

## ON THE GENUS OF SOME SUBSETS OF $G$ -SPHERES

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*Dedicated to Ky Fan*

### Introduction

The problem of estimating the genus of  $G$ -spaces ( $G$ -category) attracts considerable attention (see, for instance, [Bar1, Bar2, Fa, FaHu, Kr, LS, Šv] and others). At least two approaches to this problem exist: geometric, based on Borsuk–Ulam type theorems, and homological, based on (co)homological arguments in the study of orbit spaces.

Historically, the first result concerning this problem is the famous Lusternik–Schnirelman Theorem stating that the category of the  $n$ -dimensional real projective space equals  $n + 1$  (see [LS]). In terms of genus the Lusternik–Schnirelman Theorem can be formulated as follows: the genus of the  $n$ -dimensional sphere with respect to the antipodal action is equal to  $n + 1$ . This result was generalized by A. Fet [Fe] to the case of an arbitrary free involution on the sphere. The case of a free action of an arbitrary finite cyclic group was considered by M. Krasnosel'skiĭ [Kr] in the framework of the geometric approach. A. Švartz [Šv] was the first to consider the case of a non-free action of a cyclic group on the sphere and obtained the following result: let the finite cyclic group  $\mathbb{Z}_p$  act on the  $n$ -dimensional unit sphere  $\mathbb{S}^n$ , let  $A = \{x \in \mathbb{S}^n \mid \exists g \in \mathbb{Z}_p : g \neq 1 \text{ \& } gx = x\}$ , and suppose  $\dim A = k$ . Then  $\text{gen}(\mathbb{S}^n \setminus A) \geq n - k$ , where  $\text{gen}(\cdot)$  denotes genus.

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It is worth emphasizing that specific properties of the cohomology of the cyclic group with coefficients in a field were essentially used in [Šv].

The goal of this paper is to generalize Švartz's result to an arbitrary compact Lie group action *in the framework of the geometric approach*, and to apply the obtained result to estimating the genus of a free part of the unit sphere in the space of spherical harmonics under the natural representation of the group  $SO(n)$ .

Our study becomes natural in view of investigations of bifurcation phenomena for semilinear elliptic equations on a ball (see, for instance, [Bar1]).

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### 1. Estimate of the genus

Let  $G$  be a compact Lie group which acts freely on a metric space  $M$ . The orbit map  $p : M \rightarrow M/G$  is the projection of a locally trivial fiber bundle with fiber  $G$  (see, for instance, [Bre], p. 88).

DEFINITION (see [Šv], p. 250). The minimal cardinality of an open covering of  $M/G$  consisting of sets over which the fiber bundle is trivial, is said to be the *genus* of  $M$  (denoted by  $\text{gen}(M)$ ).

THEOREM 1. *Let  $G$  be a compact Lie group of dimension  $m$  acting smoothly on the sphere  $\mathbb{S}^n$ . Let  $A$  be a closed  $G$ -invariant subset of  $\mathbb{S}^n$  such that the  $G$ -space  $\mathbb{S}^n \setminus A$  is free. Suppose, further, that  $A$  is the image of a  $k$ -dimensional smooth compact manifold under a smooth map with  $k < n$  (if  $A$  is empty then it is viewed as the image of the  $(-1)$ -dimensional manifold under the empty map). Then*

$$\text{gen}(\mathbb{S}^n \setminus A) \geq \frac{n - k}{1 + m}.$$

By the well known properties of the Lusternik–Schnirelman category one has

COROLLARY. *Under the conditions of Theorem 1,*

$$\text{cat}((\mathbb{S}^n \setminus A)/G) \geq \frac{n - k}{1 + m}.$$

REMARK 1. The smoothness condition on  $A$  in Theorem 1 does not seem to be very restrictive. Indeed, the union of all non-principal orbits satisfies this condition.

REMARK 2. Our proof of Theorem 1 follows a geometric scheme due to M. Krasnosel'skiĭ ([Kr]). The main ingredients of Krasnosel'skiĭ's investigation of the free case are: (a) the usage of the fact that any equivariant mapping of  $\mathbb{S}^n$  into itself is essential and (b) the observation that some simple operations increase

the dimension of subsets of a sphere by at most one; this in turn allows him to use induction on dimension. We also use induction on dimension (Lemma 2), but the non-free situation we are dealing with forces us to supply the considered action and subsets with some additional structure. Instead of (a) we use a corresponding assertion for non-free actions (Lemma 1).

REMARK 3. After Theorem 1 was obtained, T. Bartsch informed us that he has a homological proof of this result (using a reduction to Švartz's result mentioned above).

For the proof of the lemma presented below, see [BB, KB, BKZ] (and also [Ko]).

LEMMA 1. *Let  $\Phi : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be an equivariant map such that  $\Phi|_A$  is the identity map. Then the degree of  $\Phi$  is not zero.*

For any  $B \subset \mathbb{S}^n$ ,  $B \neq \emptyset$ , and any  $x_0 \in \mathbb{S}^n$  with  $-x_0 \notin B$  define the (spherical) cone over  $B$  with vertex  $x_0$  by the formula

$$\text{Con}(x_0, B) = \varphi^{-1}(\{(1-t)\varphi(x) + t\varphi(x_0) \mid x \in B \text{ \& } t \in [0, 1]\}),$$

where  $\varphi$  is the stereographic projection of  $\mathbb{S}^n \setminus \{-x_0\}$  on  $\mathbb{R}^n$ . Set also  $\text{Con}(x_0, \emptyset) = x_0$ .

In the proof of Lemma 2 we use the following obvious

PROPOSITION. *Let  $B$  be a compact subset of  $\mathbb{S}^n$  and  $x_0 \notin B$ . Then for any neighborhood  $U \supset \text{Con}(x_0, B)$  there exists a compact, contractible and locally contractible set  $K$  such that  $K \subset U$ ,  $B \subset \text{Int}K$  and  $\text{Con}(x_0, B) \subset K$ .*

LEMMA 2. *Under the conditions of Theorem 1 there exists a chain of compact sets*

$$(1) \quad A \subset A_0 \subset B_0 \subset A_1 \subset B_1 \subset \dots \subset B_{p-1} \subset A_p \subsetneq \mathbb{S}^n$$

with the following properties (for  $i = 0, 1, \dots, p-1$ ):

- (a)  $p = \lceil (n-k)/(1+m) \rceil$ , where  $\lceil x \rceil = \min\{a \in \mathbb{Z} \mid a \geq x\}$ ;
- (b)  $B_i$  is contractible and locally contractible;
- (c)  $A_{i+1}$  is the union of all  $G$ -orbits passing through the points of  $B_i$ ;
- (d)  $A_0$  is an invariant neighborhood of  $A$ .

PROOF. We construct a chain of compact subsets

$$(2) \quad A = \tilde{A}_0 \subset \tilde{B}_0 \subset \tilde{A}_1 \subset \tilde{B}_1 \subset \dots \subset \tilde{B}_{p-1} \subset \tilde{A}_p \subsetneq \mathbb{S}^n$$

with the following properties:

- ( $\alpha$ )  $\tilde{B}_i$  is the cone over  $\tilde{A}_i$  with a vertex  $x_i \notin \tilde{A}_i$ ;
- ( $\beta$ )  $\tilde{A}_{i+1}$  is the union of all  $G$ -orbits passing through the points of  $\tilde{B}_i$ ;
- ( $\gamma$ )  $\mu(\tilde{A}_{i+1}) = \mu(\tilde{B}_i) = 0$ ,

where  $i = 0, 1, \dots, p-1$  and  $\mu(\cdot)$  is the Lebesgue measure. To do this we construct simultaneously by induction two chains of sets

$$(2') \quad \begin{aligned} A &= A'_0 \subset B'_0 \subset A'_1 \subset B'_1 \subset \dots \subset B'_{p-1} \subset A'_p \subsetneq \mathbb{S}^n, \\ \widehat{A}_0 &\subset \widehat{B}_0 \subset \widehat{A}_1 \subset \widehat{B}_1 \subset \dots \subset \widehat{B}_{p-1} \subset \widehat{A}_p, \end{aligned}$$

and a chain of smooth maps

$$f_0 \subset g_0 \subset f_1 \subset g_1 \subset \dots \subset g_{p-1} \subset f_p$$

such that the chain (2') has properties  $(\alpha)$ – $(\gamma)$  above (with  $A'_i, B'_i$  instead of  $\widetilde{A}_i, \widetilde{B}_i$  respectively) and

- $f_0 : \widehat{A}_0 \rightarrow A'_0 = \widetilde{A}_0 = A$  is defined by the assumptions in the lemma;
- $f_{i+1}(\widehat{A}_{i+1}) = A'_{i+1}$ ;
- $g_i(\widehat{B}_i) = B'_i$ ;
- $\dim \widehat{A}_0 = k, \dim \widehat{A}_i = k + (m+1)i, \dim \widehat{B}_i = k + (m+1)i + 1$

( $i = 0, 1, \dots, p-1$ ).

By the assumptions a smooth surjective map  $f_0 : \widehat{A}_0 \rightarrow A = A'_0 = \widetilde{A}_0$  is given. Suppose that a smooth map  $f_i : \widehat{A}_i \rightarrow A'_i$  has been constructed. Set  $\widehat{B}_i = \widehat{A}_i \times \mathbb{R}$  for  $i = 1, \dots, p-1$ . By the inductive hypothesis  $A'_i$  is of zero Lebesgue measure in  $\mathbb{S}^n$ . Hence, there exists a point  $x_i \in \mathbb{S}^n$  such that  $x_i, -x_i \notin A'_i$ . Consider the stereographic projection  $\varphi : \mathbb{S}^n \setminus \{-x_0\} \rightarrow \mathbb{R}^n$  and set

$$g_i(x, t) = \varphi^{-1}((1-t)\varphi(f_i(x)) + t\varphi(x_i)), \quad B'_i = g_i(\widehat{B}_i)$$

for all  $x \in \widehat{A}_i$  and  $t \in \mathbb{R}$ . If  $A = \emptyset$  we let  $\widehat{B}_0$  be a point and  $g_0 : \widehat{B}_0 \rightarrow \mathbb{S}^n$  be the constant map to the vertex of the cone over the empty set. Since  $\dim A'_i = k + (m+1)i, \dim B'_i = k + (m+1)i + 1 < n$  and (by the Sard Theorem)  $\mu(B'_i) = 0$ . The smoothness of  $g_i$  is obvious.

Set  $\widehat{A}_{i+1} = G \times \widehat{B}_i$  and for every  $g \in G$  and every  $x \in \widehat{B}_i$  define  $f_{i+1}(g, x) = gx$ . Clearly,  $f_{i+1}$  is a smooth map. Since  $\dim \widehat{A}_{i+1} = k + (m+1)(i+1) < n$ , we have (by the Sard Theorem)  $\mu(\widehat{A}_{i+1}) = 0$ . Thus the chain (2') is constructed.

This chain has all the required properties of (2) except compactness. Set (for all  $i = 0, \dots, p-1$ )

$$\begin{aligned} \widetilde{A}_0 &= A; \\ \widetilde{B}_i &= g_i(f_i^{-1}(A'_i) \times [0, 1]) \subset B'_i; \\ \widetilde{A}_{i+1} &= f_{i+1}(G \times g_i^{-1}(\widetilde{B}_i)) \subset A'_{i+1}. \end{aligned}$$

By construction,  $\widetilde{B}_i = \text{Con}(x_i, \widetilde{A}_i)$  and  $\widetilde{A}_{i+1} = G(\widetilde{B}_i)$ . Compactness of  $\widetilde{A}_i$  and  $\widetilde{B}_i$  can be established by induction. Notice that  $\widetilde{A}_p \neq \mathbb{S}^n$  because  $\mu(\widetilde{A}_p) = 0$ .

We now construct (1) from (2). Since  $\widetilde{A}_p$  is a proper invariant compact subset of  $\mathbb{S}^n$  there exists a closed invariant neighborhood  $\bar{A}_p$  of  $\widetilde{A}_p$  such that  $\bar{A}_p \neq \mathbb{S}^n$ .

Since  $\bar{A}_p$  is a neighborhood of  $\tilde{B}_{p-1}$  there exists a compact, contractible and locally contractible set  $B_{p-1}$  such that  $\tilde{B}_{p-1} \subset B_{p-1} \subset \bar{A}_p$  (see Proposition). By the invariance of  $\bar{A}_p$ ,  $G(B_{p-1}) \subset \bar{A}_p$ ; set  $A_p = G(B_{p-1})$ . According to Proposition,  $\tilde{A}_{p-1} \subset \text{Int}B_{p-1}$ , hence by invariance of  $\tilde{A}_{p-1}$  there exists an invariant closed neighborhood  $\bar{A}_{p-1}$  of  $\tilde{A}_{p-1}$  such that  $\bar{A}_{p-1} \subset B_{p-1}$ .

Now applying the described procedure “downward” one can construct the sets  $A_{p-1}, \dots, B_0, A_0$ .

Lemma 2 is proved.

PROOF OF THEOREM 1. To prove the theorem it suffices to show the existence of an invariant compact set  $K \subset \mathbb{S}^n \setminus A$  with  $\text{gen}(K) \geq p$ . Suppose this is false, i.e. for any invariant compact set  $K \subset \mathbb{S}^n \setminus A$ ,  $\text{gen}(K) < p$ . Consider the chain (1) from Lemma 2 and let  $K \subset \mathbb{S}^n \setminus A$  be an invariant compactum such that  $A_0 \cup K = \mathbb{S}^n$ . By the assumption  $\text{gen}(K) = \ell < p$ . Hence there exist invariant open (in the induced topology) subsets  $M_1, \dots, M_\ell$  of  $K$  such that  $\bigcup_{i=1}^\ell M_i = K$  and  $\text{gen}(M_i) = 1$ .

Consider a chain of closed invariant sets

$$A_0 = K_0 \subset K_1 \subset \dots \subset K_\ell = \mathbb{S}^n,$$

where  $K_i = (\mathbb{S}^n \setminus (M_{i+1} \cup \dots \cup M_\ell)) \cup A_0$  for  $0 \leq i \leq \ell$ . Note that for  $i \geq 1$  one has  $P_i = K_{i+1} \setminus K_i \subset M_i$ , from which it follows that the projection of  $P_i$  on  $K/G$  defines a trivial fiber bundle. Hence for any  $i \geq 1$  there exists a compact set  $L_i \subset K$  such that  $P_i = G(L_i)$  and  $g(L_i) \cap h(L_i) = \emptyset$  if  $g \neq h$  ( $g, h \in G$ ).

Now we construct an equivariant map  $\Phi : \mathbb{S}^n \rightarrow \mathbb{S}^n$  such that  $\Phi|_A$  is the identity map and  $\Phi(\mathbb{S}^n) \subsetneq \mathbb{S}^n$ ; this contradicts Lemma 1.

Set  $\Phi$  to be the identity map on  $A_0$ . The construction of  $\Phi$  is by induction. If  $\Phi$  is constructed on  $K_i$  and its image is contained in  $A_i$  then we may extend  $\Phi$  to a continuous (non-equivariant) map  $L_{i+1} \rightarrow B_i$  (this is possible since  $K_i$  is closed and  $A_i$  is an **AR**-space; see [Bo], p. 112). Next we extend  $\Phi$  over all  $M_i$  by  $\Phi gx = g\Phi x$  ( $x \in L_i, g \in G$ ). The image of the final map is contained in  $A_\ell \subset A_p \subsetneq \mathbb{S}^n$ .

Theorem 1 is proved.

## 2. Case of spherical harmonics

In this section we apply Theorem 1 to an action of the group  $SO(n)$  ( $n$  odd) on the unit sphere in a space of spherical harmonics. Denote by  $P(n, \ell)$  the space of all homogeneous polynomials of degree  $\ell$  in  $n$  variables and by  $H(n, \ell)$  the corresponding space of spherical harmonics ( $H(n, \ell) \subset P(n, \ell)$ ). Let  $x, y_1, z_1, \dots, y_m, z_m$  be an orthogonal basis in  $\mathbb{R}^n$ ,  $n = 2m + 1$ . It is well known

(see, for instance, [BtD]) that a polynomial  $f \in P(n, \ell)$  of the form

$$(3) \quad f = \sum_{k=0}^{\ell} \frac{x^k}{k!} f_k(y_1, z_1, \dots, y_m, z_m)$$

belongs to  $H(n, \ell)$  iff

$$(4) \quad \forall 0 \leq k \leq \ell - 2: \quad f_{k+2} = -\Delta f_k.$$

Moreover,  $H(n, \ell) = H_0 \oplus H_1$ , where  $H_0$  consists of harmonics containing even powers of  $x$  and  $H_1$  consists of harmonics containing odd powers of  $x$ . In addition, a polynomial  $f$  from  $H_0$  or  $H_1$  is uniquely determined by its leading term  $f_0$  or  $f_1$  respectively. It is well known that  $H_0 \cong P(n-1, \ell)$ ,  $H_1 \cong P(n-1, \ell-1)$  and

$$\dim P(n, \ell) = \binom{n + \ell - 1}{\ell},$$

$$\dim H(n, \ell) = \dim P(n-1, \ell) + \dim P(n-1, \ell-1).$$

The formula  $(gf)(u) = f(g^{-1}(u))$  defines the standard (irreducible) representation of  $SO(n)$  on  $H(n, \ell)$ .

In order to apply Theorem 1 one has to calculate

$$d(n, \ell) = \max\{\dim H(n, \ell)^g \mid g \in SO(n), g \neq 1\},$$

where  $H(n, \ell)^g = \{f \in H(n, \ell) \mid gf = f\}$ .

Let  $g$  be a non-trivial element of  $SO(n)$  and let  $x, y_1, z_1, \dots, y_m, z_m$  be an orthonormal basis in  $\mathbb{R}^n$  for which  $g$  has the following form:

$$\begin{pmatrix} 1 & & & 0 \\ & A(\varphi_1) & & \\ & & \ddots & \\ & 0 & & A(\varphi_m) \end{pmatrix},$$

where  $A(\varphi_k)$  is the matrix of rotation by the angle  $\varphi_k$  ( $1 \leq k \leq m$ ). Set  $x = x$ ,  $\bar{y}_k = y_k + iz_k$ ,  $\bar{z}_k = y_k - iz_k$ . In the new basis  $g$  is represented by the diagonal matrix

$$(5) \quad \text{diag}(1, \exp(i\varphi_1), \exp(-i\varphi_1), \dots, \exp(i\varphi_m), \exp(-i\varphi_m)).$$

It is easy to see that

$$\Delta f(\bar{y}_1, \bar{z}_1, \dots, \bar{y}_m, \bar{z}_m) = 4 \sum_{k=1}^m \frac{\partial^2 f}{\partial \bar{y}_k \partial \bar{z}_k},$$

and consequently (by (4)),

$$(6) \quad f_{k+2} = -4 \sum_{k=1}^m \frac{\partial^2 f_k}{\partial \bar{y}_k \partial \bar{z}_k}.$$

Set  $B(m, \ell) = B^{(0)}(m, \ell) \cup B^{(1)}(m, \ell)$ , where  $B^{(r)}(m, \ell)$  is the set of all non-negative integer-valued  $2m$ -vectors with sum of coordinates equal to  $\ell - r$  ( $r = 0, 1$ ). Given a vector  $b$  from  $B^{(r)}(m, \ell)$  one can consider a polynomial (uniquely determined by  $b$ )  $f^{(b)} \in H(n, \ell)$  with leading term

$$x^r \bar{y}_1^{b(1)} \bar{z}_1^{b(2)} \dots \bar{y}_m^{b(2m-1)} \bar{z}_m^{b(2m)}.$$

The family of all these polynomials forms a basis of  $H(n, \ell)$ , and (4)–(6) show the diagonality of the action of  $g$  in this basis:

$$gf^{(b)} = \exp\left(i \sum_{k=1}^m \varphi_k(b(2k-1) - b(2k))\right) f^{(b)}.$$

Hence  $\dim H(n, \ell)^g$  is equal to the number of  $g$ -fixed polynomials from our basis, i.e. it is equal to the number of vectors  $b \in B(m, \ell)$  such that

$$(7) \quad \sum_{k=1}^m t_k s_k^{(b)} \in \mathbb{Z},$$

where  $t_k = \varphi_k/(2\pi)$  and  $s_k^{(b)} = b(2k-1) - b(2k)$ . In what follows, we vary  $g$  in order to maximize  $\dim H(n, \ell)^g$  so that the matrix of  $g$  preserves the form (5) in the basis described above. Denote by  $\mathcal{D}$  the group of all these matrices. Without loss of generality, suppose the numbers  $t_k$  from (7) to be rational:  $t_k = u_k/v_k$  ( $u_k, v_k \in \mathbb{Z}$ ,  $t_k \neq 0$  if  $1 \leq k \leq \alpha$  and  $t_k = 0$  if  $k > \alpha$  for some integer  $\alpha$  ( $1 \leq \alpha \leq m$ )). Thus,  $\dim H(n, \ell)^g$  is equal to the number of vectors from  $B(m, \ell)$  satisfying the congruence

$$(8) \quad \sum_{k=1}^{\alpha} u_k v'_k s_k^{(b)} \equiv 0 \pmod{v},$$

where  $v = \text{lcm}(v_1, \dots, v_{\alpha})$  and  $v'_k$  is the corresponding complementary factor. Obviously, every vector  $b$  satisfying (8) also satisfies

$$(9) \quad \sum_{k=1}^{\alpha} a_k s_k^{(b)} \equiv 0 \pmod{p},$$

where  $p$  is a prime divisor of  $v$ , all  $a_k \neq 0$  and  $a_1 = 1$  (maybe for a smaller value of  $\alpha$ ). Fixing all the components of the vector  $b$  but the first one, it is easy to verify that the number of solutions of (9) is maximal for  $p = 2$ ; denote this number by  $c(m, \ell, \alpha)$ . Simple combinatorial arguments show that

$$(10) \quad c(m, \ell, \alpha) = \begin{cases} \sum_{k=0}^{[(\ell-1)/2]} \dim H(2m - 2\alpha, \ell - 2k) & \text{if } \alpha < m, \\ \dim P(2\alpha, 2[(\ell-1)/2]) & \text{if } \alpha = m, \end{cases}$$

where  $[\cdot]$  stands for the integer part.

Let us introduce the following notations: for  $\alpha \in [1, m] \cap \mathbb{Z}$ ,

$$B_r^\alpha = B_r^\alpha(m, \ell) = \left\{ b \in B(m, \ell) \mid \sum_{k=1}^{\alpha} s_k^{(b)} \equiv r \pmod{2} \right\},$$

and, for any  $a \in B(m, \ell)$ ,

$$B_a = B_a(m, \ell) = \{ b \in B(m, \ell) \mid \forall 3 \leq k \leq 2m - 2 : b(k) = a(k) \}.$$

Let  $\ell$  be an even number. Let us show that  $c(m, \ell, \alpha) \leq c(m, \ell, m)$  for any  $\alpha = 1, \dots, m - 1$ . Consider  $B_0^\alpha$  as the union of two disjoint sets  $B_0^\alpha \cap B_0^m$  and  $B_0^\alpha \cap B_1^m$  and define a mapping  $\varphi : B_0^\alpha \rightarrow B_0^m$  by setting  $\varphi(b) = b \in B_0^\alpha \cap B_0^m$  for  $b \in B_0^\alpha \cap B_0^m$  and  $\varphi(b) = b + (1, 0, \dots, 0) \in B_1^\alpha \cap B_0^m$  for  $b \in B_0^\alpha \cap B_1^m$ . Now the required inequality follows from the injectivity of  $\varphi$ .

Similarly, one can prove that  $c(m, \ell, \alpha) \geq c(m, \ell, m)$  for any  $\alpha = 1, \dots, m - 1$  if  $\ell$  is odd.

Now, we show that  $c(m, \ell, \alpha) \leq c(m, \ell, 1)$  for any  $\ell$  and for all  $\alpha = 2, \dots, m - 1$ . Represent  $B_0^\alpha$  as the union of two disjoint sets  $B_0^\alpha \cap B_0^1$  and  $B_0^\alpha \cap B_1^1$ . The second of these can be represented as the disjoint union of the family  $\mathcal{B}(m, \ell) = \{ B_b(m, \ell) \mid b \in B(m, \ell) \}$ . For fixed  $b$  every vector from  $B_b(m, \ell)$  is uniquely determined by its four coordinates with indices 1, 2,  $2m - 1$  and  $2m$ . The direct calculation using (10) shows that  $|B_0^1(2, k)| > |B_1^1(2, k)|$ . Hence there exists an injective mapping  $\varphi_b : B_b \cap B_1^1 \rightarrow B_b \cap B_0^1$  such that  $\varphi_b(B_0^\alpha \cap B_1^1 \cap B_b) \subset B_1^\alpha \cap B_0^1 \cap B_b$ . Taking the union of the family  $\{ \varphi_b \mid b \in B \}$  and the identity map on  $B_0^\alpha \cap B_0^1$  one obtains the required injection  $\varphi : B_0^\alpha \rightarrow B_0^1$ .

The arguments above show that the number  $c(m, \ell, \alpha)$  is maximal for  $\alpha = 1$  if  $\ell$  is odd, and for  $\alpha = m$  if  $\ell$  is even. Hence (see (10)) we obtain the following formulas:

$$(11) \quad d(n, \ell) = \begin{cases} \sum_{k=0}^{(\ell-1)/2} \dim H(2m - 2, \ell - 2k) & \text{if } \ell \text{ is odd,} \\ \dim P(2m, \ell) & \text{if } \ell \text{ is even.} \end{cases}$$

Using the natural multi-dimensional generalization of the Euler angles one can construct a smooth mapping  $\varphi$  from the torus  $\mathbb{T}^{2m^2}$  to  $SO(n)$  such that any matrix  $P \in SO(n)$  can be represented in the form  $P = TDT^{-1}$ , where  $T \in \text{Im } \varphi$  and  $D \in \mathcal{D}$ , thus  $SO(n)(H(n, \ell)^g) = (\text{Im } \varphi)(H(n, \ell)^g)$ . Now, if  $S$  is the unit sphere in  $H(n, \ell)$ , and  $A$  is the union of all non-principal orbits, then  $A$  coincides with the union of the family  $\mathcal{A}(\mathcal{D}) = \{ (\text{Im } \varphi)(H(n, \ell)^g) \mid g \in \mathcal{D} \}$  and (by the finiteness of the number of orbit types of the  $SO(n)$ -action on  $S$ , see [Bre]) is the union of some finite family  $\mathcal{A}(\mathcal{F})$ , where  $\mathcal{F}$  is a finite subset of  $\mathcal{D}$ . Now, using the manifold

$$\mathcal{M} = \bigsqcup_{g \in \mathcal{F}} (\mathbb{T}^{2m^2} \times (H(n, \ell)^g \cap S) \times \mathbb{T}^{d(g)}),$$



where  $d(g) = d(n, \ell) - \dim H(n, \ell)^g$  and  $\sqcup$  stands for disjoint union, define a surjection  $f : \mathcal{M} \rightarrow A$  as follows:

$$f(x, y, z) = \varphi(x)(y) \quad ((x, y, z) \in \mathcal{M}).$$

Applying Theorem 1 we get

**THEOREM 2.** *Let  $n = 2m + 1$  and let  $H(n, \ell)$  be the space of spherical harmonics with the natural representation of the group  $SO(n)$ . Let, further,  $S$  be the unit sphere in  $H(n, \ell)$ , and  $A \subset S$  be the union of all non-principal orbits. Then*

$$\text{gen}(S \setminus A) \geq \gamma(n, \ell),$$

where

$$\gamma(n, \ell) = \left\lceil \frac{\dim H(n, \ell) - d(n, \ell) - 2m^2}{m(2m + 1) + 1} \right\rceil,$$

and  $d(n, \ell)$  is defined by (11).

Finally, let us discuss some properties of the function  $\gamma(n, \ell)$ .

- 1)  $\gamma(3, \ell) = [(\ell + 1)/4]$ .
- 2) Let  $n \geq 3$ . For  $\ell = 1, 2$ , the set  $A$  coincides with the entire sphere  $S$ ;  $\gamma(n, 1) = \gamma(n, 2) = 0$ . One can show that  $\gamma(n, 3) = 1$  and  $\gamma(n, 4) = m - [m/3]$ .
- 3) Using simple arguments one can show that for fixed  $n$ ,  $\gamma(n, \ell)$  does not decrease in  $\ell$ . Although calculations confirm the conjecture that the function  $\gamma(n, \ell)$  never decreases, we have no proof of this statement.

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