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A COHOMOLOGY COMPLEX FOR MANIFOLDS WITH BOUNDARY

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Dedicated to Professor Ky Fan

Morse theory is an important part of critical point theory. A fashionable version of Morse theory, which implies Morse inequalities as consequences, describes a Morse function on an oriented compact differentiable manifold without boundary by a cohomology complex or a chain complex $\{C_k, \partial\}$. In J. Milnor [Mi], $C_k = \bigoplus \mathbb{Z}\langle x\rangle$, where x is a critical point with Morse index k, and ∂ is the boundary operator, i.e., $\partial^2 = 0$, determined by the matrix of intersection numbers of oriented right hand spheres with left hand spheres having oriented normal bundles. And in E. Witten [Wi], C_k is the linear space of the k-"harmonic" forms of a certain Laplacian related to the given function, and ∂ is a certain exterior differential operator. This version of Morse theory was generalized to infinite-dimensional manifolds by Floer in his study of symplectic geometry [Fl].

However, Morse inequalities for manifolds with boundary have been known to be useful in applications. The main purpose of this paper is to extend Witten's approach to that situation, i.e., we shall prove

Theorem. *Suppose that f is a Morse function defined on an oriented compact manifold M with boundary. Define*

$$d_t^p = tdf \wedge \cdot + d^p$$

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with domain $D(d_t^p) = H^1 A^p(M)$. *Define also the Laplacian*

$$\Delta_t^p = d_t^{*p} d_t^p + d_t^{p-1} d_t^{*p-1}$$

with domain

$$D(\Delta_t^p) = \{\omega \in H^2 A^p(M) : *\nu\omega|_{\partial M} = *\nu d_t\omega|_{\partial M} = 0\}.$$

Let X_t^p *be the span of the eigenvectors of* Δ_t^p *with eigenvalues* $\lambda^p(t)$ *satisfying* $\lambda^p(t) < \varepsilon t$. *Then* (X_t^p, d_t^p) *is a cohomology complex for large* t. (*See Section* 4 *for the choice of* ε *and the exact expression for the dimension of* X_t^p.)

Remark. If we take the following as domains:

$$D(d_t^p) = \{\omega \in H^1 A^p(M) : \tau\omega|_{\partial M} = 0\},$$
$$D(\Delta_t^p) = \{\omega \in H^2 A^p(M) : \tau\omega|_{\partial M} = \tau\, d_t^*\omega|_{\partial M} = 0\},$$

then the conclusion of the Theorem remains valid.

1. Preliminaries

Let M^n be a compact manifold with boundary $\Sigma = \partial M$. The following notations are used throughout this paper: $A^p(M)$ for the space of all L^2 p-forms on M, and d for the exterior differential operator.

For a p-form, we write in local coordinates,

$$\omega = \sum a_I dx^I, \qquad I = (i_1, \dots, i_p).$$

If Σ is along $x_n = 0$, and M is on the side $x_n > 0$, we call

$$\tau\omega = \sum_{n\notin I}{}' a_I dx^I \quad \text{and} \quad \nu\omega = \sum_{n\in I}{}'' a_I dx^I,$$

the *tangent part* and the *normal part* of ω respectively.

Given a Riemannian metric g on M, we introduce the Hodge star operator $* : A^p(M) \to A^{n-p}(M)$, satisfying

$$**\omega = (-1)^{p(n-p)}\omega \qquad \forall \omega \in A^p(M),$$
$$g(\omega, \theta) = \omega \wedge (*\theta) \qquad \forall \omega, \theta \in A^p(M),$$

and

$$*1 = \eta, \qquad *\eta = 1,$$

where $\eta = \sqrt{g} dx^1 \wedge \dots \wedge dx^n$. The codifferential operator d^* is defined to be $(-1)^{n(p-1)+1} {*d*}$ on $A^p(M)$.

According to Stokes' theorem,

$$\langle d\varrho, \omega\rangle = \int_M d\varrho \wedge (*\omega) = \langle \varrho, d^*\omega\rangle + \int_{\partial M} \varrho \wedge (*\omega),$$

for all $\omega \in A^{p+1}(M)$ and $\varrho \in A^p(M)$. Since

$$\int_{\partial M} \varrho \wedge (*\omega) = \int_{\partial M} (\tau\varrho) \wedge (*\nu\omega),$$

there are many ways of defining the domains of d and d^* so that they are co-adjoint, e.g.,

(1) $D(d) = H^1A^p(M)$, the H^1 section of the bundle $\bigwedge^p T^*M$,
$D(d^*) = \{\omega \in H^1A^p(M) : *\nu\omega|_{\partial M} = 0\}$;
(2) $D(d) = \{\omega \in H^1A^p(M) : \tau\omega|_{\partial M} = 0\}$,
$D(d^*) = H^1A^p(M)$;
(3) $D(d) = \{\omega \in H^1A^p(M) : \tau\omega|_{\partial M} = 0\}$,
$D(d^*) = \{\omega \in H^1A^p(M) : *\nu\omega|_{\partial M} = 0\}$.

In all these cases, we have

$$\langle d\varrho, \omega\rangle = \langle \varrho, d^*\omega\rangle \qquad \forall \varrho \in D(d),\ \forall \omega \in D(d^*).$$

Under these boundary conditions, again, we have

$$d^2 = (d^*)^2 = 0.$$

In fact, what we really want to show is the following:

Claim. *If $\omega \in D(d) \cap H^2A^p(M)$ (or $D(d^*) \cap H^2A^p(M)$), then $d\omega \in D(d)$ (or $d^*\omega \in D(d^*)$ resp.).*

It suffices to prove $\tau d\omega|_{\partial M} = 0$ from $\tau\omega|_{\partial M} = 0$. Indeed, if $\omega = \tau\omega + \nu\omega = (\sum' + \sum'')a_I\, dx^I$, then

$$\tau d\omega = \sum_{k=1}^{n-1} \sum_{n\notin I}{}' \partial_k a_I dx^k \wedge dx^I.$$

From $a_I|_{\partial M} = 0$, $n \notin I$, it follows that $\partial_k a_I|_{\partial M} = 0$ for $k \neq n$, i.e., $\tau d\omega|_{\partial M} = 0$.

Similarly, we now prove $*\nu d^*\omega|_{\partial M} = 0$ from $*\nu\omega|_{\partial M} = 0$. Noticing that $d^* = (-1)^{n(p-1)+1} *d*, *(\tau\omega) = \nu(*\omega)$ and $*(\nu\omega) = \tau(*\omega)$, from $*\nu\omega|_{\partial M} = 0$ it follows that $\tau(*\omega)|_{\partial M} = 0$, i.e., $*\omega \in D(d)$. By the previous conclusion, we have $\tau d(*\omega)|_{\partial M} = 0$, so

$$*\nu d^*\omega|_{\partial M} = (-1)^{n(p-1)+1} * \nu * (d*\omega)|_{\partial M} = 0.$$

This proves $d^*\omega \in D(d^*)$.

Now let us define the Laplacian $\Delta = d^*d + dd^*$ under various boundary conditions so that it is self-adjoint:

(1)′ $D(\Delta^p) = \{\omega \in H^2A^p(M) : *\nu\omega|_{\partial M} = *\nu d\omega|_{\partial M} = 0\}$,
(2)′ $D(\Delta^p) = \{\omega \in H^2A^p(M) : \tau\omega|_{\partial M} = \tau d^*\omega|_{\partial M} = 0\}$,
(3)′ $D(\Delta^p) = \{\omega \in H^2A^p(M) : \tau\omega|_{\partial M} = *\nu\omega|_{\partial M} = 0\}$.

Case $(1)'$ associates with (1). Indeed, for $\omega \in D(\Delta)$, both $d\omega$ and $d^*\omega$ make sense. From $*\nu d\omega|_{\partial M} = 0$, it follows that $d\omega \in D(d^*)$. And obviously $d^*\omega \in D(d)$. Similarly, case $(2)'$ associates with (2).

The self-adjointness of Δ follows from Green's formula:

$$\langle \Delta\omega, \theta\rangle = \langle d\omega, d\theta\rangle + \langle d^*\omega, d^*\theta\rangle + \int_{\partial M} \tau d^*\omega \wedge (*\nu\theta) - \tau\theta \wedge (*\nu d\omega),$$

for all $\omega, \theta \in H^2A^p(M)$.

In case $(3)'$, d^*d and dd^* do not make sense. However, Δ is defined by the bilinear form

$$[\omega, \theta] = \langle d\omega, d\theta\rangle + \langle d^*\omega, d^*\theta\rangle \qquad \forall \omega, \theta \in D(d) \cap D(d^*)$$

in case (3), and then the Friedrichs extension provides a self-adjoint operator.

In all these cases,

$$\Delta\omega = 0 \quad \text{iff} \quad d\omega = d^*\omega = 0.$$

However, in case (3), there is no nontrivial harmonic form, according to the Poincaré inequality. Therefore this is not the case of interest, and we restrict ourselves to cases (1) and (2).

We have the following Hodge Theorem:

$$\begin{aligned} &A^p(M) = N(\Delta^p) \oplus R(d^{p-1}) \oplus R((d^*)^p),\\ &N(d^p) = R(d^{p-1}) \oplus N(\Delta^p),\\ &N((d^*)^{p-1}) = R((d^*)^p) \oplus N(\Delta^p), \end{aligned}$$

where we use d^p, $(d^*)^p$ and Δ^p to indicate the associated operators.

According to various boundary conditions,

$$N(\Delta^p) = N(d^p)/R(d^{p-1}) \cong \begin{cases} H^p_{\mathrm{DR}}(M) & \text{in case (1)},\\ H^p(M, \partial M) & \text{in case (2)} \end{cases}$$

(cf. [GM], [Du], [DS], [DR]).

2. Witten complex

To a given 1-form λ, one attaches an exterior differential operator

$$d_\lambda\omega = \lambda \wedge \omega + d\omega \tag{2.1}$$

with $D(d_\lambda) = D(d)$. We have $d_\lambda^2 = 0$ if λ is exact.

Similarly, we define $d_\lambda^* = (-1)^{n(p-1)} *d_\lambda*$, thus

$$d_\lambda^*\omega = i_\lambda\omega + d^*\omega, \tag{2.2}$$

where i is the interior product, with $D(d_\lambda^*) = D(d^*)$. We have $d_\lambda^{*2} = 0$ if λ is exact. Define

$$\Delta_\lambda = d_\lambda^* d_\lambda + d_\lambda d_\lambda^*, \tag{2.3}$$

where $D(\Delta_\lambda) = D(\Delta)$. We have the expression

$$\Delta_{t\lambda} = \Delta + t^2 g(\lambda, \lambda) + tP_\lambda, \tag{2.4}$$

where

$$P_\lambda \omega = i_\lambda d\omega + d(i_\lambda \omega) + d^*(\lambda \wedge \omega) + \lambda \wedge d^* \omega.$$

It is known that P_λ commutes with multiplication, i.e., for all $\varphi \in C^\infty(M)$,

$$P_\lambda(\varphi\omega) = \varphi(P_\lambda \omega). \tag{2.5}$$

Now, for a function $f \in C^2(M, \mathbb{R}^1)$, we use the shorthand

$$d_t \omega = d_{tdf}\omega,$$

and similarly for d_t^* and Δ_t.

In a conformal metric,

$$P_{df}\omega = \sum_{k,l} \frac{\partial^2 f}{\partial x_l \partial x_k}[dx^l \wedge i_{dx^k}]\omega.$$

Now, for the pair of differential operators d_t, d_t^*, we call the complex

$$E = \{A^p(M) : p = 0, 1, \dots, n\}, \quad d_t = \{d_t^p : p = 0, 1, \dots, n-1\}$$

with

$$0 \to A^0(M) \to \dots \to A^p(M) \xrightarrow{d_t^p} A^{p+1}(M) \to \dots \to 0$$

the *Witten complex*. With the given domains as boundary conditions, again we have the Hodge decomposition:

$$\begin{aligned} A^p(M) &= N(\Delta_t^p) \oplus R(d_t^{p-1}) \oplus R((d_t^*)^p), \\ N(d_t^p) &= R(d_t^{p-1}) \oplus N(\Delta_t^p), \\ N((d_t^*)^{p-1}) &= R((d_t^*)^p) \oplus N(\Delta_t^p), \end{aligned}$$

and

$$N(\Delta_t^p) = N(d_t^p)/R(d_t^{p-1}) \cong N(\Delta^p). \tag{2.6}$$

Indeed, only the last relation is to be verified. By looking at the complex

$$\begin{array}{ccccccccc} 0 & \to A^0(M) & \to \dots & \to A^p(M) & \xrightarrow{d^p} & A^{p+1}(M) & \to \dots & \to 0 \\ & \downarrow e^{-tf} & & \downarrow e^{-tf} & & \downarrow e^{-tf} & & \\ 0 & \to A^0(M) & \to \dots & \to A^p(M) & \xrightarrow{d_t^p} & A^{p+1}(M) & \to \dots & \to 0 \end{array}$$

one sees that $R(d_t^{p-1})$ and $R((d_t^*)^p)$ are isomorphic to $R(d^{p-1})$ and $R((d^*)^p)$ resp. This proves $N(\Delta_t^p) \cong N(\Delta^p)$ for all t.

In the following, we assume that f satisfies the general boundary conditions, i.e., f has no critical point on ∂M, and both f and $\widehat{f} = f|_{\partial M}$ are Morse functions.

Let

$$\Sigma_{\mp} = \{x \in \Sigma : \pm\langle df(x), n(x)\rangle \leq 0\},$$

where $n(x)$ is the unit normal vector on Σ, and let

$$\Sigma_* = \begin{cases} \Sigma_- & \text{in case (1),} \\ \Sigma_+ & \text{in case (2).} \end{cases}$$

In a local chart about x, we take $x' = (x_1, \dots, x_{n-1})$ along $T_x(\Sigma)$, and the y axis directed opposite to $n(x)$.

Let $K(f) = \{x_1^*, \dots, x_s^*\}$ and $K_*(\widehat{f}) = \{y_1^*, \dots, y_w^*\}$ be the critical sets of f and $\widehat{f}|_{\Sigma_*}$ respectively. We have the Morse lemmas

$$f(x) = f(x^*) + \frac{1}{2}\sum_{k=1}^{n} \mu_k x_k^2, \qquad \mu_k = \pm 1, \tag{2.7}$$

in a local chart about x^*, and

$$f(x) = \widehat{f}(y^*) + \frac{1}{2}\sum_{k=1}^{n-1} \mu_k x_k^2 \pm y, \qquad \mu_k = \pm 1, \tag{2.8}$$

in a local chart about $y^* \in \Sigma_{\mp}$.

In a local chart about x^* (and y^*) under the flat metric, the Laplacian Δ_t^p is expressed as follows:

$$\sum_{k=1}^{n} \mathcal{H}_{k,t} \qquad \bigg(\text{and } \sum_{k=1}^{n-1} \mathcal{H}_{k,t} + \bigg(-\frac{\partial^2}{\partial y^2} + t^2\bigg) \text{ resp.}\bigg), \tag{2.9}$$

where

$$\mathcal{H}_{k,t} = -\frac{\partial^2}{\partial x_k^2} + t^2 x_k^2 + t\mu_k[dx^k \wedge i_{dx^k}].$$

For all $x^* \in K(f)$, we define a self-adjoint operator Δ_{t,x^*}^p on $A^p(\mathbb{R}^n)$ with the same expression as in (2.9). Thus, $N(\Delta_{t,x^*}^p)$ is spanned by all p-forms of the form

$$\varphi_I^t = t^{n/4} \exp\bigg(-\frac{t}{2}\sum_{k=1}^{n} x_k^2\bigg) dx^I,$$

where I is a p-multiindex such that $\mu_k < 0 < \mu_{k'}$ for $k \in I$ and $k' \notin I$.

Similarly, for $y^* \in K_*(\widehat{f}), \Delta_{t,y^*}^p$ is defined on $A^p(\mathbb{R}_+^n)$ with the same expression as in (2.9) and with boundary conditions either $*(\nu\omega) = *(\nu d_t\omega) = 0$ on $y = 0$, or $\tau\omega = \tau d_t^*\omega = 0$ on $y = 0$.

Again Δ_{t,y^*}^p so defined is self-adjoint.

We are going to find the kernel $N(\Delta_{t,y^*}^p)$.

Lemma 2.1. *$N(\Delta^p_{t,y^*})$ is spanned by all p-forms of the form*

$$\varphi^t_I = t^{(n-1)/4} \exp\left\{ -t\left(y + \frac{1}{2}\sum_{k=1}^{n-1} x_k^2\right)\right\} dx^I,$$

where I is a p-multiindex in $\{1,\dots,n\}$ such that $\mu_k < 0 < \mu_{k'}$ for $k \in I$ and $k' \notin I$ and $n \notin I$ in case (1), *while $n \in I$ in case* (2).

Proof. We only discuss the case where the boundary condition for Δ^p_{t,y^*} reads

$$*\nu\omega = *(\nu d_t\omega) = 0 \qquad \text{on } y = 0,$$

where $d_t = d_{tdf}$ and f is as in (2.8). Set

$$E_1 = \{\omega = e^{-ty}\omega_1 : \omega_1 \in H^2A^p(\mathbb{R}^{n-1})\},$$

$$E_2 = \left\{\omega = \sum_{n\notin J} a_J(x',y)dx^J : a_J \in H^2(\mathbb{R}^n_+),\ \int_0^\infty a_J(x',y)e^{-ty}\,dy = 0\right.$$

$$\left. \text{and } \partial_y a_J(x',0) + t a_J(x',0) = 0;\ J \text{ is a } p\text{-multiindex}\right\},$$

$$E_3 = \{\omega \in H^2A^p(\mathbb{R}^n_+) : \omega = \omega_1 \wedge dy,\ \nu\omega_1 = 0 \text{ and } \omega_1|_{y=0} = 0\}.$$

We shall prove

$$D(\Delta^p_{t,y^*}) = E_1 \oplus E_2 \oplus E_3. \tag{2.10}$$

Firstly, all E_i, $i = 1,2,3$, are in $D(\Delta^p_{t,y^*})$, i.e., the boundary condition is satisfied.

Indeed, for $\omega \in E_1$, $\nu\omega = 0$ so $*(\nu\omega)|_{y=0} = 0$. Further, on $y = 0$,

$$\nu d_t\omega = -te^{-ty}dy \wedge \omega_1 + te^{ty}dy \wedge \omega_1 = 0.$$

For $\omega \in E_2$, again $\nu\omega = 0$. Moreover,

$$\nu d_t\omega = \sum(\partial_y a_J + t a_J)dy \wedge dx^J$$

so $*(\nu d_t\omega)|_{y=0} = 0$.

For $\omega \in E_3$, $*(\nu\omega)|_{y=0} = *\omega_1|_{y=0} = 0$. From

$$\nu d_t\omega = (tdf \wedge \omega_1 + d\omega_1) \wedge dy$$

and $\omega_1|_{y=0} = 0$ it follows that $*(\nu\, d_t\omega)|_{y=0} = 0$.

Secondly, E_1, E_2 and E_3 are mutually orthogonal with respect to the inner product of $A^p(\mathbb{R}^n_+)$.

Thirdly, E_1, E_2 and E_3 span $D(\Delta^p_{t,y^*})$.

Similarly, for the boundary condition

$$\tau\omega = \tau d_t^*\omega = 0 \qquad \text{on } y = 0,$$

we set

$$E_1 = \{\omega = e^{-ty}\omega_1 \wedge dy : \omega_1 \in H^2A^p(\mathbb{R}^{n-1})\},$$
$$E_2 = \bigg\{\omega = \sum_{n\in J} a_J(x',y)dx^J : a_J \in H^2(\mathbb{R}^n_+),\ \int_0^\infty a_J(x',y)e^{-ty}\,dy = 0$$
$$\text{and } \partial_y a_J(x',0) + ta_J(x',0) = 0;\ J \text{ is a } p\text{-multiindex}\bigg\},$$

and

$$E_3 = \{\omega \in H^2A^p(\mathbb{R}^n_+) : \nu\omega = 0, *\omega|_{y=0} = 0\}.$$

The verification of (2.10) is the same.

Now we show that for $\omega = a_I dx^I$,

$$\langle \Delta^p_{t,y^*}\omega,\omega\rangle \geq t^2\|\omega\|^2 + t\sum_{k=1}^{n-1}(1+\varepsilon^I_k)\|\omega\|^2 \tag{2.11}$$

if $\omega \in E_2 \oplus E_3$, and

$$\langle \Delta^p_{t,y^*}\omega,\omega\rangle \geq t\sum_{k=1}^{n-1}(1+\varepsilon^I_k)\|\omega\|^2 \tag{2.12}$$

if $\omega \in E_1$, where

$$\varepsilon^I_k = \begin{cases} 1 & \text{if } k \in I,\\ -1 & \text{if } k \notin I.\end{cases}$$

That (2.11) holds for $\omega \in E_3$ is verified by using the Hermite operators in separate variables and the boundary condition. For $\omega \in E_2$, since

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty a_J(x',y)(-\partial_y^2+t^2)a_J(x',y)\,dx'\wedge dy$$
$$= \int_{\mathbb{R}^n_+}((\partial_y a_J)^2 + t^2\,a_J^2)\,dx'\wedge dy + \int_{\mathbb{R}^{n-1}} a_J(x',0)\partial_y a_J(x',0)\,dx',$$

and

$$\int_0^\infty e^{-ty}\partial_y a_J(x',y)\,dy = -a_J(x',0),$$

we have

$$\bigg|\int_0^\infty e^{-ty}\partial_y a_J(x',y)\,dy\bigg|^2 \leq \bigg(\int_0^\infty |\partial_y a_J(x',y)|^2\,dy\bigg)\bigg(\int_0^\infty e^{-2ty}\,dy\bigg)$$
$$= (2t)^{-1}\int_0^\infty |\partial_y a(x',y)|^2\,dy.$$

Therefore

$$\int_{\mathbb{R}^{n-1}} a_J(x',0)\partial_y a_J(x',0)\,dx' = -t\int_{\mathbb{R}^{n-1}} |a_J(x',0)|^2\,dx' \geq -\frac{1}{2}\int_{\mathbb{R}^n_+} |\partial_y a|^2\,dx'\wedge dy.$$

This proves (2.11).

For $\omega \in E_1$, since e^{-ty} is a solution of the equation

$$(-\partial_y^2 + t^2)v = 0,$$

(2.12) follows directly.

Combining (2.11) and (2.12), we see that $N(\Delta^p_{t,y^*})$ is spanned by the forms φ^t_I.

Let us introduce a Hilbert space $H = A^p(\mathbb{R}^n)^s \oplus A^p(\mathbb{R}^n_+)^w$, where X^s denotes the s-fold product of a Banach space X. Moreover, let

$$A^p_t = (\Delta^p_{t,x_1^*}, \dots, \Delta^p_{t,x_s^*}, \Delta^p_{t,y_1^*}, \dots, \Delta^p_{t,y_w^*})$$

be a self-adjoint operator on H, where Δ^p_{t,x^*} (and Δ^p_{t,y^*}) is defined as above. We obtain

Theorem 1.

$$\dim N(A^p_t) = m_p + \begin{cases} n_p & \text{in case } (1),\\ n_{p-1} & \text{in case } (2), \end{cases}$$

where

$$m_p = \#\{x^* \in K(f) : \operatorname{ind}(f,x^*) = p\},$$
$$n_p = \#\{y^* \in K_*(\widehat{f}) : \operatorname{ind}(\widehat{f},y^*) = p\}.$$

Remark 2.1. The operator A^p_t may have continuous spectrum.

3. The relationship between eigenvalues of Δ^p_t and A^p_t

We arrange the eigenvalues of Δ^p_t and A^p_t as follows:

$$0 \leq \lambda^p_1(t) \leq \lambda^p_2(t) \leq \dots, \qquad 0 \leq te^p_1 \leq te^p_2 \leq \dots,$$

but ignore the continuous spectrum of A^p_t. We shall prove

Theorem 2.

$$\lim_{t\to\infty} \frac{\lambda^p_k(t)}{t} = e^p_k.$$

The proof is divided into two parts:

(i) $\limsup_{t\to\infty} t^{-1}\lambda^p_k(t) \leq e^p_k$,

(ii) $\liminf_{t\to\infty} t^{-1}\lambda^p_k(t) \geq e^p_k$.

The proof of (i) is quite similar to that for manifolds without boundary. Write down the eigenforms of Δ^p_{t,x^*} and Δ^p_{t,y^*}:

$$\varphi^t_{N,I} = \prod_{k=1}^{n} \sqrt{t}\, H_{N_k}(\sqrt{t}\, x_k)dx^I \qquad \text{for } x^* \in K(f),$$

$$\varphi^t_{N',I} = e^{-ty} \prod_{k=1}^{n-1} \sqrt{t}\, H_{N_k}(\sqrt{t}\, x_k)dx^I \qquad \text{for } y^* \in K_-(\widehat{f}),$$

where $H_j(\dot{x})$ is the jth Hermite function, $N = (N_1, \dots, N_n)$, $N' = (N_1, \dots, N_{n-1})$ and I is a multiindex. Let $\varrho \in C^\infty(\mathbb{R}^n)$, with $0 \le \varrho \le 1$, satisfy $\varrho(y) = 1$ if $|y| \le 1/2$ and $\varrho(y) = 0$ if $|y| \ge 1$.

We pull back these functions, and glue them up to define a form on M:

$$\psi^t_\alpha = \bigg(\sum_{j'=1}^{s} + \sum_{j''=1}^{w} \bigg) \varrho(t^{2/5}\eta_j(x))(\varphi^t_\alpha)^j \circ \eta_j(x),$$

where φ^t_α is an eigenform of A^p_t, and $(\varphi^t_\alpha)^j$ is its jth component, $\alpha = (N^1, \dots, N^s; N'^1, \dots, N'^w)$, and η_j is the coordinate function in a neighborhood of $x^*_{j'}$ (or $y^*_{j''}$), $j = j'$ or j''.

As in [An], [Ch], we have

$$|\langle \psi^t_\alpha, \psi^t_\beta \rangle - \delta_{\alpha\beta}| \le C_{\alpha\beta} \exp\big(-\tfrac{1}{2} t^{1/5}\big),$$
$$\big|\langle \psi^t_\alpha, \Delta^p_t \psi^t_\beta \rangle - \tfrac{1}{2} t (e^p_\alpha + e^p_\beta) \langle \psi^t_\alpha, \psi^t_\beta \rangle\big| \le C_{\alpha\beta} \exp\big(-\tfrac{1}{2} t^{1/5}\big)$$

as $t \to \infty$, where e^p_α and e^p_β are the eigenvalues of $t^{-1} A^p_t$ associated with φ^t_α and φ^t_β resp.

Applying the Rayleigh–Ritz Principle, it follows that

$$\limsup_{t\to\infty} t^{-1} \lambda^p_k(t) \le e^p_k. \tag{3.1}$$

Next we turn to the proof of the reverse inequality (ii).

Let $U_{j'}$ (or $U_{j''}$) denote a neighborhood of $x^*_{j'}$ (or $y^*_{j''}$) on which the Morse lemma holds, and suppose a metric g is constructed in such a manner that $g|_{U_j}$ is conformal.

Set (for t large)

$$J^t_j(x) = \begin{cases} 0, & x \notin U_j, \\ \varrho(t^{2/5}\eta_j(x)), & x \in U_j, \end{cases}$$

where $j = j'$ or j'', and set

$$J^t_0 = \bigg(1 - \sum_j (J^t_j)^2 \bigg)^{1/2}.$$

By direct computation, one has

$$\Delta_t^p = J_0^t \Delta_t^p J_0^t + \sum_j J_j^t \Delta_j^t J_j^t - \sum_j (\nabla J_j^t)^2, \tag{3.2}$$

where we use j to denote j' and j''.

We have

$$(\nabla J_j^t)^2 = O(t^{4/5}), \tag{3.3}$$

and for $\omega \in D(\Delta_t^p)$,

$$\begin{aligned} \sum_j \langle J_j^t \Delta_t^p J_j^t \omega, \omega \rangle &= \langle A_t^p \omega_t, \omega_t \rangle \\ &\geq t e_{k+1}^p \sum_j \| J_j^t \omega \|^2 + \langle F_k(t) \omega, \omega \rangle, \end{aligned} \tag{3.4}$$

where $\omega_t = \varrho(t^{2/5} x) \cdot \omega \circ \eta_j^{-1}(x) \in H$, and

$$F_k(t) = \sum_j J_j^t \widetilde{P}_k (\Delta_t^p - t e_{k+1}^p) \widetilde{P}_k J_j^t, \tag{3.5}$$

$\widetilde{P}_k$ being the pull back of P_k, which is the orthogonal projection onto the subspace spanned by the first k eigenvectors of A_t^p.

It remains to estimate $\langle J_0^t \Delta_t^p J_0^t \omega, \omega \rangle$. A new difficulty is the lack of positive definiteness of Δ^p on $D(\Delta_t^p)$. Indeed,

$$\langle \Delta^p \omega, \omega \rangle = \|d\omega\|^2 + \|d^*\omega\|^2 + \int_{\partial M} (\tau d^* \omega \wedge (*\nu\omega) - \tau\omega \wedge (*\nu d\omega)).$$

For instance, if $*\nu\omega|_{\partial M} = *(\nu d_t \omega)|_{\partial M} = 0$, one has

$$\langle \Delta^p \omega, \omega \rangle = \|d\omega\|^2 + \|d^*\omega\|^2 + t \int_{\partial M} \tau\omega \wedge (*\nu(df \wedge \omega));$$

and if $\tau\omega|_{\partial M} = \tau d_t^* \omega|_{\partial M} = 0$, then

$$\langle \Delta^p \omega, \omega \rangle = \|d\omega\|^2 + \|d^*\omega\|^2 - t \int_{\partial M} \tau i_{df} \omega \wedge (*\nu\omega).$$

Since

$$\tau\omega \wedge (*\nu(df \wedge \omega)) = g(\tau\omega, \tau\omega) \partial_n f \cdot \eta$$

and

$$\tau i_{df} \omega \wedge (*\nu\omega) = g(\nu\omega, \nu\omega) \partial_n f \cdot \eta,$$

where

$$\partial_n f(x) = \langle df(x), n(x) \rangle \qquad \forall x \in \partial M,$$

and η is the volume form on ∂M, Δ^p might be positive definite on $D(\Delta_t^p)$ if $\pm \partial_n f \geq 0$ on ∂M in case (1) and (2) resp. However, generally speaking, this is not true.

We only investigate case (1).

To overcome this difficulty, let us define a 1-form λ as follows. We choose $U_{j'}$ and $U_{j''}$ as above; let $U' = \bigcup_{j'} U_{j'}$, $U'' = \bigcup_{j''} U_{j''}$, and let W be a neighborhood of $\Sigma_-\backslash U''$ with $W \cap U' = \emptyset$ for which there exists $\varepsilon_0 > 0$ such that

$$g(\tau df, \tau df)|_x \geq \varepsilon_0 \qquad \forall x \in W.$$

The existence of ε_0 is due to the fact that $\tau df \neq 0$ on $\Sigma_-\backslash U''$.

One may choose an open set V such that $V \cap \Sigma_- = \emptyset$ and $\{U'', W, V\}$ is a covering of M.

Let χ_1, χ_2 and χ_3 be a C^∞-partition of unity on M associated with $\{U'', W, V\}$, i.e., $\operatorname{supp} \chi_1 \subset U''$, $\operatorname{supp} \chi_2 \subset W$ and $\operatorname{supp} \chi_3 \subset V$. Set

$$[\nu df]_- = \begin{cases} \nu df & \text{if } \partial_n f < 0, \\ 0 & \text{if } \partial_n f \geq 0, \end{cases}$$

and

$$\lambda = \chi_1(x)\sqrt{1-y^2}\, dy + \chi_2(x)[\nu df]_-,$$

where $(x', y) = \eta_{j''}(x)$ for all $x \in U_{j''}$ and all j''.

Lemma 3.1.

$$\langle df - \lambda, n(x)\rangle \geq 0 \qquad \forall x \in \Sigma, \tag{3.6}$$

$$g(df, df) - g(\lambda, \lambda)|_x \geq \varepsilon_0 > 0 \qquad \forall x \in M\backslash(U' \cup U''). \tag{3.7}$$

Proof. For $x \in \operatorname{supp} \chi_1 \cap \Sigma_-$,

$$\partial_n f = -[\nu df]_- = -1,$$

and $\chi_1 + \chi_2 = 1$, therefore $\langle df - \lambda, n\rangle = -1 + \chi_1 + \chi_2 = 0$.

For $x \in \Sigma_- \backslash \operatorname{supp} \chi_1$,

$$\partial_n f = -[\nu df]_- = -1,$$

therefore $\langle df - \lambda, n\rangle = -1 + 1 = 0$.

For $x \in \Sigma\backslash\Sigma_-$,

$$\partial_n f \geq 0, \qquad [\nu df]_- = 0,$$

therefore $\langle df - \lambda, n\rangle = \partial_n f \geq 0$. Thus (3.6) is proved.

One may choose U' suitably such that

$$g(df, df)|_x \geq \varepsilon_0 \qquad \forall x \in M\backslash U'.$$

This is due to the fact that $K(f) \subset U'$. Since

$$\lambda = 0 \qquad \forall x \in V\backslash(W \cup U'' \cup U'),$$

for such x we have

$$g(df, df) - g(\lambda, \lambda)|_x \geq \varepsilon_0.$$

Finally, for $x \in W\backslash U''$,

$$\begin{aligned} g(df, df) - g(\lambda, \lambda) &= g(\tau df, \tau df) + g(\nu df, \nu df) - \chi_2^2 g([\nu df]_-, [\nu df]_-) \\ &\geq g(\tau df, \tau df) \geq \varepsilon_0, \end{aligned}$$

and (3.7) is proved.

Lemma 3.2. *Suppose $e_k^p < r < e_{k+1}^p$. Then for large t, there is a finite rank operator $F_k(t) : A^p(M) \to A^p(M)$ with $\dim R(F_k(t)) \leq k$ and*

$$\Delta_t^p \geq rt.\mathrm{Id} + F_k(t).$$

Proof. Obviously, the operator $F_k(t)$ defined in (3.5) is of finite rank, and $\dim R(F_k(t)) \leq k$. As we have seen in (3.2)–(3.4), it suffices to estimate

$$\langle J_0^t \Delta_t^p J_0^t \omega, \omega\rangle \geq t e_{k+1}^p \|J_0^t \omega\|^2$$

for large t.

Indeed, from (2.4),

$$\begin{aligned} \langle J_0^t \Delta_t^p J_0^t \omega, \omega\rangle &= \langle \Delta_t^p J_0^t \omega, J_0^t \omega\rangle \\ &= \langle \Delta^p J_0^t \omega, J_0^t \omega\rangle + t^2 \langle g(df, df) J_0^t \omega, J_0^t \omega\rangle + t\langle P_{df} J_0^t \omega, J_0^t \omega\rangle \\ &= \langle \Delta_{t\lambda}^p J_0^t \omega, J_0^t \omega\rangle + t^2 \langle (g(df, df) - g(\lambda, \lambda)) J_0^t \omega, J_0^t \omega\rangle \\ &\quad + t\langle P_{df-\lambda} J_0^t \omega, J_0^t \omega\rangle \\ &= T_1 + T_2 + T_3. \end{aligned}$$

Now, for all $\omega \in D(\Delta_t^p)$,

$$\begin{aligned} T_1 &= \|d_{t\lambda}\omega\|^2 + \|d_{t\lambda}^*\omega\|^2 + \int_{\partial M} (-\tau\omega \wedge (*\nu d_{t\lambda}\omega)) \\ &\quad + \tau d_{t\lambda}^*\omega \wedge (*\nu\omega) \\ &= \|d_{t\lambda}\omega\|^2 + \|d_{t\lambda}^*\omega\|^2 + t\int_{\partial M} \tau\omega \wedge (*(df - \lambda) \wedge \omega) \geq 0 \end{aligned}$$

by Lemma 3.1, if $*(\nu\omega)|_{\partial M} = *(\nu d_t\omega)|_{\partial M} = 0$. Similarly one reasons for the other boundary condition by using

$$[\nu df]_+ = \begin{cases} 0 & \text{if } \partial_n f \leq 0, \\ \nu df & \text{if } \partial_n f > 0, \end{cases}$$

in place of $[\nu df]_-$.

Obviously, since P_λ commutes with multiplication (cf. (2.5)), it is a bounded operator on $A^p(M)$, and we have a constant $C > 0$ such that

$$T_3 \geq -Ct\|J_0^t\omega\|^2.$$

We turn to estimating T_2. By Lemma 3.1, for $x \in M\backslash(U' \cup U'')$, we have

$$g(df, df) - g(\lambda, \lambda) \geq \varepsilon_0.$$

For $x \in U'$, we have $\lambda = 0$, and

$$g(df, df) = |x|^2 \geq \tfrac{1}{4} t^{-4/5}.$$

For $x \in U''$, we have $\lambda = (1 - y^2) dy$, and

$$g(df, df) - g(\lambda, \lambda) = |x'|^2 + y^2 \geq \tfrac{1}{4} t^{-4/5}.$$

Therefore

$$T_2 \geq t^2 \min\big(\varepsilon_0, \tfrac{1}{4} t^{-4/5}\big) \|J_0^t \omega\|^2 \geq \tfrac{1}{4} t^{6/5} \|J_0^t \omega\|^2 \qquad \text{for } t \text{ large.}$$

This proves the lemma.

The rest of the proof of Theorem 2 is the same as the proof for manifolds without boundary; we refer the reader to [An], [Ch].

4. Cohomology complex

We introduce a new cohomology complex

$$X^p = X_t^p = \{\omega \in A^p(M) : \omega \text{ an eigenvector of } \Delta_t^p \text{ with eigenvalue } \lambda_m^p(t) \text{ satisfying } \lambda_m^p(t) \leq \varepsilon t\},$$

where

$$0 < \varepsilon < \min\{e^p_{M_{p+1}} : p = 0, 1, \dots, n\},$$

and

$$M_p = \dim N(A_t^p) = m_p + \begin{cases} n_p & \text{in case (1),} \\ n_{p-1} & \text{in case (2)} \end{cases}$$

(cf. Theorem 1). Thus, by Theorem 2,

$$\dim X^p = m_p + \begin{cases} n_p & \text{in case (1),} \\ n_{p-1} & \text{in case (2).} \end{cases}$$

We are going to show that

$$0 \to X^0 \xrightarrow{d_t^0} X^1 \to \dots \xrightarrow{d_t^{n-1}} X^n \to 0 \tag{4.1}$$

is a cohomology complex.

Claim 1. $d_t^p : X^p \to X^{p+1}$.

For all $\omega \in X^p$, we have $d_t^p \omega = \lambda_t^p(t) \omega$, and

$$*\nu\omega|_{\partial M} = *(\nu d_t^p \omega)|_{\partial M} = 0 \qquad (\text{or } \tau\omega|_{\partial M} = \tau (d_t^*)^p \omega|_{\partial M} = 0),$$

so that

$$*(\nu d_t^p \omega)|_{\partial M} = *\nu(d_t^{p+1} d_t^p \omega)|_{\partial M} = 0.$$

The last equality follows from $d_t^2 = 0$. Moreover, we have

$$\Delta_t^{p+1} d_t^p = (d_t^{*p+1} d_t^{p+1} + d_t^p d_t^{*p}) d_t^p = d_t^p d_t^{*p} d_t^p = d_t^p \Delta_t^p.$$

This proves the claim.

Claim 2. $d_t^{*p-1} : X^p \to X^{p-1}$.

Indeed, $\forall \omega \in X^p$, we have $*(\nu\omega)|_{\partial M} = *(\nu d_t^p \omega)|_{\partial M} = 0$.

Set $\theta = d_t^{*p-1}\omega$. Then $*\nu\theta|_{\partial M} = *\nu d_t^{*p-1}\omega|_{\partial M} = 0$ since $*(\nu\omega)|_{\partial M} = 0$ (see §1). Moreover,

$$d_t^{p-1}\theta = d_t^{p-1} d_t^{*p-1}\omega = (\lambda_m^p(t) - d_t^{*p} d_t^p)\omega.$$

Therefore

$$*(\nu d_t^{p-1}\theta)|_{\partial M} = - *(\nu d_t^{*p}(d_t^p\omega))|_{\partial M} = 0 \qquad \text{since } *(\nu d_t^p \omega)|_{\partial M} = 0,$$

i.e., we proved $\theta \in D(\Delta_t^{p-1})$. Moreover,

$$\Delta_t^{p-1} d_t^{*p-1} = (d_t^{*p-1} d_t^{p-1} + d_t^{p-2} d_t^{*p-2}) d_t^{*p-1} = d_t^{*p-1} d_t^{p-1} d_t^{*p-1} = d_t^{*p-1} \Delta_t^p.$$

Again, this proves the claim. Similarly, we verify the case (2).

Claim 3. $N(d_t^p) = R(d_t^{p-1}) \oplus N(\Delta_t^p)$.

It is easily seen that $N(\Delta_t^p) \subset X^p \cap N(d_t^p)$. Now, for $\omega \in X^p \cap N(d_t^p) \cap N(\Delta_t^p)^\perp$, we have

$$d_t^p\omega = 0, \qquad d_t^{p-1} d_t^{*p-1}\omega = \lambda_m^p(t)\omega,$$

where $\lambda_m^p(t) \neq 0$. Define $\theta = d_t^{*p-1}\omega$. By Claim 2, $\theta \in X^{p-1}$. It follows that

$$\omega = \frac{1}{\lambda_m^p(t)} d_t^{p-1}\theta \in R(d_t^{p-1}).$$

Finally, we have shown that the smaller cohomology complex (4.1) has the following properties:

$$\dim X^p = m_p + \begin{cases} n_p & \text{in case (1)}, \\ n_{p-1} & \text{in case (2)}, \end{cases}$$

$$\dim N(d_t^p)/R(d_t^{p-1}) = \begin{cases} \beta_p & \text{in case (1)}, \\ \beta_p^* & \text{in case (2)}, \end{cases}$$

where

$$\beta_q = \operatorname{rank} H_{\mathrm{DR}}^q(M) \quad \text{and} \quad \beta_q^* = \operatorname{rank} H^q(M, \partial M).$$

As a consequence, Morse inequalities for M with boundary conditions hold, i.e., for all t,

$$\sum_{q=0}^{\infty}(m_q + n_q - \beta_q)t^q \quad \left(\text{and } \sum_{q=0}^{\infty}(m_q + n_{q-1} - \beta_q^*)t^q\right) \quad = (1+t)Q(t),$$

where Q is a formal series with nonnegative coefficients.

Remark 4.1. The two boundary conditions yield two different cohomology complexes. However, anyone is the dual of the other, in the sense that the second complex can be obtained by considering the first complex for the function $-f$ via the Poincaré duality theorem, and vice versa.

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