{\partial \Phi}{\partial \eta} = \frac{T}{K_2} + \underline{\varepsilon L_4 \omega}. \tag{1.19}$$

Here

$$\begin{cases} L_1 = -\dfrac{1}{Re_1}\Delta - \dfrac{1}{Re_2}\dfrac{\partial}{\partial\eta}\left(K_1\dfrac{\partial}{\partial\eta}\right), \\ L_2 = -\dfrac{1}{Rt_1}\Delta - \dfrac{1}{Rt_2}\dfrac{\partial}{\partial\eta}\left(K_1\dfrac{\partial}{\partial\eta}\right), \\ L_4 = -\dfrac{1}{Rv_1}\Delta - \dfrac{1}{Rv_2}\dfrac{\partial}{\partial\eta}\left(K_1\dfrac{\partial}{\partial\eta}\right). \end{cases} \tag{1.20}$$

The nondimensional pseudo-spatial domain is

$$M = \mathcal{O}\times(0,1), \qquad \mathcal{O} = (0,2\pi)^2. \tag{1.21}$$

The nondimensional boundary conditions are

$$\begin{cases} \omega = 0, \quad \dfrac{\partial T}{\partial \eta} = \alpha_s(T - T_s), \quad \text{at } \eta = 0, \\ \dfrac{\partial v}{\partial \eta} = 0, \quad \omega = 0, \quad \dfrac{\partial T}{\partial \eta} = 0, \quad \text{at } \eta = 1, \\ v, \omega, \Phi, T \text{ are periodic with respect to } x \\ \qquad \text{and } y \text{ with period } 2\pi, \end{cases} \tag{1.22}$$

and

$$v = 0 \qquad \text{at } \eta = 0 \text{ in case (1.13a)} \tag{1.23a}$$

or

$$\frac{\partial v}{\partial \eta} = \gamma_s(v - v_s) \qquad \text{at } \eta = 0 \text{ in case (1.13b).} \tag{1.23b}$$

Here $\alpha_s, \gamma_s, v_s, T_s$ are the nondimensionalized forms of $\widetilde{\alpha}_s, \widetilde{\gamma}_s, \widetilde{v}_s$ and $\widetilde{T}_s$. Integrating the diagnostic equations (1.18) and (1.19) and taking into account (1.22) ($\omega = 0$ at $\eta = 1$), we find

$$
\text{(1.24)} \qquad \begin{cases} \omega = W(v) = -\operatorname{div} \mathcal{M}^* v, \\ \operatorname{div} \displaystyle\int_0^1 v \, d\eta = 0, \\ \Phi = \Phi_s + \mathcal{M}\left(\dfrac{T}{K_2}\right) - \underline{\varepsilon \mathcal{M} L_4 \operatorname{div} \mathcal{M}^* v}, \end{cases}
$$

where the operator $\mathcal{M}$ and its adjoint (in the L^2 sense) $\mathcal{M}^*$ are given by

$$
\text{(1.25)} \qquad \mathcal{M}\psi = \int_0^\eta \psi \, d\eta', \qquad \mathcal{M}^*\psi = \int_\eta^1 \psi d\eta'.
$$

In (1.24), the function Φ_s, depending only on x and y, is the unknown value of $\phi = gz$ at the isobar $p = P$ ($\eta = 0$).[2] Then we can rewrite the PEs (without the underlined terms) and the PEV^{2}s (including the underlined terms) as follows (see [24] and [26]):

$$
\text{(1.26)} \qquad \varepsilon\left[\frac{\partial v}{\partial t} + v \cdot \nabla v - W(v)\frac{\partial v}{\partial \eta}\right] + fk \times v + \nabla \Phi_s + \nabla \mathcal{M}\left(\frac{T}{K_2}\right) - \underline{\varepsilon \nabla \mathcal{M} L_4 \operatorname{div} \mathcal{M}^* v} + \varepsilon L_1 v = 0,
$$

$$
\text{(1.27)} \qquad \varepsilon\alpha\left[\frac{\partial}{\partial t} + v \cdot \nabla - W(v)\frac{\partial}{\partial \eta}\right] T - \frac{W(v)}{K_2} + \varepsilon L_2 T = \varepsilon Q,
$$

$$
\text{(1.28)} \qquad \operatorname{div} \int_0^1 v \, d\eta = 0.
$$

Space periodicity in the horizontal variables x and y is assumed. The initial and the other boundary conditions are

$$
\text{(1.29)} \qquad \begin{cases} \dfrac{\partial T}{\partial \eta} = \alpha_s (T - T_s), & \text{for } \eta = 0, \\ \dfrac{\partial v}{\partial \eta} = 0, \quad \dfrac{\partial T}{\partial \eta} = 0, & \text{for } \eta = 1, \end{cases}
$$

and, as in (1.23),

$$
\text{(1.30a)} \qquad v = 0 \qquad \text{at } \eta = 0 \text{ in case (1.13a)},
$$

[2] Therefore Φ_s/g "gives" the geometrical altitude of the isobar $p = P$. But this is not completely precise, because $\Phi_s = \Phi_s(x, y, t)$ is defined up to an additive constant depending on t (as usual with Lagrange multipliers). In order to remove this type of difficulty (which is not at all essential for the goals of the present paper) we have in [29] introduced the equations governing the altitude Φ_s and the altitude of the ocean with respect to the reference $z = 0$ in the coupled system ocean-atmosphere of [28].

or

$$\frac{\partial v}{\partial \eta} = \gamma_s(v - v_s) \qquad \text{at } \eta = 0 \text{ in case (1.13b).} \tag{1.30b}$$

$$(v, T)\,|_{t=0} = (v_0, T_0). \tag{1.31}$$

REMARK 1.1. As we have already said the model (1.26)–(1.31) is essentially similar to the one in [24], [26] with mainly two differences:

(i) The boundary conditions in the horizontal variables are different. We have considered for the sake of simplicity *periodic* boundary conditions. We could as well consider other boundary conditions. Of course this does not change the proof, given in [24], of the existence of a (weak) solution global in time (see Remark 3.2).

(ii) We have not introduced here q, the humidity. However, this could be done with minor modifications since the equation for q is essentially uncoupled from the other equations for velocity and temperature.

It would be interesting to introduce beside q the modified state equations

$$p = R\rho T(1 + 0.61q)$$

(cf. [31]). But this leads to a nonlinear constraint (instead of the linear constraints (2.3) hereafter) and the difficulty is much more important than the previous remark related to the sole introduction of q, when the state equation is still $p = R\rho T$.

2. Geostrophic asymptotics

2.1. The formal asymptotic expansion of the equations. First of all, we study the asymptotic expansion of the PEs and PEV^{2}s given by (1.26)–(1.28), with respect to the Rossby number $\varepsilon = \text{Ro}$. As we shall see, this leads to the *geostrophic* and the *quasi-geostrophic equations*. All the expansions are formal at this stage; more precise results are given in Section 3.

We write

$$\begin{cases} v = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \dots, \\ T = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \dots, \\ \Phi_s = \Phi_s^0 + \varepsilon \Phi_s^1 + \varepsilon^2 \Phi_s^2 + \dots, \\ f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots, \\ Q = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2 + \dots \end{cases} \tag{2.1}$$

The expansion of $f = \widetilde{f}/\widetilde{f}^0$ is obtained from (1.11). Therefore, we have

$$f^0 = \frac{\widetilde{f}^0}{\widetilde{f}^0} = 1, \qquad f^n = \frac{\widetilde{f}^n}{\widetilde{f}^0}, \qquad n = 1, 2, \ldots$$

Similarly we expand given functions $u_s = (v_A, T_A)$, $u_0 = (v_0, T_0)$ in the form

$$\text{(2.2)} \qquad \begin{cases} v_s = v_s^0 + \varepsilon v_s^1 + \varepsilon^2 v_s^2 + \ldots, \\ T_s = T_s^0 + \varepsilon T_s^1 + \varepsilon^2 T_s^2 + \ldots, \\ v_0 = v_0^0 + \varepsilon v_0^1 + \varepsilon^2 v_0^2 + \ldots, \\ T_0 = T_0^0 + \varepsilon T_0^1 + \varepsilon^2 T_0^2 + \ldots \end{cases}$$

Substituting (2.1) into (1.26)–(1.28) we have the following approximate equations of the PEs:

a. Zeroth order approximation. At zeroth order, equations (1.26)–(1.28) reduce to

$$\text{(2.3)} \qquad \begin{cases} f^0 k \times v^0 + \nabla \Phi_s^0 + \nabla \mathcal{M}(T^0/K_2) = 0, \\ \operatorname{div} \mathcal{M}^* v^0 = 0. \end{cases}$$

Note that the first equation (2.3) amounts to saying that

$$\text{(2.3}'\text{)} \qquad -v^0 = \operatorname{curl}\left(\frac{\Phi^0}{f^0}\right) = \frac{1}{f^0}\left\{\frac{\partial \Phi^0}{\partial y}, -\frac{\partial \Phi^0}{\partial x}\right\}, \qquad \Phi^0 = \Phi_s^0 + \mathcal{M}(T^0/K_2),$$

or

$$\text{(2.4)} \qquad f^0 k \times v^0 + \nabla \Phi^0 = 0, \qquad T^0 = K_2 \frac{\partial \Phi^0}{\partial \eta}.$$

In fact, (2.3) is equivalent to the existence of some function Φ^0 such that (2.4) holds. We call the motions governed by (2.4) the *geostrophic motions.*

b. First order approximation. At the first order (1.26)–(1.28) and (2.1) yield

$$\text{(2.5)} \quad \frac{\partial v^0}{\partial t} + v^0 \cdot \nabla v^0 + f^1 k \times v^0 + f^0 k \times v^1 + \nabla \Phi_s^1 + \nabla \mathcal{M}(T^1/K_2) + L_1 v^0 = 0,$$

$$\text{(2.6)} \qquad \alpha\left[\frac{\partial}{\partial t} + v^0 \cdot \nabla\right] T^0 + \frac{\operatorname{div} \mathcal{M}^* v^1}{K_2} + L_2 T^0 = Q^0,$$

$$\text{(2.7)} \qquad \operatorname{div} \int_0^1 v^1 \, d\eta = 0.$$

Regrouping (2.4)–(2.7), we obtain the following *quasi-geostrophic equations* (*QGs*):

$$(2.8)\qquad \begin{cases} \dfrac{\partial v^0}{\partial t} + v^0 \cdot \nabla v^0 + f^1 k \times v^0 + f^0 k \times v^1 + \nabla \Phi^1 + L_1 v^0 = 0, \\ \alpha \left[\dfrac{\partial}{\partial t} + v^0 \cdot \nabla \right] T^0 + \dfrac{\operatorname{div} \mathcal{M}^* v^1}{K_2} + L_2 T^0 = Q^0. \end{cases}$$

$$(2.9)\qquad \begin{cases} \operatorname{div} v^0 = 0, \\ f^0 k \times v^0 + \nabla \Phi^0 = 0, \\ T^0 = K_2 \dfrac{\partial \Phi^0}{\partial \eta}. \end{cases}$$

Here v^1 and Φ^1 are unknown functions of x, y, η and t satisfying

$$(2.10)\qquad \operatorname{div} \int_0^1 v^1 \, d\eta = 0.$$

Of course all functions are periodic in the horizontal variables x and y; and we also infer from (1.29)–(1.31) the following initial and boundary conditions for the QGs equations:

$$(2.11)\qquad \begin{cases} \dfrac{\partial T^0}{\partial \eta} = \alpha_s (T^0 - T_s^0), & \text{at } \eta = 0, \\ \dfrac{\partial v^0}{\partial \eta} = 0, \quad \dfrac{\partial T^0}{\partial \eta} = 0, & \text{at } \eta = 1, \\ v^0 = 0 & \text{at } \eta = 0 \text{ in case (1.13a)}, \\ \dfrac{\partial v^0}{\partial \eta} = \gamma_s (v^0 - v_s^0) & \text{at } \eta = 0 \text{ in case (1.13b)}. \end{cases}$$

$$(2.12)\qquad (v^0, T^0)|_{t=0} = (v_0^0, T_0^0).$$

We call the motions governed by (2.8)–(2.11) the *quasi-geostrophic motions.*

REMARK 2.1. We shall show below that problem (2.8)–(2.11) admits a unique solution $\{v^0, T^0\}$, v^1 and Φ^1 playing the role of Lagrange multipliers attached to the linear constraints (2.9). This is, in fact, an unusual set of equations but the role of v^1 and Φ^1 will be made clear in Sections 3.3.2 and 3.3.3, after we introduce the appropriate function spaces.

REMARK 2.2. By the second equation (2.3), we find

$$\nabla \mathcal{M} L_4 \operatorname{div} \mathcal{M}^* v^0 = 0.$$

Therefore, at the zeroth and first orders, the PEV^{2}s and the PEs lead to the same geostrophic and quasi-geostrophic equations.

2.2. Quasi-geostrophic equations in vorticity form. We now derive from (2.8)–(2.10) the vorticity form of the quasi-geostrophic equations. We do not intend to make use of these equations in our analytical study, but our aim in deriving the vorticity form of the QG equations is to show that following our procedure we obtain anew (in a more rigorous fashion) what is already known in [4], [33] and that we obtain some further results. First of all, applying the (horizontal) curl operator to both sides of $(2.8)_1$ and using the formulas after (2.13), we obtain

$$\frac{\partial}{\partial t}\Big(\frac{1}{f^0}\Delta\Phi_0\Big) + \nabla_{v^0}\Big(\frac{1}{f^0}\Delta\Phi^0 + f^1\Big) + f^0 \operatorname{div} v^1 + L_1\Big(\frac{1}{f^0}\Delta\Phi_0\Big) = 0. \tag{2.13}$$

In deriving (2.13), we used the following relations:

$$\operatorname{curl}(k \times v^1) = \operatorname{div} v^1,$$
$$\operatorname{curl}(f^1 k \times v^0) = \operatorname{div}(f^1 v^0) = \nabla_{v^0} f^1.$$

To eliminate v^1 from (2.13), we multiply $(2.8)_2$ by K_2, and differentiate the resulting equation with respect to η. Since K_2 depends only on η, we obtain

$$\begin{aligned} \operatorname{div} v^1 = {} & \Big[\frac{\partial}{\partial t} + v^0 \cdot \nabla\Big] \cdot \Big[\frac{\partial}{\partial \eta}\Big(\alpha K_2^2 \frac{\partial \Phi_0}{\partial \eta}\Big)\Big] \\ & - \frac{1}{Rt_1}\Delta\Big[\frac{\partial}{\partial \eta}\Big(K_2^2 \frac{\partial \Phi^0}{\partial \eta}\Big)\Big] \\ & - \frac{1}{Rt_2}\frac{\partial}{\partial \eta}\Big\{K_2 \frac{\partial}{\partial \eta}\Big[K_1 \frac{\partial}{\partial \eta}\Big(K_2 \frac{\partial \Phi^0}{\partial \eta}\Big)\Big]\Big\} \\ & - \frac{\partial}{\partial \eta}[K_2 Q^0]. \end{aligned} \tag{2.14}$$

Using (2.14), we infer from (2.13) that

$$\Big[\frac{\partial}{\partial t} + \nabla_{v^0}\Big] \cdot \Big[\frac{1}{f^0}\Delta\Phi^0 + \frac{\partial}{\partial \eta}\Big(\alpha f^0 K_2^2 \frac{\partial \Phi^0}{\partial \eta}\Big) + f^1\Big] - L\Phi^0 = F, \tag{2.15}$$

where

$$\begin{aligned} L\Phi^0 = {} & L_1\Big(\frac{1}{f^0}\Delta\Phi^0\Big) + \frac{1}{Rt_1}\Delta\Big[\frac{\partial}{\partial \eta}\Big(f^0 K_2^2 \frac{\partial \Phi^0}{\partial \eta}\Big)\Big] \\ & + \frac{1}{Rt_2}\frac{\partial}{\partial \eta}\Big\{f^0 K_2 \frac{\partial}{\partial \eta}\Big[K_1 \frac{\partial}{\partial \eta}\Big(K_2 \frac{\partial \Phi^0}{\partial \eta}\Big)\Big]\Big\}, \end{aligned} \tag{2.16}$$

$$F = f^0 \frac{\partial}{\partial \eta}[K_2 Q^0]. \tag{2.17}$$

Equation (2.15) is the *quasi-geostrophic vorticity equation.*

REMARK 2.3. The linear operator L given by (2.16) is a fourth order elliptic operator, representing the diabatic heating and friction of the atmosphere. Usually for short term weather prediction one studies the quasi-geostrophic equation without dissipation [5], i.e.

$$\left[\frac{\partial}{\partial t} + v^0 \cdot \nabla\right] \cdot \left[\frac{1}{f^0}\Delta\Phi^0 + \frac{\partial}{\partial \eta}\left(\alpha f^0 K_2^2 \frac{\partial \Phi^0}{\partial \eta}\right) + f^1\right] = 0. \tag{2.18}$$

As Charney stated [5], however, when prediction is extended to three or more days, the dissipation has to be taken into consideration. Therefore we study in this article the quasi-geostrophic equation (2.15) with dissipation. We would also like to mention that, as far as we know, for the first time in this article the dissipation for the quasi-geostrophic motion is written in such an explicit form.

REMARK 2.4. As we mentioned before, usually one uses $\zeta = p/P$ as the vertical coordinate. We prefer here to use $\eta = (P - p)/(P - p_0)$ since this variable is more convenient for the equations coupling the atmosphere and the oceans [26]–[28]. If we use ζ it is easy to see (cf. [40]) that the vorticity form of the quasi-geostrophic equations becomes, using (2.22),

$$\left[\frac{\partial}{\partial t} + v^0 \cdot \nabla\right] \cdot \left[\frac{1}{f^0}\Delta\Phi^0 + \frac{\partial}{\partial \zeta}\left(\alpha f^0 K_2^2 \frac{\partial \Phi^0}{\partial \zeta}\right) + f^1\right] = \overline{L}\Phi^0 + \overline{F}, \tag{2.19}$$

where

$$\begin{aligned} \overline{L}\Phi^0 = &\left[\frac{1}{Re_1}\Delta + \frac{1}{Re_2}\left(\frac{P - p_0}{P}\right)^2 \frac{\partial}{\partial \zeta}\left(K_1 \frac{\partial}{\partial \zeta}\right)\right]\left(\frac{1}{f^0}\Delta\Phi^0\right) \\ &+ \frac{1}{Rt_1}\Delta\left[\frac{\partial}{\partial \zeta}\left(f^0 K_2^2 \frac{\partial \Phi^0}{\partial \zeta}\right)\right] + \frac{1}{Rt_2}\left(\frac{P - p_0}{P}\right)^2 \\ &\times \frac{\partial}{\partial \zeta}\left\{f^0 K_2 \frac{\partial}{\partial \zeta}\left[K_1 \frac{\partial}{\partial \zeta}\left(K_2 \frac{\partial \Phi^0}{\partial \zeta}\right)\right]\right\}, \end{aligned} \tag{2.20}$$

$$\overline{F} = f^0 \frac{\partial}{\partial \zeta}[\zeta Q^0]. \tag{2.21}$$

Equation (2.19) can be obtained from (2.15) using the following relationship between ζ and η:

$$\eta = \frac{P}{P - p_0}[1 - \zeta], \qquad K_2 = \frac{P}{P - p_0}\zeta. \tag{2.22}$$

Once again, usually one considers the quasi-geostrophic equation without viscosity in the ζ-coordinate system, namely $v^0 = -\operatorname{curl}\Phi^0/f^0$:

$$\left[\frac{\partial}{\partial t} + v^0 \cdot \nabla\right] \cdot \left[\frac{1}{f^0}\Delta\Phi^0 + \frac{\partial}{\partial \zeta}\left(\alpha f^0 K_2^2 \frac{\partial \Phi^0}{\partial \zeta}\right) + f^1\right] = 0. \tag{2.23}$$

REMARK 2.5. The mathematical study of the quasi-geostrophic vorticity equation was made in [40].

3. Mathematical setting of the equations

3.1. The primitive equations. The mathematical analysis of the primitive equations has been studied in [24] for the whole sphere and therefore *without* boundary conditions in the θ and φ variables. We recall here the main points of the method followed in [24] and we show how it is adapted to the present situation.

3.1.1. Some function spaces. We consider the PEs in the form (1.26)–(1.28) with initial and boundary conditions (1.29)–(1.31). For simplicity, we call u the pair $\{v, T\}$,

$$u = (v, T). \tag{3.1}$$

We write

$$\begin{cases} \Gamma = \Gamma_b \cup \Gamma_u, \\ \Gamma_b = \mathcal{O} \times \{0\}, \quad \Gamma_u = \mathcal{O} \times \{1\}, \quad \mathcal{O} = (0, 2\pi)^2. \end{cases} \tag{3.2}$$

As we did in [24], we introduce the following function spaces:

$$\begin{cases} H = H_1 \times H_2, \quad V = V_1 \times V_2, \\ H_i = \text{the closure of } \mathcal{V}_i \text{ for the } L^2\text{-norm, } i = 1, 2, \\ V_i = \text{the closure of } \mathcal{V}_i \text{ for the } H^1\text{-norm, } i = 1, 2, \end{cases} \tag{3.3}$$

where

$$\begin{cases} \mathcal{V}_1 = \left\{ v \in (C^\infty_{b,0}(M_p))^2 \,\middle|\, \operatorname{div} \int_0^1 v \, d\eta = 0 \right\} \quad \text{in case (1.13a)}, \\ \mathcal{V}_1 = \left\{ v \in (C^\infty(M_p))^2 \,\middle|\, \operatorname{div} \int_0^1 v \, d\eta = 0 \right\} \quad \text{in case (1.13b)}, \\ \mathcal{V}_2 = C^\infty(M_p), \\ \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2. \end{cases} \tag{3.4}$$

Here $C^\infty(M_p)$ is the set of all C^∞ functions on M which are restrictions to M of C^∞ functions on $\mathbb{R}^2 \times [0, 1]$, 2π-periodic in directions x and y; $C^\infty_{b,0}(M_p)$ consists of the functions in $C^\infty(M_p)$ vanishing near the lower boundary Γ_b. The L^2-norms and inner products in H_i $(i = 1, 2)$ and H are denoted by $|\cdot|$ and $(\cdot, \cdot)$, whereas the H^1-norms and inner products in V_i and V are all denoted by $\|\cdot\|$ and $((\cdot, \cdot))$:

$$(f, g) = \int_M f g \, dM, \qquad ((f, g)) = \int_M \nabla f \cdot \nabla g \, dM.$$

Hence $H_2 = L^2(M), V_2 = H^1(M_p)$ and it can be proved as in Lemma 2.1 of [24] that

$$H_1 = \left\{ v = (v_1, v_2) \in L^2(M)^2 \,\middle|\, \operatorname{div} \int_0^1 v\, d\eta = 0, \right.$$
$$\left. \int_0^1 v\, d\eta \text{ is } 2\pi\text{-periodic with respect to } x \text{ and } y \right\},$$

$$V_1 = V_{1a} = \{v \in V_{1b},\ v = 0 \text{ on } \Gamma_b\} \qquad \text{in case (1.13a)},$$
$$V_1 = V_{1b} = \left\{ v \in H^1(M_p)^2 \,\middle|\, \operatorname{div} \int_0^1 v\, d\eta = 0 \right\} \qquad \text{in case (1.13b)}.$$

Here $H^1(M_p)$ denotes the subspace of $H^1(M)$ of functions periodic in directions x and y.[3]

Moreover, we also need the equivalent norms and inner products $|A_0^{1/2}\cdot|$ and $(A_0\cdot,\cdot)$ for H_i $(i = 1, 2)$ and H. Here the operator A_0 is defined by

$$A_0 u = (v, \alpha T), \qquad A_0 v = v, \qquad A_0 T = \alpha T. \tag{3.5}$$

3.1.2. Some functionals. We define some functionals and their associated operators as follows; the terms underlined appear in case (1.13a) but not in case (1.13b) (where they vanish):

$$\begin{aligned} a(u, \tilde{u}) = (Au, \tilde{u}) = \int_M \Big\{ &\frac{1}{Re_1} \nabla v \cdot \nabla \tilde{v} + \frac{K_1}{Re_2} \frac{\partial v}{\partial \eta} \cdot \frac{\partial \tilde{v}}{\partial \eta} \\ &+ \frac{1}{Rt_1} \nabla T \cdot \nabla \tilde{T} + \frac{K_1}{Rt_2} \frac{\partial T}{\partial \eta} \cdot \frac{\partial \tilde{T}}{\partial \eta} \Big\} dM \\ &+ \frac{1}{Rt_2} \int_{\Gamma_b} \alpha_s K_1 T \tilde{T}\, d\Gamma. \end{aligned} \tag{3.6}$$

$$a_1(u_s; \tilde{u}) = \underline{-\frac{1}{Re_2} \int_{\Gamma_b} \gamma_s K_1 v_s \cdot \tilde{v}\, d\Gamma_b} - \frac{1}{Rt_2} \int_{\Gamma_b} \alpha_s K_1 T_s \tilde{T}\, d\Gamma. \tag{3.7}$$

$$\begin{aligned} b(u, \overline{u}, \tilde{u}) &= (B(u, \overline{u}), \tilde{u}) \\ &= \int_M \Big\{ \Big[v \cdot \nabla \overline{v} + (\operatorname{div} \mathcal{M}^* v) \frac{\partial \overline{v}}{\partial \eta} \Big] \cdot \tilde{v} \\ &\quad + \alpha \Big[v \cdot \nabla \overline{T} + (\operatorname{div} \mathcal{M}^* v) \frac{\partial \overline{T}}{\partial \eta} \Big] \tilde{T} \Big\} dM, \end{aligned} \tag{3.8}$$

[3]The trace on Γ_b of functions v in H_1 is not defined and is not required to vanish; see, however, a related but different situation in Lemma 3.3 below.

$$(3.9)\quad e(f;u,\widetilde{u}) = (E(f;u),\widetilde{u}) = \int_M \Big\{(fk\times v)\cdot\widetilde{v} + \nabla\mathcal{M}(T/K_2)\cdot\widetilde{v} + \frac{1}{K_2}(\operatorname{div}\mathcal{M}^* v)\cdot\widetilde{T}\Big\}\,dM.$$

The following lemmas, borrowed from [24] and [26], describe the basic properties of these functionals and operators:

LEMMA 3.1.

$$(3.10)\quad \begin{cases} a(u,\widetilde{u}) \le \dfrac{1}{R_{\min}}\|u\|\cdot\|\widetilde{u}\|, \\ a(u,u) \ge \dfrac{1}{R_{\max}}\|u\|^2, \end{cases}$$

where $R_{\max}$ and $R_{\min}$ are defined by

$$(3.11)\quad \begin{cases} R_{\max} = C\ \max\{Re_j, Rt_j \mid j=1,2\}, \\ R_{\min} = C'\ \min\{Re_j, Rt_j \mid j=1,2\}, \end{cases}$$

C and C' being two absolute constants.

3.1.3. Weak formulation of the PEs. We now state the weak formulation of the PEs as follows:

PROBLEM 3.1. *For any given $u_0 = (v_0, T_0) \in H$, find $u = (v,T)$ such that*

$$(3.12)\quad u \in L^2(0,t_1;V)\cap L^\infty(0,t_1;H), \qquad \forall t_1 > 0,$$

$$(3.13)\quad \varepsilon\frac{d}{dt}(A_0u,\widetilde{u}) + \varepsilon a(u,\widetilde{u}) + \varepsilon a_1(u_s;\widetilde{u}) + \varepsilon b(u,u,\widetilde{u}) + e(f;u,\widetilde{u}) = \varepsilon(Q,\widetilde{T}), \qquad \forall\widetilde{u} = (\widetilde{v},\widetilde{T}) \in V,$$

$$(3.14)\quad u|_{t=0} = u_0.$$

REMARK 3.1. As shown in [24], the function Φ_s, depending only on x and y, is the Lagrange multiplier of the constraint

$$\operatorname{div}\int_0^1 v\,d\eta = 0.$$

It disappears in (3.13) and Problem 3.1 is indeed the weak formulation of the PEs.

3.1.4. Existence of global weak solutions of the PEs. We now recall from [24] an existence theorem for global weak solutions of the PEs. This result can be proved by constructing an approximate solution with the Galerkin method and using a priori estimates on the approximate solutions.

THEOREM 3.1. *There exists at least one solution for Problem 3.1, the weak formulation of the PEs.*

REMARK 3.3. The method of proof of existence of a weak solution global in time in [24] (or for the present system with different boundary conditions, see Remark 1.1) relies on the general ideas introduced by J. Leray in [18]-[20]. Because of technical differences with the "usual" Navier-Stokes equations we have to rely on the technique of estimating fractional derivatives in time, as introduced in [21]. Uniqueness of solution is still an open question but there is at this time little evidence of nonuniqueness for this initial value problem as well as for the three-dimensional Navier-Stokes equations.

3.2. The primitive equations with vertical viscosity.

3.2.1. Weak formulation of the PEV^2s. We recall that the PEV^2s are obtained by retaining in (1.26) the underlined viscosity terms for the vertical velocity. We set

$$\text{(3.15)} \qquad \begin{cases} V^\omega = V_1^\omega \times V_2, \\ V_1^\omega = \text{the closure of } \mathcal{V}_1^\omega \text{ for the norm } \|\cdot\|_w, \end{cases}$$

where

$$\text{(3.16)} \qquad \begin{cases} \mathcal{V}^\omega = \mathcal{V}_1^\omega \times \mathcal{V}_2, \\ \mathcal{V}_1^\omega = \{v \in \mathcal{V}_1 \mid \operatorname{div} \mathcal{M}^* v \in C_{b,0}^\infty(M)\}, \\ \|v\|_w = \{\|v\|^2 + \|\operatorname{div} \mathcal{M}^* v\|_{H_0^1}^2\}^{1/2}, \end{cases}$$

the corresponding inner product being denoted by $((\cdot,\cdot))_w$. The induced norm and inner product for V^ω are still denoted by the same notations $\|\cdot\|_w$ and $((\cdot,\cdot))_w$. As before, the definition of $\mathcal{V}_1$, V_1 and thus of $\mathcal{V}_1^\omega$, V_1^ω and V^ω is not the same in case (1.13a) and in case (1.13b).

We then introduce

$$\text{(3.17)} \qquad a_w(u, \widetilde{u}) = a(u, \widetilde{u}) + I(u, \widetilde{u}),$$

where $a(\cdot,\cdot)$ is defined by (3.6) and $I = I(u, \widetilde{u})$ is given by

$$\text{(3.18)} \qquad I = I(u, \widetilde{u}) = \int_M \left\{ \frac{1}{Rv_1} \nabla(\operatorname{div} \mathcal{M}^* v) \cdot \nabla(\operatorname{div} \mathcal{M}^* \widetilde{v}) + \frac{K_1}{Rv_2} \frac{\partial(\operatorname{div} \mathcal{M}^* v)}{\partial \eta} \cdot \frac{\partial(\operatorname{div} \mathcal{M}^* \widetilde{v})}{\partial \eta} \right\} dM.$$

The weak formulation of the PEV^2s can then be stated as

PROBLEM 3.2 (Weak formulation of the PEV²s). *For $u_0 = (v_0, T_0) \in H$, find $u = (u, T)$ such that*

$$u \in L^2(0, t_1; \mathcal{V}^\omega) \cap L^\infty(0, t_1; H), \qquad \forall t_1 > 0, \tag{3.19}$$

$$\varepsilon \frac{d}{dt}(A_0 u, \widetilde{u}) + \varepsilon a_w(u, \widetilde{u}) + \varepsilon a_1(u_s; \widetilde{u}) + \varepsilon b(u, u, \widetilde{u}) + e(f; u, \widetilde{u}) = \varepsilon(Q, \widetilde{T}), \qquad \forall \widetilde{u} = (\widetilde{v}, \widetilde{T}) \in \mathcal{V}^\omega, \tag{3.20}$$

$$u|_{t=0} = u_0. \tag{3.21}$$

REMARK 3.3. We have the same nonlinear functional $b(\cdot,\cdot,\cdot)$ here as in the weak formulation of the PEs. However, we have now the following *improved a priori* estimates for $b(\cdot,\cdot,\cdot)$ in terms of the norm $\|\cdot\|_w$:

$$|b(u, \widetilde{u}, \overline{u})| \le c\|u\|_w \cdot |\widetilde{u}|^{1/2} \cdot \|\widetilde{u}\|^{1/2} \cdot \|\overline{u}\|. \tag{3.22}$$

as compared to the estimate

$$|b(u, \widetilde{u}, \overline{u})| \le c\|u\|_{H^{3/2}} \cdot \|\widetilde{u}\| \cdot \|\overline{u}\|. \tag{3.23}$$

3.2.2. Solutions of the PEV²s. As for the PEs, we have the following existence result of global weak solutions of the PEV²s borrowed from [24]:

THEOREM 3.2. *There exists at least one solution for Problem 3.2, the weak formulation of the PEV²s.*

3.3. The quasi-geostrophic equations. We now consider the quasi-geostrophic equations (2.8)–(2.10) with initial and boundary conditions (2.11) and (2.12), and we start with some comments and results concerning the constraints (2.9).

We first observe that if $u^0 = (v^0, T^0) \in H$ satisfies (2.3) for some function Φ_s^0 independant of η, then according to (2.4) it satisfies (2.9); moreover by, (2.3), (2.3′) and a theorem of J. Deny and J.-L. Lions [11], Φ^0 is in $H^1(M_p)$. Conversely any Φ^0 in $H^1(M_p)$ defines with (2.3′) and (2.4), $u^0 = (v^0, T^0) \in H$ satisfying (2.9) (and (2.3)). The remarks hold for $u^0 \in V$, but for such a u^0, we find that $\Phi^0 \in H^2(M_p)$ in case (1.13b), $\Phi^0 \in H^2(M_p)$ with $\partial\Phi^0/\partial x$ and $\partial\Phi^0/\partial y$ vanishing on Γ_b in case (1.13a); like $H^1(M_p)$, $H^2(M_p)$ is defined as the set of functions in $H^2(M)$ periodic in directions x and y together with all their first derivatives.

Now assume that $u^0 = (v^0, T^0) \in V$ satisfies (2.9) for some distribution Φ^0. Then (2.3) and (2.4) hold and Φ^0 is as above ($\Phi^0 \in H^2(M_p)$ in case (1.13b),

$\Phi^0 \in H^2(\mathcal{M}_p)$ with $\partial\Phi^0/\partial x$ and $\partial\Phi^0/\partial y$ vanishing on Γ_b in case (1.13a)). It is then obvious that u^0 satisfies

$$e(f^0; u^0, \tilde{u}) = 0, \qquad \forall \tilde{u} \in V, \tag{3.24}$$

or equivalently

$$E(f^0; u^0) = 0 \qquad \text{in } V'. \tag{3.25}$$

Conversely, if $u^0 = (v^0, T^0) \in V$ satisfies (3.25), then the second equation (2.3) follows readily and the first equation (2.3) follows from Lemma 2.1 in [24]. Hence (2.3) and (2.9) are satisfied and we conclude that (2.9) is equivalent to (3.25).

For $u^0 \in H$, $E(f^0; u^0)$ is not defined and (3.25) does not make sense; however, we can and will consider, equivalently, functions u^0 in H satisfying (2.9) or (2.3) (2.4), for some Φ in $H^1(\mathcal{M}_p)$.

3.3.1. Some function spaces. Based on the above observations, we define the function spaces needed for the quasi-geostrophic equations as follows:

$$\begin{cases} V_G = \text{the closure of } \mathcal{V}_G \text{ for the } H^1\text{-norm}, \\ H_G = \text{the closure of } \mathcal{V}_G \text{ for the } L^2\text{-norm}. \end{cases} \tag{3.26}$$

The function space $\mathcal{V}_G$ is defined by

$$\begin{aligned} \mathcal{V}_G &= \{u = (v, T) \in \mathcal{V} \mid E(f^0; u) = 0\} \\ &= \{u \in \mathcal{V} \mid e(f^0; u, \tilde{u}) = 0, \forall \tilde{u} \in V\}. \end{aligned} \tag{3.27}$$

As before, $\mathcal{V}$ is not the same in cases (1.13a) and (1.13b).

We have the following lemma, which characterizes these function spaces:

LEMMA 3.3.

(i) *The space of functions $u = (v, T)$ in H such that* (2.9) *holds for some distribution Φ is closed in H.*

(ii) *If $u = (v, T) \in H$ and satisfies* (2.9) *for some distribution Φ, the trace of v on $\Gamma = \Gamma_b \cup \Gamma_u$ is defined and belongs to $H^{-1/2}(\Gamma)^2$.*

(iii) *The spaces H_G and V_G satisfy*

$$H_G = \begin{cases} \{u \in H \mid u \text{ satisfies (2.9) and } v \text{ vanishes on } \Gamma_b\} & \text{in case (1.13a)}, \\ \{u \in H \mid \text{ satisfies (2.9)}\} & \text{in case (1.13b)}, \end{cases} \tag{3.28}$$

$$V_G = \{u = (v, T) \in V \mid E(f^0; u) = 0\}. \tag{3.29}$$

REMARK 3.1. From the above definition and Lemma 3.3, we observe that the function spaces H_G and V_G describe exactly the geostrophic motion of the atmosphere.

PROOF OF LEMMA 3.3. (i) Consider a sequence of functions $u_n = (v_n, T_n)$ which satisfy (2.9) for some Φ_n, namely

$$f^0 k \times v_n + \nabla\Phi_n = 0, \qquad T_n = K_2 \frac{\partial \Phi_n}{\partial \eta},$$

and assume that u_n converges to some limit u in H. By the theorem of [11] quoted above, we conclude that Φ_n belongs to $H^1(M_p)$, and that, as $n \to \infty$, Φ_n converges in $H^1(M_p)/\mathbb{R}$ to some limit $\Phi \in H^1(M_p)$, with

$$f^0 k \times v + \nabla\Phi = 0, \qquad T = K_2 \frac{\partial \Phi}{\partial \eta}.$$

Hence the result.

(ii) As observed above $\Phi \in H^1(M_p)$; hence the trace of Φ on $\Gamma_{\eta_0} = \mathcal{O} \times \{\eta = \eta_0\}$ is defined and belongs to $H^{1/2}(\Gamma_{\eta_0})$ for every $\eta_0, 0 \le \eta_0 \le 1$. Thus, after modification of v on a subset of measure 0, we find that $v(\cdot, \eta_0)$ is defined and belongs to $H^{-1/2}(\Gamma_{\eta_0})^2$ for every $\eta_0, 0 \le \eta_0 \le 1$. In particular, for $\eta_0 = 0, 1$, we obtain the trace of v on Γ; of course the trace depends linearly on u and continuously for the H-norm in $\{u \in H \mid u \text{ satisfies } (2.9)\}$.

(iii) We start with the case (1.13b) and first show that H_G is included in the right-hand side of (3.28).

If u belongs to H_G, then u is the limit in H of a sequence of functions $u_n \in \mathcal{V}_G$, i.e. $u_n = (v_n, T_n) \in \mathcal{V}$ and $E(f^0; u_n) = 0$. According to (i) and to the previous remarks, u_n satisfies (2.9) and so does u.

Conversely, if u belongs to the space on the right-hand side of (3.28), then the corresponding function $\Phi \in H^1(M_p)$ can be approximated in $H^1(M_p)$ by functions $\Phi_n \in C^\infty(M_p)$. The functions u_n associated with Φ_n belong to $\mathcal{V}_G$ and converge to u; therefore $u \in H_G$.

In case (1.13a), the proof is the same, except that $\Phi \in H^1(M_p)$ vanishes on Γ_b and is approximated in $H^1(M_p)$ by functions in $\mathcal{C}^\infty_{b,0}(M_p)$.

The proof of (3.29) is similar but simpler since, in this case, the trace of v on Γ_b is obviously defined. For $u = (v,T) \in \mathcal{V}_G$ we see that (2.9) is satisfied. By the result of [11] already referred to, the corresponding function ϕ belongs to $H^2(M)$, and even to $H^2(M_p)$, the function ϕ being periodic in directions x and y. Furthermore, in case (1.13a), ϕ vanishes on Γ_b whereas no further condition in ϕ appears in case (1.13b). □

Consider now $H_G^\perp$, the orthogonal of H_G in $L^2(H)^3$. We have the following characterization of $H_G^\perp$.

LEMMA 3.4.

$$H_G^\perp = \{u = (v,T) \in L^2(M)^3 \mid \mathrm{div}_3(-k \times v/f^0, K_2T) = 0,\ T|_{\Gamma_u} = 0\}$$

in case (1.13a),

$$H_G^\perp = \{u = (v,T) \in L^2(M)^3 \mid \mathrm{div}_3(-k \times v/f^0, K_2T) = 0,\ T|_{\Gamma_b \cup \Gamma_u} = 0\}$$

in case (1.13b).

PROOF. Let us first assume that $u = (v,T)$ belongs to the space on the right-hand side of the above relation. We want to show that $u \in H_G^\perp$, and it suffices to show that

$$(u,\widetilde{u}) = (v,\widetilde{v}) + (T,\widetilde{T}) = 0, \tag{3.30}$$

for every $\widetilde{u} = (\widetilde{v}, \widetilde{T}) \in \mathcal{V}_G$.

According to part (iii) in the proof of Lemma 3.3, for every $u \in \mathcal{V}_G$, there exists $\widetilde{\phi} \in H^2(M_p)$ such that

$$f^0 k \times \widetilde{v} + \nabla\widetilde{\phi} = 0, \qquad \widetilde{T} = K_2 \frac{\partial\widetilde{\phi}}{\partial\eta}, \tag{3.31}$$

and $\widetilde{\phi}$ vanishes on Γ_b in case (1.13a).

Hence (3.30) becomes

$$\int_M \{v(k \times \nabla\widetilde{\phi}/f^0) + (K_2T) \cdot \partial\widetilde{\phi}/\partial\eta\}\, dM = 0,$$

i.e.

$$\int_M \left\{ -(k \times v/f^0) \cdot \nabla\widetilde{\phi} + (K_2T)\frac{\partial\widetilde{\phi}}{\partial\eta} \right\} dM = 0, \tag{3.32}$$

and upon integration by parts

$$\int_M \{\mathrm{div}_3(k \times v/f^0, -K_2T)\}\widetilde{\phi}\, dM + \int_{\mathcal{O}} (K_2T\widetilde{\phi})|_{\eta=1}\, dx\, dy - \int_{\mathcal{O}} (K_2T\widetilde{\phi})|_{\eta=0}\, dx\, dy = 0. \tag{3.33}$$

Hence (3.30).

Conversely, if u belongs to $H_G^\perp$, then (3.30) holds for every $\widetilde{u} \in \mathcal{V}_G$, i.e. (3.32) holds for all the $\widetilde{\phi}$ described above. Writing first (3.32) with ϕ compactly supported in M, we conclude that

$$\mathrm{div}_3(-k \times v/f^0, K_2T) = 0.$$

Then we integrate by parts and (3.33) shows that $T|_{\Gamma_b} = 0$ in case (1.13b), and $T|_{\Gamma_b \cup \Gamma_u} = 0$ in case (1.13b). □

3.3.2. Lagrange multipliers. Now that the appropriate function spaces have been introduced, we can make clear the role of v^1, T^1 and ϕ_s^1 as Lagrange multipliers for the constraints (2.9) and explain the structure of (2.8).

The key point is to study the structure of functions (ψ, Θ) such that

$$(\psi, v) + (\Theta, T) = 0,$$

for every (v, T) satisfying (2.3) or (2.9). This has been done in Lemma 3.4 and, with a change of notations we conclude (assuming that $(\psi, \Theta) \in L^2(M)^3$) that

$$\operatorname{div}_3(-k \times \psi/f^0, K_2\Theta) = 0, \tag{3.34}$$

with

$$\Theta|_{\Gamma_u} = 0 \quad \text{in case (1.13a)}, \qquad \Theta|_{\Gamma_u \cup \Gamma_b} = 0 \quad \text{in case (1.13b)}.$$

According to [35, Proposition 1.3, Appendix I], (3.34) implies that the vector $(-k \times \psi/f^0, k_2\Theta)$ belongs to Curl $H^1(M)^3$; note that since M is simply connected, the conditions in that proposition reduce to (3.34). Hence there exists a vector of $H^1(M)^3$, which we write in the form $(w, \sigma), w \in H^1(M)^2, \sigma \in H^1(M)$, such that

$$(-k \times \psi/f^0, K_2\Theta) = \operatorname{curl}(w, \sigma) = -k \times \nabla\sigma + k \times \frac{\partial w}{\partial \eta} + k \operatorname{curl} w.$$

Therefore

$$\frac{1}{f^0}\psi = \nabla\sigma - \frac{\partial w}{\partial \eta}, \qquad \Theta = \frac{1}{K_2}\operatorname{curl} w. \tag{3.35}$$

We now set

$$v^1 = -k \times \frac{\partial w}{\partial \eta}, \qquad \phi^1 = -f^0\sigma. \tag{3.36}$$

We observe that curl w vanishes on Γ_u since Θ does, and therefore

$$\operatorname{curl} w = -\mathcal{M}^*\operatorname{curl}\frac{\partial w}{\partial \eta} = -\mathcal{M}^*\operatorname{curl}(k \times v^1) = -\mathcal{M}^*\operatorname{div} v^1.$$

Finally,

$$\psi = -k \times v^1 - \nabla\phi^1, \qquad \Theta = -\frac{1}{K_2}\mathcal{M}^*\operatorname{div} v^1. \tag{3.37}$$

This explains the structure of (2.8).

REMARK 3.2. The proof above uses simple algebraic manipulations and de Rham's Theorem, which is the main ingredient in the proof of the above quoted propositions of [35].

3.3.3. Weak formulation of the quasi-geostrophic equations.

PROBLEM 3.3. *For any given* $u_0^0 = (v_0^0, T_0^0) \in H_G$, *find* $u^0 = (v^0, T^0)$ *such that*

$$u^0 \in L^2(0,t_1;V_G) \cap L^\infty(0,t_1;H_G), \qquad \forall t_1 > 0, \tag{3.38}$$

$$\begin{aligned}\frac{d}{dt}(A_0 u^0, \widetilde{u}) + a(u^0,\widetilde{u}) + a_1(u_s^0;\widetilde{u})\\ + b(u^0,u^0,\widetilde{u}) + e^0(f^1;u^0,\widetilde{u}) = (Q^0,\widetilde{T}), \qquad \forall \widetilde{u} = (\widetilde{v},\widetilde{T}) \in V_G,\end{aligned} \tag{3.39}$$

$$u^0|_{t=0} = u_0^0. \tag{3.40}$$

In (3.39) *above,* $e^0(f^1;u^0,\widetilde{u})$ *is defined by*

$$e^0(f^1;u^0,\widetilde{u}) = \int_M (f^1 k \times v^0)\cdot \widetilde{v}\, dM. \tag{3.41}$$

3.3.4. Solutions of the quasi-qeostrophic equations. As for the PEs, we also have the following existence of global weak solutions of the quasi-geostrophic equations. This theorem can be proved using the Galerkin approximation, and we omit the details.

THEOREM 3.3. *There exists at least one solution for Problem* 3.3, *the weak formulation of the quasi-geostrophic equations.*

4. Elements of the mathematical justification of the Rossby asymptotics

In this section we study in a rigorous manner the asymptotic expansions of the PEs and the PEV^{2}s, starting from the functional form of the equations. The convergence to the geostrophic equations is fully proved. For the convergence to the quasi-geostrophic equations we encounter some difficulties due perhaps to the existence of persistent oscillations (see Remark 4.3); we obtain partial asymptotic results and open questions (see Section 4.2.3).

4.1. Formal asymptotics of the weak formulations of the equations. First of all, we repeat briefly the formal asymptotics studied in Section 2 in terms of the weak formulations. We set

$$\begin{cases} u = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots \\ u^j = (v^j, T^j), \quad j = 0,1,\dots, \end{cases} \tag{4.1}$$

$$
(1.2)\qquad \begin{cases} T_s = T_s^0 + \varepsilon T_s^1 + \varepsilon^2 T_s^2 + \dots\,, \\ Q = Q^0 + \varepsilon Q^1 + \varepsilon^2 Q^2 + \dots\,, \\ f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots \quad (f^0 = 1). \end{cases}
$$

Substituting the above relations into (3.15) and (3.22), i.e. the weak formulations of the PEs and the PEV^{2}s, we obtain the following:

a. *Zeroth order approximation:*

$$
(4.3)\qquad e(f^0; u^0, \widetilde{u}) = 0, \qquad \forall \widetilde{u} \in \mathcal{V},
$$

which provides the *geostrophic motions*. As we indicated in Section 3, the equation (4.3) is equivalent to (2.2), i.e. equations governing the geostrophic motions.

b. *First order approximation:*

$$
(4.4)\qquad \frac{d}{dt}(A_0 u^0, \widetilde{u}) + a(u^0, \widetilde{u}) + a_1(u_s^0; \widetilde{u}) + b(u^0, u^0, \widetilde{u}) \\ + e(f^0; u^1, \widetilde{u}) + e^0(f^1; u^0, \widetilde{u}) = (Q^0, \widetilde{T}), \qquad \forall \widetilde{u} \in \mathcal{V}.
$$

Equations (4.3) and (4.4) are called the quasi-geostrophic equations. Due to the constraint (4.3), we only have to find solutions of (4.4) in the function space with elements satisfying (4.3). This is accomplished by introducing the function spaces $\mathcal{V}_G, V_G$ and H_G. Choosing $\widetilde{u} \in \mathcal{V}_G$ in (4.4), we can see that

$$
e(f^0; u^1, \widetilde{u}) = -e(f^0; \widetilde{u}, u^1) = 0.
$$

Therefore (4.3) and (4.4) amount to finding a solution u^0 in $L^2(0, t_1, V_G) \cap L^\infty(0, t_1; H_G)$ satisfying

$$
(4.5)\qquad \frac{d}{dt}(A_0 u^0, \widetilde{u}) + a(u^0, \widetilde{u}) + a_1(u_s^0; \widetilde{u}) + b(u^0, u^0, \widetilde{u}) + e^0(f^1; u^0, \widetilde{u}) \\ = (Q^0, \widetilde{T}), \qquad \forall \widetilde{u} \in \mathcal{V}_G.
$$

Namely, we obtain here the weak formulation of the quasi-geostrophic equations given by Problem 3.3.

4.2. Convergence of the PEs to the geostrophic and quasi-geostrophic equations.

4.2.1. A priori estimates. First of all, we have the following lemma, giving some a priori estimates independent of ε of the solutions of the PEs; hence they provide some stability of the solutions of the PEs with respect to the Rossby number $\varepsilon = \mathrm{Ro} = U/f^0\Omega$.

LEMMA 4.1 (Stability). *For any solution $u = u_\varepsilon$ of the weak formulation of the PEs, Problem* 3.1, *obtained by the Galerkin method* (*see Theorem* 3.1), *we have*

$$u_\varepsilon \text{ belongs to a subset of } L^2(0,t_1;V)\cap L^\infty(0,t_1;H), \tag{4.6}$$

bounded independently of ε.

PROOF. Here we only present the formal *a priori* estimates of the solutions, which can be made precise by using the Galerkin procedure. Set $\widetilde{u} = u$ in (3.15). We see easily that

$$\frac{1}{2}\varepsilon\frac{d}{dt}|A_0^{1/2}u|^2 + \varepsilon a(u,u) + \varepsilon a_1(u_s;u) \le \varepsilon(Q,T),$$

i.e.

$$\frac{1}{2}\frac{d}{dt}|A_0^{1/2}u|^2 + \frac{1}{R_{\max}}\|u\|^2 \le (Q,T) - a_1(u_s;u) \le c + \frac{1}{2R_{\max}}\|u\|^2.$$

Hence

$$\frac{d}{dt}|A_0^{1/2}u|^2 + \frac{1}{R_{\max}}\|u\|^2 \le c,$$

and (4.6) follows easily. □

4.2.2. Convergence of the PEs to the geostrophic equations. We now study the convergence of the PEs to the geostrophic equations in the sense that the solutions of the PE converge to solutions of the geostrophic equations as the Rossby number $\varepsilon = \mathrm{Ro}$ goes to zero. To this end, we only consider the solutions of Problem 3.1 (the weak formulation of the PEs) obtained by the Galerkin method. The main convergence result is then:

THEOREM 4.1. *There exists a sequence $u_{\varepsilon'}$ of solutions of Problem* 3.1 (*weak formulation of the PEs*) ($\varepsilon' \to 0$) *such that*

$$u_{\varepsilon'} \to u^0 \quad \begin{cases} \text{in } L^2(0,t_1;V) \text{ weakly,} \\ \text{in } L^\infty(0,t_1;H) \text{ weak-star,} \end{cases} \tag{4.7}$$

with u^0 a solution of the geostrophic equations

$$E(f^0;u^0) = 0. \tag{4.8}$$

Moreover, for any sequence $u_{\varepsilon'}$ of solutions of Problem 3.1 *such that* (4.7) *holds true, u^0 is a solution of* (4.26).

PROOF. First of all, by virtue of the *a priori* estimates (4.6) independent of ε, there exists a sequence $u_{\varepsilon'}$ satisfying (4.7). Now we only have to prove that u^0 is a solution of the geostrophic equations (4.8).

By (4.7), we can see easily that in the sense of distributions

$$\varepsilon' \frac{d}{dt}(A_0 u_{\varepsilon'}; \widetilde{u}) + \varepsilon' a(u_{\varepsilon'}, \widetilde{u}) + \varepsilon' a_1(u_s; \widetilde{u}) - \varepsilon'(Q, \widetilde{T}) \to 0 \qquad \text{as } \varepsilon' \to 0.$$

We now consider the nonlinear term. By definition, for any $\widetilde{u} \in \mathcal{V}$,

$$|b(u_{\varepsilon'}, u_{\varepsilon'}, \widetilde{u})| = |b(u_{\varepsilon'}, \widetilde{u}, u_{\varepsilon'})| \leq c|u_{\varepsilon'}| \cdot \|u_{\varepsilon'}\|,$$

with a constant c independent of ε'. Hence thanks to (4.7), it is easy to see that $b(u_{\varepsilon'}, u_{\varepsilon'}, \widetilde{u})$ belongs to a set in $L^2(0, t_1)$ bounded independently of ε'. This yields that

$$\varepsilon' b(u_{\varepsilon'}, u_{\varepsilon'}, \widetilde{u}) \to 0 \qquad \text{as } \varepsilon' \to 0. \tag{4.10}$$

Thanks to the expansion

$$f = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \dots,$$

we find

$$e(f; u_{\varepsilon'}, \widetilde{u}) = e(f^0; u_{\varepsilon'}, \widetilde{u}) + e^0(\varepsilon' f^1 + \dots; u_{\varepsilon'}, \widetilde{u}). \tag{4.11}$$

Passing to the limit, we deduce that

$$\begin{cases} e(f^0; u_{\varepsilon'}, \widetilde{u}) \to b_1(f^0; u^0, \widetilde{u}) & \text{weakly in } L^2(0, t_1), \\ e^0(\varepsilon^1 f^1 + \dots; u_{\varepsilon'}, \widetilde{u}) \to 0 & \text{strongly in } L^2(0, t_1). \end{cases} \tag{4.12}$$

In summary, we obtain

$$(E(f^0; u^0), \widetilde{u}) = e(f^0; u^0, \widetilde{u}) = 0.$$

The proof is complete. □

4.2.3. Convergence of the linearized PEs to the linearized QGs. We have obtained in the previous section the convergence of the PEs to the geostrophic equations. At this time, however, we are not able to consider the next step, namely, to pass to the limit from the PEs to the quasi-geostrophic equations (the first order approximation). This difficulty may possibly be due to the existence of persistent oscillations (see Remark 4.3 below). Technically the main difficulty is that we are not able to establish some compactness properties of the solutions of the PEs. The compactness we need is usually derived from *a priori* estimates of the time derivatives of the solutions, which are not at our disposal at this time.

Due to the lack of compactness, we cannot pass to the limit in the nonlinear terms $b(\cdot,\cdot,\cdot)$. Hence we handle here some simpler cases.

First of all, dropping the nonlinear terms in (3.15) and (4.4), we consider the following linearized primitive equations:

$$\text{(4.13)} \qquad \begin{cases} \varepsilon \dfrac{d}{dt}(A_0 u, \widetilde{u}) + \varepsilon a(u, \widetilde{u}) + \varepsilon a_1(u_s; \widetilde{u}) + e(f^1; u, \widetilde{u}) \\ \qquad = \varepsilon(Q, \widetilde{T}), \quad \forall \widetilde{u} \in \mathcal{V}, \\ u|_{t=t_0} = u_0, \end{cases}$$

and the linearized quasi-geostrophic equations

$$\text{(4.14)} \qquad \begin{cases} \dfrac{d}{dt}(A_0 u^0, \widetilde{u}) + a(u^0, \widetilde{u}) + a_1(u_s^0; \widetilde{u}) + e^0(f^1; u^0, \widetilde{u}) \\ \qquad = (Q^0, \widetilde{T}), \quad \forall \widetilde{u} \in \mathcal{V}_G, \\ u|_{t=0} = u_0^0. \end{cases}$$

It is easy to see that for Q given in $L^2(0, t_1; H)$ and u_0 given in H, there exists a unique solution $u = u_\varepsilon$ of (4.13). Similarly for Q^0 given in $L^2(0, t_1; H_G)$ (or $L^2(0, t_1; H)$), and u_0^0 given in H_G (*but not in* H), there exists a unique solution u^0 of (4.14).

The component of u_0 in the orthogonal of H_G in H produces a boundary layer at $t = 0$ which will be discussed elsewhere ($\varepsilon \to 0$). At this point we prove the following.

THEOREM 4.2. *Assume* $u_0 = u_0^0 + \varepsilon u_0^1 + \dots$ *Then the solutions* $u = u_\varepsilon$ *of* (4.13) *converge to the solution* u^0 *of* (4.14) *in the following sense:*

$$\text{(4.15)} \qquad u_\varepsilon \to u^0 \qquad \begin{cases} \textit{in } L^2(0, t_1; V) \textit{ weakly}, \\ \textit{in } L^\infty(0, t_1; H) \textit{ weak-star}. \end{cases}$$

PROOF. Obviously, the stability estimates (4.6) hold true for solutions $u = u_\varepsilon$ of the linearized PEs (4.13). By virtue of these estimates, there exists a sequence $\varepsilon' \to 0$ such that

$$u_{\varepsilon'} \to u^0 \qquad \begin{cases} \text{in } L^2(0, t_1; V) \text{ weakly}, \\ \text{in } L^\infty(0, t_1; H) \text{ weak-star}. \end{cases}$$

Then we prove that u^0 is a solution of the linearized QGs. To see this, first of all, we notice that

$$\varepsilon \frac{d}{dt}(A_0 u_\varepsilon, \widetilde{u}) + \varepsilon a(u_\varepsilon(t), \widetilde{u}) + \varepsilon a_1(u_s; \widetilde{u}) - \varepsilon(Q, \widetilde{T}) \to 0$$

in the distribution sense. Then by definition

$$e(f;u_{\varepsilon'},\widetilde{u}) = e(f^0;u_{\varepsilon'},\widetilde{u}) + e^0(f-f^0;u_{\varepsilon'},\widetilde{u}) \overset{\varepsilon'\to 0}{\longrightarrow} e(f^0;u^0,\widetilde{u}), \qquad \forall \widetilde{u}\in V.$$

Hence we obtained

$$e(f^0;u^0,\widetilde{u}) = 0, \qquad \forall \widetilde{u}\in \mathcal{V}.$$

Namely, u^0 provides a geostrophic motion and $u^0 \in V_G$.

On the other hand, for any $\widetilde{u}\in \mathcal{V}_G$ and for any continuously differentiable scalar function $\Psi \in \mathcal{C}^1([0,t_1])$ with $\psi(t_1)=0$, we multiply (4.13) by $\psi(t)$ and integrate in t. By integration by parts, we obtain

$$\begin{aligned}(4.16)\qquad -\int_0^{t_1}(A_0u_{\varepsilon'}(t),\widetilde{u}\psi'(t))\,dt + \int_0^{t_1} a(u_{\varepsilon'}(t),\widetilde{u}\psi(t))\,dt\\ + \int_0^{t_1} a_1(u_s;\psi(t)\widetilde{u})\,dt + \frac{1}{\varepsilon'}\int_0^{t_1} e(f;u_{\varepsilon'}(t),\widetilde{u}\psi(t))\,dt\\ = \int_0^{t_1}(Q,\widetilde{T}\psi(t))\,dt + (A_0u_0,\widetilde{u})\psi(0).\end{aligned}$$

By definition, we also have

$$\begin{aligned}&\frac{1}{\varepsilon'}\int_0^{t_1} e(f;u_{\varepsilon'}(t),\widetilde{u}\psi(t))\,dt\\ &=\frac{1}{\varepsilon'}\int_0^{t_1}(1;u_{\varepsilon'}(t),\widetilde{u})\psi(t)\,dt + \frac{1}{\varepsilon'}\int_0^{t_1} e^0(\varepsilon' f^1 + (\varepsilon')^2 f^2 + \dots; u_{\varepsilon'}(t),\widetilde{u})\psi(t)\,dt\\ &= -\frac{1}{\varepsilon'}\int_0^{t_1} e(f^0;\widetilde{u},u_{\varepsilon'}(t)),\psi(t)\,dt + \int_0^{t_1} e^0(f^1+\varepsilon' f^2+\dots;u_{\varepsilon'}(t),\widetilde{u})\psi(t)\,dt\\ &= \int_0^{t_1} e^0(f^1+\varepsilon' f^2+\dots;u_{\varepsilon'}(t),\widetilde{u})\psi(t)\,dt\\ &\to \int_0^{t_1} e^0(f^1,u^0(t),\widetilde{u})\psi(t)\,dt \qquad \text{as } \varepsilon'\to 0.\end{aligned}$$

Finally, in terms of (4.15), we deduce from (4.16) that

$$\begin{aligned}(4.17)\qquad -\int_0^{t_1}(A_0u^0(t),\widetilde{u})\psi(t)\,dt + \int_0^{t_1} a(u^0(t),\widetilde{u}\psi(t))\,dt\\ + \int_0^{t_1} a_1(u_s^0;\widetilde{u}\psi(t))\,dt + \int_0^{t_1} e^0(f^1;u^0(t),\widetilde{u}\psi(t))\,dt\\ \int_0^{t_1}(Q^0,\widetilde{T}\psi(t))\,dt + (A_0u_0^0,\widetilde{u})\psi(0).\end{aligned}$$

It is obvious to see that (4.17) is equivalent to (4.14). □

REMARK 4.1 (An open problem). It is still unknown if the weak solutions of the original (nonlinear) PEs converge to the solution of the QGs. Even though we have the *a priori* estimates (4.6) for the original PEs, we need some *a priori* estimates for the time derivatives of the solutions independent of ε. For steady-state solutions, however, we are able to prove the convergence of the solutions of the PEs to some solutions of the QGs; see Section 4.2.4 hereafter.

4.2.4. Convergence of steady-state solutions of the PEs to steady-state solutions of the QGs. We now consider the (nonlinear) steady-state PEs:

$$\varepsilon a(u,\widetilde{u}) + \varepsilon a_1(u_s;\widetilde{u}) + \varepsilon b(u,u,\widetilde{u}) + e(f;u,\widetilde{u}) = \varepsilon(Q,\widetilde{T}), \qquad \forall \widetilde{u} \in \mathcal{V}, \tag{4.18}$$

and the (nonlinear) steady-state QGs:

$$a(u^0,\widetilde{u}) + a_1(u^0_s;\widetilde{u}) + b(u^0,u^0,\widetilde{u}) + e^0(f^1;u^0,\widetilde{u}) = (Q^0,\widetilde{T}), \qquad \forall \widetilde{u} \in \mathcal{V}_G. \tag{4.19}$$

As for the time dependent case, there exist solutions $u = u_\varepsilon$ of (4.18) and u^0 of (4.19) such that

$$u = u_\varepsilon \in V, \qquad u^0 \in V_G. \tag{4.20}$$

The following theorem establishes the convergence of the solutions $u = u_\varepsilon$ of the steady-state PEs (4.18) to solutions of the steady-state QGs (4.19):

THEOREM 4.3. *There exists a sequence $u_{\varepsilon'}$ of solutions of* (4.18) *such that*

$$u_{\varepsilon'} \xrightarrow{\varepsilon' \to 0} u^0 \qquad \text{weakly in } V \text{ and strongly in } H, \tag{4.21}$$

u^0 being a solution of (4.19)*. For any sequence $u_{\varepsilon'}$ of solutions of* (4.18) *satisfying* (4.21)*, u^0 is a solution of* (4.19).

PROOF. As in the evolution case, it is easy to prove that for any solution $u = u_\varepsilon$ of (4.18), we have

$$\|u_\varepsilon\| \le c, \tag{4.22}$$

c being a constant independent of ε. Thanks to the compactness of the embedding of V into H, we readily obtain a sequence $u_{\varepsilon'}$ of solutions of (4.18) satisfying (4.21). We then only have to prove that u^0 is a solution of (4.19). To this end, we infer from (4.18) that

$$a(u_{\varepsilon'},\widetilde{u}) + a_1(u_s;\widetilde{u}) + b(u_{\varepsilon'},u_{\varepsilon'},\widetilde{u}) + \frac{1}{\varepsilon'} e(f;u_{\varepsilon'},\widetilde{u}) = (Q,\widetilde{T}), \qquad \forall \widetilde{u} \in \mathcal{V}_G. \tag{4.23}$$

Obviously $b(u_{\varepsilon'},u_{\varepsilon'},\widetilde{u}) = -b(u_{\varepsilon'},\widetilde{u},u_{\varepsilon'}) \to -b(u^0,\widetilde{u},u^0)$ as $\varepsilon' \to 0$, and

$$\begin{aligned}\frac{1}{\varepsilon'} e(f;u_{\varepsilon'},\widetilde{u}) &= \frac{1}{\varepsilon'} e(f^0;u_{\varepsilon'},\widetilde{u}) + \frac{1}{\varepsilon'} e^0(\varepsilon' f^1 + \dots; u_{\varepsilon'},\widetilde{u}) \\ &= e^0(f^1 + \varepsilon' f^2 + \dots; u_{\varepsilon'},\widetilde{u}) \xrightarrow{\varepsilon' \to 0} e^0(f^1;u^0,\widetilde{u}).\end{aligned}$$

Hence passing to the limit $\varepsilon' \to 0$ in (4.23) implies immediately (4.19).

The proof is complete. □

REMARK 4.2. With obvious notations (see e.g. [24]), equation (3.13) can be written in the form of a differential operational equation

$$\frac{d}{dt}A_0 u_\varepsilon + Au_\varepsilon + B(u_\varepsilon, u_\varepsilon) + \frac{1}{\varepsilon}E(f; u_\varepsilon) = Q - A_1(T_s). \tag{4.24}$$

For $\varepsilon \to 0$, equation (4.24) appears as a penalized equation, the penalty operator being the antisymmetric operator $u \mapsto E(u)$. Penalization has been extensively studied and is well understood when the penalty operator is either linear positive definite or nonlinear and monotone (see R. Courant [7], J.-L. Lions [21], R. Temam [35], [36]). However, little is known about other forms of penalty. The following very simple differential system in $\mathbb{R}^2$ shows (by explicit calculations) that persistent oscillations can appear ($\lambda_1, \lambda_2 > 0$):

$$\begin{cases} \dfrac{du_{1,\varepsilon}}{dt} + \lambda_1 u_{1,\varepsilon} - \dfrac{1}{\varepsilon}u_{2,\varepsilon} = f_1, \\ \dfrac{du_{2,\varepsilon}}{dt} + \lambda_2 u_{2,\varepsilon} + \dfrac{1}{\varepsilon}u_{1,\varepsilon} = f_2. \end{cases} \tag{4.25}$$

The corresponding geostrophic solution u^0 is

$$u_1^0 = u_2^0 = 0, \tag{4.26}$$

and the corresponding quasi-geostrophic limit is the system

$$\begin{cases} \dfrac{du_1^0}{dt} + \lambda_1 u_1^0 - u_2^1 = f_1^0, \\ \dfrac{du_2^0}{dt} + \lambda_2 u_2^0 + u_1^1 = f_2^0, \\ u_2^1 = -f_1^0, u_1^1 = f_2^0. \end{cases} \tag{4.27}$$

By explicit calculation when $f_1 = f_2 = 0$, we find that, for $\varepsilon \to 0$,

$$u \sim e^{-\alpha t}\mathrm{Re}\left\{ \begin{pmatrix} u_{01} - iu_{02} \\ (iu_{01}) + u_{02} \end{pmatrix} e^{i\beta t/\varepsilon} \right\}.$$

Here $\alpha = (\lambda_1 - \lambda_2)/2 > 0, u_0 = (u_{0_1}, u_{0_2})$ is the initial data for (4.25) and $\beta > 0$ is a real number close to 1.

Therefore it appears that, as $\varepsilon \to 0$, the solutions of (4.25) cannot converge strongly (persistent oscillations) to the solution of (4.26)–(4.27), namely to 0. Since the functional structure of (4.25) is the same as that of the linearized equation (4.24), one can speculate that such oscillations also occur in (4.24) or that more specific informations on the structure of (3.13), (4.24) are needed for passing to the limit $\varepsilon \to 0$.

REMARK 4.3. The reader is referred to [14], [15] and [16] for related oscillatory problems involving different timescales.

4.3. Convergence of the PEV²s to the geostrophic equations. We now study the convergence, as $\varepsilon \to 0$, of the solutions of the PEV²s to the solutions of the geostrophic and quasi-geostrophic equations.

4.3.1. A priori estimates. In this subsection, we present some *a priori* estimates of the solutions[4] $u = u_\varepsilon$ of the PEV²s independently of ε. As in Section 4.2.1, it is easy to establish the following estimates:

$$\|u_\varepsilon\|_{L^2(0,t_1;V^\omega)\cap L^\infty(0,t_1;H)} \leq c, \tag{4.28}$$

the constant c being independent of ε.

4.3.2. Convergence of the PEV²s to the geostrophic equations. We now state the main theorem in this section; the proof can be carried out in the same fashion as we did in Section 4.2:

THEOREM 4.5. *There exists a sequence $u_{\varepsilon'}$ of solutions of Problem* 3.2 (*weak formulation of the PEV²s) such that as $\varepsilon' \to 0$,*

$$u_{\varepsilon'} \to u^0 \quad \begin{cases} \text{in } L^2(0,t_1;V^\omega) \text{ weakly,} \\ \text{in } L^\infty(0,t_1;H) \text{ weak-star,} \end{cases} \tag{4.29}$$

with u^0 a solution of the geostrophic equation

$$E_1(f^0;u^0) = 0. \tag{4.30}$$

For any sequence $u_{\varepsilon'}$ of solutions of Problem 3.2 *such that* (4.29) *holds true, u^0 is a solution of* (4.26).

REMARK 4.4. As in Sections 4.2.2–4.2.3, we can also study the linearized PEV²s and steady-state PEV²s. The corresponding results are also true here. We omit the details.

REMARK 4.5. In a forthcoming article, we shall study the Rossby asymptotics of the PEs and the PEV²s for the whole globe, leading to the global geostrophic equations and quasi-geostrophic equations. Similar mathematical justifications as we did here in Section 4 hold true in that case as well. □

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Manuscript received November 18, 1994

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