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INVARIANT MEANS AND FIXED POINT PROPERTIES FOR NON-EXPANSIVE REPRESENTATIONS OF TOPOLOGICAL SEMIGROUPS

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To Professor Ky Fan with admiration and respect

1. Introduction

Let S be a semitopological semigroup, i.e. S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S into S are continuous. Let C be a non-empty subset of a Banach space E and $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings from C to C, i.e. the map $S \times C \to C$ defined by $(s, x) \to T_s x$, $s \in S$, $x \in C$, is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set of common fixed points for $\mathcal{S}$ in C.

It is well known that if S is left reversible (i.e. any two closed right ideals in S have non-void intersection), and each T_s, $s \in S$, is a non-expansive self-map of C, then each of the following conditions implies $F(\mathcal{S})$ is non-empty (see [15]):

(a) C is compact and convex (see [21] and [9]);
(b) C is weakly compact, convex, and has normal structure (see [19]);
(c) S is discrete, C is weakly compact, convex, and each T_s is weakly continuous (see [10]);
(d) C is a weak*-compact convex subset of ℓ^1 ([20, Theorem 4]).

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It is also known that if $AP(S)$, the space of continuous almost periodic functions on S, has a left invariant mean, C is compact, convex, and each T_s, $s \in S$, is a non-expansive self-map of C, then $F(\mathcal{S}) \neq \emptyset$ (see [26] and [12]).

It is the purpose of this paper to study fixed point properties for non-expansive or asymptotically non-expansive representations of a semitopological semigroup S when S is left amenable or left subamenable (i.e. the space of bounded left uniformly continuous real-valued functions on S has a left invariant mean or submean). We prove (Theorem 4.5), among other things, that if $CB(S)$, the space of bounded continuous functions on S, has a left invariant submean (which is the case when S is left reversible as a discrete semigroup), then S has a certain fixed point property for asymptotically non-expansive representations on non-empty (but not necessarily convex) subsets of a Hilbert space. We also prove (Theorem 5.3) that if S is left amenable or S is a left subamenable discrete semigroup, then whenever $\mathcal{S} = \{T_s : s \in S\}$ is a weak*-jointly continuous non-expansive representation of S on a norm-separable weak*-compact convex subset C of a dual Banach space, C contains a common fixed point for $\mathcal{S}$.

This paper is organized as follows: In Section 3, we introduce the notion of left invariant submean and the class of left subamenable semigroups. In Section 4, we study elements in $F(\mathcal{S})$ determined by left invariant submeans when S is an asymptotically non-expansive representation of S acting on a non-empty (not necessarily convex) subset of a Hilbert space. Finally (in Section 5), we shall establish a fixed point property for S when $\mathcal{S}$ is a representation of S as non-expansive self-maps of a weak*-compact norm-separable subset of a dual Banach space and S is left amenable or S is a left subamenable discrete semigroup.

2. Some preliminaries

All topologies in this paper are assumed to be Hausdorff. If E is a Banach space and E^* its continuous dual, then the value of $f \in E^*$ at $x \in E$ will be denoted by $f(x)$ or $\langle f, x\rangle$. Also if $A \subseteq E$, then $\bar{A}$ and $\overline{\text{co}}\, A$ will denote the closure of A and the closed convex hull of A in E, respectively.

Given a non-empty set S, we denote by $\ell^\infty(S)$ the Banach space of bounded real-valued functions on S with the supremum norm. Let S be a semigroup. Then a subspace X of $\ell^\infty(S)$ is *left* (resp. *right*) *translation invariant* if $\ell_a(X) \subseteq X$ (resp. $r_a(X) \subseteq X$) for all $a \in S$, where $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, $s \in S$. If S is a semitopological semigroup, we denote by $CB(S)$ the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions. Let $LUC(S)$ (resp. $RUC(S)$) be the space of *left* (resp. *right*) *uniformly continuous functions* on S, i.e. all $f \in CB(S)$ such that the mapping from S into $CB(S)$ defined by $s \to \ell_s f$ (resp. $s \to r_s f$) is continuous when $CB(S)$ has the sup norm topology. Then as is known [22] (see also [3]), $LUC(S)$ and $RUC(S)$ are left and right translation invariant closed subalgebras of $CB(S)$ containing constants. Note that when S is a topological group, then $LUC(S)$ is precisely the space of right uniformly continuous functions on S defined in [6]. Also let $AP(S)$ (resp. $WAP(S)$) denote the space of almost periodic (resp. weakly almost periodic) functions f in $CB(S)$, i.e. all $f \in CB(S)$ such that $\{\ell_a f : a \in S\}$ is relatively compact in the norm (resp. weak) topology of $CB(S)$, or equivalently $\{r_a f : a \in S\}$ is relatively compact in the norm (resp. weak) topology of $CB(S)$. Then as is known [3, p. 164], $AP(S) \subseteq LUC(S) \cap RUC(S)$, and $AP(S) \subseteq WAP(S)$. When S is a group, then $WAP(S) \subseteq LUC(S) \cap RUC(S)$ (see [3, p. 167]).

A function $f \in CB(S)$ is called *asymptotically left uniformly continuous* if for any $s \in S$, $\epsilon > 0$, there exist a neighbourhood U of s and a *right* ideal J of S such that

$$\|\ell_u f - \ell_s f\|_J = \sup\{|f(ut) - f(st)| : t \in J\} < \epsilon \tag{2.1}$$

for all $u \in U$. The closed linear span of the set of asymptotically left uniformly continuous functions on S is denoted by $ALUC(S)$. Similarly we define the closed subspace $ARUC(S)$ of $CB(S)$ with left and right interchanged. Clearly $ALUC(S) \supseteq LUC(S)$, and $ARUC(S) \supseteq RUC(S)$.

PROPOSITION 2.1. *For any semitopological semigroup S, the subspaces $ALUC(S)$ and $ARUC(S)$ are left and right translation invariant. Furthermore, if S is left reversible* (*resp. right reversible*), *then each function in $ALUC(S)$* (*resp. $ARUC(S)$*) *is asymptotically left* (*resp. right*) *uniformly continuous. In this case, $ALUC(S)$* (*resp. $ARUC(S)$*) *is even an algebra.*

PROOF. We will only consider the space $ALUC(S)$. Let $a \in S$ be fixed, and f be asymptotically left uniformly continuous. Then for any $\epsilon > 0$ and $s \in S$, choose a neighbourhood U of s and a right ideal J such that $\|\ell_u f - \ell_s f\|_J < \epsilon$ for all $u \in U$. Now (since left and right translations commute),

$$\|\ell_u(r_a f) - \ell_s(r_a f)\|_J = \|r_a(\ell_u f - \ell_s f)\|_J \leq \|\ell_u f - \ell_s f\|_J < \epsilon$$

(since J is a right ideal). Hence $r_a f \in ALUC(S)$. To show that $\ell_a f \in ALUC(S)$, we choose a neighbourhood V of as and a right ideal J such that

$$\|\ell_v f - \ell_{as} f\|_J < \epsilon \qquad \text{for all } v \in V.$$

Now let $U = a^{-1}V = \{u \in S : au \in V\}$. Then if $u \in U$,

$$\|\ell_u(\ell_a f) - \ell_s(\ell_a f)\|_J = \|\ell_{au} f - \ell_{as} f\|_J < \epsilon.$$

It is easy to see that the set $\mathcal{L}$ of asymptotically left uniformly continuous functions on S is norm-closed; also if $f \in \mathcal{L}$ and $\alpha \in \mathbb{R}$, then $\alpha f \in \mathcal{L}$. Suppose S is left reversible and $f, g \in \mathcal{L}$. Let $s \in S$. Choose neighbourhoods U_f and U_g

and right ideals J_f, J_g such that (2.1) holds for f and g. Let $U = U_f \cap U_g$, and $J = \bar{J}_f \cap \bar{J}_g$. Then J is a right ideal of S, and for any $u \in U$,

$$\|\ell_u(f+g) - \ell_s(f+g)\|_J \le \|\ell_u f - \ell_s f\|_{J_f} + \|\ell_u g - \ell_s g\|_{J_g} < 2\epsilon,$$

i.e. $f + g \in \mathcal{L}$. Similarly we show that $f \cdot g \in \mathcal{L}$.

Proposition 2.2.

(a) *If S has no proper right (resp. left) ideal, then $LUC(S) = ALUC(S)$ (resp. $RUC(S) = ARUC(S)$).*
(b) *If S has jointly continuous multiplication and contains a compact right (resp. left) ideal, then $CB(S) = ALUC(S)$ (resp. $CB(S) = ARUC(S)$).*

Proof. (a) is trivial.

(b) Let J be a compact right ideal of S and $f \in CB(S)$. Then, to show $f \in ALUC(S)$, it is sufficient to show that for any $\epsilon > 0$ and $s \in S$, there exists a neighbourhood U of s such that

$$\|\ell_u f - \ell_s f\|_J < \epsilon \qquad \text{for all } u \in U.$$

If not, there exists a net $\{u_\alpha\}$ such that $u_\alpha \to s$ and

$$\|\ell_{u_\alpha} f - \ell_s f\|_J \ge \epsilon \qquad \text{for each } \alpha.$$

For α, pick $t_\alpha \in J$ such that

$$\|\ell_{u_\alpha} f - \ell_s f\|_J = |(\ell_{u_\alpha} f - \ell_s f)(t_\alpha)|.$$

By compactness of J, and by passing to a subnet, we may assume that $t_\alpha \to t_0$ for some $t_0 \in J$. Then

$$\begin{aligned} 0 < \epsilon \le \|\ell_{u_\alpha} f - \ell_s f\|_J &= |f(u_\alpha t_\alpha) - f(s t_\alpha)| \\ &\le |f(u_\alpha t_\alpha) - f(s t_0)| + |f(s t_0) - f(s t_\alpha)| \to 0 \end{aligned}$$

by joint continuity of multiplication in S.

Let S be a non-empty set and X be a subspace of $\ell^\infty(S)$ containing constants. Then $\mu \in X^*$ is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s) \qquad \text{for each } f \in X.$$

By a *submean* on X, we shall mean a real-valued function μ on X with the following properties:

(1) $\mu(f+g) \le \mu(f) + \mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \ge 0$;
(3) for $f, g \in X$, $f \le g$ implies $\mu(f) \le \mu(g)$;
(4) $\mu(c) = c$ for every constant function c.

REMARK 2.3. (a) Clearly every mean is a submean. The notion of submean was first introduced by Mizoguchi and Takahashi in [23].

(b) Let SM denote the set of submeans on X. For each $\phi \in SM$, $-\|f\| \leq \phi(f) \leq \|f\|$ by (3) and (4). Hence SM may be identified as a subset of the product space $\prod_{f\in X}[-\|f\|, \|f\|]$, which is compact by Tikhonov's Theorem. Hence SM is a compact convex subset of the product topological vector space $\prod_{f\in X} \mathbb{R}_f$, where each $\mathbb{R}_f = \mathbb{R}$.

Depending on time and circumstances, the value of a submean (or mean) μ at $f \in X$ will also be denoted by $\mu(f)$, $\langle \mu, f\rangle$ or $\mu_t f(t)$.

3. Subamenability and reversibility

In this section, we study the relation between invariant submeans on subspaces of $CB(S)$ of a semitopological semigroup S and reversibility of S.

If S is a semigroup, and $X \subseteq \ell^\infty(S)$ is a left translation invariant subspace of $\ell^\infty(S)$ containing constants, a submean μ on X is *left invariant* if $\mu(\ell_a f) = \mu(f)$ for each $a \in S$ and $f \in X$.

We abbreviate left invariant submean = $LISM$ and left invariant mean = LIM.

LEMMA 3.1. *Let S be a semitopological semigroup and X be a left translation invariant subspace of $CB(S)$ containing constants and which separates closed subsets of S. If X has a $LISM$, then S is left reversible.*

PROOF. Let μ be a $LISM$ of X, and I_1 and I_2 be disjoint non-empty closed right ideals of S. By assumption, there exists $f \in X$ such that $f \equiv 1$ on I_1 and $f \equiv 0$ on I_2. Now if $a_1 \in I_1$, then $\ell_{a_1} f = 1$. So $\mu(f) = \mu(\ell_{a_1} f) = 1$. But if $a_2 \in I_2$, then $\ell_{a_2} f \equiv 0$. So $\mu(f) = \mu(\ell_{a_2} f) = 0$, which is impossible.

COROLLARY 3.2. *If S is normal and $CB(S)$ has a $LISM$, then S is left reversible.*

COROLLARY 3.3. *If S is normal and $CB(S)$ has a $LISM$, then $AP(S)$ has a LIM.*

PROOF. This follows from Corollary 3.2 and [12, Corollary 3.3].

REMARK 3.4. Corollary 3.2 is false without normality. Indeed, let S be the topological space which is regular and Hausdorff and $CB(S)$ consists of constant functions only ([5]). Define on S the multiplication $st = s$ for $s, t \in S$. Let $a \in S$ be fixed. Define $\mu(f) = f(a)$ for all $f \in CB(S)$. Then μ is a $LISM$ on $CB(S)$, but S is not left reversible.

If S is a left reversible semitopological semigroup, then $(S, \preceq)$ is a directed system when the binary relation $\preceq$ on S is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{aS} \supseteq \{b\} \cup \overline{bS}$, $a, b \in S$.

Lemma 3.5. *Let S be a semitopological semigroup, J be a non-empty subset of S and $f \in LUC(S)$. If $\sup\{f(t) : t \succeq u\} \geq \beta$ for each $u \in J$, then $\sup\{f(t) : t \succeq p\} \geq \beta$ for each $p \in \bar{J}$.*

Proof. Let $p \in \bar{J}$ and $\sup\{f(t) : t \succeq p\} \leq \beta - \delta$, $\delta > 0$. Then

$$f(ps) \leq \beta - \delta \qquad \text{for each } s \in S \cup \{e\},$$

where $xe = ex = e$. Let $u_\alpha \in J$ be a net such that $u_\alpha \to p$. Hence $\|\ell_{u_\alpha} f - \ell_p f\| \to 0$. Consequently, there exists α_0 such that

$$f(u_\alpha s) \leq \beta - \delta/2 \qquad \text{for each } s \in S \cup \{e\}, \alpha \geq \alpha_0.$$

Hence for $\alpha \geq \alpha_0$, we have $\sup\{f(t) : t \succeq u_\alpha\} \leq \beta - \delta/2$, which contradicts the assumption.

A semitopological semigroup S is *left subamenable* if $LUC(S)$ has a $LISM$.

Proposition 3.6. *Let S be a semitopological semigroup. If S is left reversible, then S is left subamenable.*

Proof. For each $f \in CB(S)$, define

$$\mu(f) = \inf_s \sup_{t \succeq s} f(t).$$

Then μ is a submean on $CB(S)$. Indeed, if $f, g \in CB(S)$, and $\epsilon > 0$, choose $a, b \in S$ such that

$$\sup_{t \succeq a} f(t) \leq \mu(f) + \epsilon \qquad \text{and} \qquad \sup_{t \succeq b} g(t) \leq \mu(g) + \epsilon.$$

Let $c \in \overline{aS} \cap \overline{bS}$ (which is non-empty by left reversibility). Then $c \succeq a$ and $c \succeq b$. Hence

$$\sup_{t \succeq c} f(t) \leq \mu(f) + \epsilon \qquad \text{and} \qquad \sup_{t \succeq c} g(t) \leq \mu(g) + \epsilon.$$

So

$$\sup_{t \succeq c}(f(t) + g(t)) \leq \sup_{t \succeq c} f(t) + \sup_{t \succeq c} g(t) \leq \mu(f) + \mu(g) + 2\epsilon.$$

Consequently, $\mu(f + g) \leq \mu(f) + \mu(g) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, condition (1) for a submean holds. The proofs of conditions (2), (3) and (4) are routine.

To see that μ is left invariant, let $f \in LUC(S)$ and $a \in S$. Then

$$\begin{aligned}
\mu(\ell_a f) &= \inf_s \sup_{t \succeq s} f(at) = \inf_s \{\sup\{f(at) : t \in \overline{sS} \cup \{s\}\} \\
&= \inf_s \{\sup\{f(at) : t \in sS \cup \{s\}\} \\
&\quad \text{(by continuity of } f \text{ and multiplication in } S) \\
&= \inf_s \{\sup\{f(ast) : t \in S \cup \{e\}\} \quad (\text{where } se = s) \\
&= \inf_s \{\sup\{f(t) : t \in asS \cup \{as\}\}\} = \inf_s \sup_{t \succeq as} f(t) \geq \mu(f).
\end{aligned}$$

To prove the reverse inequality, let $\alpha = \mu(f)$ and $\beta = \mu(\ell_a f)$, $f \in LUC(S)$. Then for each $s \in S$, $\sup_{t \succeq as} f(t) \geq \beta$. Hence, by Lemma 3.5,

$$\sup_{t \succeq p} f(t) \geq \beta \qquad \text{for all } \ p \in \overline{aS}. \tag{3.1}$$

If $\alpha < \beta$, let $\epsilon = (\beta - \alpha)/2$. Choose s_0 such that $\sup_{t \succeq s_0} f(t) < \alpha + \epsilon$. Then for each $s \succeq s_0$, $\sup_{t \succeq s} f(t) < \alpha + \epsilon$. Let $p \in \overline{s_0 S} \cap \overline{aS}$. Then $p \succeq s_0$; so $\sup_{t \succeq p} f(t) < \alpha + \epsilon$, contradicting (3.1).

COROLLARY 3.7. *Let S be a discrete semigroup. Then S is left reversible if and only if S is left subamenable. In this case $WAP(S)$ has a LIM.*

PROOF. The first statement follows from Corollary 3.2 and Proposition 3.6, and the last statement follows from [10] (see also [13] and Remark 5.7).

PROPOSITION 3.8. *Let S, T be semitopological semigroups, and $\theta : S \to T$ a continuous homomorphism of S onto T. If S is left subamenable, then T is left subamenable.*

PROOF. Let $\tilde{\theta} : LUC(T) \to LUC(S)$ be defined by $\tilde{\theta}(f)(s) = f(\theta(s))$. Let μ be a left invariant submean on $LUC(S)$. Then $\tilde{\mu}(f) = \mu(\tilde{\theta}(f))$ is a submean, and $\tilde{\mu}(\ell_t f) = \mu(\tilde{\theta}(\ell_t f)) = \mu(\ell_s \tilde{\theta}(f)) = \mu(\tilde{\theta}(f)) = \tilde{\mu}(f)$, where $s \in S$ is such that $\theta(s) = t$.

REMARK 3.9. A subsemigroup of a left subamenable (even amenable) semigroup need not be left subamenable. Indeed, there is a solvable group G which contains a free subsemigroup S on 2-generators. Clearly G is amenable, and S is not left subamenable by Corollary 3.7 (see [7]).

PROPOSITION 3.10. *Let G be an amenable group, and $S \subseteq G$ be a subsemigroup of G. Then S is left amenable if and only if S is left subamenable.*

PROOF. If S is left subamenable, then S is left reversible (Corollary 3.2), and so S must be left amenable [16, Theorem 1]. The converse is obvious.

PROPOSITION 3.11. *Let S be a semitopological semigroup and $\{S_\alpha : \alpha \in I\}$ be subsemigroups of S with the induced topology such that $\bigcup\{S_\alpha : \alpha \in I\} = S$ and for each $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $S_\gamma \supseteq S_\alpha \cup S_\beta$. If for each $\alpha \in I$, S_α is left subamenable, then S is left subamenable.*

PROOF. Partially order I by $\alpha \succeq \beta$ if and only if $S_\alpha \supseteq S_\beta$. Then "$\succeq$" makes I into a directed set. For $f \in LUC(S)$, $\alpha \in I$, define a function $P_\alpha f$ on S_α by $(P_\alpha f)(s) = f(s)$ if $s \in S_\alpha$. One readily checks that $P_\alpha f \in LUC(S_\alpha)$ and $\ell_a(P_\alpha f) = P_\alpha(\ell_a f)$ for $a \in S_\alpha$. For each $\alpha \in I$, let μ_α be a $LISM$ on $LUC(S_\alpha)$. Define a submean $\overline{\mu}_\alpha$ on $LUC(S)$ by $\overline{\mu}_\alpha(f) = \mu_\alpha(P_\alpha f)$. Then $\overline{\mu}_\alpha(\ell_a f) = \overline{\mu}_\alpha(f)$ for each $a \in S_\alpha$. Since the set SM of submeans on $LUC(S)$ is compact in the topology of pointwise convergence (see Remark 2.3(b)), by passing to a subnet if necessary, we may assume that $\overline{\mu}_\alpha \to \mu$ for a submean μ on $LUC(S)$. Then, as is readily checked, μ is a $LISM$ on $LUC(S)$.

4. Asymptotically non-expansive representations

Let S be a semigroup and C be a non-empty subset of a Banach space E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as mappings from C into E. We say that $\mathcal{S}$ is *left asymptotically non-expansive* if for any $\epsilon > 0$ and $x \in C$, there exists a *left* ideal J of S such that

$$\|T_s x - T_s y\| \le \|x - y\| + \epsilon$$

for each $s \in J$ and $y \in C$.

Note that our notion of left asymptotic non-expansiveness differs from a similar notion used in [8]. It coincides with the notion of asymptotic non-expansiveness defined in [11] for the commutative semigroups $\mathbb{R}^+ \cup \{0\}$ and $\mathbb{N} \cup \{0\}$ with addition.

Proposition 4.1. *Let S be a semigroup and let C be a closed convex subset of a uniformly convex Banach space E. Let $\mathcal{S} = \{T_t : t \in S\}$ be a left asymptotically non-expansive semigroup on C such that for each $s \in S$, T_s is continuous. Then $F(\mathcal{S})$ is closed and convex.*

Proof. It is sufficient to show $z = (x+y)/2 \in F(\mathcal{S})$ if $x, y \in F(\mathcal{S})$. We first show that for any $\epsilon > 0$, there exists $t_0 \in S$ such that

$$\|T_{tt_0} z - z\| < \epsilon \qquad \text{for every } t \in S.$$

If not, there exists $\epsilon > 0$ such that for each $s \in S$, there is $t_s \in S$ with $\|T_{t_s s} z - z\| \ge \epsilon$. For such ϵ, choose $\epsilon_0 > 0$ such that

$$\left(\frac{1}{2}\|x - y\| + \epsilon_0\right)\left(1 - \delta\left(\frac{\|x - y\|}{\frac{1}{2}\|x - y\| + \epsilon_0}\right)\right) < \epsilon,$$

where δ is the modulus of convexity of E. Then choose $u \in S$ such that

$$\sup_t \sup_{f \in C} (\|T_{tu} z - T_{tu} f\| - \|z - f\|) < \epsilon_0.$$

Hence, we have

$$\|T_{tu} z - x\| < \|z - x\| + \epsilon_0 \quad \text{and} \quad \|T_{tu} z - y\| < \|z - y\| + \epsilon_0$$

for every $t \in S$. Therefore, for each $t \in S$,

$$\begin{aligned}
\|T_{tu} z - z\| &= \left\| \frac{T_{tu} z - x + T_{tu} z - y}{2} \right\| \\
&\le \left(\frac{1}{2}\|x - y\| + \epsilon_0\right)\left(1 - \delta\left(\frac{\|x - y\|}{\frac{1}{2}\|x - y\| + \epsilon_0}\right)\right) < \epsilon.
\end{aligned}$$

On the other hand, for such $u \in S$, there exists $t_u \in S$ such that $\|T_{t_u u} z - z\| \geq \epsilon$. This is a contradiction.

Suppose S is a non-empty set, and let X be a subspace of $\ell^\infty(S)$ containing constants. Let μ be a submean on X, E be a Banach space, $\Phi : S \to E$ be a bounded function, and K be a closed convex subset of E. Suppose that for each $x \in K$, the real-valued function f on S defined by

$$f_x(t) = \|\Phi(t) - x\|^2 \qquad \text{for all } t \in S$$

belongs to X. Then setting

$$r(x) = \langle \mu, f_x \rangle \qquad \text{for all } x \in K,$$

we define $r = \inf_{x \in K} r(x)$ and $M_\mu = \{y \in K : r(y) = r\}$.

Lemma 4.2. *The non-negative real-valued function r on K is continuous, convex and $r(x_n) \to \infty$ as $\|x_n\| \to \infty$. If E is reflexive or K is weakly compact, then M_μ is a non-empty closed convex subset of K. Furthermore, if E is a Hilbert space, then M_μ contains a unique element y and $r + \|y - x\| \leq r(x)$ for all $x \in K$.*

Proof. We first observe that r is continuous and convex on K. Indeed, if $x, y \in K$, then for each $t \in S$,

$$\begin{aligned} (4.1) \quad \|\Phi(t) - y\|^2 - \|\Phi(t) - x\|^2 & \\ &= (\|\Phi(t) - y\| + \|\Phi(t) - x\|)(\|\Phi(t) - y\| - \|\Phi(t) - x\|) \\ &\leq \gamma(\|\Phi(t) - y\| - \|\Phi(t) - x\|) \leq \gamma\|x - y\| \end{aligned}$$

where $\gamma = 2\alpha + \|x\| + \|y\|$, with $\alpha = \{\|\Phi(t)\| : t \in S\} < \infty$ by boundedness of Φ. Also we have by (4.1),

$$\|\Phi(t) - y\|^2 \leq \|\Phi(t) - x\|^2 + \gamma\|x - y\|.$$

Hence

$$\langle \mu, f_y \rangle \ \leq \ \langle \mu, f_x \rangle + \ \gamma\|x - y\|.$$

Similarly

$$\langle \mu, f_x \rangle \ \leq \ \langle \mu, f_y \rangle + \ \gamma\|x - y\|.$$

So $|r(x) - r(y)| \leq \gamma\|x - y\|$. This implies that r is continuous on K. Also, if $0 \leq \lambda \leq 1$ and $x, y \in K$, then

$$\|\Phi(t) - (\lambda x + (1 - \lambda)y)\|^2 \leq \lambda\|\Phi(t) - x\|^2 + (1 - \lambda)\|\Phi(t) - y\|^2.$$

Hence $f_{\lambda x + (1-\lambda)y}(t) \leq \lambda f_x(t) + (1 - \lambda) f_y(t)$. So, by the properties of a submean,

$$r(\lambda x + (1 - \lambda)y) \leq \lambda r(x) + (1 - \lambda)r(y),$$

i.e. r is a convex function. Finally, since $\|\Phi(t)-x\|^2 \geq (\|x\|-\alpha)^2$ for $\|x\| \geq \alpha+1$, we have $r(x_n) \geq (\|x_n\| - \alpha)^2 \to \infty$ as $n \to \infty$.

That M_μ is closed and convex follows from continuity and convexity of r. Also, if E is reflexive, then M_μ is non-empty by [2, p. 89]. If K is weakly compact, for each n, let $K_n = \{x \in K : r(x) \leq r + 1/n\}$. Then each K_n is norm closed and convex by continuity and convexity of r. Hence K_n is also weakly closed. Since $\{K_n : n = 1, 2, \dots\}$ has finite intersection property, it follows that the set $M_\mu = \bigcap K_n$ is closed, convex and non-empty. The last statement was proved in [23, Lemma 1].

LEMMA 4.3. *Let S be a semitopological semigroup, and let $\mathcal{S} = \{T_s : s \in S\}$ be a left asymptotically non-expansive continuous representation of S as self-maps of a non-empty subset C of a Banach space E. If C contains an element z of bounded orbit, then the function $f_x(t) = \|T_t z - x\|^2$, $t \in S$, belongs to $ARUC(S)$ for each $x \in E$. Furthermore, if $\mathcal{S}$ is non-expansive, then each $f_x \in RUC(S)$.*

PROOF. Clearly the functions f_x, $x \in E$, are bounded and continuous. To see that $f = f_x \in ARUC(S)$, let $\gamma = 2\sup_{t\in S} \|T_t z - x\|$. Then for $s \in S$ and $\epsilon > 0$, choose a neighbourhood U of s and a left ideal J of S such that

(i) $\|T_u z - T_s z\| < \epsilon$ for all $u \in U$,

(ii) $\|T_t(T_s z) - T_t y\| \leq \|T_s z - y\| + \epsilon$ for all $t \in J$ and $y \in C$.

Then for $u \in U$,

$$\begin{aligned}
\|r_u f - r_s f\|_J &= \sup_{t\in J}\{|r_u f(t) - r_s f(t)|\} \\
&= \sup_{t\in J} |f(tu) - f(ts)| = \sup_{t\in J} |\|T_{tu} z - x\|^2 - \|T_{ts} z - x\|^2| \\
&= \sup_{t\in J} |(\|T_{tu} z - x\| + \|T_{ts} z - x\|) \cdot (\|T_{tu} z - x\| - \|T_{ts} z - x\|)| \\
&\leq \gamma \sup_{t\in J} \|T_{tu} z - T_{ts} z\| = \gamma \sup_{t\in J} \|T_t(T_u z) - T_t(T_s z)\| \\
&\leq \gamma(\|T_u z - T_s z\| + \epsilon) \qquad \text{(by (ii))} \\
&\leq 2\gamma\epsilon \qquad \text{(by (i))},
\end{aligned}$$

i.e. $f_x \in ARUC(S)$.

The proof of the second statement is similar.

THEOREM 4.4. *Let S be a semitopological semigroup, and C be a closed convex subset of a Banach space E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a left asymptotically non-expansive continuous representation of S as self-maps of C. If C contains an element z such that $\{T_t z : t \in S\}$ is bounded, let $M_\mu = \{y \in C : r(y) = r\}$, where $r = \inf_{x\in C} r(x)$, $r(x) = \mu_t \|T_t z - x\|^2$ and μ is a submean on $ARUC(S)$.*

(a) *If E is reflexive or C is weakly compact, then M_μ is a non-empty closed convex subset of C.*

(b) *If μ is a LISM on $ARUC(S)$, then for any $y \in M_\mu$ and $\epsilon > 0$, there exists a left ideal J of S such that $r(T_s y) < r + \epsilon$ for all $s \in J$.*

(c) *If $\mathcal{S}$ is non-expansive, and μ is a LISM on $RUC(S)$, then M_μ is $\mathcal{S}$-invariant.*

PROOF. (a) is a consequence of Lemmas 4.2 and 4.3.

(b) If μ is a $LISM$ on $ARUC(S)$, $y \in M$, and $\epsilon > 0$, choose a left ideal $J \subseteq S$ such that

$$\|T_s y - T_s y'\| \le \|y - y'\| + \delta \qquad \text{for all } s \in J,\ y' \in C,$$

where $\delta > 0$, $\delta^2 + 2\delta\gamma < \epsilon$, and $\gamma = \sup_{t\in S} \|T_t z - y\|^2$. Then for any $t \in S$,

$$\begin{aligned}\mu_t(\|T_t z - T_s y\|^2) &= \mu_t\|T_{st} z - T_s y\|^2 && \text{(by invariance of } \mu)\\ &\le \mu_t(\|T_t z - y\| + \delta)^2 && (\text{since } T_t z \in C)\\ &\le \mu_t(\|T_t z - y\|^2 + \delta^2 + 2\delta\gamma) \le r + \epsilon,\end{aligned}$$

i.e. $r(T_s y) < r + \epsilon$ for all $s \in J$.

(c) If $\mathcal{S}$ is non-expansive, then each $f_x \in RUC(S)$ (Lemma 4.3). Hence if μ is a $LISM$ on $RUC(S)$, $y \in M_\mu$, and $s \in S$, we have

$$\mu_t\|T_t z - T_s y\|^2 = \mu_t\|T_{st} z - T_s y\|^2 \le \mu_t\|T_t z - y\|^2.$$

Hence $T_s y \in M_\mu$.

THEOREM 4.5. *Let S be a semitopological semigroup. If $ARUC(S)$ has a LISM, then S has the following fixed point property:*

(H) *Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a left asymptotically non-expansive continuous representation of S on a non-empty subset C of a Hilbert space such that for some $z \in C$, $\{T_t z : t \in S\}$ is bounded and*

$$\bigcap_{s\in S} \overline{\text{co}}\,\{T_{st} z : t \in S\} \subseteq C,$$

then C contains a common fixed point for $\mathcal{S}$.

REMARK 4.6. Note that the condition $\bigcap_{s\in S} \overline{\text{co}}\,\{T_{st} z : t \in S\} \subseteq C$ is automatically satisfied when C is closed, convex and $\mathcal{S}$-invariant.

PROOF OF THEOREM 4.5. Let μ be a $LISM$ on $ARUC(S)$. By Lemma 4.3, for each $x \in H$, the function $f_x(t) = \|T_t z - x\|^2$, $t \in S$, is in $ARUC(S)$. Let $M_\mu = \{y \in H : r(y) = r\}$, where $r = \inf\{r(x) : x \in H\}$ and $r(x) = \langle \mu, f_x\rangle$. By Lemma 4.2, M_μ contains a unique element y such that

$$r + \|y - x\|^2 \le r(x) \qquad \text{for all } x \in H. \tag{4.2}$$

For each $s \in S$, let Q_s be the metric projection of H onto $\overline{\text{co}}\,\{T_{st}z : t \in S\}$. Then by [24], Q_s is non-expansive, and for each $t \in S$,

$$\|T_{st}z - Q_s y\|^2 = \|Q_s T_{st}z - Q_s y\|^2 \le \|T_{st}z - y\|^2. \tag{4.3}$$

So, we have

$$\begin{aligned}\mu_t\|T_t z - Q_s y\|^2 &= \mu_t\|T_{st}z - Q_s y\|^2\\ &\le \mu_t\|T_{st}z - y\|^2 \qquad \text{(by (4.3))}\\ &= \mu_t\|T_t z - y\|^2\end{aligned}$$

and thus $Q_s y = y$. This implies $y \in \overline{\text{co}}\,\{T_{st}z : t \in S\}$ for every $s \in S$ and hence $y \in \bigcap_{s\in S} \overline{\text{co}}\,\{T_{st}z : t \in S\} \subseteq C$. We shall now show that $T_s y = y$ for all $s \in S$. In fact, since $T_t z \in C$ for each $t \in S$ and $\{T_t z : t \in S\}$ is bounded, for any $\epsilon > 0$ there exists $s_0 \in S$ such that

$$\|T_{ss_0}y - T_{ss_0}T_t z\|^2 < \|y - T_t z\|^2 + \epsilon^2$$

for all $s, t \in S$. Then

$$\begin{aligned}\mu_t\|T_{ss_0}y - T_t z\|^2 &= \mu_t\|T_{ss_0}y - T_{ss_0t}z\|^2\\ &= \mu_t\|T_{ss_0}y - T_{ss_0}T_t z\|^2 \le \mu_t\|y - T_t z\|^2 + \epsilon^2\end{aligned} \tag{4.4}$$

for all $s \in S$. On the other hand, since

$$\|y - x\|^2 \le \mu_t\|T_t z - x\|^2 - \mu_t\|T_t z - y\|^2 \qquad \text{(by (4.2))}$$

for all $x \in H$, we have for each $s \in S$,

$$\begin{aligned}\|y - T_{ss_0}y\|^2 &\le \mu_t\|T_t z - T_{ss_0}y\|^2 - \mu_t\|T_t z - y\|^2\\ &\le \mu_t\|y - T_t z\|^2 + \epsilon^2 - \mu_t\|T_t z - y\|^2 = \epsilon^2.\end{aligned} \tag{4.5}$$

Fix $s \in S$, and let $\epsilon > 0$. Then, from continuity of T_s at y, there exists $\delta > 0$ such that

$$\|y - f\| < \delta \;\Rightarrow\; \|T_s y - T_s f\| < \epsilon/2,\; f \in C. \tag{4.6}$$

By (4.5), we may choose $s_0 \in S$ such that $\|T_{ts_0}y - y\| < \min\{\epsilon/2, \delta\}$ for every $t \in S$. Then by (4.6), we have for each $t \in S$,

$$\|T_s y - y\| \le \|T_s y - T_s T_{ts_0}y\| + \|T_{sts_0}y - y\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $T_s y = y$ for every $s \in S$. This completes the proof.

COROLLARY 4.7. *Any discrete left subamenable semigroup has the fixed point property* (H).

PROOF. This follows from Corollary 3.2.

Let $\mathcal{S} = \{T_s : s \in S\}$ be a left asymptotically non-expansive continuous representation of S on a non-empty subset C of a Hilbert space and $z \in C$ such that $\{T_t z : t \in S\}$ is bounded and $\bigcap_{s\in S} \overline{\text{co}}\,\{T_{st}z : t \in S\} \subseteq C$. Then for each $x \in H$, the function $h(t) = \langle T_t z, x\rangle$ is in $ARUC(S)$. If μ is a mean on $ARUC(S)$, by the Riesz representation theorem, there exists $z_\mu \in H$ such that $\mu_t\langle T_t z, x\rangle = \langle z_\mu, x\rangle$ for each $x \in H$ [27].

A net of means $\{\mu_\alpha\}$ on $ARUC(S)$ is called *asymptotically invariant* ([24], [29]) if for each $f \in ARUC(S)$ and $a \in S$,

$$\mu_\alpha(r_a f) - \mu_\alpha(f) \to 0 \qquad \text{and} \qquad \mu_\alpha(\ell_a f) - \mu_\alpha(f) \to 0.$$

THEOREM 4.8. *Let S be a semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ be a left asymptotically non-expansive continuous representation of S on a non-empty subset C of a Hilbert space. Assume that there exists $z \in C$ such that $\{T_t z : t \in S\}$ is bounded and $\bigcap_{s\in S} \overline{\text{co}}\,\{T_{st}z : t \in S\} \subseteq C$. If μ is a left invariant mean on $ARUC(S)$, then z_μ is a common fixed point for $\mathcal{S}$ such that*

$$r(z_\mu) = \inf_{y\in H} r(y), \quad \textit{where} \quad r(y) = \mu_t\|T_t z - y\|^2.$$

Furthermore, if μ is an invariant mean on $ARUC(S)$, then for any asymptotically invariant net $\{\mu_\alpha\}$ of means on $ARUC(S)$, the net z_{μ_α} converges weakly to z_μ. In particular, if ψ is another invariant mean on $ARUC(S)$, then $z_\mu = z_\psi$.

PROOF. Observe that if for any $x \in H$ and $t \in S$,

$$\|z_\mu - x\|^2 = \|T_t z - x\|^2 - \|T_t z - z_\mu\|^2 - 2\,\langle T_t z - z_\mu, z_\mu - x\rangle\,,$$

then

$$\begin{aligned} 0 \le \|z_\mu - x\|^2 &= \mu_t(\|T_t z - x\|^2 - \|T_t z - z_\mu\|^2 - 2\,\langle T_t z - z_\mu, z_\mu - x\rangle) \\ &= \mu_t\|T_t z - x\|^2 - \mu_t\|T_t z - z_\mu\|^2 - 2\,\langle z_\mu - z_\mu, z_\mu - x\rangle \\ &= \mu_t\|T_t z - x\|^2 - \mu_t\|T_t z - z_\mu\|^2. \end{aligned}$$

This implies that M_μ consists of the single point z_μ. So, by the proof of Theorem 4.5, z_μ is a common fixed point for $\mathcal{S}$ and $r(z_\mu) = r$.

If μ is an invariant mean on $ARUC(S)$, then

$$\mu_t\|T_t z - x\|^2 \le \inf_s \sup_t \|T_{ts}z - x\|^2 \qquad \text{(by right invariance of } \mu),$$

for each $x \in H$ [28]. On the other hand, for any $y \in F(\mathcal{S})$ and $s \in S$,

$$\inf_u \sup_t(\|y - T_{tu}T_s z\|^2 - \|y - T_s z\|^2) \le 0$$

and hence

$$\inf_u \sup_t \|T_{tu}z - y\|^2 \le \inf_u \sup_t \|T_{tus}z - y\|^2 = \inf_u \sup_t \|T_{tu}T_s z - y\|^2 \le \|T_s z - y\|^2.$$

So, we have

$$\inf_u \sup_t \|T_{tu}z - y\|^2 \le \mu_s \|T_s z - y\|^2.$$

Therefore, for each $y \in F(\mathcal{S})$,

$$\mu_t \|T_t z - y\|^2 = \inf_s \sup_t \|T_{ts}z - y\|^2.$$

Hence if ψ is another invariant mean on $ARUC(S)$, then $z_\psi \in F(\mathcal{S})$; hence

$$\begin{aligned}\mu_t \|T_t z - z_\mu\|^2 &= \inf_s \sup_t \|T_{ts}z - z_\mu\|^2 \le \mu_t \|T_t z - z_\psi\|^2 \\ &= \inf_s \sup_t \|T_{ts}z - z_\psi\|^2 = \psi_t \|T_t z - z_\psi\|^2 \le \psi_t \|T_t z - z_\mu\|^2 \\ &= \inf_s \sup_t \|T_{ts}z - z_\mu\|^2 = \mu_t \|T_t z - z_\mu\|^2.\end{aligned}$$

Hence $\mu_t \|T_t z - z_\mu\|^2 = \mu_t \|T_t z - z_\psi\|^2$. By uniqueness of the element in M_μ, we have $z_\mu = z_\psi$.

Finally, if $\{\mu_\alpha\}$ is an asymptotically invariant net, and μ is a cluster point of $\{\mu_\alpha\}$ in the weak*-topology, then μ is an invariant mean on $ARUC(S)$. Hence if $\{z_{\mu_\beta}\}$ is a subnet of the net $\{z_{\mu_\alpha}\}$ such that z_{μ_β} converges weakly to some y in H, then, since a cluster point ψ of $\{\mu_{\alpha_\beta}\}$ is also a cluster point of $\{\mu_\alpha\}$, ψ is an invariant mean. So, $y = z_\psi = z_\mu$ by the above. This implies that z_{μ_α} converges weakly to z_μ.

5. Weak*-compact convex sets

In this section, we shall establish a fixed point property for representations of a semitopological semigroup S as non-expansive self-maps of norm-separable and weak*-compact convex sets of a dual Banach space when S is *left amenable*, i.e. $LUC(S)$ has a LIM (see [14] for various properties of such semigroups), or a discrete left subamenable semigroup.

Lemma 5.1. *Let S be a left amenable semitopological semigroup or a discrete left subamenable semigroup. Let X be a compact Hausdorff space such that $S \times X \to X$, $(s, x) \to s \cdot x$, is a jointly continuous action of S on X. Then there exists a compact S-invariant subset K of X satisfying:*

(1) $\overline{S(x)} = K$ *for each* $x \in K$,
(2) $s(K) = K$ *for every* $s \in S$.

Proof. We first assume that S is amenable. By Zorn's lemma, there exists a non-empty closed subset K of X which is minimal with respect to being closed and invariant under each element of S. Let $y \in K$. Define $(T_y f)(s) = f(s \cdot y)$,

$s \in S$, $f \in C(K)$. Then $T_y f \in LUC(S)$. Indeed, $T_y f \in CB(S)$. If $a_\alpha \to a$ and $\|\ell_{a_\alpha} T_y f - \ell_a T_y f\| \to 0$, we may assume, by passing to a subnet if necessary, that there exists $\epsilon > 0$ such that

$$\|\ell_{a_\alpha} T_y f - \ell_a T_y f\| \geq \epsilon \qquad \text{for any } \alpha.$$

Now

$$\|\ell_{a_\alpha} T_y f - \ell_a T_y f\| = \sup_{s \in S}\{|f(a_\alpha s y) - f(a s y)|\} = \sup_{z \in \overline{O(y)}}\{|f(a_\alpha z) - f(a z)|\}.$$

Since $\overline{O(y)}$ is compact, where $O(y) = \{t \cdot y : t \in S\}$, and $z \to |f(a_\alpha z) - f(az)|$ is continuous on $\overline{O(y)}$, we may find $z_\alpha \in \overline{O(y)}$ such that $\|\ell_{a_\alpha} T_y f - \ell_a T_y f\| = |f(a_\alpha z_\alpha) - f(a z_\alpha)|$ for each α. Again by passing to a subnet, we may assume that $z_\alpha \to z_0$. So

$$\begin{aligned} \epsilon = \|\ell_{a_\alpha} T_y f - \ell_a T_y f\| &= |f(a_\alpha z_\alpha) - f(a z_\alpha)| \\ &\leq |f(a_\alpha z_\alpha) - f(a z_0)| + |f(a z_0) - f(a z_\alpha)| \to 0 \end{aligned}$$

by joint continuity of the action of $\mathcal{S}$ on X. Let m be a LIM on $LUC(S)$. Define a positive norm one functional ϕ on $C(K)$ by $\phi(f) = m(T_y f)$ for all $f \in C(K)$. Then, as is readily checked, $\phi({}_s f) = \phi(f)$ for all $s \in S$ and $f \in C(K)$, where ${}_s f(x) = f(s \cdot x)$, $x \in K$, $s \in S$. Let μ be the probability measure on K corresponding to ϕ. Then $\mu(B) = \mu(a^{-1}B)$ for all $a \in S$ and for each Borel subset B of K. Let $\Im$ be the family of all closed subsets B of K such that $\mu(B) = 1$, and let $K_0 = \bigcap \Im$. Then K_0 is non-empty. Also if $B \in \Im$ and $s \in S$, then $s^{-1}B \in \Im$. Hence $s^{-1}K_0 \supseteq K_0$ or $K_0 \supseteq sK_0$. By minimality of K, $K = K_0$. Since $\mu(aK) = \mu(a^{-1}(aK)) = \mu(K) = 1$, $aK \in \Im$ for each $a \in S$. Therefore $K \supseteq aK \supseteq K_0 = K$; hence $aK = K$. So (2) holds; (1) follows by minimality of K.

If S is a discrete left subamenable semigroup, then S is left reversible (Corollary 3.2). Hence by Lemma 2 in [10, Chapter 2], any minimal invariant subset K of X satisfies (1) and (2).

Lemma 5.2. *Let E be a Banach space, and τ be a Hausdorff locally convex topology on E weaker than the norm topology; let K be a τ-compact norm-separable subset of E and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S as non-expansive and τ-τ-continuous self-maps of K such that for each $x \in K$, $\{T_t x : t \in S\}$ is τ-dense in K. Then for any $z \in K$ and any τ-neighbourhood V of 0, there exist $t_1, \ldots, t_p \in S$ such that $K = \bigcup_{j=1}^{p}\{T_{s_j}^{-1}[(z+V) \cap K]\}$ where $s_j = t_j t_{j-1} \ldots t_1$. Furthermore, if each T_s is onto and $\{x \in E : \|x\| \leq 1\}$ is τ-closed, then the τ-topology agrees with the norm topology on K. In particular, K is norm-compact.*

PROOF. We follow an idea of Hsu in [10, Chapter 2, Lemma 3]. Fix $z \in K$ and a τ-neighbourhood V of 0. For $\epsilon > 0$, let $N_\epsilon = \{x \in E : \|x\| < \epsilon\}$. Choose a τ-open neighbourhood V_1 of 0 such that $V_1 + V_1 \subseteq V$. Since V_1 is also a norm neighbourhood of 0, there exists $\delta > 0$ such that $N_\delta \subseteq V_1$. Cover K by countably many sets $x_i + N_\delta$, $x_i \in K$. Since $\{T_t x_1 : t \in S\}$ is τ-dense in K, we can choose $t_1 \in S$ such that $T_{t_1} x_1 \in (z + V_1) \cap K$. By induction, we can choose a sequence $\{t_j\}$, $j = 1, 2, \dots$, in S such that $T_{s_j} x_j \in (z + V_1) \cap K$ where $s_j = t_j t_{j-1} \dots t_1$. Since each T_s is non-expansive, we have

$$T_{s_j}[(x_j + N_\delta) \cap K] \subseteq (z + N_\delta + V_1) \cap K \subseteq (z + V) \cap K.$$

Consequently, $\{T_{s_j}^{-1}[(z + V) \cap K]\}_{j=1}^{\infty}$ is a τ-open covering of K. Since K is τ-compact, there exists p such that $K = \bigcup_{j=1}^{p} T_{s_j}^{-1}[(z + V) \cap K]$.

Now if each T_s is onto and $\{x \in E : \|x\| \leq 1\}$ is τ-closed, let $\epsilon > 0$ be fixed. Cover K by countably many sets $y_i + \frac{1}{2} N_\epsilon$, $y_i \in K$; as K is τ-compact, hence second category in itself, there is a point $y \in K$ and a τ-open set W such that

$$K \cap (\tfrac{1}{2} N_\epsilon + y) \supseteq W \cap K \neq \emptyset.$$

Let $z \in W \cap K$ and V be a τ-open neighbourhood of 0 such that $z + V \subseteq W$. So we have $(z + V) \cap K = \emptyset$ and

$$(z + V) \cap K \subseteq (y + \tfrac{1}{2} N_\epsilon) \cap K \subseteq (z + N_\epsilon) \cap K. \tag{5.1}$$

By the above, we can find $t_1, \dots, t_p \in S$ such that $K = \bigcup_{j=1}^{p} T_{s_j}^{-1}[(z + V) \cap K]$, where $s_j = t_j t_{j-1} \dots t_1$. Since each T_s is onto, we have

$$\begin{aligned} K = T_{s_p} K = T_{s_p} \Big\{ \bigcup_{j=1}^{p} T_{s_j^{-1}}[(z + V) \cap K] \Big\} &\subseteq \bigcup_{j=1}^{p} \{T_{t_p t_{p-1} \dots t_{j+1}}[(z + V) \cap K]\} \\ &\subseteq \bigcup_{j=1}^{p} \{T_{t_p t_{p-1} \dots t_{j+1}}[(z + N_\epsilon) \cap K]\} \qquad \text{(by (5.1))} \\ &\subseteq \bigcup_{j=1}^{p} \{T_{t_p t_{p-1} \dots t_{j+1}}(z) + N_\epsilon\} \end{aligned}$$

by non-expansiveness of T_s, $s \in S$. Consequently, K is totally bounded. So K is norm-compact. Since the topology τ on K is Hausdorff and weaker than the norm topology, it follows that they must agree on K.

THEOREM 5.3. *Let S be a semitopological semigroup. If either S is left amenable or S is a left subamenable discrete semigroup, then S has the following fixed point property:*

(F) *Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm non-expansive mappings of a norm-separable weak*-compact convex subset C of a dual Banach space such that the map $S \times C \to C$, $(s, x) \to T_s x$, $s \in S$, $x \in C$, is jointly continuous when C has the weak*-topology, then there exists a common fixed point for $\mathcal{S}$ in C.*

PROOF. We shall prove the theorem for the case of $LUC(S)$ having a LIM. The proof of the left subamenable case is similar using Theorem 5.1 and Corollary 3.7.

By Zorn's lemma, there exists a non-empty weak*-compact convex subset X of C which is minimal with respect to being weak*-closed, convex and invariant under each element of $\mathcal{S}$. A second application of Zorn's lemma shows that there is a non-empty subset F of X which is minimal with respect to being weak*-closed and invariant under each element of $\mathcal{S}$. By Lemma 5.2, F is norm-compact. If F consists of one point, we are done. Otherwise, let $r = \operatorname{diam}(F)$. Then by [4, Lemma 1], there is $u \in \overline{\operatorname{co}}\, F \subseteq X$ such that

$$r_0 = \sup\{\|u - x\| : x \in F\} < r.$$

Let $X_0 = X \cap \bigcap_{x \in F} B[x, r_0]$, where $B[x, r_0] = \{y \in E : \|x - y\| \leq r_0\}$ (which is weak*-closed). Then $u \in X_0$ and X_0 is a non-empty weak*-closed convex proper subset of X. Furthermore, if $x \in X_0$, then $x \in X$ and $F \subseteq B[x, r_0]$. Hence for any $a \in S$, $F = a \cdot F \subseteq B[a \cdot x, r_0]$ by non-expansiveness of S on X. It follows that $aX_0 \subseteq X_0$, contradicting the minimality of X. Consequently, F must consist of a single point.

REMARK 5.4. (a) Let (F$'$) denote the same fixed point property as (F) with the separability condition removed. Then an argument similar to the proof of Theorem 1 of [22] shows that (F$'$) $\Rightarrow$ $LUC(S)$ has a LIM. In particular, S left subamenable $\Rightarrow$ (F$'$) in general. However, we do not know if $LUC(S)$ has a $LIM \Rightarrow$ (F$'$). (See [12, Problem 5].)

(b) T. C. Lim [20, Theorem 4] shows that if S is left reversible (topologically) and $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as non-expansive self-maps of a weak*-compact convex subset C of ℓ^1 (which is separable), then C contains a common fixed point for $\mathcal{S}$ *without* the assumption that the map $\psi : (s, x) \to T_s x$ from $S \times C$ to C is jointly continuous when C has the weak*-topology. However, this weak*-continuity condition on ψ cannot be entirely dropped in general. Indeed, it follows from Alspach's example [1] that there exists a representation of the commutative semigroup $S = (\mathbb{N}, +)$, $\mathbb{N} = \{1, 2, \dots\}$, as non-expansive mappings of a weakly compact convex subset C of the separable Banach space $L_1[0,1]$. Then C, regarded as a subset of $L_1[0,1]^{**}$, is norm-separable, weak*-compact, and convex.

Corollary 5.5. *Let S be a semitopological semigroup. If S is left amenable or if S is a left subamenable discrete semigroup, then S has the following fixed point property:*

(G) *Whenever $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as norm non-expansive mappings on a norm-separable weakly compact convex subset of a Banach space E such that the map $S \times C \to C$, $(s,x) \to T_s x$, $s \in S$, $x \in C$, is jointly continuous when C has the weak topology, then there exists a common fixed point for $\mathcal{S}$ in C.*

Proof. Embed C in E^{**}. Then C is norm-separable, weak*-compact and convex.

Remark 5.6. (a) Corollary 5.5 follows from Hsu [10] for the case when S is discrete and left subamenable (using Corollary 3.2).

(b) We do not know whether a left subamenable semitopological semigroup would have fixed point properties (F) or (G).

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