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EQUIVARIANT MAPS BETWEEN COHOMOLOGY SPHERES

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Dedicated to Ky Fan

Introduction

Let G be a compact Lie group. Let X be a Hausdorff compact G-space which is a cohomology sphere over a ring $\mathcal{R}$ (cf. Def. 1.1).

In Section 1 we define the Euler class of a locally trivial bundle with X as fiber. Using this definition we show that the main result of [7] still holds if the sphere of an orthogonal representation is replaced by a G-space which is a cohomology sphere over $\mathcal{R}$ (Ths. 1.8, 1.9).

In the second section we consider actions of a finite cyclic group $G = C_k$. For this group, and X as above, an index of X has been defined in [1] and [2]. First, with the use of the Euler class we define an index of X (cf. [7] for the case where X is the sphere of an orthogonal representation; see Def. 2.2). Next we prove that this index is equal to that introduced by Borisovich and Izrailevich (Th. 2.3), and consequently we obtain a simple geometrical interpretation of the index defined in [1]. Finally, using our definition of the index (via the Euler class) we compute its value for $X = S(V)$, the sphere of an orthogonal representation V of $G = C_k$ (Th. 3.3). This allows us to find for which V this index is different from 0. From the results of [7] and those quoted above we derive a formula on the degree (modulo k) of a G-equivariant map $f : S(W) \to S(V)$ between the spheres of representations W and V of $G = C_k$ (Prop. 3.11) (cf. [3], [8]).

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The presented material is a part of the authors' preprint [5]. In [6] it is shown that the vanishing of the Euler class of V is the only obstruction to the existence of an equivariant map from the sphere of another representation W of dimension greater than the dimension of V. This is connected with the problem of estimating the G-category of X ([6]) and consequently can be applied to invariant variational problems.

1. The Euler class of a cohomology sphere

Assume that X is a compact Hausdorff G-space with finitely generated Čech cohomology with coefficients in a ring $\mathcal{R}$.

We denote by CX the cone over X, $CX = X \times I/X \times \{0\}$, with the natural action of G. Note that $(CX)^G = CX^G$, with the convention that $CX^G = [X,0] = *$ if $X^G = \emptyset$. Also X embeds equivariantly in CX. Moreover, CX is G-contractible. The *Borel space* of X is the orbit space $X_G = EG \times_G X = (EG \times X)/G$, where EG is the universal space of G. X_G, $(CX)_G$ and $((CX)_G, X_G)$ are the total spaces of fibrations over the classifying space BG with fibers X, CX and (CX,X) respectively. Since the fibration $EG \to BG = EG/G$ is locally trivial, all the above fibrations are locally trivial. We denote $(CX)_G$ by CX_G. We shall use the same letter p to denote the projections of all the above fibrations.

Let $f : X \to Y$ be a G-equivariant map between two G-spaces X and Y. Clearly, f induces the map $f_G = \mathrm{id} \times_G f : X_G \to Y_G$. The cone map $Cf : CX \to CY$, $Cf([x,t]) = [f(x),t]$, is a G-map which maps (CX,X) into (CY,Y), and consequently induces a map $Cf_G : (CX_G, X_G) \to (CY_G, Y_G)$. Moreover, the restrictions of f_G and Cf_G to the fibers X_b and $(CX,X)_b$, $b \in BG$, are equal to f and Cf, respectively.

We shall denote by j the inclusion of $CX_G = (CX_G, \emptyset)$ into the pair (CX_G, X_G) and by $s_0 : BG \to CX_G$ the map $s_0 = [b,*]$. We use the same letters for restrictions of these maps.

1.1. Definition. A G-space X is called *m-acyclic* over $\mathcal{R}$ if X is path-connected and $H_i(X;\mathcal{R}) = H^i(X;\mathcal{R}) = 0$ for $1 \le i \le m$.

A G-space X is called a *cohomology sphere* of dimension n over $\mathcal{R}$ if X is path-connected and $H^*(X;\mathcal{R}) = H^*(S^n;\mathcal{R})$, $H_*(X;\mathcal{R}) = H_*(S^n;\mathcal{R})$.

1.2. Definition. Assume that a G-space X is an $\mathcal{R}$-cohomology sphere of dimension n. A fibration X_G is said to be $\mathcal{R}$-orientable if there exists a cohomology class $\mathcal{U} \in H^{n+1}(CX_G, X_G;\mathcal{R})$, called the *Thom class*, such that for every $b \in BG$ the restriction $\mathcal{U}_{|b}$ of $\mathcal{U}$ to $H^{n+1}(CX,X;\mathcal{R})$ is a fixed generator of the module $H^{n+1}(CX,X;\mathcal{R}) \simeq \mathcal{R}$. We say that X is *orientable* if the fibration X_G is orientable (cf. [9], [10]).

1.3. PROPOSITION. *A fibration X_G is $\mathcal{R}$-orientable if the action of G on $H^n(X;\mathcal{R}) \simeq \mathcal{R} \simeq H^n(X;\mathbb{Z}) \otimes \mathcal{R}$ is trivial. This means that X is always orientable if G acts trivially on $H^n(X;\mathbb{Z})$ and is also orientable if G acts nontrivially on $\mathbb{Z} = H^n(X;\mathbb{Z})$ but the characteristic of $\mathcal{R}$ equals 2.*

1.4. COROLLARY. *Let $G = C_k$ be the cyclic group of order k. If k is odd then every cohomology sphere over $\mathbb{Z}$, or $\mathbb{Z}_k$, is $\mathbb{Z}$-orientable, respectively $\mathbb{Z}_k$-orientable. Every cohomology sphere is $\mathbb{Z}_2$-orientable.*

One can prove Proposition 1.3 by adapting the standard proof for the vector bundle ([9]) and using the following lemma.

1.5. LEMMA. *Let $G_0 \subset G$ be the component of identity of G and $\omega : I \to BG$ a loop at $b \in BG/G_0$. Then the automorphism*

$$h^*_{|[\omega]} : H^{n+1}(CX, X;\mathcal{R}) \to H^{n+1}(CX, X;\mathcal{R})$$

induced by ω is equal to g^*, where $g \in G/G_0$ corresponds to $[\omega] \in \pi_1(B(G/G_0)) \simeq G/G_0$. □

1.6. DEFINITION. Suppose that a G-space X is an $\mathcal{R}$-orientable cohomology sphere of dimension n over $\mathcal{R}$. The class

$$\mathbf{e}(X) = (p^*)^{-1} j^*(\mathcal{U}) \in H^{n+1}(BG;\mathcal{R})$$

is called the *Euler class* of X in $\mathcal{R}$. If V is an orthogonal representation of dimension n then we denote by $\mathbf{e}(V) \in H^*(BG;\mathcal{R})$ the Euler class of its sphere $S(V)$.

1.7. REMARK. Since $ps_0 : BG \to BG$ is the identity on BG, the homomorphism $(p^*)^{-1} : H^*(CX_G;\mathcal{R}) \to H^*(BG;\mathcal{R})$ is equal to s_0^*. Consequently, $\mathbf{e}(X) = s_0^* j^*(\mathcal{U})$.

If $X^G \neq \emptyset$ then the bundle X_G has a section $s(b) = [b, *]$ which shows that $X^G \neq \emptyset$ implies $\mathbf{e}(X) = 0$ for any coefficient ring $\mathcal{R}$.

The following theorems can be shown by the same arguments as for the analogous statements of [7].

1.8. THEOREM. *Let Y be a G-space which is an $\mathcal{R}$-orientable cohomology sphere of dimension n. Assume that $\mathbf{e}(Y) \neq 0$. Suppose that X is a G-space which is l-acyclic over $\mathcal{R}$. If $l \geq n$ then there is no G-equivariant map $f : X \to Y$.* □

Suppose that $f : X \to Y$ is a G-map between two cohomology spheres of dimension n over $\mathcal{R}$. The *degree* of f in the ring $\mathcal{R}$, denoted by $\deg_{\mathcal{R}} f$, is defined to be the remainder of $\deg f$ modulo $\operatorname{char} \mathcal{R}$. For example, if $\mathcal{R} = \mathbb{Z}_k$ then $\deg_{\mathcal{R}}$ is the remainder of $\deg f$ modulo k.

1.9. Theorem. *Suppose that G-spaces X and Y with $X^G = Y^G = \emptyset$ are $\mathcal{R}$-orientable cohomology spheres of dimension n. Then for every G-equivariant map $f : X \to Y$, $\deg_{\mathcal{R}} f \cdot \mathbf{e}(X) = \mathbf{e}(Y)$ in $H^*(BG;\mathcal{R})$.* □

1.10. Remark. One can get rid of the $\mathcal{R}$-orientability assumption but then we have to work with a local coefficient system on BG given by the action of $\pi_1(B(G/G_0)) = G/G_0$ on $H^*(X;\mathcal{R})$.

2. Equality of indices for the cyclic group

From now on we assume that $G = C_k$ is a cyclic group of order $k > 1$, and all representations considered are $\mathbb{Z}_k$-orientable.

Using the Euler class and periodicity of cohomology of C_k we define a numerical index (in $\mathbb{Z}_k$) of a cohomology sphere. Next we show that this index is equal to another one defined earlier by Borisovich and Izrailevich. As a matter of fact this is a simple exercise in algebraic topology, but requires a little background on the cohomology of cyclic groups, which we include for the convenience of the reader. On the other hand, this fact gives a geometric interpretation of the index, allows us to derive an explicit formula for it in the case when X is the sphere of an orthogonal representation, and allows extending the definition of the index to the case of a finite group with periodic cohomology.

First we recall some facts on the cohomology groups of C_k with coefficients in $\mathbb{Z}_k$ (cf. [4]). It is known that $H^i(C_k;\mathbb{Z}_k) = \mathbb{Z}_k$ for $i \geq 0$. Moreover, there is a periodicity

$$H^q(C_k;\mathbb{Z}_k) = H^{q+2}(C_k;\mathbb{Z}_k), \qquad q \geq 0,$$

given by multiplication by an element $u \in H^2(C_k;\mathbb{Z}_k)$ if $k \neq 2$ and $u \in H^1(C_2;\mathbb{Z}_2)$ if $k = 2$. The element u is defined as follows. Let V^1 be the one-dimensional (complex if $k \neq 2$ and real if $k = 2$) representation of C_k given by the inclusion $C_k \subset S^1$ (resp. $C_2 = S^0 = \{-1,1\} \subset \mathbb{R}$). Take $u = \mathbf{e}(V^1) \in H^2(C_k;\mathbb{Z}) = H^2(C_k;\mathbb{Z}_k) = \mathbb{Z}_k$ (resp. $\mathbf{e}(V^1) \in H^1(C_2;\mathbb{Z}) = H^1(C_2;\mathbb{Z}_2) = \mathbb{Z}_2$). Note that $\mathbf{e}((V^1)^k) = k \cdot \mathbf{e}(V) = 0$, and $m \cdot \mathbf{e}(V^1) \neq 0$ if $0 < m < k$, which means that u is a generator of $H^2(C_k;\mathbb{Z}_k)$ (resp. $H^1(C_2;\mathbb{Z}_2)$).

Let X be a G-space which is a $\mathbb{Z}_k$-orientable cohomology sphere of dimension n over $\mathbb{Z}_k$. We make the following choice of generators. Let $\delta_0 \in H^0(X;\mathbb{Z}_k)$ be the generator given by $\delta_0 = c^*(1)$, where $c : X \to *$ is the map to the point. Denote by $\widetilde{\delta}_0$ the corresponding generator of $H^0(BG; H^0(X;\mathbb{Z}_k)) \simeq H^0(X;\mathbb{Z}_k)$. Since X is orientable, there exists a generator $\mu \in H^n(X;\mathbb{Z}_k)$ such that $g^*(\mu) = \mu$ for every $g \in G$. We denote by $\widetilde{\mu}$ the corresponding generator of $H^0(BG; H^n(X;\mathbb{Z}_k)) \simeq H^n(X;\mathbb{Z}_k) \simeq \mathbb{Z}_k$. We can assume that $\widetilde{\delta}_0 = \widetilde{\mu}$.

Next, let $\nu \in H^1(BG;\mathbb{Z}_k)$ be an element such that $u\delta_0 = \beta(\nu)$, where $\beta : H^1(BG;\mathbb{Z}_k) \to H^2(BG;\mathbb{Z}_k)$ is the Bockstein homomorphism. Since β is an isomorphism, ν is a generator. Finally, let ξ_i be a generator of $H^i(BG;\mathbb{Z}_k)$ defined by

$$\xi_i = \begin{cases} u^i\delta_0 & \text{if } i = 2n, \\ u^i\nu & \text{if } i = 2n+1,\ n \geq 0. \end{cases}$$

We can now formulate the definitions of both the indices mentioned above.

First note that the Borel–Serre spectral sequence of X_G with coefficients in $\mathbb{Z}_k$ exists and converges to $H^*(X_G;\mathbb{Z}_k)$ since X is orientable. We have $E_2^{p,q}(X_G) = H^p(BG; H^q(X;\mathbb{Z}_k))$ and therefore

$$E^{p,q}(X) = \begin{cases} \mathbb{Z}_k & \text{if } q = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, d_{n+1} is the only nontrivial differential of this sequence. Let $\alpha : H^n(X;\mathbb{Z}_k) \to H^0(X;\mathbb{Z}_k)$ be the composition of the following functorial homomorphisms:

$$H^n(X;\mathbb{Z}_k) \simeq E_2^{p,n} \simeq E_{n+1}^{0,n} \xrightarrow{d_{n+1}} E_{n+1}^{n+1,0} \simeq E_2^{n+1,0} \simeq H^0(X;\mathbb{Z}_k),$$

where the first (resp. last) homomorphism is given by $\mu \mapsto \widetilde{\nu}$ (resp. $\xi_{n+1} \mapsto \delta_0$).

The following definition is a simplified version of that of [1] since we assume $X^G = \emptyset$. If $X^G \neq \emptyset$ one has to use the relative cohomology of the pair (X, X^G).

2.1. Definition. Let $G = C_k$ be a cyclic group. Let X be a G-space which is a $\mathbb{Z}_k$-orientable cohomology sphere of dimension n over $\mathbb{Z}_k$. The number (modulo k) $\operatorname{ind}(X) \in \mathbb{Z}_k$ is defined by

$$\alpha(\mu) = \operatorname{ind}(X) \cdot \delta_0,$$

where α, ν, δ_0 are as above.

2.2. Definition. Let $G = C_k$ be a cyclic group. Let X be a G-space which is a $\mathbb{Z}_k$-orientable cohomology sphere of dimension n over $\mathbb{Z}_k$. The number (modulo k) $\operatorname{Ind}(X) \in \mathbb{Z}_k$ is defined by

$$\mathbf{e}(X) = \operatorname{Ind}(X) \cdot \xi_{n+1} \in H^{n+1}(BG;\mathbb{Z}_k).$$

Our aim is to show the following theorem:

2.3. Theorem. *Suppose that X is a cohomology sphere of dimension n over $\mathbb{Z}_k$ equipped with an orientation preserving action of the cyclic group C_k. Assume that $X^G = \emptyset$. Then both* $\operatorname{Ind}(X)$ *and* $\operatorname{ind}(X)$ *are well defined and are equal.*

Proof. We refer the reader to Switzer's book ([10]). First note that the isomorphism $H^n(X;\mathbb{Z}_k) \to H^0(BG; H^n(X;\mathbb{Z}_k))$ maps μ onto $\widetilde{\mu}$, by its definition. Set $\widetilde{\mathbf{e}}(X) = d_{n+1}(\widetilde{\mu})$. We have $\widetilde{\mathbf{e}}(X) = \operatorname{ind}(X)\cdot\xi_{n+1}$, since the homomorphism $H^{n+1}(BG; H^0(X;\mathbb{Z}_k)) \to H^0(BG; H^0(X;\mathbb{Z}_k))$ maps ξ_{n+1} onto δ_0 by its definition. On the other hand, $d_{n+1} = \Psi$, where $\Psi = (p^*)^{-1} j^*\Phi$, and $\Phi(z) = p^*(z)\cup\mathcal{U}$ is the Thom isomorphism (see [10], Ex. 15.31). This gives

$$\begin{aligned} d_{n+1}(\widetilde{\mu}) &= (p^*)^{-1} j^*(\widetilde{\mu}) \cup (p^*)^{-1} j^*(\mathcal{U}) = \widetilde{\mu}\cup\mathbf{e}(X) = \widetilde{\mu}\cdot\operatorname{Ind}(X)\cdot\xi_{n+1} \\ &= \operatorname{Ind}(X)\cdot\xi_{n+1}\widetilde{\delta}_0 = \operatorname{Ind}(X)\cdot\xi_{n+1} \end{aligned}$$

since $\widetilde{\delta}_0 = \widetilde{\mu}$ and this is a ring generator of $H^0(BG;\mathbb{Z}_k)\simeq\mathbb{Z}_k$, and $s_0^0 j^0 p^0 = \mathrm{id}$ on $H^0(BG;\mathbb{Z}_k)$. This proves the theorem. $\square$

The problem of equality of the two indices was posed to the authors by H. Steinlein.

3. Computations of the Euler class and degree of C_k-equivariant maps

First we will derive the Euler class of any orthogonal $\mathbb{Z}_k$-orientable representation of C_k.

Note that V^1 can be considered as a 2-dimensional real orthogonal representation of C_k. Denote by V^i, $1\le i\le k/2$, its ith tensor power (over $\mathbb{C}$) and by V^0 the 1-dimensional (real) trivial representation of G. If $k=2m$, then for $i=m=k/2$, we denote by $V_{\mathbb{R}}^m$ the 1-dimensional real representation of C_k given by the epimorphism $C_k\to C_2\simeq\{-1,1\}=\mathbb{O}(1)$.

Since a representation $(V,\varrho : G\to GL(V))$ is $\mathbb{Z}_k$-orientable iff the map $\operatorname{sgn}\det\varrho : G\to\{-1,1\}$ is trivial modulo k, we have the following

3.1. Fact. Every real orthogonal representation of C_k is of the form

$$V = \begin{cases} \displaystyle\bigoplus_{0\le i\le (k-1)/2} l_i V^i & \text{if } k \text{ is odd},\\ \displaystyle\bigoplus_{0\le i\le m-1} l_i V^i \oplus l_m V_{\mathbb{R}}^m & \text{if } k = 2m. \end{cases}$$

Moreover, for $k=2m$, $k\ne 2$, V is orientable iff $V=\bigoplus_{i=1}^m l_i V^i$ (note that $V^m = 2V_{\mathbb{R}}^m$), and every representation $V = l_0V^0\oplus l_iV_{\mathbb{R}}^i$ of C_2 is $\mathbb{Z}_2$-orientable. Every orthogonal representation of C_k with k odd is $\mathbb{Z}_k$-orientable.

As a consequence we get

3.2. Corollary. *Every orientable orthogonal representation V of C_k, $k\ne 2$, can be written as*

$$V = \bigoplus_{0\le i\le [k/2]} m_i V^i.$$

In particular, if $V^G = \{0\}$ *then* $\dim_{\mathbb{R}} V$ *is even and* V *admits a complex structure.* $\square$

Using the facts that $\mathbf{e}(V \oplus W) = \mathbf{e}(V) \cdot \mathbf{e}(W)$ (cf. [3]) and $e(V^i) = i \cdot \mathbf{e}(V) = i \cdot u \in H^2(C_k; \mathbb{Z}_k)$ (cf. e.g. [7]), we get the following theorem:

3.3. Theorem. *Let* $V = \bigoplus_{\leq i \leq [k/2]} m_i V^i$ *be an orientable orthogonal representation of* C_k. *Set* $r = \sum m_i$. *Then* $\mathbf{e}(V) = \prod i^{m_i} \cdot u^r$. *The last means that* $\mathbf{e}(V) \neq 0$ *iff the integer* $\mathbf{h}(V) = \prod i^{m_i}$ *is not divisible by* k (*with the convention that* $0^0 = 1$ *and* $0^r = 0$ *for* $r > 0$). *In particular,* $\mathbf{e}(V) = 0$ *if* $m_0 \neq 0$. $\square$

Now we turn to the derivation of $\deg_{\mathbb{Z}_k} f$ of a C_k-equivariant map. Let W, V be a pair of $\mathbb{Z}_k$-orientable orthogonal representations of $G = C_k$ such that $n = \dim W = \dim V$ and $n_1 = \dim W^G = \dim V^G$. Denote by $W_\perp^G$, $V_\perp^G$ the orthogonal complements of W^G, V^G in W, V respectively. The following relative analogue of Theorem 1.9 holds for equivariant maps of spheres of orthogonal representations of G.

For any G-equivariant map $f : S(W) \to S(V)$ we have

$$\deg_{\mathbb{Z}_k} f \cdot \mathbf{e}(W_\perp^G) = \deg_{\mathbb{Z}_k} f^G \cdot \mathbf{e}(V_\perp^G) \tag{3.4}$$

(cf. [3], 4.28, for $G = \mathbb{Z}_p$, p prime, and [7] for any compact Lie group).

Theorem 3.3 and (3.4) lead to the following theorem.

3.5. Theorem. *Let* W *and* V *be* $\mathbb{Z}_k$*-orientable orthogonal representations of a cyclic group* $G = C_k$. *Assume that* $\dim W = \dim V$ *and* $\dim W^G = \dim V^G$. *Let* $V_\perp^G = \bigoplus_{1 \leq i \leq [k/2]} m_i V^i$ *and* $W_\perp^G = \bigoplus_{1 \leq i \leq [k/2]} m_i' V^i$. *Then for any* G*-equivariant map* $f : S(W) \to S(V)$,

$$\deg_{\mathbb{Z}_k} f \cdot \Big(\prod i^{m_i'}\Big) = \deg_{\mathbb{Z}_k} f^G \cdot \Big(\prod i^{m_i}\Big).$$

In particular, if $\prod i^{m_i'} \not\equiv 0 \pmod{k}$ *then* $\deg_{\mathbb{Z}_k} f$ *is uniquely determined by* $\deg_{\mathbb{Z}_k} f^G$, $V_\perp^G$, *and* $W_\perp^G$ *via the formula*

$$\deg_{\mathbb{Z}_k} f = \deg_{\mathbb{Z}_k} f^G \cdot \Big(\prod i^{m_i}\Big) \cdot \Big(\prod i^{m_i'}\Big)^{-1}.$$

$\square$

3.6. Corollary. *Suppose that* $\dim W^G = \dim V^G = 0$. *Then the degree modulo* k *of any* C_k*-equivariant map* $f : S(W) \to S(V)$ *is equal to* $(\prod i^{m_i}) \cdot (\prod i^{m_i'})^{-1}$ *and thus depends on* W *and* V *only.*

Note that in the special case $W = \bigoplus_{i=1}^n V^{t_i}$, $V = \bigoplus_{i=1}^n V^{s_i}$, $(t_i, k) = (s_i, k) = 1$, Corollary 3.6 gives 4.12 of [3].

3.7. Problem. Theorem 3.5 and Corollary 3.6 give some necessary condition on W and V for the existence of a G-equivariant map $f : S(W) \to S(V)$ (cf. [7] for such a condition for $G = T^k$, $\mathbb{Z}_p^k$, p prime). It is natural to ask whether it is also a sufficient condition.

To end this section we describe $\mathbf{e}(V)$ and $\deg_{\mathbb{Z}_k} f$ of a representation of C_k and a C_k-map respectively.

First we recall the description of the cohomology groups of C_k, $k = p^r$, with coefficients in the ring $\mathbb{Z}_{p^l}$. It is known that (cf. [12], 15.7)

$$H^0(G; A) = A/kA, \qquad H^1(G; A) = \mathrm{Hom}(G; A)$$

if $|G| = k$ and A is a trivial G-module. Since $G = C_{p^r}$ is cyclic, it is 2-periodic. Altogether, the above gives

$$H^0(C_{p^r}; \mathbb{Z}_{p^l}) = \left\{ \begin{matrix} \mathbb{Z}_{p^r} & \text{if } r \le l \\ \mathbb{Z}_{p^l} & \text{if } r > l \end{matrix} \right\} = H^1(C_{p^r}; \mathbb{Z}_{p^l}),$$

and consequently, for $n \ge 0$,

$$H^{2n}(C_{p^r}; \mathbb{Z}_{p^l}) = H^{2n+1}(C_{p^r}; \mathbb{Z}_{p^l}) = \begin{cases} \mathbb{Z}_{p^r} & \text{if } r \le l, \\ \mathbb{Z}_{p^l} & \text{if } r > l. \end{cases}$$

For $l \le r$, let $C_{p^l} \subset C_{p^r}$ be the canonical embedding. Then the restriction homomorphism

$$\mathrm{res}^* : H^i(C_{p^r}; \mathbb{Z}_{p^r}) \simeq \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^l} = H^i(C_{p^l}; \mathbb{Z}_{p^l})$$

is the quotient map $\mathbb{Z}_{p^r} \to \mathbb{Z}_{p^r}/p^l\mathbb{Z}_{p^r} \simeq \mathbb{Z}_{p^l}$. Furthermore, for $l \le r$ the homomorphism $H^*(C_{p^r}; \mathbb{Z}_{p^l}) \to H^*(C_{p^r}; \mathbb{Z}_{p^r})$ is induced by the embedding $\mathbb{Z}_{p^l} \subset \mathbb{Z}_{p^r}$ given by multiplication by p^{r-l}. Let $k = p_1^{r_1} \ldots p_N^{r_N}$. We have $H^i(C_k; \mathbb{Z}_k) = \mathbb{Z}_k = \bigoplus_{j=1}^N \mathbb{Z}_{p_j^{r_j}}$, thus its p-torsion, for $p \in \{p_1, \ldots, p_N\}$, is equal to

$$H^i(C_k; \mathbb{Z}_k)_{(p)} \simeq \mathbb{Z}_{p^r} \simeq H^i(C_{p^r}; \mathbb{Z}_k) \simeq H^i(C_{p^r}; \mathbb{Z}_{p^r}).$$

The functoriality of the Euler class and the above yield

3.8. Proposition. *Let V be an orthogonal representation of a cyclic group C_k, $k = p_1^{r_1} \ldots p_N^{r_N}$. Let $H = C_{p^r}$, $p \in \{p_1, \ldots, p_N\}$, be its maximal p-subgroup. Then $\mathrm{res}_H^*(\mathbf{e}(V)) = \mathbf{e}(\mathrm{res}_H V)$, and consequently $\mathbf{e}(V) \neq 0$ iff there exists $p \,|\, k$ such that $\mathbf{e}(\mathrm{res}_H V) \neq 0$ in $H^*(BH; \mathbb{Z}_{p^r})$.*

Let V be an orthogonal representation of C_k and p a prime dividing k. Assume that p is odd. For simplicity we use the same symbol V for the restriction of V to $H = C_{p^r} \subset C_k$. Expanding V into irreducible factors with respect to H, we have

$$V = \bigoplus_{0 \le i \le [p^r/2]} l_i V^i, \qquad l_i \in \mathbb{N} \cup \{0\}.$$

From Theorem 3.3 we have

$$\mathbf{h}(V) = \prod_{0\le i\le [p^r/2]} i^{l_i} \in \mathbb{Z}_{p^r}.$$

For $1 \le \alpha \le r-1$, we set $\mathcal{A}_\alpha = \{i \in \mathbb{Z}_{p^r} : i \equiv p^\alpha \pmod{p^r} \text{ and } i \not\equiv p^{\alpha+1} \pmod{p^r}\}$, and $\mathcal{A}_0 = \{0\}$, $\mathcal{A} = \{i \in \mathbb{Z}_{p^r} : (p,i) = 1\}$. Note also that $\prod_{i\in\mathcal{A}} i^{l_i}$ is an invertible element of $\mathbb{Z}_{p^r}$. Writing $\mathbf{h}(V)$ as the product

$$\mathbf{h}(V) = \prod_{i\in\mathcal{A}} i^{l_i} \cdot \prod_{i\in\mathcal{A}_0} i^{l_i} \cdot \prod_{i\in\mathcal{A}_1} i^{l_i} \cdot \ldots \cdot \prod_{i\in\mathcal{A}_{r-1}} i^{l_i}$$

and observing that for $i \in \mathcal{A}_\alpha$ we have $i^{l_i} = p^{\alpha l_i} c$, where $c \in \mathbb{Z}^*_{p^r}$, we get the following proposition.

3.9. PROPOSITION. *Let $V = \bigoplus_{0\le i\le[p^r/2]} l_i V_i$ be an orthogonal representation of a cyclic group C_{p^r}, p odd. Then $\mathbf{e}(V) \ne 0$ in $H^*(BC_{p^r};\mathbb{Z}_{p^r})$ if p^r does not divide $\prod_{i\in\mathcal{A}_0} i^{l_i} \cdot \prod_{i\in\mathcal{A}_1} i^{l_i} \cdot \ldots \cdot \prod_{i\in\mathcal{A}_{r-1}} i^{l_i}$, or equivalently if*

$$l_0 r + \sum_{i\in\mathcal{A}_1} l_i + 2\sum_{i\in\mathcal{A}_2} l_i + \ldots + (r-1)\sum_{i\in\mathcal{A}_{r-1}} l_i < r. \tag{+}$$

We are left with the case $k = 2$. Every orthogonal representation of C_2 is isomorphic to $m_0V^0 \oplus m_1V^1$, where V^1 is the one-dimensional nontrivial real representation of $C_2 \subset \{-1,1\} = \mathbb{O}(1)$. Also $\mathbf{h}(V) = 1$ iff $m_0 = 0$. Note also that the statement of Proposition 3.9 still holds if $p = 2$ but V is orientable over $\mathbb{Z}_{p^r}$ (or equivalently over $\mathbb{Z}$ if $r > 1$). Obviously, $l_0 \ne 0$ yields $\mathbf{e}(V) = 0$.

From Proposition 3.9 it follows that the Euler class $\mathbf{e}(V)$ (or equivalently $\mathbf{h}(V)$, cf. Theorem 3.3) is uniquely determined by all its p-torsions. Moreover, we can write down a more explicit formula for the modulo k number $\mathbf{h}(V)$. As follows from the Chinese remainder theorem, $\mathbf{h}(V) \in \mathbb{Z}_k$ is uniquely given by $\{\mathbf{h}(V)_{(p)}\}$, $p \mid k$, as the solution of the system of congruences

$$x \equiv \mathbf{h}(V)_{(p)} \pmod{p^r} \tag{$*$}$$

(cf. [11], IV, 3a). For $p_j \mid k$, $1 \le j \le N$, let $q_j = k/p_j^{r_j}$, and $\{q_j\}^{-1}$ be the inverse of q_j in $\mathbb{Z}_{p_j^{r_j}} \subset \mathbb{Z}_k$. Then the unique (modulo k) solution $x_0 = \mathbf{h}(V)$ of $(*)$ is given by the formula

$$x_0 = \sum_{j=1}^{N} q_j\{q_j\}^{-1}\mathbf{h}(V)_{(p_j)}. \tag{3.10}$$

Analogously, we can describe $\deg_{\mathbb{Z}_k} f$ by its p-torsions, $\{\deg_{\mathbb{Z}_k} f_{(p)}\}$, $p \mid k$. For $k = p_1^{r_1}\ldots p_N^{r_N}$, let $J_1 = \{1 \le j \le N : \mathbf{e}(W_\perp^G)_{(p_j)} \ne 0\}$ and $J_2 = \{1 \le j \le N : \mathbf{e}(W_\perp^G)_{(p_j)} = 0\}$. Combining 3.3, 3.5 and 3.9 we get the following statement.

3.11. Proposition. *Let W and V be $\mathbb{Z}_k$-orientable orthogonal representations of a cyclic group $G = C_k$, where $k = p_1^{r_1} \dots p_N^{r_N}$. Assume that* $\dim W = \dim V$ *and* $\dim W^G = \dim V^G$. *Then for any G-equivariant map $f : S(W) \to S(V)$, $\deg_{\mathbb{Z}_k} f$ is uniquely determined by $\deg_{\mathbb{Z}_k} f^G$, up to the ideal $I = (\prod_{j \in J_2} p_j^{r_j})\mathbb{Z}_k$, by the formula*

$$\deg_{\mathbb{Z}_k} f = \sum_{j \in J_1} q_j \{q_j\}^{-1} \deg_{\mathbb{Z}_k} f^G_{(p_j)} \mathbf{h}(V_\perp^G)_{(p_j)} \cdot \mathbf{h}(W_\perp^G)^{-1}_{(p_j)}.$$

3.12. Remark. Using the relative Euler class of a pair (X, Y) of $\mathbb{Z}_k$-cohomology spheres one can extend our results to the relative case. In particular, if $W \subsetneq V$ is a pair of orthogonal, $\mathbb{Z}_k$-orientable representations of C_k then $\operatorname{Ind}(S(V), S(W)) = \operatorname{ind}(S(V), S(W)) = \mathbf{e}(S(V), S(W))$ is equal to $\mathbf{e}(S(W_\perp^G)) \in H^*(C_k; \mathbb{Z}_k)$.

3.13. Remark. The formula of Proposition 3.11 has been shown by Shchelokova [8] for G-equivariant maps of $\mathbb{Z}_k$-cohomology spheres. In her formulas the index $\operatorname{ind}(X)$ appears instead of the Euler classes. We wish to emphasize that in all our formulas the coefficients $\mathbf{h}(V)$ are explicitly given by the representations, which is not the case in [8].

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