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NONLINEAR INTEGRAL INCLUSIONS OF HAMMERSTEIN TYPE

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Dedicated to Professor Ky Fan

1. Integral inclusions of Hammerstein type

Let Ω be a bounded domain in Euclidean space, $k : \Omega\times\Omega \to \mathbb{R}^{N\times N}$ a matrix-valued kernel function, and $f : \Omega \times \mathbb{R}^N \to \mathrm{CpCv}(\mathbb{R}^N)$ a (multivalued) nonlinear function, where $\mathrm{CpCv}(\mathbb{R}^N)$ denotes the system of all nonempty compact convex subsets of $\mathbb{R}^N$. Consider the linear integral operator

$$Ky(s) = \int_\Omega k(s,t)y(t)\,dt \tag{1}$$

generated by k, and the (multivalued) superposition operator (see e.g. [3])

$$N_f x(t) = \{y(t) : y \text{ measurable selection of } f(\,\cdot\,, x(\,\cdot\,))\} \tag{2}$$

generated by f. The present paper is concerned with the integral inclusion of Hammerstein type

$$x \in KN_f x. \tag{3}$$

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Evidently, if the nonlinearity f is singlevalued, i.e. $f(t,u) = \{g(t,u)\}$ for some function $g : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ which generates a (singlevalued) superposition operator

$$N_g x(t) = g(t, x(t)), \tag{4}$$

the inclusion (3) reduces to the classical integral equation (system) of Hammerstein type

$$x(s) = \int_\Omega k(s,t) g\big(t, x(t)\big)\, dt. \tag{5}$$

There exist various motivations for studying inclusions of type (3); let us mention some of them.

First of all, when investigating boundary value problems in physics, mechanics, or control theory which define the state x of a system by an acting force h, one is led to equations of the form

$$Lx = h, \tag{6}$$

where L is a linear operator on an appropriate function space. Now, if the force h is perturbed, i.e. is subject to both the state x and an "undetermined noise", (6) has to be replaced by the equation with multivalued right-hand side

$$Lx \in Nx, \tag{7}$$

where N is some multivalued nonlinear operator (for example, the operator (2)). In many cases L is some differential operator which admits a Green function on a space determined by suitable boundary conditions. In this case the problem (7) may be written in the form (3) by putting $K = L^{-1}$.

The second motivation is related to "nonsmooth" calculus of variations (see e.g. the monograph [16]). Suppose that we are interested in minimizing the energy functional

$$\Psi x = \int_\Omega \{h(x(s)) - g(s, x(s))\}\, ds, \tag{8}$$

where h denotes the kinetic energy of the system, and g is a potential energy generating a (singlevalued) superposition operator (4). Assume further that the function (8) is not differentiable in the usual sense, due to some lack of regularity of the operator (4), but admits a generalized gradient or subgradient in the sense, for instance, of Clarke's generalized gradient, Aubin's contingent cone, Ioffe's fan, etc. (see e.g. [6, 8, 16, 18, 19]). Consequently, the problem of minimizing (8) leads to the study of boundary value problems for the "Euler–Lagrange inclusion"

$$Lx \in \partial N_g x, \tag{9}$$

where ∂N_g is one of the generalized gradients or subgradients mentioned above. The problem (9) in turn is in various function spaces equivalent to the Hammerstein inclusion (3).

Finally, we mention another typical situation where the inclusion (3) arises quite naturally. Suppose that $g : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a (singlevalued) function which, however, is so "badly behaved" that one cannot apply the usual solvability criteria to the Hammerstein equation (5). In nonlinear multivalued analysis it is then a standard device to pass from g to another nonlinearity f which is usually multivalued and has "nicer" properties. A useful choice is often the convexification

$$g^{\square}(t,u) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{g(t,v) : |v-u| \le \varepsilon\} \tag{10}$$

of the function $g(t,\,\cdot\,)$ and, similarly, the convexification

$$N_g^{\square}x = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{N_g z : \|z-x\| \le \varepsilon\} \tag{11}$$

of the operator (4). As was shown in [5] (see also [28, Theorem 27.1]), the equality

$$N_{g^{\square}} = N_g^{\square} \tag{12}$$

holds in many function spaces, i.e. the operator generated by the convexification of g coincides with the convexification of the operator generated by g. Thus, putting $f(t,u) = g^{\square}(t,u)$ one arrives again at the Hammerstein inclusion (3). Moreover, it is then possible to apply classical fixed point principles for multivalued operators to (3), since the operator $N_f = N_g^{\square}$ has nicer properties than the operator N_g (for example, N_f is always closed and "often" upper semicontinuous). An example of an application, where these properties of the convexification are useful, may be found in [21].

The aim of this paper is to prove a fairly general solvability theorem for the Hammerstein inclusion (3), and to illustrate this theorem by means of three applications, namely boundary value problems for elliptic differential inclusions (i.e. elliptic equations with multivalued right hand side), forced periodic oscillations in nonlinear control problems with "noise", and critical points of nonsmooth energy functionals. We remark that some existence theorems have been given in several papers in the last 20 years. These existence theorems are mainly based on fixed point principles of Nadler [31], Kakutani [24], and Bohnenblust–Karlin [10] which may be considered as "multivalued analogues" of the classical fixed point principles of Banach–Caccioppoli, Brouwer, and Schauder, respectively. More sophisticated results have been obtained in [17], where eigenvalue problems for

(3) are studied by means of a topological characteristic (called "genus") for multivalued operators. In what follows, we shall apply (a multivalued variant of) the classical Leray–Schauder continuation principle to derive an existence theorem for (3). More precisely, we shall prove the existence of solutions in so-called ideal spaces of vector functions which embrace various classes of measurable vector functions arising in applications. The interested reader may find applications of the Hammerstein inclusion (3) to specific problems in mechanics and physics in [15, 21, 22, 32].

2. A multivalued continuation principle

Given a Banach space X, we write as before $\mathrm{CpCv}(X)$ for the family of all nonempty compact convex subsets of X. Recall that a (multivalued) operator $A : X \to \mathrm{CpCv}(X)$ is called *compact* if $A(U) = \bigcup\{Ax : x \in U\}$ is precompact for every bounded $U \subset X$. The following is a multivalued analogue to the well-known Leray–Schauder continuation principle for nonlinear compact operators.

Theorem 1. *Let X be a Banach space and $A : X \to \mathrm{CpCv}(X)$ a compact upper semicontinuous operator. Suppose that there exists an $r > 0$ such that the a priori estimate*

$$x \in \lambda Ax \ (0 < \lambda \leq 1) \Rightarrow \|x\| \leq r \tag{13}$$

holds. Then the inclusion $x \in Ax$ has at least one solution in the ball $\{x : \|x\| \leq r\}$.

Proof. Let U be any open ball around zero of radius $R > r$. Then the multivalued vectorfield Φ defined by $\Phi x = x - Ax$ is nondegenerate on ∂U, by (13). Consequently, the rotation $\gamma(\Phi; \partial U)$ of Φ on ∂U satisfies $\gamma(\Phi; \partial U) = 1$ (see e.g. [12, Theorem 2.3.44]), and thus the operator A has a fixed point. □

We remark that Theorem 1 is also true for more general operators such as condensing or limit-compact multivalued operators (see e.g. [11, 18]). The crucial point in applying Theorem 1 is of course the verification of the a priori estimate (13). Roughly speaking, the a priori estimate (13) may often be verified if the operator A grows so "rapidly" that one cannot find invariant balls for A. Thus, the Leray–Schauder continuation principle applies to operators of fast growth in rather the same way as the Schauder fixed point principle does to operators of slow growth (see also the remark at the end of Section 3 below).

To apply Theorem 1 to the Hammerstein operator $A = KN_f$, suitable choices for X are the space C of continuous functions, the Hölder spaces C^α, the Lebesgue spaces L_p, the Orlicz spaces L_Φ, or, more generally, ideal spaces of vector functions. Recall [37] that a Banach space X of measurable vector functions $x : \Omega \to \mathbb{R}^N$ is called an *ideal space* if $x \in X$ and $\alpha \in L_\infty(\Omega, \mathbb{R})$ implies that $\alpha x \in X$ and $\|\alpha x\|_X \leq \|\alpha\|_{L_\infty} \|x\|_X$. In the scalar case $N = 1$, ideal spaces are just Banach lattices with monotone norm (see e.g. [30, 35, 36]). In the vector case $N > 1$, the theory of ideal spaces is more involved and requires tools from Convex Analysis. A prominent example of an ideal space is the Orlicz space $L_\Phi = L_\Phi(\Omega, \mathbb{R}^N)$ with the Luxemburg norm

$$\|x\|_{L_\Phi} = \inf\left\{k : k > 0,\ \int_\Omega \Phi[x(s)/k]\, ds \leq 1\right\}, \tag{14}$$

where Φ is a given Young function (see e.g. [33, 38]).

Let X be an ideal space with the property that there exists a sequence $x_1, x_2, \dots$ in X such that the linear hull of $\{x_1(s), x_2(s), \dots\}$ is dense in $\mathbb{R}^N$ for almost all $s \in \Omega$. Consider the set X' of all measurable functions $x' : \Omega \to \mathbb{R}^N$ for which the pairing

$$\langle x, x' \rangle = \int_\Omega (x(s), x'(s))\, ds \tag{15}$$

is finite, where $(\,\cdot\,,\,\cdot\,)$ is the usual scalar product in $\mathbb{R}^N$. Equipped with the natural norm

$$\|x'\|_{X'} = \sup\{\langle x, x' \rangle : \|x\|_X \leq 1\}, \tag{16}$$

this set is then also an ideal space, called the *associate space* of X. The space X' coincides with the dual space X^* of X if and only if all elements $x \in X$ have absolutely continuous norms; in this case the ideal space X is called *regular*. For example, if X is the Orlicz space L_Φ generated by the Young function Φ, then X' is the Orlicz space $L_{\Phi'}$ generated by the associate Young function

$$\Phi'(v) = \inf\{(u, v) - \Phi(u) : u \in \mathbb{R}^N\};$$

moreover, the space L_Φ is regular if and only if Φ satisfies the Δ_2 condition [33].

3. The main theorem

Let Y be an ideal space which contains the Lebesgue space $L_2(\Omega, \mathbb{R}^N)$. Following M. A. Krasnosel'skiĭ [26], we call a linear operator $K : Y \to Y'$ *positive* if

$$\langle Ky, Ky \rangle \leq \mu \langle y, Ky \rangle \qquad (y \in Y) \tag{17}$$

for some $\mu > 0$; the smallest μ with this property will be denoted by $\mu(K; Y)$ in the sequel. By $\mathfrak{B}(K; Y)$ we denote the family of all ideal spaces X such that $K(Y) \subseteq X$ and

$$\|Ky\|_X^2 \leq \delta \langle y, Ky \rangle \qquad (y \in Y) \tag{18}$$

for some $\delta > 0$. Examples of spaces $X \in \mathfrak{B}(K; Y)$ will be given below.

Theorem 2. *Let $k : \Omega \times \Omega \to \mathbb{R}^{N\times N}$ be a matrix function, $f : \Omega \times \mathbb{R}^N \to \mathrm{CpCv}(\mathbb{R}^N)$ a multivalued nonlinearity, and X and Y two ideal spaces over Ω with the following properties:*

(a) *The* (*singlevalued*) *operator* (1) *acts from Y into X, the* (*multivalued*) *operator* (2) *acts from X into Y, and the composition $KN_f : X \to \mathrm{CpCv}(X)$ is upper semicontinuous and compact.*
(b) *The operator K is positive, and $X \in \mathfrak{B}(K;Y)$.*
(c) *The function f satisfies the unilateral estimate*

$$\sup_{v\in f(t,u)} (u,v) \leq a|u|^2 + b(t) \tag{19}$$

for some $a \geq 0$ and $b \in L_1(\Omega,\mathbb{R})$.

Then the Hammerstein inclusion (3) *has at least one solution in X if $a\mu(K;Y) < 1$.*

Proof. We apply Theorem 1 to the operator $A = KN_f$ in X. To this end, suppose that $x \in \lambda KN_f x$ for some $\lambda \in (0,1]$ and $x \in X$, i.e. $x = \lambda Ky$ for some $y \in N_f x$. By (c) this implies that

$$\langle x, y\rangle \leq a\langle x, x\rangle + \|b\|_{L_1}. \tag{20}$$

By the positivity condition (17) we have in turn

$$\langle x, x\rangle = \lambda^2\langle Ky, Ky\rangle \leq \lambda^2\mu(K;Y)\langle y, Ky\rangle = \lambda\mu(K;Y)\langle y, x\rangle. \tag{21}$$

Combining (20) and (21) yields

$$\langle x, y\rangle \leq \frac{\|b\|_{L_1}}{1 - a\mu(K;Y)} < \infty.$$

Now, the hypothesis $X \in \mathfrak{B}(K;Y)$ implies that

$$\delta(K) = \sup\{\|Ky\|_X^2 : y \in Y,\ \langle y, Ky\rangle \leq 1\} < \infty. \tag{22}$$

We conclude that the a priori estimate (13) holds with

$$r = \left(\frac{\delta(K)\|b\|_{L_1}}{1 - a\mu(K;Y)}\right)^{1/2},$$

and the assertion follows from the hypothesis (a) on the operator $A = KN_f$. $\square$

We make some remarks on condition (17). In the Russian literature, this condition is usually attributed to M. A. Krasnosel'skiĭ (see [26], where also a special variant of Theorem 2 for singlevalued f in Lebesgue spaces is given). However, essentially the same condition, as well as the condition (18), had been introduced 3 years before Krasnosel'skiĭ by P. Hess [23]. In [23] it is also proved that every angle-bounded operator in the sense of H. Amann [1] is positive in the sense of (17). The first papers where these conditions are discussed in the setting of general ideal spaces seem to be [39, 40].

The question arises how to verify the hypotheses (a)–(c) of Theorem 2. A natural growth condition on f under which the upper semicontinuity of the function $f(t,\cdot)$ implies the upper semicontinuity of the operator N_f between Lebesgue spaces can be found in [13]. We give a more general set of sufficient conditions; for the proof and the terminology see [4].

PROPOSITION. *Let $f:\Omega\times\mathbb{R}^N\to \mathrm{CpCv}(\mathbb{R}^N)$ be a superpositionally measurable function, and assume that the corresponding superposition operator* (2) *acts between two ideal spaces X and Y. Suppose that one of the following three conditions is satisfied:*

(α) *f is a Carathéodory function, and the space Y is regular.*

(β) *$f(t,\cdot)$ is upper semicontinuous for almost all $t\in\Omega$, and N_f maps U-bounded sets in X into U-bounded sets in Y.*

(γ) *$f(t,\cdot)$ has closed graph in $\mathbb{R}^N\times\mathbb{R}^N$, and both spaces X and Y are regular.*

Then the superposition operator N_f generated by f is upper semicontinuous between X and Y.

Condition (b) of Theorem 2 holds, for example, if K maps Y into Y' and is normal (in particular, self-adjoint) and positive definite in $L_2(\Omega,\mathbb{R}^N)$. In this case one may put

$$\mu=\|K\|,\qquad \delta=\|K^{1/2}\|^2$$

in (17) and (18), respectively.

Condition (c) of Theorem 2 is a multivalued version of the classical one-sided estimate

$$uf(t,u)\le a|u|^2+b(t)$$

for a singlevalued scalar function $f:\Omega\times\mathbb{R}\to\mathbb{R}$. The meaning of our estimate (19) may be illustrated by a comparison with analogous hypotheses in fixed point principles for multivalued operators. In fact, to apply the fixed point principle of Bohnenblust–Karlin, say, one has to guarantee the existence of an invariant ball for the operator $A=KN_f$. Introducing the growth function

$$\varphi(r)=\sup\{\|y\|:y\in N_fx,\ \|x\|\le r\}$$

of the multivalued superposition operator (2), the existence of an invariant ball for $A=KN_f$ means that $\varphi(\|K\|r)\le r$ for some $r>0$; this condition is obviously more restrictive than the growth condition (19).

We point out that the hypotheses (a)–(c) of Theorem 2, as well as the hypotheses (α)–(γ) of the above proposition are easily verified if X and Y are Orlicz spaces (in particular, Lebesgue spaces).

4. First application: Multivalued elliptic systems

Let Ω be a bounded domain in Euclidean space with smooth boundary $\partial\Omega$, $f : \Omega \times \mathbb{R}^N \to \mathrm{CpCv}(\mathbb{R}^N)$ a Carathéodory function, and L a uniformly elliptic linear differential operator of order $2k$ in divergence form, i.e.

$$Lx(s) = \sum_{|\alpha|,|\beta|\le k} D^\alpha(a_{\alpha\beta}(s)D^\beta)x(s) \qquad (s \in \Omega) \tag{23}$$

with matrixvalued smooth coefficient functions $a_{\alpha\beta} : \Omega \to \mathbb{R}^{N\times N}$. Consider the system

$$Lx(s) \in f(s, x(s)) \qquad (s \in \Omega), \tag{24}$$

subject to the Dirichlet boundary condition

$$D^\gamma x(s) = 0 \qquad (s \in \partial\Omega,\ |\gamma| \le k). \tag{25}$$

Suppose that the linear problem

$$Lx(s) = y(s) \qquad (s \in \Omega)$$

with boundary condition (25) has a unique generalized solution $x = Ky$, where the (integral) operator K maps the Sobolev space $H^{-k} = H^{-k}(\Omega, \mathbb{R}^N)$ into the Sobolev space $H_0^k = H_0^k(\Omega, \mathbb{R}^N)$ and is bounded. Sufficient conditions for the existence and boundedness of the operator K may be found in a vast literature on linear elliptic operators (see e.g. [20, 29]). For our purpose, the classical Gårding inequality

$$\langle Lx, x\rangle \ge \alpha\|x\|^2_{H_0^k} \qquad (x \in H_0^k) \tag{26}$$

is sufficient. Define an ideal space Z by

$$Z = \begin{cases} L_{2N/(N-2k)} & \text{if } N > 2k, \\ L_\Phi & \text{if } N = 2k, \\ L_\infty & \text{if } N < 2k; \end{cases} \tag{27}$$

here L_Φ is the Orlicz space generated by the Young function $\Phi(u) = e^{|u|^2} - 1$ ($u \in \mathbb{R}^N$). By classical imbedding theorems of Sobolev, Pokhozhaev and Trudinger (see e.g. [20]), the operator K acts then also between the ideal spaces $Y = Z'$ and $Y' = Z$. Moreover, if $X \supseteq Z$ is any ideal space with the property that the unit ball of Z is an absolutely bounded subset of X (for example, $X = L_p$ with $1 \le p < 2N/(N - 2k)$ for $N > 2k$ and $1 \le p < \infty$ for $N \le 2k$), then H_0^k is compactly imbedded in X, and hence K is compact and self-adjoint as an operator from X' into X. From the continuity of the imbeddings $H_0^k \subseteq L_2$ and $H_0^k \subseteq Z \subseteq X$ it follows that

$$\|x\|_{H_0^k} \geq c \max\{\|x\|_{L_2}, \|x\|_X\} \qquad (x \in H_0^k)$$

for some constant $c > 0$. Combining this with Gårding's inequality (26) we get

$$\langle Lx, x \rangle \geq \alpha c^2 \max\{\|x\|_{L_2}^2, \|x\|_X^2\} \qquad (x \in H_0^k),$$

which shows that the operator $K = L^{-1}$ is positive in the sense of (17), and that $X \in \mathfrak{B}(K; Y)$ in the sense of (18).

Finally, the inequality (19) leads here to the condition

$$\sup\left\{\sum_{j=1}^N u_j v_j : v_j \in f_j(s, u_1, \ldots, u_N)\right\} \leq a \sum_{j=1}^N u_j^2 + b(s) \qquad (b \in L_1(\Omega, \mathbb{R})).$$

If this is satisfied, we may apply Theorem 2 and get an existence result for the elliptic system (24) with boundary condition (25). The simplest example is, of course, the case $k = 1$ and $L = -\Delta$.

We remark that the papers [14] and [34] contain existence results for the scalar equation

$$-\Delta x(s) = g(s, x(s)) \qquad (s \in \Omega),$$

where the discontinuous nonlinearity g is supposed to satisfy the growth condition

$$\limsup_{u \to \pm\infty} \frac{g(s,u)}{u} < \infty \tag{28}$$

uniformly with respect to $s \in \Omega$. Putting $f(s, u) = g^{\square}(s, u)$ (see (10)), it is clear that the growth condition (28) for g implies the growth condition (19) for f, but not vice versa. Thus, the existence results in [14] and [34] follow from our Theorem 2 by putting $f = g^{\square}$.

5. Second application: Forced oscillations in control systems

Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathrm{CpCv}(\mathbb{R}^N)$ be a Carathéodory function which is 2π-periodic in the first argument. For $j = 1, \ldots, N$ consider the polynomials

$$L_j(\xi) = \xi^{p_j} + a^j_{p_j-1}\xi^{p_j-1} + \ldots + a^j_2\xi^2 + a^j_1\xi + a^j_0 \tag{29}$$

and

$$M_j(\xi) = \xi^{q_j} + b^j_{q_j-1}\xi^{q_j-1} + \ldots + b^j_2\xi^2 + b^j_1\xi + b^j_0, \tag{30}$$

where $\deg L_j = p_j > q_j = \deg M_j$. We are interested in finding 2π-periodic solutions (so-called forced periodic oscillations) in the nonlinear control system with multivalued right-hand side (so-called nonlinearity with indetermined noise) described by

$$L_j\left(\frac{d}{ds}\right)x_j(s) \in M_j\left(\frac{d}{ds}\right)f_j(s, x_1(s), \dots, x_N(s)) \qquad (j = 1, \dots, N). \tag{31}$$

Let

$$\alpha(L_j, M_j) = \inf_k \frac{\operatorname{Re}[L_j(-ik)M_j(ik)]}{|M_j(ik)|^2},$$

where the infimum is taken over all indices k such that $M_j(ik) \neq 0$; moreover, fix numbers $\alpha_j \in (-\infty, \alpha(L_j, M_j))$. By [27, Theorem 26.1], the numbers ik are then not roots of the equations $L_j(z) = \alpha_j M_j(z)$ $(j = 1, \dots, N)$; consequently, the system (31) is equivalent to the system

$$\left[L_j\left(\frac{d}{ds}\right) - \alpha_j M_j\left(\frac{d}{ds}\right)\right]x_j(s) \in M_j\left(\frac{d}{ds}\right)\widetilde{f}_j(\alpha_j; s, x_1(s), \dots, x_N(s)), \tag{32}$$

where

$$\widetilde{f}_j\,(\alpha_j; t, u) = f_j(t, u) - \alpha_j u_j. \tag{33}$$

The system (32) in turn is equivalent to the Hammerstein inclusion

$$x_j(s) \in \int_0^{2\pi} h_j(\alpha_j; s - t)\,\widetilde{f}_j[\alpha_j; t, x(t)]\,dt, \tag{34}$$

where $h_j(\alpha_j; \,\cdot\,)$ is the so-called impulse-frequency characteristic of the nonlinear link f_j with respect to the transfer function

$$W(\alpha_j; z) = \frac{M_j(z)}{L_j(z) - \alpha_j M_j(z)}. \tag{35}$$

The linear integral operator K_j defined by the characteristic $h_j(a_j; \,\cdot\,)$ is, in general, not self-adjoint, but normal. Moreover, K_j is bounded as an operator from $Y = L_p([0, 2\pi], \mathbb{R}^N)$ into $Y' = L_{p/(p-1)}$ if $1 \leq p \leq 2$ and even compact if $p > 1$. In [25] it is shown that the operator $K = (K_1, \dots, K_N) : Y^N \to (Y')^N$ is positive in the sense of (17) and satisfies $Y' \in \mathfrak{B}(K; Y)$ if the additional condition

$$\deg P_j > 2 \deg M_j \qquad (j = 1, \dots, N)$$

holds, where $P_j(z) = \operatorname{Re}[L_j(-iz)M_j(iz)]$. Finally, the inequality (19) leads here to the condition

$$\sup\left\{\sum_{j=1}^N u_j v_j : v_j \in f_j(s, u_1, \dots, u_N)\right\} \leq \sum_{j=1}^N a_j u_j^2 + b(s) \qquad (b \in L_1([0, 2\pi], \mathbb{R}))$$

with $a_j < \alpha(L_j, M_j)$. If this is satisfied, we may apply Theorem 2 and get an existence result for forced 2π-periodic oscillations in the control system (31).

We remark that the inequality (19) is satisfied, for example, in control systems with a simple circuit governed by one (singlevalued) nonlinear link g and some set U of admissible controls u. Here (31) reduces to the single integral inclusion

$$L\left(\frac{d}{ds}\right)x(s) \in M\left(\frac{d}{ds}\right)f(s,x(s)),$$

where $f(s,x(s)) = \{g(x(s)) + u(s) : u \in U\}$ satisfies

$$\sup_{u\in U} |x(s)|\,|g(x(s)) + u(s)| \le a|x(s)|^2 + b(s)$$

(see [27, Section 26]), and hence (19). A detailed discussion of the control system (31) for singlevalued f may be found in the recent book [9].

6. Third application: Critical points of non-smooth energy functionals

Let $L : H_0^k \to H^{-k}$ be again a uniformly elliptic operator (23) which satisfies Gårding's inequality (26). Suppose that $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a (singlevalued) Carathéodory function such that $g(s,\cdot\,)$ is locally Lipschitz for almost all $s \in \Omega$. Finally, we assume that the corresponding superposition operator (4) is bounded from the ideal space Z defined in (27) into $L(\Omega,\mathbb{R})$. Under these assumptions, the energy functional

$$\Psi x = \frac{1}{2}\langle Lx, x\rangle - \int_\Omega g(s,x(s))\,ds \tag{36}$$

is correctly defined on the space $H_0^k = H_0^k(\Omega,\mathbb{R}^N)$. Moreover, the functional

$$\Gamma x = \int_\Omega g(s,x(s))\,ds \tag{37}$$

is locally Lipschitz both from Z into $\mathbb{R}$ and from H_0^k into $\mathbb{R}$. By [16, Theorem 2.1.2], the generalized gradient $\partial\Gamma x$ of the functional (37) acts both from Z into $\mathrm{Cv}(Z')$ and from H_0^k into $\mathrm{Cv}(H^{-k})$, and hence the same is true for the generalized gradient

$$\partial\Psi x = Lx - \partial\Gamma x \tag{38}$$

of the functional (36). A critical point of (36) is, by definition, any element $x \in H_0^k$ such that $0 \in \partial\Psi x$; by (38), this may be written equivalently as

$$Lx \in \partial\Gamma x. \tag{39}$$

To reduce (39) to the form (24), we have to find a multivalued Carathéodory function $f : \Omega \times \mathbb{R}^N \to \mathrm{CpCv}(\mathbb{R}^N)$ such that $\partial\Gamma x = N_f x$. This problem was solved in [7] (see also [16, Theorem 2.7.3 and Theorem 2.7.5]). In fact, if we put $f(s,u) = \partial_u g(s,u)$ (the generalized gradient of the function $g(s,\cdot\,)$) we always have $\partial\Gamma x \subseteq N_f x$, and equality holds if $g(s,\cdot\,)$ is "regular" in the sense of [16]; in particular, $\partial\Gamma x = N_f x$ if the function $g(s,\cdot\,)$ is convex for almost all $s \in \Omega$.

Combining the previous assumptions and the assumptions made in Section 4, we get an existence result for critical points of the energy functional (36) by means of Theorem 2. The crucial condition (19) in Theorem 2 is here nothing else than the coercivity of the functional Ψ (see again [16]).

We remark that other existence results for critical points of (36) have been obtained by means of "nonsmooth variants" of classical minimax principles (e.g., mountain pass lemmas) in [14], and of classical dual variational principles in [2].

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