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THE YAMABE PROBLEM ON SUBDOMAINS OF EVEN-DIMENSIONAL SPHERES

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

We prove the existence of complete conformally flat metrics of constant positive scalar curvature on the complement in $\mathbb{S}^n$ of a finite number of $(n-2)/2$-dimensional smooth submanifolds, provided $n \geq 4$ is even.

1. Introduction and results

The singular Yamabe problem is concerned with the following question:

Question. Given a subset Λ included in the n-dimensional sphere $\mathbb{S}^n$, does there exist on $\mathbb{S}^n \setminus \Lambda$ a complete metric with constant positive scalar curvature which is conformally equivalent to the standard metric of $\mathbb{S}^n \setminus \Lambda$?

This problem has been extensively studied since the pioneering work of R. Schoen [10], who proved that the answer to the above question is positive when Λ consists of finitely many points (at least 2). Moreover, in the same paper, the existence of solutions to the singular Yamabe problem is proven for some sets Λ whose Hausdorff dimension is not an integer, taking away all hope to prove any regularity results for the singular set itself. Later on, R. Mazzeo and N. Smale [7] have proved that the answer is positive in the case where Λ is a perturbation of a k-dimensional sphere, for any $0 < k < (n-2)/2$. More recently, R. Mazzeo, D. Pollack and K. Uhlenbeck [5] have studied the moduli space of solutions to this problem when Λ is a finite set of points. The technique

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they have developed also allowed them to build solutions when Λ consists of a finite number of dipoles (pairs of points).

Since the work of R. Schoen and S. T. Yau [11], we know that a necessary condition for the singular Yamabe problem to have a solution is that the Hausdorff dimension of Λ must be less than or equal to $(n-2)/2$.

Let us denote by g_0 the standard metric on $\mathbb{S}^n$. The conformal Laplacian on the sphere is given by

$$L_0 U \equiv \Delta_{g_0} U - \frac{n(n-2)}{4} U, \tag{1}$$

where Δ_{g_0} denotes the Laplacian with respect to the metric g_0. The singular Yamabe problem (see for example the book of T. Aubin [1]) reduces to the existence of positive solutions for the nonlinear equation

$$-L_0 U = U^{(n+2)/(n-2)} \tag{2}$$

on $\mathbb{S}^n \setminus \Lambda$, where we require that the metric defined by $g \equiv U^{4/(n-2)} g_0$ is a complete metric on $\mathbb{S}^n \setminus \Lambda$.

The aim of this paper is to prove the following result:

Theorem 1. *Assume that $n \geq 4$ is even. Given any finite disjoint union Λ of compact $(n-2)/2$-dimensional submanifolds of $\mathbb{S}^n$ without boundaries, there exist on $\mathbb{S}^n \setminus \Lambda$ infinitely many complete metrics with constant positive scalar curvature which are conformally equivalent to the standard metric on $\mathbb{S}^n \setminus \Lambda$.*

This theorem gives a positive answer to the singular Yamabe problem in the case where Λ is any finite disjoint union of $(n-2)/2$-dimensional submanifolds of $\mathbb{S}^n$. This work generalizes our previous work on the same subject [8] in dimension 4 (or in dimension 6) when Λ is a finite number of circles (or 2-spheres when $n = 6$). In [8], solutions to the singular Yamabe problem were obtained by a variational method, starting from an approximate solution.

In the case where $n = 6$, the result of Theorem 1 was announced without proof in [9].

2. Outline of the proof

Our proof of the existence of solutions of (2) is very similar to the proof of R. Mazzeo and N. Smale (see Section 3 of [7]).

Our first aim will be to describe the construction of the approximate solutions. In order to achieve this goal, we will need technical tools that are stated in Section 3. This will also be the opportunity to introduce, in Section 4, the weighted Hölder spaces we will work on in the subsequent sections. The construction of the approximate solution itself is carried out in Section 5. After having rephrased the nonlinear problem as a fixed point problem, the solution is obtained by using some contraction mapping argument similar to the one used in [7]; this is done in Section 8. To be able to do it, we will need some estimates in weighted Hölder spaces which are derived in Section 6 and we will also need some estimates for solutions of elliptic equations in weighted Hölder spaces; this will be the purpose of Section 7.

Let us emphasize the main differences between our work and the work of R. Mazzeo and N. Smale. The main difference appears in the construction of the approximate solutions. Our approximate solutions are, in some sense, much closer to an exact solution than theirs since they allow us to prove the existence of global solutions to our problem; in contrast with the work of R. Mazzeo and N. Smale in which, given a singular set, one can only produce local solutions. Moreover, in the case where the dimension of the singular set Λ is $k = (n-2)/2$, it appears that the analysis is much simpler than the one involved in [7], when the singular set has dimension $k < (n-2)/2$. In particular, we do not need the full theory of singular elliptic operators in weighted Hölder spaces as developed in [7].

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3. Well known properties

We start the proof by recalling some well known properties which will be needed in the sequel. The equation (2) enjoys the following property, which is proved for example in [4]:

Proposition 1 ([4]). *Assume that (M, g) and (N, h) are two Riemannian manifolds of dimension $n \geq 3$. In addition, assume that there exists a conformal diffeomorphism f from M to N, i.e. there exists a positive regular function ϕ defined on M such that $f^* h = \phi^{4/(n-2)} g$. Set $\widetilde{\phi} = \phi \circ f^{-1}$. If $u \in \mathcal{C}^2(M)$ satisfies the equation*

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g(x) u + F(x, u) = 0,$$

then $U \equiv (u \circ f^{-1})/(\phi \circ f^{-1})$ is a solution of the equation

$$\Delta_h U - \frac{n-2}{4(n-1)} R_h(y) U + \widetilde{\phi}(y)^{-(n+2)/(n-2)} F(f^{-1}(y), \widetilde{\phi}(y) U) = 0,$$

where R_g and R_h are the scalar curvatures of (M, g) and (N, h) respectively.

We will often use this proposition in the case where f is the inverse of $\pi : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$, the stereographic projection from the north pole N of $\mathbb{S}^n$. Then (see [4]) f is a conformal diffeomorphism from $\mathbb{R}^n$ into $\mathbb{S}^n \setminus \{N\}$. Thus, applying the result of Proposition 1, we obtain

LEMMA 1. *Define* $\phi(x) \equiv (2/(1+|x|^2))^{(n-2)/2}$. *Then the formula*

$$U = (u \circ \pi)/(\phi \circ \pi) \tag{3}$$

establishes a one-to-one correspondence between solutions of

$$\Delta_{\delta_{ij}^n} u + F(x,u) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^n$$

and solutions of

$$\Delta_{g_0} U - \frac{n(n-2)}{4} U + \widetilde{\phi}(y)^{-(n+2)/(n-2)} F(f^{-1}(y), \widetilde{\phi}(y) U) = 0 \qquad \text{in } \pi^{-1}(\Omega) \subset \mathbb{S}^n.$$

In this lemma and in the sequel, $\Delta_{\delta_{ij}^n}$ denotes the Laplacian in $\mathbb{R}^n$. As a particular case, we may take $F(x,u) = u^{(n+2)/(n-2)}$. Lemma 1 states that the function U defined by (3) satisfies

$$\Delta_{g_0} U - \frac{n(n-2)}{4} U + U^{(n+2)/(n-2)} = 0, \tag{4}$$

whenever u is a solution of

$$\Delta_{\delta_{ij}^n} u + u^{(n+2)/(n-2)} = 0. \tag{5}$$

Another case of interest will be $F(x,u) \equiv g(x)$. In this case, if we define

$$F(y) \equiv \left(\frac{1+|\pi(y)|^2}{2} \right)^{(n+2)/2} g(\pi(y)), \tag{6}$$

we deduce that the function U given by (3) is a solution of

$$\Delta_{g_0} U - \frac{n(n-2)}{4} U + F(y) = 0, \tag{7}$$

whenever u is a solution of

$$\Delta_{\delta_{ij}^n} u + g(x) = 0. \tag{8}$$

4. The function spaces

Assume that $\Lambda \subset \mathbb{S}^n$ satisfies all the assumptions of Theorem 1. Using the conformal invariance of our problem, we may always perform some rotation and assume that the north pole of $\mathbb{S}^n$ does not belong to Λ. After the stereographic projection from the north pole, our problem reduces to the resolution in $\mathbb{R}^n \setminus \Lambda'$ of the equation

$$-\Delta_{\delta_{ij}^n} u = u^{(n+2)/(n-2)} \tag{9}$$

(see for example (4) and (5)). Here and in the sequel, Λ' will denote the stereographic projection of the set Λ. Thus, by assumption, the set Λ' is a finite disjoint union of $(n-2)/2$-dimensional compact submanifolds of $\mathbb{R}^n$ without boundaries. We want to look for solutions of this new problem satisfying the additional condition that $u(x)$ tends fast enough to ∞ as x converges to Λ' in order to ensure the completeness of the metric defined by $g \equiv u^{4/(n-2)}\delta_{ij}$ in the neighborhood of Λ'. In addition, we must require that $u(x)$ tends fast enough to 0 as $|x|$ tends to ∞. To fix ideas, we require that $\lim_{|x|\to\infty} |x|^{n-2}u(x)$ exists. In view of (3), which allows us to pass from u to U, the last condition will ensure that U is bounded near the north pole, and therefore, by classical regularity theory, U being a solution of (4), will be regular near this point.

Moreover, we may verify that equation (9) is invariant undeer the group of dilations

$$\mathbb{R}^+ \ni \varepsilon \to \varepsilon^{(n-2)/2}u(\varepsilon x).$$

Therefore, up to dilation, we may always assume that Λ' is included in $B_n(1) \subset \mathbb{R}^n$, the unit ball in $\mathbb{R}^n$.

From now on, we assume that Λ and therefore Λ' are fixed. We denote by $\mathcal{T}_\sigma$ the union of tubular neighborhoods of the different connected components of Λ, whose diameters are given by $\sigma > 0$. And naturally, $\mathcal{T}'_\sigma$ will denote the stereographic projection of $\mathcal{T}_\sigma$ in $\mathbb{R}^n$. We assume in the sequel that the parameter $\sigma > 0$ is chosen small enough in order to ensure that all the tubular neighborhoods of the different connected components of Λ are disjoint.

In the sequel, $\Phi(X)$ will denote a $\mathcal{C}^{2,\alpha}(\mathbb{S}^n \setminus \Lambda)$ function such that, for all $X \in \mathbb{S}^n \setminus \Lambda$,

$$\tfrac{1}{2}\operatorname{dist}(X,\Lambda) \leq \Phi(X) \leq 2\operatorname{dist}(X,\Lambda) \tag{10}$$

and also

$$|\nabla\Phi(X)| \leq c \tag{11}$$

for some constant $c > 0$ only depending on Λ. As in the paper of R. Mazzeo and N. Smale [7], we define some weighted Hölder spaces as follows.

Definition 1 ([7]). Given some positive parameter $\nu > 0$, we define, for any function $U \in \mathcal{C}^{0,\alpha}(\mathbb{S}^n \setminus \Lambda)$, the norm

$$\|U\|_\nu = \sup_{r\leq \pi/2} |\Phi^\nu U|_{0,\alpha,[r,2r]}.$$

We will denote by $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ the space of functions U satisfying $\|U\|_\nu < \infty$.

Here $|\cdot|_{0,\alpha,[r,2r]}$ denotes the norm in the Hölder space $\mathcal{C}^{0,\alpha}$, which is defined by

$$|u|_{0,\alpha,[r,2r]} = \sup_{\operatorname{dist}(x,\Lambda)\in[r,2r]} |u(x)| + r^\alpha \sup_{\operatorname{dist}(x_i,\Lambda)\in[r,2r]} \frac{|u(x_1)-u(x_2)|}{|x_1-x_2|^\alpha}.$$

The parameter α is assumed to be fixed once for all and to satisfy $0 < \alpha < 1$.

Remark 1. Given some function ϕ defined in $\mathbb{R}^n \setminus \Lambda'$ and satisfying inequalities like (10) and (11) in $B_n(2)$ and such that $\phi(x)|x|^{2-n}$ tends to 1 as $|x|$ tends to ∞, we can define a norm similar to $\| \cdot \|_\nu$ for functions defined in $\mathbb{R}^n \setminus \Lambda$. We will use the same notations for the two norms. The corresponding space will be denoted by $\mathcal{C}^{0,\alpha}_\nu(\mathbb{R}^n \setminus \Lambda')$.

Remark 2. The functions that belong to the space $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ can roughly be described as the functions which can blow up at every point of Λ not faster than $\Phi^{-\nu}$.

The following lemma is straightforward but essential:

Lemma 2. *If $\nu \leq (n+2)/2$ and if*

$$1 \leq p < \frac{n+2}{2\nu},$$

then, for all $U \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, we have $U \in L^p(\mathbb{S}^n)$.

We now give a series of technical properties which lead to a calculus in the spaces $\mathcal{C}^{0,\alpha}_\nu$.

Proposition 2. *The following properties hold:*

1. *Given $U \in \mathcal{C}^{0,\alpha}_\nu$ and $V \in \mathcal{C}^{0,\alpha}_{\nu'}$, we have $UV \in \mathcal{C}^{0,\alpha}_{\nu+\nu'}$ and*

$$\|UV\|_{\nu+\nu'} \leq 2\|U\|_\nu \|V\|_{\nu'}.$$

2. *Given $U \in \mathcal{C}^{0,\alpha}_\nu$ such that $U \geq 0$ and given $p > 1$, we have $U^p \in \mathcal{C}^{0,\alpha}_{p\nu}$ and*

$$\|U^p\|_{p\nu} \leq p\|U\|^p_\nu.$$

3. *Assume that $\Phi^\nu U \in \mathcal{C}^0$ and $\Phi^{\nu+1}|\nabla U| \in \mathcal{C}^0$. Then $U \in \mathcal{C}^{0,\alpha}_\nu$ for every $\alpha \in (0,1)$.*

Notice that, in this proposition, the set on which the functions are defined is not specified since all these properties hold in $\mathcal{C}^{0,\alpha}_\nu(\mathbb{R}^n \setminus \Lambda')$ as well as in $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$. Our next proposition shows the influence of the stereographic projection on elements of $\mathcal{C}^{0,\alpha}_\nu(\mathbb{R}^n \setminus \Lambda')$ *via* the transformations given by (3) and (6).

Proposition 3.

1. *Given $u \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{R}^n \setminus \Lambda')$ with compact support in $\mathbb{R}^n$, the function U defined in* (3) *belongs to $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}_n \setminus \Lambda)$. In addition,*

$$\|U\|_\nu \leq c\|u\|_\nu$$

for some constant c which does not depend on u.

2. *Given* $f \in \mathcal{C}^{0,\alpha}_{\nu}(\mathbb{R}^n \setminus \Lambda')$ *with compact support in* $\mathbb{R}^n$*, the function* F *defined in* (6) *is in* $\mathcal{C}^{0,\alpha}_{\nu}(\mathbb{S}^n \setminus \Lambda)$ *and*

$$\|F\|_\nu \leq c\|f\|_\nu$$

for some constant c *which does not depend on* f*.*

5. Construction of an approximate solution

We denote by $B_k(R)$ the ball in $\mathbb{R}^k$ with radius $R > 0$, centered at the origin. In some neighborhood $\mathcal{U}' \subset \Lambda'$ of a point $z_0 \in \Lambda'$, there exists a diffeomorphism $\psi_{z_0} : \mathcal{T}'_\sigma|_{\mathcal{U}'} \to B_{(n+2)/2}(\sigma) \times \mathcal{U}'$. In the coordinate system induced by ψ_{z_0}, we can write as in [7]:

Proposition 4 ([7]). *In some neighborhood of* $z_0 \in \Lambda'$*, let the coordinates be given by* $(y, z) \in B_{(n+2)/2}(\sigma) \times \mathcal{U}'$*. Then for any function* u *we can write*

$$\Delta_{\delta^n_{ij}} u = \Delta_{\delta^{(n+2)/2}_{ij}} u + \Delta_{g(\Lambda')} u + e_1 \cdot \nabla^2 u + e_2 \cdot \nabla u.$$

In addition, e_1 *and* e_2 *satisfy the estimate*

$$\|e_1\|_{-1} + \|e_2\|_0 \leq c$$

for some constant $c > 0$ *which does not depend on* α*,* σ *nor on* y, z*.*

Here, we have adopted the notation

$$e_1 \cdot \nabla^2 u \equiv \sum_{i,j} e_1^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad e_2 \cdot \nabla u \equiv \sum_i e_2^i \frac{\partial u}{\partial x_i},$$

and naturally

$$\|e_1\|_{-1} + \|e_2\|_0 \equiv \sum_{i,j} \|e_1^{ij}\|_{-1} + \sum_i \|e_2^i\|_0.$$

Reducing σ if necessary, we can assume that the result of Proposition 4 holds in all $\mathcal{T}'_\sigma$.

We now recall the following existence result, a proof of which can be found in P. Aviles' papers [2], [3]:

Theorem 2 ([2], [3]). *For all dimensions* $m \geq 3$*, there exists a positive weak solution of*

$$-\Delta_{\delta^m_{ij}} \overline{u} = \overline{u}^{m/m-2}, \tag{12}$$

defined on the unit ball of $\mathbb{R}^m$*, which is regular except at the origin. In addition, there exists some constant* $c > 0$ *only depending on* m *such that for* y *near the origin we have*

$$\overline{u}(y) = (c + o(1))(-|y|^2 \log |y|)^{2-m/2}.$$

Let χ be a regular function from $\mathbb{R}^{(n+2)/2}$ into $\mathbb{R}^+$ such that

$$\chi \equiv 1 \quad \text{in } B_{(n+2)/2}(1), \quad \chi \equiv 0 \quad \text{in } \mathbb{R}^{(n+2)/2} \setminus B_{(n+2)/2}(2).$$

For all $\tau > 0$ we define the cut-off function $\chi_\tau(y) \equiv \chi(y/\tau)$ for all $y \in \mathbb{R}^m$. Given $0 < \varepsilon < 1$ and $\tau < \sigma/2$, we define, in some neighborhood of $z_0 \in \Lambda'$, the function

$$u_0(y,z) \equiv u_0(y) \equiv \varepsilon^{(n-2)/2}\overline{u}(\varepsilon y)\chi_\tau(y).$$

The above construction allows one to define locally the value of the function u_0, near any point of $\mathcal{T}'_\sigma$. Taking into account the result of Proposition 2, we can compute in $B_{(n+2)/2}(\sigma) \times \mathcal{U}'$ the error function f_0 given by

$$f_0(y,z,\varepsilon,\tau) = \Delta_{\delta^n_{ij}} u_0 + u_0^{(n+2)/(n-2)},$$

where we get, after some computation,

$$\begin{aligned} f_0(y,z,\varepsilon,\tau) \equiv{}& e_1(y,z)\cdot\nabla^2 u_0(y) + e_2(y,z)\cdot\nabla u_0(y) \\ &+ \varepsilon^{(n-2)/2}\overline{u}(\varepsilon y)\Delta_{\delta_{ij}^{(n+2)/2}}\chi_\tau(y) + 2\nabla\chi_\tau(y)\nabla(\varepsilon^{(n-2)/2}\overline{u}(\varepsilon x)) \\ &+ (\chi_\tau^{(n+2)/(n-2)}(y) - \chi_\tau(y))\varepsilon^{(n+2)/2}\overline{u}^{(n+2)/(n-2)}(\varepsilon y). \end{aligned}$$

We may now come back to our problem on $\mathbb{S}^n$ using the result of Proposition 1. After an inverse stereographic projection, we get from the functions u_0 and f_0 defined on $\mathbb{R}^n \setminus \Lambda'$, some functions U_0 and F_0 defined by (3) and (6) on $\mathbb{S}^n \setminus \Lambda$ which satisfy the equation

$$-L_0 U_0 = U_0^{(n+2)/(n-2)} + F_0 \qquad \text{in } \mathbb{S}^n \setminus \Lambda.$$

The function F_0 depends on τ, which we will assume to be fixed, and also depends on ε, which we may take as small as we want. In the next section we are going to estimate the norm of the different quantities we are going to deal with, when the parameter ε tends to 0.

6. Estimates in the space $\mathcal{C}^{0,\alpha}_\nu$

Our first lemma is concerned with the norm of the function F_0 which has been defined in the previous section. Let us recall that F_0 depends on the scaling factor $\varepsilon > 0$.

LEMMA 3. *Assume that $\nu > (n-4)/2$ and that the Hölder exponent α is chosen to satisfy $0 < \alpha < \nu - (n-4)/2$. For any $\eta > 0$, there exists some $\theta > 0$, depending on η and ν, such that*

$$\|F_0\|_{\nu+2} \le \eta \qquad \text{if } \varepsilon < \theta.$$

Proof. In this proof we define $m = (n+2)/2$. The first step is to prove some estimates on $\overline{u}$. We already know that $\overline{u}$ is bounded by

$$\overline{u}(y) \le c(|y|^2(-\log|y|))^{(2-m)/2}$$

for all $|y| \le 1$. Since $\overline{u}$ is radial, from equation (12) we get

$$\frac{d}{dr}\left(r^{m-1}\frac{d\overline{u}}{dr}\right) = -r^{m-1}\overline{u}^{m/(m-2)}.$$

Therefore, we have

$$\left|\frac{d}{dr}\left(r^{m-1}\frac{d\overline{u}}{dr}\right)\right| \le cr^{-1}(-\log r)^{-m/2}$$

for all $r \le 1$. Integrating the last inequality from 0 to $r < 1$ we get the upper bound

$$r^{m-1}\left|\frac{d\overline{u}}{dr}\right| \le c(-\log r)^{(2-m)/2}.$$

Therefore

$$|\nabla\overline{u}(y)| \le c|y|^{1-m}(-\log|y|)^{(2-m)/2}. \tag{13}$$

Finally, using once more the equation (12), we derive the estimate

$$|\nabla^2\overline{u}(y)| \le c|y|^{-m}(-\log|y|)^{(2-m)/2}, \tag{14}$$

and also

$$|\nabla^3\overline{u}(y)| \le c|y|^{-m-1}(-\log|y|)^{(2-m)/2}. \tag{15}$$

Now using (13)–(15) we easily get the desired estimate for f_0 from the properties stated in Proposition 2. Namely, if $\operatorname{dist}(x, \Lambda') \in [r, 2r]$ and $\operatorname{dist}(\overline{x}, \Lambda') \in [r, 2r]$, then we have, for $r \le \tau/2$,

$$|\phi(x)^{\nu+2}f_0(x)| \le cr^{3-m+\nu}(\log(\varepsilon r))^{(2-m)/2},$$

and also

$$|\phi(x)^{\nu+2}f_0(x) - \phi(\overline{x})^{\nu+2}f_0(\overline{x})| \le cr^{3-m+\nu-\alpha}(\log(\varepsilon r))^{(2-m)/2}|x-\overline{x}|,$$

where the constant $c > 0$ depends on τ, n, α and ν. Therefore, applying the result of Proposition 3, we see that if $\operatorname{dist}(X, \Lambda) \in [r, 2r]$ and $\operatorname{dist}(\overline{X}, \Lambda) \in [r, 2r]$, then

$$|\Phi(X)^{\nu+2}F_0(X)| \le cr^{3-m+\nu}(\log(\varepsilon r))^{(2-m)/2}.$$

and also

$$|\Phi(X)^{\nu+2}F_0(X) - \Phi(\overline{X})^{\nu+2}F_0(\overline{X})| \le cr^{3-m+\nu-\alpha}(\log(\varepsilon r))^{(2-m)/2}|X-\overline{X}|^{\alpha}.$$

The result follows at once since, for all $n \ge 4$, we have $m \equiv (n+2)/2 > 2$ and also since we have assumed that $\nu + 2 > n/2 \equiv m - 1$. □

Our next goal will be to estimate the norm of $U_0^{4/(n-2)}V$ in the weighted Hölder spaces. This is the aim of the next lemma:

Lemma 4. *For all $\eta > 0$, there exists some $\theta > 0$, only depending on ν, such that*

$$\|U_0^{4/(n-2)}V\|_{\nu+2} \leq \eta\|V\|_\nu,$$

provided $\varepsilon < \theta$.

Proof. Outside $\mathcal{T}_\sigma(\Lambda)$, the estimate can be derived easily. Now, for all $X \in \mathcal{T}_\sigma(\Lambda)$, notice as above that there exists some constant $c > 0$ independent of $\varepsilon \in (0,1)$ such that

$$\frac{1}{c}(\mathrm{dist}(X,\Lambda))^{(2-n)/2}(\log(\varepsilon\,\mathrm{dist}(X,\Lambda)))^{(2-n)/4} \leq U_0(X) \leq c(\mathrm{dist}(X,\Lambda))^{(2-n)/2}(\log(\varepsilon\,\mathrm{dist}(X,\Lambda)))^{(2-n)/4}$$

and also

$$|\nabla U_0|(X) \leq c(\mathrm{dist}(X,\Lambda))^{-n/2}(\log(\varepsilon\,\mathrm{dist}(X,\Lambda)))^{(2-n)/4}.$$

From these inequalities and from the result of Proposition 2, we find that the norm of U_0 can be taken as small as we want in the space $\mathcal{C}^{0,\alpha}_{(n-2)/2}(\mathbb{S}^n \setminus \Lambda)$, provided ε is small enough. Therefore, $U_0^{4/(n-2)}$ can be taken as small as we want in $\mathcal{C}^{0,\alpha}_2(\mathbb{S}^n \setminus \Lambda)$. If we assume that $V \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, we can easily derive the result from Proposition 2. □

Finally, we will have to estimate the norm of nonlinear terms like $V^{(n+2)/(n-2)}$ in the weighted Hölder spaces. This is the purpose of the last lemma of this section:

Lemma 5. *Assume that $\nu \leq (n-2)/2$. For all $\eta > 0$, there exists some $\delta > 0$, only depending on ν and η, such that*

$$\|V^{(n+2)/(n-2)}\|_{\nu+2} \leq \eta\|V\|_\nu,$$

provided $\|V\|_\nu < \delta$.

Proof. The proof relies on the simple fact that

$$\|V^{(n+2)/(n-2)}\|_{(n+2)\nu/(n-2)} \leq c\eta^{4/(n-2)}\|V\|_\nu.$$

But, as we have chosen $\nu \leq (n-2)/2$, we obtain

$$\frac{n+2}{n-2}\nu \leq \nu + 2,$$

and the result follows immediately. □

7. The conformal Laplacian in weighted Hölder spaces

We begin this section by the following important proposition:

Proposition 5. *Assume that* $0 < \nu < (n-2)/2$. *For all* $V \in \mathcal{C}_{\nu+2}(\mathbb{S}^n \setminus \Lambda)$, *there exists a unique weak solution* $U \in \mathcal{C}^{0,\alpha}_{\nu}(\mathbb{S}^n \setminus \Lambda)$ *to the equation*

$$-L_0 U = V \qquad \text{in } \mathbb{S}^n. \tag{16}$$

In addition, there exists some constant $c > 0$, *only depending on* n *and* ν, *such that*

$$\|U\|_{\nu} \leq c\|V\|_{\nu+2}.$$

Proof. The existence of a weak solution of (16) is standard as we have assumed that $\nu \in ((n-4)/2, (n-2)/2)$. More precisely, applying the result of Lemma 2, we find that $V \in L^p(\mathbb{S}^n)$ for all $1 < p < (n+2)/(2(\nu+2))$. Therefore, one can apply the classical L^p existence theory to get the existence and uniqueness of a weak solution $U \in W^{2,p}(\mathbb{S}^n)$ of (16). In addition, we claim that

$$\sup_X |\Phi^{\nu+1}|\nabla U||(X) + \sup_X |\Phi^{\nu} U|(X) \leq c\|V\|_{\nu+2}, \tag{17}$$

where the constant $c > 0$ does not depend on V. In order to prove the above estimates, let us emphasize that we are dealing with the conformal Laplacian on the sphere. Therefore, by (6)–(8), the resolution of (16) is equivalent to the resolution of the equation

$$-\Delta_{\delta^n_{ij}} u = v \qquad \text{in } \mathbb{R}^n,$$

where the relations between u, v and U, V are given by (3) and (6). But since $V \in \mathcal{C}^{0,\alpha}_{\nu+2}(\mathbb{S}^n \setminus \Lambda)$ we have the estimates

$$|v(x)| \leq c\|V\|_{\nu+2}\phi^{-\nu-2}(x) \qquad \text{for all } x \in B_n(2) \tag{18}$$

and

$$|v(x)| \leq c\|V\|_{\nu+2}|x|^{2-n} \qquad \text{for all } x \in \mathbb{R}^n \setminus B_n(2). \tag{19}$$

(We recall that, by assumption, $\Lambda' \subset B_n(1)$.)

In $\mathbb{R}^n \setminus \mathcal{T}'_\sigma$ the estimate (17) follows from classical regularity theory using (18) and (19). In $\mathcal{T}'_\sigma$ we can easily build some supersolution by considering the function

$$\widetilde{u} : x \to \operatorname{dist}(x, \Lambda')^{-\nu}.$$

By Proposition 4, we see that

$$-\Delta_{\delta^n_{ij}} \widetilde{u} = \nu\left(\frac{n-2}{2} - \nu\right) \operatorname{dist}(x, \Lambda')^{-\nu-2} + o(\operatorname{dist}(x, \Lambda')^{-\nu-2})$$

near Λ'. Moreover, from the choice on ν we see that $\nu((n-2)/2-\nu) > 0$. Therefore, if τ is taken small enough and the constant $k > 0$ large enough, then $k\|V\|_{\nu+2}\widetilde{u}$ is a supersolution for u in $\mathcal{T}'_\sigma$. And the estimate on U in (17) follows easily from Proposition 3. The estimate for the gradient in (17) follows immediately using classical regularity theory. □

Remark 3. Notice that we only need to know that $\Phi^{\nu+2}V$ is bounded in order to get the estimate of Proposition 5.

8. The proof of Theorem 1 completed

In this last section, in view of the previous propositions, we restrict the set in which the parameter ν is chosen to

$$\nu \in ((n-4)/2, (n-2)/2).$$

In order to prove Theorem 1, it is sufficient to solve in the space $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ the nonlinear equation

$$-L_0 W = (|W + U_0|^{(n+2)/(n-2)} - U_0^{(n+2)/(n-2)}) - F_0. \tag{20}$$

Since $\nu < (n-2)/2$, we will see below that this will imply that $W + U_0$ is singular at every point of Λ.

To achieve this, we are going to define some continuous operator K from $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ into itself. Then, we are going to prove that, when restricted to a small ball in $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, this operator is a contraction. The existence of a solution W of (20) will then follow from a classical fixed point theorem.

Assume that we have already obtained a solution to (20). Then $U_0 + V$ is a weak solution of

$$-L_0(V + U_0) = |V + U_0|^{(n+2)/(n-2)} \qquad \text{in } \mathbb{S}^n.$$

We claim that $V + U_0$ is positive in $\mathbb{S}^n \setminus \Lambda$. In fact, we first have the estimate

$$U_0(x) \geq c(\operatorname{dist}(x,\Lambda))^{(2-n)/2}(-\log(\operatorname{dist}(x,\Lambda)))^{(2-n)/4} \qquad \text{in } \mathcal{T}_\sigma. \tag{21}$$

On the other hand, since $V \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, there exists some constant $c > 0$ such that

$$V(x) \leq c(\operatorname{dist}(x,\Lambda))^{-\nu}. \tag{22}$$

Using the fact that $\nu \in ((n-4)/2, (n-2)/2)$, one finds immediately that $V + U_0 > 0$ near Λ and, by the maximum principle, we conclude that $V + U_0 > 0$ everywhere in $\mathbb{S}^n \setminus \Lambda$. So, $U_0 + V$ is a positive solution of (2).

Finally, the fact that the metric $(V + U_0)^{4/(n-2)} g_0$ is complete is also a consequence of the estimates (21) and (22), which imply that

$$U_0(x) + V(x) \geq c(\mathrm{dist}(x,\Lambda))^{(2-n)/2}(-\log(\mathrm{dist}(x,\Lambda)))^{(2-n)/4},$$

and the completeness of the metric follows at once.

It remains to prove the existence of a solution of (20) (see also [7], Propositions 3.4 and 3.5). We begin by

Lemma 6. *There exists some constant $c > 0$ such that, for all $x > 0$ and all $y \in \mathbb{R}$,*

$$||x+y|^{(n+2)/(n-2)} - x^{(n+2)/(n-2)}| \leq c(|y||x|^{4/(n-2)} + |y|^{(n+2)/(n-2)}). \tag{23}$$

Let $V \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ be given. We define $K(V) \equiv W \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ to be the solution of

$$-L_0 W = (|V + U_0|^{(n+2)/(n-2)} - U_0^{(n+2)/(n-2)}) - F_0.$$

Using (23) and the results of Lemmas 3–5, we see that, for all $V \in \mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, the right hand side of (20) belongs to $\mathcal{C}_{\nu+2}(\mathbb{S}^n \setminus \Lambda)$. Therefore, by Proposition 4, the operator K is well defined. The next lemma will enable us to estimate it:

Lemma 7. *There exists some constant $c > 0$ such that, for all $x > 0$ and all $y, z \in \mathbb{R}$,*

$$||x+y|^{(n+2)/(n-2)} - |x+z|^{(n+2)/(n-2)}| \leq c|y-z|(|x+y|^{4/(n-2)} + |x+z|^{4/(n-2)}).$$

Using this last lemma, we see that, by Lemmas 4 and 5, and also considering Remark 3, we can choose ε small enough and V, V' in a small ball of $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$, in order to get

$$\|K(V) - K(V')\|_\nu \leq \tfrac{1}{2}\|V - V'\|_\nu.$$

Therefore, if ϱ and ε are chosen small enough, the operator K sends the ball of radius ϱ in $\mathcal{C}^{0,\alpha}_\nu(\mathbb{S}^n \setminus \Lambda)$ into itself and, when restricted to this ball, the operator K is a contraction. This ends the proof of Theorem 1.

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