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QUASILINEAR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENTS

Philippe Clément — Djairo Guedes de Figueiredo — Enzo Mitidieri

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Introduction

It was proved by Pokhozhaev [PO1] (using what it is now known as Pokhozhaev's identity) that the problem

$$(\mathrm{P}_0)\qquad \begin{cases} -\Delta u = u^{2^*-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

has no solution if $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is bounded and starshaped with respect to some point, and $2^* = 2N/(N-2)$. In (P_0) the nonlinear term is a power of u with the critical exponent $(N+2)/(N-2)$. This terminology comes from the fact that the continuous Sobolev imbeddings $H_0^1(\Omega) \subset L^p(\Omega)$, for $p \leq 2^*$ and Ω bounded, are also compact except when $p = 2^*$. This loss of compactness reflects in that the functional whose Euler–Lagrange equation is (P_0) fails to satisfy the Palais–Smale condition. Later Brezis and Nirenberg [BN] observed that the Palais–Smale condition fails at certain levels only. Then they proved that if the nonlinear term is slightly perturbed, the new problem has a solution.

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More precisely, the problem

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution if $0 < \lambda < \lambda_1$ and $N \geq 4$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Also, if $N = 3$, then there is a $\lambda^* > 0$ such that (P_λ) has a solution if $\lambda^* \leq \lambda < \lambda_1$. In the case when Ω is a ball, we have $\lambda^* = \lambda_1/4$.

These results have been partly extended to the case when the Laplacian is replaced by a p-Laplacian operator, with $1 < p < N$. Existence and nonexistence theorems are contained in [GP], [GV], [EG2], [EG3] in the case of general domains Ω. In the case $\Omega = B_R, R > 0$, sharp results were obtained by Knaap and Peletier [KP] (see also [KN]) by using Emden–Fowler's approach generalizing previous work in [AP1], [AP2].

Existence and nonexistence theory for a more general class of operators was developed by Egnell in [EG1]. He considered homogeneous (nonlinear) differential operators of "weighted p-Laplacian" type and critical nonlinearities containing (possible) singular weights. See [EG1] for the precise statement of these results.

Our aim here is to study the Brezis–Nirenberg problems for a class of quasilinear elliptic equations of the type

$$(\mathrm{Q}_0) \qquad \begin{cases} Lu := -(r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q-2} u, \\ u'(0) = u(R) = 0, \\ u > 0 \quad \text{in } (0, R), \end{cases}$$

and

$$(\mathrm{Q}_\lambda) \qquad \begin{cases} Lu = \lambda r^\delta |u|^\beta u + r^\gamma |u|^{q-2} u, \\ u'(0) = u(R) = 0, \\ u > 0 \quad \text{in } (0, R). \end{cases}$$

This class of problems, when specialized to radial solutions, is the same as Egnell's [EG1].

The motivation for our study is twofold. The first motivation relies on a classical inequality proved by Bliss in 1930 (see [BL]). Bliss's (see (1.3) below) inequality is a *Sobolev type* inequality for scalar absolutely continuous functions of one variable. In the statement of this inequality the dimension N does not (necessarily) appear as in the usual Sobolev inequality. However, when rewritten with appropriate change of variables, Bliss's inequality may be interpreted as a *weighted Sobolev type* inequality (see Proposition 1.3 below) with *fractional dimensions*. This has important consequences for the computation of the *best* embedding constant (e.g. best Sobolev constant) related to the spaces under consideration.

The second motivation is that "k-Hessian type operators" (see [CNS], [R], [TS1], [TS2]) are included in this class. Results for k-Hessian equations, related to the subject of this paper, have been obtained in [TS1] and [TS2]. In particular, Tso [TS1], among other things, introduced the notion of critical exponent associated with a k-Hessian operator. He proved then some existence and nonexistence theorems for problems involving subcritical and supercritical nonlinearities. When specialized to radial solutions, these results are strictly related to Bliss's inequality.

It is then apparent that Bliss's inequality is one of the basic tools for the study of quasilinear elliptic problems containing critical nonlinearities and (possible) singular coefficients.

We point out that some of the results obtained in this paper overlap with earlier results obtained by Egnell [EG1]. Nevertheless we believe that our point of view may be useful for more general problems and more importantly may be used to clarify the different notions of criticality that appear when dealing with various differential operators.

One of the basic assumptions that we shall make throughout this paper is the following:

$$\text{(P1)} \qquad q-1>\beta+1>0, \quad \gamma+1>\alpha-\beta-1 \quad \text{and} \quad \delta+1\geq\alpha-\beta-1.$$

We remark that in view of the results proved in [CM], under the assumption $\alpha>\beta+1$, the condition $\gamma+1>\alpha-\beta-1$ is necessary for the existence of a positive solution of

$$\begin{cases} -(r^{\alpha}\,|u'|^{\beta}u')' \geq r^{\gamma}|u|^{q-2}u & \text{in } (0,R), \\ u>0 & \text{in } (0,R), \end{cases}$$

and $\delta+1\geq\alpha-\beta-1$ is necessary for the existence of a positive solution of the problem

$$\begin{cases} -(r^{\alpha}|u'|^{\beta}u')' \geq r^{\delta}|u|^{\beta}u & \text{in } (0,R), \\ u>0 & \text{in } (0,R). \end{cases}$$

Let $N\geq 1$ be an integer. Let $B_R=\{x\in\mathbb{R}^N : \|x\|<R\}$. The following operators, when considered as acting on functions defined on B_R, are included in our class:

(i) Laplacian: $\alpha=\gamma=\delta=N-1$, $\beta=0$,
(ii) p-Laplacian: $\alpha=\gamma=\delta=N-1$, $\beta=p-2$,
(iii) k-Hessian ($1\leq k\leq N$): $\alpha=N-k,\ \gamma=\delta=N-1$, $\beta=k-1$.

Observe that the N-Hessian is the Monge–Ampère operator.

Our analysis will throw a light on the questions of nonexistence, of critical exponent and of the so-called *critical dimensions*, which are the dimensions N where (Q_λ) has solution only if λ is larger than a certain $\lambda^* > 0$. In the case of the Laplacian the only critical dimension is $N = 3$, as seen above. For our class of operators, statements on the dimension N will be replaced by statements on the parameters appearing in equation (Q_λ). We shall see in Section 5 that critical dimensions will correspond here to a condition on the parameters, namely

$$(\mathrm{P2}) \qquad (\delta+1)(\beta+1) - (\alpha-\beta-1)(\beta+2) > 0.$$

In the case of the Laplacian, problems involving powers may exhibit a loss of compactness only when $N \geq 3$. This is explained by the Sobolev imbeddings. In dimension $N = 2$ the nonlinearities leading to a loss of compactness are of exponential type. In this case the Pokhozhaev–Trudinger (see [LP], [M1], [M2], [PO2], [TR]) imbedding appears. To emphasize these distinct behaviors, we refer to them as the *Sobolev case* and the *Pokhozhaev–Trudinger case*, respectively.

This paper is organized as follows: in Section 1 we shall recall some imbedding theorems for the *Sobolev case*. For our class of differential operators L this corresponds to the assumption

$$(\mathrm{P3}) \qquad \alpha - \beta - 1 > 0,$$

while the *Pokhozhaev–Trudinger case*, that is, $\alpha - \beta - 1 = 0$, will appear only in Section 6, where a related Gelfand type problem will be studied.

As a consequence of the imbedding theorems of Section 1, we define the critical exponent associated with (Q_0) to be the number

$$q^* = \frac{(\gamma+1)(\beta+2)}{\alpha-\beta-1}.$$

In Section 2 we introduce the notions of *weak solution* and *integral solution* and show that every such solution is classical if besides (P1) we assume

$$(\mathrm{P4}) \qquad \gamma, \delta > \alpha - 1.$$

As usual a *classical solution* of (Q_0) or (Q_λ) is a function $u \in C^2(0,T) \cap C^1[0,T]$ satisfying the equation and the boundary conditions.

In Section 3 we shall discuss the eigenvalue problem

$$(\mathrm{EP}) \qquad \begin{cases} Lu = \lambda r^\delta |u|^\beta u & \text{in } (0,R), \\ u'(0) = u(R) = 0. \end{cases}$$

As we shall see, certain compact imbeddings related to (P_1) will play an important role in obtaining the existence of a first eigenvalue $\lambda_1(R) > 0$ for (EP). Indeed, this will require

$$(\mathrm{P5}) \qquad \alpha - \beta - 2 < \delta.$$

In Section 4 we shall prove some nonexistence results by using a special form of a variational identity proved by Pucci and Serrin [PS2]. Section 5 contains a generalized version of an inequality by Atkinson and Peletier [AP2] (see also [KN]). This result is a consequence of a monotonicity property shared by certain functionals evaluated along the possible positive solutions (see Theorem 5.1 below) of the equation $Lu = r^\gamma f(u)$.

One of the consequences of this inequality is that it can be used to prove sharp results for existence and nonexistence of positive solutions and related asymptotic estimates for (Q_λ) in the spirit of [AP1], [AP2], [KN], [KP]. This shows that the Atkinson–Peletier approach works for more general problems.

In Section 6, by using essentially the same approach used in the preceding section, we study a Gelfand type problem associated with L. We will show that the bifurcation diagram of the solution set in the case of $\Omega = B_R$ is qualitatively the same for all operators L. In particular, in the case when L is the p-Laplacian and $\Omega = B_R$ we will show that it is possible to write explicitly the possible positive solutions of the problem (see (6.16) below).

The main results of this paper are:

THEOREM 3.1. *Assume that* (P1), (P3), (P4), (P5) *hold. If* $\lambda \geq \lambda_1(R)$ *and* $q = q^*$ *then* (Q_λ) *has no solution.*

THEOREM 4.1. *Assume that* (P1), (P3), (P4), (P5) *hold. If* $\lambda \leq 0$ *and* $q = q^*$ *then* (Q_λ) *has no solution.*

THEOREM 4.2. *Assume that* (P1)–(P5) *hold,* $\beta \geq 0$ *and* $q = q^*$*. Then there is* $\lambda^* > 0$ *such that* (Q_λ) *has no solution for* $\lambda \leq \lambda^*$*.*

THEOREM 7.1. *Assume that* (P1)–(P5) *hold and* $q = q^*$*. Then there exists* $\lambda^{**} > 0$ *such that* (Q_λ) *has a solution for* $\lambda^{**} < \lambda < \lambda_1$*.*

THEOREM 7.2. *Assume that* (P1), (P3), (P4), (P5) *hold and* $q = q^*$*. If*

$$(\delta+1)(\beta+1) - (\alpha-\beta-1)(\beta+2) \leq 0$$

then (Q_λ) *has a solution for* $0 < \lambda < \lambda_1(R)$*.*

1. Variational preliminaries

The solutions of problems (Q_λ) will be obtained as critical points of appropriate functionals. For that purpose we introduce the following function spaces. For $0 < R < \infty$, $\alpha > 0$ and $\beta > -1$, let X_R be the set of absolutely continuous functions $u : (0, R] \to \mathbb{R}$ such that $u(R) = 0$ and

$$\int_0^R r^\alpha |u'(r)|^{\beta+2}\, dr < \infty.$$

Observe that in order to have a less cumbersome notation for the space X_R we have omitted in it its dependence on α and β. The space X_R becomes a Banach space if we define a norm $\| \ \|_{X_R}$ by

$$\|u\|_{X_R}^{\beta+2} = \int_0^R r^\alpha |u'(r)|^{\beta+2}\,dr.$$

We could also introduce X_R as follows: for $0 < R < \infty$, $\alpha > 0$ and $\beta > -1$, let $\widetilde{X}_R$ be the set of L^1_{loc} real functions defined in $(0,R)$ with distributional derivatives in L^1_{loc} and such that

$$\int_0^R r^\alpha |u(r)|^{\beta+2}\,dr < \infty, \qquad \int_0^R r^\alpha |u'(r)|^{\beta+2}\,dr < \infty.$$

Then $\widetilde{X}_R$ is a Banach space with the norm $\| \ \|_{\widetilde{X}_R}$ defined by

$$\|u\|_{\widetilde{X}_R}^{\beta+2} = \int_0^R r^\alpha |u|^{\beta+2} + \int_0^R r^\alpha |u'|^{\beta+2}.$$

It follows that $u \in \widetilde{X}_R$ is necessarily absolutely continuous in $(0,R]$. Thus, we can consider the subspace X_R of $\widetilde{X}_R$ consisting of functions u such that $u(R) = 0$. We see, by using Proposition 1.0 below, that for $u \in X_R$ we have

$$\int r^\alpha |u|^{\beta+2} \le C \int r^\alpha |u'|^{\beta+2}.$$

Consequently, $\| \ \|_{X_R}$ and $\| \ \|_{\widetilde{X}_R}$ are equivalent norms on X_R. For different values of α and β, the spaces X_R are Sobolev spaces with weight [KO]. When $R = \infty$, X_∞ can be defined similarly provided we replace the boundary condition $u(R) = 0$ by $\lim_{r\to\infty} u(r) = 0$.

Let $q \ge 1$ and $\gamma > 0$. Let R be such that $0 < R < \infty$. Denote by $L^q_\gamma = L^q_\gamma(0,R)$ the Banach space of Lebesgue measurable functions $u : [0,R] \to \mathbb{R}$ such that

$$\|u\|_{L^q_\gamma} = \left(\int_0^R r^\gamma |u(r)|^q\,dr\right)^{1/q} < \infty.$$

Associated with each space X_R and each weight γ we define the *critical exponent*:

$$q^* = \frac{(\gamma+1)(\beta+2)}{\alpha-\beta-1}, \tag{1.1}$$

under the assumption that

$$\alpha - \beta - 1 > 0. \tag{1.2}$$

If this condition is violated we say that there is no critical exponent. In such a case (Q_0) is solvable for all q.

The critical exponent in the case of the p-Laplacian is

$$p^* = \frac{Np}{N-p} \quad \text{if } p < N.$$

There is no critical exponent, as defined above, if $p \geq N$. In particular, the critical exponent for the Laplacian is given by $2^* = 2N/(N-2)$.

The critical exponent in the case of the k-Hessian [TS1] is given by

$$q^* = \frac{N(k+1)}{N-2k} \quad \text{if } k < \frac{N}{2},$$

and there is no critical exponent if $k \geq N/2$. In particular, there is no critical exponent for the Monge–Ampère operator.

The following result is well known and in a form similar to the one below appears in Kufner–Opic [KO].

Proposition 1.0. *Let $u : (0, R] \to R$ be an absolutely continuous function. If $u(R) = 0$ and*

(i) *for $1 \leq \beta + 2 \leq q < \infty$ one has*
 (a) $\alpha > \beta + 1$, $\gamma \geq \alpha \dfrac{q}{\beta+2} - \dfrac{q(\beta+1)}{\beta+2} - 1$, *or*
 (b) $\alpha \leq \beta + 1$, $\gamma > -1$,

(ii) *for $1 \leq q < \beta + 2 < \infty$ one has*
 (c) $\alpha > \beta + 1$, $\gamma > \alpha \dfrac{q}{\beta+2} - \dfrac{q(\beta+1)}{\beta+2} - 1$, *or*
 (d) $\alpha \leq \beta + 1$, $\gamma > -1$,

then

$$\left(\int_0^R x^\gamma |u(x)|^q \, dx\right)^{1/q} \leq c\left(\int_0^R x^\alpha |u'(x)|^{\beta+2} \, dx\right)^{1/(\beta+2)}.$$

Remark 1.1. The above result expresses the fact that the imbedding $X_R \subset L^q_\gamma$ is continuous if $q \leq q^*$ and $\alpha - \beta - 1 > 0$. If $\alpha - \beta - 1 \leq 0$ this imbedding holds for all $q < \infty$. If $\alpha - \beta - 1 > 0$ we see, by using an Arzelà–Ascoli type argument, that such an imbedding is compact if $q < q^*$. Consequently, we have the following:

Proposition 1.1. *Assume that condition* (1.2) *holds. The space X_R is continuously imbedded in $L^{q^*}_\gamma$, and it is compactly imbedded in L^q_γ if $q < q^*$ and $0 < R < \infty$.*

For each $u \in X_R$ and $0 < R < \infty$, $q \leq q^*$ (or $R = \infty$ and $q = q^*$) we define

$$S_0(u; q, R) = \frac{\int_0^R r^\alpha |u'(r)|^{\beta+2} \, dr}{(\int_0^R r^\gamma |u(r)|^q \, dr)^{(\beta+2)/q}}$$

and

$$S_0(q, R) = \inf\{S_0(u; q, R) : u \in X_R \setminus \{0\}\}.$$

Proposition 1.0 says that in any of the above cases $S_0(q, R) > 0$. It is clear that

$$S_0(q, R) = \inf\{\|u\|^{\beta+2}_{X_R} : u \in X_R,\ \|u\|_{L^q_\gamma} = 1\}.$$

Proposition 1.2. *$S_0(q,R)$ is independent of R if and only if $q=q^*$.*

Proof. Let $R_1<R_2$. For each $u\in X_{R_1}$, the function $v(r):=u((R_1/R_2)r)$ belongs to X_{R_2}. This establishes an isomorphism between these two spaces. It is an easy calculation to prove that

$$S_0(u;q,R_1)=S_0(v;q,R_2)$$

if $q=q^*$. This implies that $S_0(q,R_1)=S_0(q,R_2)$ if $q=q^*$. Now if $q<q^*$, it follows from Proposition 1.1 that $S_0(q,R)$ is attained for some $u_R\in X_R$, for each $0<R<\infty$. Let $\varrho>R$ and define v_ϱ by $v_\varrho(r)=u_R(Rr/\varrho)$. Clearly v_ϱ belongs to X_ϱ and

$$S_0(v_\varrho;q,\varrho)=(R/\varrho)^{(\alpha-\beta-1)(q^*/q-1)}S_0(u_R;q,R).$$

Therefore, since $R<\varrho$ and $q<q^*$, we conclude that $S_0(q,\varrho)<S_0(q,R)$. $\square$

Remark 1.2. We denote by S the common value of $S_0(q^*,R)$ for all $R>0$. It is related to the best Sobolev constant, which of course depends on α,β and γ. Indeed, the best Sobolev constant for the imbedding of X_R into $L^{q^*}_\gamma$ will be $S^{-1/(\beta+2)}$. Its precise value comes from Bliss [B] (see Proposition 1.3 and the remarks following it). For the computation of S in the case of the p-Laplacian we also refer to Talenti [TA] and Rodemich [RO].

The following inequality was proved back in 1930 by Bliss [B]; see also Hardy–Littlewood–Pólya [HLP; p. 195].

Proposition 1.3 (Bliss). *For all $v:(0,\infty)\to\mathbb{R}$ absolutely continuous with $v'\in L^k(0,\infty)$ and $v(0)=0$ one has*

$$\int_0^\infty\frac{|v(x)|^l}{x^{l-h}}\,dx\le K\bigg(\int_0^\infty|v'(x)|^k\,dx\bigg)^{l/k} \tag{1.3}$$

where $l>k>1$, $h=l/k-1$ and

$$K=\frac{1}{l-h-1}\bigg[\frac{h\Gamma(l/h)}{\Gamma(1/h)\Gamma((l-1)/h)}\bigg]^h.$$

Moreover, equality holds in (1.3) *if and only if*

$$v(x)=\frac{1}{(a+bx^{-h})^{1/h}}, \tag{1.4}$$

for arbitrary positive constants a and b.

Next, we use the above proposition to transform (1.3) into a form exhibiting the imbedding in Proposition 1.1. Let v be a function defined by (1.4) with $a,b>0$ and $\sigma\neq0$. Define

$$u(r)=v(x),\quad r=x^\sigma.$$

Then (1.3) becomes

$$\int_0^\infty r^{(h-l-\sigma+1)/\sigma}|u(r)|^l\,dr \le K|\sigma|^{1+l-l/k}\bigg(\int_0^\infty r^{(k-1)(\sigma-1)/\sigma}|u'(r)|^k\,dr\bigg)^{l/k}. \tag{1.5}$$

Now we select the parameters as follows:

$$\frac{h-l-\sigma+1}{\sigma}=\gamma,\qquad \frac{(k-1)(\sigma-1)}{\sigma}=\alpha,\quad k=\beta+2. \tag{1.6}$$

Using these relations we get

$$l=q^*,\quad h=\frac{\beta+\gamma+2-\alpha}{\alpha-\beta-1}\quad\text{and}\quad \sigma=\frac{\beta+1}{\beta+1-\alpha}. \tag{1.7}$$

With this choice of parameters, (1.5) becomes

$$\int_0^\infty r^\gamma|u(r)|^{q^*}\,dr\le \widetilde{K}\bigg(\int_0^\infty r^\alpha|u'(r)|^{\beta+2}\,dr\bigg)^{q^*/(\beta+2)}, \tag{1.8}$$

where

$$\widetilde{K}=K|\sigma|^{(\alpha+\gamma(\beta+1))/(\alpha-\beta-1)}. \tag{1.9}$$

Thus the value of the constant S (see Remark 1.2) is given by

$$S=K^{(\beta+1-\alpha)/(\gamma+1)}|\sigma|^{-(\alpha+\gamma(\beta+1))/(\gamma+1)},$$

where K is given in Proposition 1.3 and h,l and σ are defined by (1.7).

Moreover, equality holds in (1.8) if and only if

$$u(r)=\frac{1}{(a+br^n)^{1/m}}, \tag{1.10}$$

where

$$m=\frac{\gamma+\beta+2-\alpha}{\alpha-\beta-1},\quad n=-\frac{m}{\sigma}=\frac{\gamma+\beta+2-\alpha}{\beta+1}, \tag{1.11}$$

and a,b are arbitrary positive constants. For future references we call the function u defined by (1.10)–(1.11) the *Bliss function*.

These exponents for the model operators presented in the introduction are: for the p-Laplacian ($p<N$)

$$m=\frac{p}{N-p},\quad n=\frac{p}{p-1},$$

and for the k-Hessian ($k<N/2$)

$$m=\frac{2k}{N-2k},\quad n=2.$$

Let $b,c>0$. A straightforward computation shows that

$$u(r)=c(b+r^n)^{-1/m}$$

is a solution of

$$-(r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q^*-2}u \quad \text{in } (0,\infty) \tag{1.12}$$

if and only if

$$\left(\frac{\alpha-\beta-1}{\beta+1}\right)^{\beta+1}(\gamma+1)b = c^{(\beta+2)(\beta+\gamma+2-\alpha)/(\alpha-\beta-1)}.$$

Consider then the following special solution of (1.11);

$$\widehat{u}_1(r) = \widehat{c}\,(1+r^n)^{-1/m},$$

where

$$\widehat{c} = \left[\left(\frac{\alpha-\beta-1}{\beta+1}\right)^{\beta+1}(\gamma+1)\right]^{(\alpha-\beta-1)/((\beta+2)(\beta+\gamma+2-\alpha))}. \tag{1.13}$$

Consequently, for each $\varepsilon > 0$, the function

$$\widehat{u}_\varepsilon(r) = \varepsilon^{\widetilde{s}}\widehat{u}_1(r/\varepsilon) = \widehat{c}\varepsilon^{\widetilde{s}+n/m}(\varepsilon^n + r^n)^{-1/m}$$

is also a solution of (1.12) provided

$$\widetilde{s} = \frac{\beta+1-\alpha}{\beta+2}.$$

Using relations (1.11) we can rewrite $\widehat{u}_\varepsilon$ as

$$\widehat{u}_\varepsilon(r) = \widehat{c}\varepsilon^s(\varepsilon^n + r^n)^{-1/m}, \quad s = \frac{\alpha-\beta-1}{(\beta+1)(\beta+2)}, \tag{1.14}$$

with $\widehat{c}$ given in (1.13).

We have proved the following

Proposition 1.4. *$S_0(q^*, R)$ is achieved when $R = \infty$. Moreover, the constant S satisfies*

$$S = \left(\int_0^\infty r^\alpha |\widehat{u}_\varepsilon'|^{\beta+2}\right)^{(\beta+\gamma+2-\alpha)/(\gamma+1)} = \left(\int_0^\infty r^\gamma |\widehat{u}_\varepsilon|^{q^*}\right)^{(\beta+\gamma+2-\alpha)/(\gamma+1)}$$

2. Solutions of (Q_λ) and their regularity

In this section we shall consider the equation

$$\begin{cases} Lu = r^\theta f(r,u) & \text{in } (0,R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2.1}$$

where $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$|f(r,u)| \le c|u|^{p-1} + c \quad \text{for any } u \in \mathbb{R},\ 0 \le r \le R, \tag{2.2}$$

with p satisfying the condition

$$\beta + 2 \le p \le \frac{(\theta+1)(\beta+2)}{\alpha-\beta-1}. \tag{2.3}$$

and $c > 0$.

All along this section we shall assume that (P1) and (P3) hold with $\min(\gamma,\delta)$ replaced by θ.

A *weak solution* of (2.1) is a critical point of the functional

$$\Phi(u) = \frac{1}{\beta+2}\int_0^R r^\alpha |u'(r)|^{\beta+2}\,dr - \int_0^R r^\theta F(r,u(r))\,dr$$

where $F(r,s) = \int_0^s f(r,t)\,dt$. This functional is well defined for $u \in X_R$ in view of the condition (2.2)–(2.3) and the imbedding $X_R \subset L^p_\theta$. So $u \in X_R$ is a weak solution of (2.1) if and only if

$$\int_0^R r^\alpha |u'|^\beta u'v' = \int_0^R r^\theta f(r,u)v \quad \text{for any } v \in X_R. \tag{2.4}$$

A function $u \in X_R$ is an *integral solution* of (2.1) if

$$-r^\alpha |u'(r)|^\beta u'(r) = \int_0^r s^\theta f(s,u(s))\,ds \tag{2.5}$$

for $r \in [0,R]$ (a.e.). This expression makes sense since f satisfies (2.2)–(2.3).

PROPOSITION 2.1. *Assume that conditions* (2.2)–(2.3) *hold. A function* $u \in X_R$ *is a weak solution of* (2.1) *if and only if it is an integral solution.*

PROOF. Let $u \in X_R$ satisfy (2.5). Multiplying it by v', where $v \in X_R$, and integrating on $(0,R)$ we obtain (2.4). To handle the right side we use Fubini's theorem.

Conversely, suppose that $u \in X_R$ satisfies (2.4). For each $r \in (0,R)$ and each $\varepsilon > 0$ consider the following continuous function:

$$v_\varepsilon(s) = \begin{cases} 1 & \text{if } 0 \le s \le r, \\ 0 & \text{if } s \ge r+\varepsilon, \\ \text{linear} & \text{between } r \text{ and } r+\varepsilon. \end{cases}$$

Since $v_\varepsilon \in X_R$, we can use it as a test function in (2.4) to obtain

$$\frac{1}{\varepsilon}\int_r^{r+\varepsilon} s^\alpha |u'|^\beta u'\,ds = \int_0^r s^\theta f(u(s))\,ds + \int_r^{r+\varepsilon} s^\theta f(u(s))v_\varepsilon(s)\,ds.$$

Letting $\varepsilon \to 0$ in the above identity we obtain (2.5). □

Consider now the problem of the regularity of weak solutions of (2.5). For more general results see [TO].

For simplicity we shall write (2.5) as

$$-r^{\alpha}|u'|^{\beta}u' = \int_0^r s^{\theta} f(s)\, ds, \tag{2.6}$$

with the assumption that f is continuous in $[0,R]$. We observe that this includes equation (Q_λ) with $\theta = \min(\gamma,\delta)$. Recall that in our model problems $\theta=\gamma=\delta$.

Let $\xi(t) = |t|^{1/(\beta+1)-1}t$, $t \in \mathbb{R}$. This function is the inverse of the function $t \mapsto |t|^{\beta}t$. [Observe that if $\beta > 0$ then ξ is not differentiable at $t=0$.] So, from (2.6) we obtain

$$-u'(r) = \xi(g(r)),$$

where

$$g(r) := \frac{1}{r^{\alpha}}\int_0^r s^{\theta} f(s)\, ds.$$

Clearly g is continuous in $(0,R]$, and in fact differentiable there. It follows that $u \in C^2(0,R]$. So we have just to worry about its regularity at $r=0$.

Lemma 2.1. *If $\theta \geq \alpha - 1$ then the function g is continuous at $r=0$. Moreover, if $\theta > \alpha-1$, then $g(0)=0$, and if $\theta = \alpha - 1$ then $g(0) = f(0)/\alpha$. Consequently, if $\theta > \alpha - 1$ then u' is continuous in $[0,R]$ and $u'(0) = 0$.*

Proof. The result follows readily by the use of L'Hôspital's rule

$$\lim_{r\to 0} g(r) = \lim_{r\to 0}\frac{r^{\theta}f(r)}{\alpha r^{\alpha-1}}.$$

Lemma 2.2. *If $\theta > \alpha - 1$ and $\theta \geq \alpha+\beta$ then the function u' is differentiable up to $r=0$. Moreover, if $\theta > \alpha+\beta$ then $u''(0)=0$.*

Proof. It suffices to prove that the limit $\lim_{r\to 0}\xi(g(r))/r$ exists. This in turn is equivalent to the existence of $\lim_{r\to 0} g(r)/r^{\beta+1}$. By L'Hôspital's rule this will be case if $\theta \geq \alpha+\beta$. This limit is zero if $\theta > \alpha+\beta$. This completes the proof.

We see then that $u \in C^2[0,R]$ in the case of the p-Laplacian for $p \leq 2$ and it is always $C^2[0,R]$ in the case of the k-Hessian for $1 \leq k \leq N$.

The conclusion of Lemma 2.1 can be improved to get that the solution u is in some Hölder space $C^{1+\mu}[0,R]$. This will be particularly useful in the case of the p-Laplacian for $p > 2$.

Lemma 2.3. *Let $\theta > \alpha - 1$. Then the function u' given in (2.5) is Hölder continuous up to $r = 0$.*

Proof. Since ξ is Hölder continuous, it suffices to prove that g is Hölder continuous. Let $0 < s < r < R$. Since g is differentiable in $(0,R)$ we obtain

$$g(r) - g(s) = \int_s^r g'(t)\, dt.$$

Using Hölder's inequality we have

$$|g(r) - g(s)| \le |r - s|^{1/q} \left(\int_s^r |g'(t)|^{q'} \, dt \right)^{1/q'}. \tag{2.7}$$

Next we estimate g'. Differentiating g we get

$$g'(r) = -\alpha r^{-\alpha-1} \int_0^r t^\theta f(t) \, dt + r^{-\alpha+\theta} f(r),$$

which implies that $g'(r) = O(r^{\theta-\alpha})$ as $r \to 0$.

Hence the integral in (2.7) is bounded if $(\theta - \alpha)q' > -1$ or $\theta - \alpha + 1 > 1/q$. By choosing $q > 1$ satisfying this inequality, it follows that the function u' is then Hölder continuous with exponent $\mu < (\theta - \alpha + 1)/(\beta + 1)$. □

Summarizing, we can state the following basic result where the last statement is a consequence of the theorem on existence and uniqueness of solution of the initial value problem in ODE.

Proposition 2.2. *Assume that conditions* (2.2), (2.3) *and $\theta > \alpha - 1$ hold. Then any weak solution of* (2.1) *belongs to $C^2(0, R) \cap C^{1,\mu}[0, R]$ and it has only simple zeros.*

3. The eigenvalue problem

The problem consists in looking for $\lambda \in \mathbb{R}$ such that the problem

$$\begin{cases} Lu = \lambda r^\delta |u|^\beta u & \text{in } (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{EP}$$

has a solution $u \in X_R \setminus \{0\}$. We observe that the imbedding $X_R \subset L_\delta^{\beta+2}$ is continuous if

$$\beta + 2 \le \frac{(\delta + 1)(\beta + 2)}{\alpha - \beta - 1},$$

that is, if $\alpha - \beta - 2 \le \delta$, and it is compact if

$$\alpha - \beta - 2 < \delta. \tag{P5}$$

In all our examples this last condition is satisfied since $\alpha \le \delta$. All along this section we shall assume that (P1), (P3), (P4) and (P5) hold.

Proposition 3.1. *The infimum*

$$\lambda_1(R) := \inf_{u \in X_R \setminus \{0\}} \frac{\int_0^R r^\alpha |u'(r)|^{\beta+2} \, dr}{\int_0^R r^\delta |u(r)|^{\beta+2} \, dr}$$

is achieved by a function $\varphi_1 \in X_R$*, and* $\lambda_1(R)$ *in an eigenvalue of problem* (EP)*. Moreover,* $\varphi_1(r) \neq 0$ *for* $r \in [0, R)$ [*so we can choose one* $\varphi_1 > 0$ *in* $[0, R)$], $\lambda_1(R)$ *is the smallest eigenvalue of* (EP) *and it is simple.*

Proof. The first part of the statement is clear in view of the compact imbedding. Suppose by contradiction that φ_1 vanishes at $r_0 \in [0, R)$. If $r_0 = 0$, it follows that $\varphi_1 \equiv 0$. So suppose that $r_0 > 0$. Since the function $|\varphi_1|$ also minimizes the above ratio, it follows that $|\varphi_1|$ is a solution of (EP), and so by Proposition 2.2 it is C^2 in $(0, R)$. Thus $|\varphi_1|(r_0) = |\varphi_1'|(r_0) = 0$ implies $\varphi_1 \equiv 0$, a contradiction. The fact that $\lambda_1(R)$ is the smallest eigenvalue of (EP) follows readily from

$$\lambda_1(R) \int_0^R r^\delta |u(r)|^{\beta+2}\, dr \leq \int_0^R r^\alpha |u'(r)|^{\beta+2}\, dr.$$

Finally, suppose that φ_1 and φ_2 are two eigenfunctions corresponding to $\lambda_1(R)$. We know that $\varphi_1'(R)$ and $\varphi_2'(R)$ are different from zero. So there exists a constant $a \neq 0$ such that $\varphi_2'(R) = a\varphi_1'(R)$. Thus, by the existence and uniqueness theorem for the initial value problem, we have $\varphi_2(r) = a\varphi_1(r)$ for $r \in [0, R]$. □

For later reference, we state the next lemma, whose proof follows easily by Hölder's inequality.

Lemma 3.1. *Assume that* (P1), (P3) *and* (P5) *hold. Then*

$$L_\gamma^{q^*} \hookrightarrow L_\delta^{\beta+2}.$$

Lemma 3.2. *Let* $f \in L^\infty[0, R]$ *be nonnegative and not identically zero. Let* $u \in C^1[0, R] \cap C^2(0, R)$ *be a solution of*

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = f(r) & \text{in } (0, R), \\ u'(0) = u(R) = 0. \end{cases}$$

Then $u'(r) \leq 0$ *for all* $r \in [0, R]$ *and* $u > 0$ *in* $[0, R)$.

Proof. Integrate the equation from 0 to r:

$$-r^\alpha |u'(r)|^\beta u'(r) = \int_0^r f(\theta)\, d\theta =: \chi(r).$$

Observe that $\chi(r) \geq 0$ and $\chi(R) > 0$. Then

$$u'(r) = \left(-\frac{\chi(r)}{r^\alpha}\right) \left|\frac{-\chi(r)}{r^\alpha}\right|^{-\beta/(\beta+1)}.$$

This implies $u'(r) \leq 0$. Since $u(R) = 0$ and $u'(R) < 0$, it follows that $u(r) > 0$ for r near R, and consequently $u(r) > 0$ for all $r > 0$. □

LEMMA 3.3. *Let* $f, g \in L^\infty([0,T],\mathbb{R})$. *Assume that* $g \geq 0$ *with* $g \not\equiv 0$ *and* $f \geq g$ *with* $f \not\equiv g$. *Then* $u'(R) < v'(R)$ *and* $u(r) > v(r)$ *for* $0 \leq r \leq R$, *where* u *and* v *are solutions, respectively, of the equations*

$$-(r^\alpha |u'(r)|^\beta u'(r))' = f(r), \tag{3.1}$$

$$-(r^\alpha |v'(r)|^\beta v'(r))' = g(r), \tag{3.2}$$

subject to the boundary conditions $u'(0) = v'(0) = u(R) = v(R) = 0$.

PROOF. Subtracting (3.2) from (3.1) we can write the difference as

$$-\bigg(r^\alpha \int_0^1 \frac{d}{d\theta}\{|v'(\theta)+\theta(u'(\theta)-v'(\theta))|^\beta(v'(\theta)+\theta(u'(\theta)-v'(\theta)))\}d\theta\bigg)' = f(r)-g(r).$$

This can be rewritten as

$$-(r^\alpha a(r)(u(r)-v(r))')' = f(r) - g(r). \tag{3.3}$$

where

$$a(r) = (\beta+1)\int_0^1 |v'(\theta) + \theta(u'(\theta) - v'(\theta))|^\beta \, d\theta.$$

From Lemma 3.2 it follows that $a(r) > 0$ for r near R. So, integrating (3.3) we find

$$(u(r) - v(r))' = -\frac{1}{r^\alpha a(r)} \int_0^r (f(\theta) - g(\theta))\, d\theta \quad \text{for } r \text{ near } R.$$

This implies $u'(R) < v'(R)$ and $u(r) > v(r)$ for r near R. For the other values of r we integrate (3.3) and obtain

$$-r^\alpha a(r)(u(r)-v(r))' \geq 0 \quad \text{for } 0 \leq r < R.$$

This implies that $u'(r) \leq v'(r)$ if $a(r) \neq 0$. On the other hand, from the expression of a the statement $a(r) = 0$ implies $u'(r) = v'(r)$. So in any case $u'(r) \leq v'(r)$ for all $r \in [0,R]$. Consequently $u(r) > v(r)$ for $r \in [0,R)$. □

THEOREM 3.1. *Assume that* (P1), (P3), (P4) *and* (P5) *hold. Then the problem*

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = \lambda r^\delta |u(r)|^\beta u(r) + r^\gamma |u(r)|^{q^*-2} u(r) & \text{in } (0,R), \\ u'(0) = u(R) = 0, \\ u(r) > 0 \quad \text{for } r \in [0,R), \end{cases}$$

has no solution if $\lambda \geq \lambda_1(R)$.

PROOF. Suppose that the above problem has a solution u. Since $u'(R) < 0$, the set

$$\Lambda = \{\varepsilon > 0 : u(r) > \varepsilon\varphi_1(r),\ \forall r \in (0,R)\}$$

is nonempty. Let $\varepsilon_0 = \sup \Lambda$ and

$$g(r) := \lambda_1(R) r^\delta \varepsilon_0^{\beta+1} \varphi_1^{(\beta+1)}(r), \quad f(r) := \lambda r^\delta |u(r)|^\beta u(r) + r^\gamma |u(r)|^{q^*-2} u(r).$$

We see that f and g satisfy the hypotheses of Lemma 3.3. Thus $u'(R) < \varepsilon_0 \varphi'(R)$. Let a be such that $u'(R) = a\varphi_1'(R)$. It follows that $a > \varepsilon_0$ and this implies that there exists $\eta > 0$ such that

$$u(r) > \frac{a+\varepsilon_0}{2}\varphi_1(r) \quad \text{for } r \in [R-\eta, R). \tag{3.4}$$

On the other hand, by using Lemma 3.3 again we see that

$$u(r) \geq b\varepsilon_0\varphi_1(r) \quad \text{for } r \in [0, R-\eta] \tag{3.5}$$

with some $b > 1$. The estimates (3.4) and (3.5) give

$$u(r) \geq \min\left(b\varepsilon_0, \frac{a+\varepsilon_0}{2}\right)\varphi_1(r).$$

Since this minimum is strictly larger than ε_0, we come to a contradiction. This completes the proof. □

4. Nonexistence results

The aim of this section is to prove some nonexistence results for problem (Q_λ). Throughout this section we assume that (P1), (P3), (P4) and (P5) hold. The proof will be based on a general identity of the Pokhozhaev–Pucci–Serrin type for solutions of the equation

$$-(r^\alpha |u'|^\beta u)' = f(r,u) \quad \text{in } (0,\infty) \tag{Q}$$

subject to the condition $u'(0) = 0$, where α and β are as in Section 1 and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0,s) = 0$ for all $s \in \mathbb{R}$. General variational identities were proved in [PO3], [PS1] and [PS2]. For completeness sake we shall derive here the most suitable one for our purposes by following the original idea of Rellich [RE].

Proposition 4.1. *Let $a, b \in C^1[0,\infty)$. Let $u \in C^2(0,\infty) \cap C^1[0,\infty)$ be a solution of* (Q)*. Then for $R > 0$ we have*

$$\begin{aligned}\left[-r^\alpha u'|u'|^\beta\left(au + \frac{\beta+1}{\beta+2}bu'\right)\right]_{r=R} &+ \int_0^R r^\alpha a'uu'|u'|^\beta \\ &+ \int_0^R r^\alpha\left(a + \frac{\beta+1}{\beta+2}b' - \frac{\alpha}{\beta+2}\,\frac{b}{r}\right)|u'|^{\beta+2} \\ &= [bF(r,u)]_{r=R} + \int_0^R (auf(r,u) - bD_1F(r,u) - b'F(r,u)),\end{aligned} \tag{4.1}$$

where $F(r,t) = \int_0^t f(r,s)\,ds$, and D_1 denotes the derivative with respect to the first variable.

Proof. After multiplying the equation (Q) by $au + bu'$ and integrating by parts on (ε, R) with $\varepsilon > 0$, the left hand side of the equation becomes

$$\begin{aligned} I_L &:= -\int_\varepsilon^R (r^\alpha u'|u'|^\beta)'(au + bu') \\ &= [-r^\alpha u'|u'|^\beta(au + bu')]_{r=\varepsilon}^R + \int_\varepsilon^R r^\alpha u'|u'|^\beta(a'u + au' + b'u' + bu''). \end{aligned}$$

The integral involving u'' can be treated as follows:

$$\begin{aligned} \int_\varepsilon^R r^\alpha u'|u'|^\beta bu'' &= \int_\varepsilon^R \frac{r^\alpha b}{\beta+2}(|u'|^{\beta+2})' \\ &= \left[\frac{r^\alpha b}{\beta+2}|u'|^{\beta+2}\right]_{r=\varepsilon}^R - \int_\varepsilon^R \frac{(r^\alpha b)'}{\beta+2}|u'|^{\beta+2}. \end{aligned} \tag{4.2}$$

Now, since $\alpha \geq 0$, $u' \in C^0[0,\infty)$ and $u'(0) = 0$, we see that the boundary terms above computed at $r = \varepsilon$ are of the order of ε. Using (4.2) in $I_{\rm L}$ we get

$$\begin{aligned} I_{\rm L} = {} & \left[-r^{-\alpha}u'|u'|^\beta\left(au + \frac{\beta+1}{\beta+2}bu'\right)\right]_{r=R} + \int_\varepsilon^R r^\alpha a'uu'|u'|^\beta \\ & + \int_\varepsilon^R r^\alpha\left(a + \frac{\beta+1}{\beta+2}b' - \frac{\alpha}{\beta+2}\,\frac{b}{r}\right)|u'|^{\beta+2} + O(\varepsilon). \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ we get the left side of (4.1).

Now we examine the right hand side of the equation after multiplication by $au + bu'$. We have

$$I_{\rm R} := \int_\varepsilon^R f(r,u)(au + bu') = \int_\varepsilon^R f(r,u)au + \int_\varepsilon^R f(r,u)bu'.$$

Since

$$\frac{d}{dr}F(r,u(r)) = D_1F(r,u(r)) + f(r,u(r))u'(r),$$

the last integral in the expression of $I_{\rm R}$ becomes

$$\begin{aligned} \int_\varepsilon^R f(r,u)bu' &= \int_\varepsilon^R \frac{d}{dr}F(r,u(r)).b - \int_\varepsilon^R bD_1F(r,u(r)) \\ &= [bF(r,u)]_{r=\varepsilon}^R - \int (b'F(r,u) + bD_1F(r,u)). \end{aligned} \tag{4.3}$$

Next, since $f(0,s) = 0$ for all $s \in \mathbb{R}$, we have

$$\lim_{r\to 0} F(r,u(r)) = \lim_{r\to 0}\int_0^{u(r)} f(r,s)\,ds = \int_0^{u(0)} f(0,s)\,ds = 0. \tag{4.4}$$

Replacing (4.3) in the expression of $I_{\rm R}$ and passing to the limit as $\varepsilon \to 0$ we get the right side of the identity (4.1). □

As a first consequence of identity (4.1) we prove a result on the nonexistence of solutions of (Q_λ) for $\lambda \leq 0$ and $q = q^*$.

Theorem 4.1. *Assume that* (P1), (P3), (P4) *and* (P5) *hold. If* $\lambda \le 0$ *and* $q = q^*$, *then* (Q_λ) *has no solution.*

Remark 4.1. The search of a solution for (Q_0) is done through a minimization with constraint. Namely,

$$\inf\left\{\int_0^R r^\alpha |u'(r)|^{\beta+2}\,dr : u \in X_R,\ \int_0^R r^\gamma |u(r)|^q\,dr = 1\right\}.$$

This infimum is precisely $S_0(q,R)$ defined in Section 1. We have two cases to consider:

(i) If $q < q^*$, then, in view of the compact imbedding $X_R \subset L^q_\gamma$, the above infimum is attained. Let $u \in X_R$ be the function that realizes it. So, u satisfies the Euler–Lagrange equation

$$\int_0^R r^\alpha |u'(r)|^\beta u'v' = \mu \int_0^R r^\gamma\,|u(r)|^{q-2}uv \quad \text{for any } v \in X_R,$$

where μ is the Lagrange multiplier. It is then easy to see that $w = \mu^{1/(q-\beta-2)}u$ is a (weak) solution of (Q_0).

(ii) If $q = q^*$ the above theorem states that there is no solution.

Proof of Theorem 4.1. Without loss of generality assume $R = 1$. Let u be a solution of (Q_λ). Using the identity (4.1) with $a = \mathrm{const}$, $b(r) = r$ and

$$f(r,u) = r^\delta u|u|^\beta + \lambda r^\gamma u|u|^{q-2},$$

we get

$$\begin{aligned}(4.5)\quad \left[-\frac{\beta+1}{\beta+2}|u'|^{\beta+2}\right]_{r=1} &+ \int_0^1 r^\alpha\left[a + \frac{\beta+1-\alpha}{\beta+2}\right]|u'|^{\beta+2} \\ &= \int_0^1 r^\delta\left[a - \frac{\delta+1}{\beta+2}\right]\lambda|u|^{\beta+2} + \int_0^1 r^\gamma\left[a - \frac{\gamma+1}{q}\right]|u|^q.\end{aligned}$$

Now we choose

$$a = \frac{\alpha-\beta-1}{\beta+2}.$$

This implies that the integral on the left side of (4.5) vanishes. Since $q = q^*$, the same holds true for the last integral in (4.5). Further with our choice of a, we see that the coefficient appearing in the first integrand of the right side of (4.5) becomes

$$a - \frac{\delta+1}{\beta+2} = \frac{\alpha-\beta-\delta-2}{\beta+2}.$$

Since $\lambda \le 0$, the right side of (4.5) is nonnegative, while, on the other hand, the left side is negative. Observe that by the theorem of the existence and uniqueness of solution for the initial value problem we must have $u'(1) \ne 0$. $\square$

We say that the parameters α, β, δ belong to the *critical range* (see [PS3], [M]) if

$$(\delta+1)(\beta+1)-(\alpha-\beta-1)(\beta+2)>0. \tag{P2}$$

In the sequel we will show that if the parameters α, β, δ belong to the critical range then there exists a $\lambda^*>0$, depending on α, β, δ, such that (Q_λ) has no solution if $\lambda \le \lambda^*$.

For the model operators presented in the introduction, their parameters being in the critical range implies certain values of the dimension N, which were called *critical dimensions* by Pucci and Serrin [PS3]. Indeed, if the parameters α, β and δ of the p-Laplacian are in the critical range, then $N<p^2$. In particular, in the case of the Laplacian, $N<4$. Since $\alpha-\beta-1>0$ in this case implies $N\ge 3$, we see that in the case of the Laplacian the only critical dimension is $N=3$, which was proved first in Brezis–Nirenberg [BN]. In the case of the p-Laplacian the critical dimensions are the integers N such that $p<N<p^2$. Finally, for the case of the k-Hessian, the critical dimensions are the integers in the range $2k<N<2k(k+1)$.

Theorem 4.2. *Assume that the parameters α, β, δ belong to the critical range with $\beta\ge 0$. Suppose that* (P1), (P3), (P4), (P5) *hold. Then there exists $\lambda^*>0$ such that (Q_λ) has no solution if $\lambda\le\lambda^*$.*

Proof. Let us use identity (4.1) with $R=1$ and

$$a=a_1+a_2r^m, \quad b=-r+r^{m+1}$$

and

$$f(r,u)=\lambda r^\delta u|u|^\beta + r^\gamma u|u|^{q-2},$$

where a_1, a_2 and m are constants which will be determined as we proceed.

Since $b(1)=0$ and $u(1)=0$, all the boundary terms vanish. Now we choose a_1 and a_2 such that the integral containing $|u'|^{\beta+2}$ vanishes:

$$a_1=-\frac{\alpha-\beta-1}{\beta+2}, \quad a_2=\frac{\alpha-(m+1)(\beta+1)}{\beta+2}.$$

Next we see that the integral on the right side of (4.1) is equal to

$$\begin{aligned}
&\lambda\int_0^1\left[a_1+\frac{\delta+1}{\beta+2}\right]r^\delta|u|^{\beta+2}+\lambda\int_0^1\left[a_2+\frac{\delta+m+1}{\beta+2}\right]r^{\delta+m}|u|^{\beta+2}\\
&+\int_0^1\left[a_1+\frac{\gamma+1}{q}\right]r^\gamma|u|^q+\int_0^1\left[a_2+\frac{\gamma+m+1}{q}\right]r^{\gamma+m}|u|^q\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=I_1+I_2+I_3+I_4.
\end{aligned}$$

Since $q=q^*$, the integral I_3 vanishes. $I_1>0$ in view of hypothesis (P5).

Our aim now is to choose m in such a way that I_2 and I_4 are ≤ 0 and the remaining integral I_5 (see below) on the left side is bounded from below by a positive multiple of I_1. If this is achieved we will have completed the proof. Let us work with I_5. First we observe from equation (Q_λ) that

$$-r^\alpha u'(r)|u'(r)|^\beta = \int_0^r [\lambda r^\delta u|u|^\beta + r^\gamma u|u|^{q-2}] > 0$$

for positive solutions of (Q_λ). Hence $u'(r) < 0$ for all $0 < r \le 1$. So, assuming $a_2 < 0$ (the choice of m will imply this) we obtain

$$I_5 := \int_0^1 r^\alpha a' u u' |u'|^\beta = m|a_2| \int_0^1 r^{\alpha+m-1} u|u'|^{\beta+1}.$$

Viewing to apply Proposition 1.0 we write the integral in I_5 as

$$\int_0^1 r^{\alpha+m-1}[(u^{(\beta+2)/(\beta+1)})']^{\beta+1} \ge c \int r^\delta (u^{(\beta+2)/(\beta+1)})^{\beta+1}, \tag{4.6}$$

and the estimate is correct if $m \le \delta - \alpha + \beta + 2$. We then choose

$$m = \delta - \alpha + \beta + 2$$

which is a positive number in view of the hypothesis of the theorem.

Now we check if with this choice of m, the coefficient a_2 is negative. That is,

$$\alpha - (\delta - \alpha + \beta + 3)(\beta + 1) < 0,$$

which is precisely the condition for the parameters α, β and δ to be in the critical range.

Finally, we observe that I_2 and I_4 are nonpositive, since the terms in brackets are nonpositive. This completes the proof. $\square$

5. First integrals and existence of ground states

In this section we introduce certain functionals that have some monotonicity properties when evaluated along the possible positive solutions of

$$\begin{cases} -(r^\alpha |u'|^\beta u')' = r^\gamma f(u) & \text{in } (0, R), \\ u'(0) = 0, \end{cases} \tag{5.1}$$

where $f : \mathbb{R} \to [0, \infty)$ is a C^1 function, and conditions (P1) and (P3) are assumed to hold. More precisely, we are interested in finding functionals

$$\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$$

such that

$$\varphi(r) := \Phi(r, u(r), u'(r)) \tag{5.2}$$

is a continuous monotonic function, where u is a positive solution of (5.1).

Under appropriate assumptions on the function f, the existence of such a Φ is assured. As we shall see timely, using such functionals we infer interesting qualitative properties of the set of positive solutions of (5.1). These properties are strictly related to earlier results of Atkinson and Peletier [AP1], [AP2].

We say that a functional Φ is a *first integral* if φ, as defined in (5.2), is constant. The use of a first integral for a certain equation will give us easily the Bliss function introduced in Section 1 (see Remark 5.1 below).

Our main result in this section is the following:

THEOREM 5.1. *Assume that conditions* (P1) *and* (P3) *hold. Suppose that*

$$f : \mathbb{R} \to \mathbb{R} \text{ is } C^1 \quad \text{and} \quad f(r) > 0 \quad \text{for } r > 0. \tag{F}$$

Let $u \in C^2(0,R) \cap C^1[0,R]$ *be a positive solution of* (5.1). *If* f *satisfies one of the following conditions:*

(i) $tf'(t) \le \varrho f(t)$, $t > 0$,
(ii) $tf'(t) \ge \varrho f(t)$, $t > 0$,
(iii) $tf'(t) = \varrho f(t)$, $t > 0$,

where

$$\varrho = q^* - 1 \quad [q^* \text{ is defined in } (1.1)], \tag{5.3}$$

then the function

$$\varphi(r) := -\sigma r^{-\theta} u(r)^{-(\sigma+1)} u'(r), \tag{5.4}$$

where

$$\theta = \frac{1+\gamma-\alpha}{\beta+1} \quad \text{and} \quad \sigma = \frac{2+\beta+\gamma-\alpha}{\alpha-\beta-1}, \tag{5.5}$$

is respectively (i) *nondecreasing,* (ii) *nonincreasing,* (iii) *constant.*

REMARK 5.1. Observe that (iii) holds if and only if $f(t) = K\,t^{q^*-1}$, $K > 0$. Hence the above result says that $\varphi(r) = \text{const}$ along the positive solution u of the corresponding equation (5.1). Since we can write $\varphi(r)$ as $r^{-\theta}(u^{-\sigma})'$, a simple integration in this case leads to an expression for u exactly as in (1.10). Observe that $\sigma = m$ and $\theta + 1 = n$, where m, n are defined in (1.11).

REMARK 5.2. The above choice of the parameters θ and σ may seem sort of mysterious. Of course, a motivation is the Bliss function itself! However, the choice of θ can be inferred from the requirement that $\varphi(r)$ has to be continuous at $r = 0$. Indeed, since $u(0) > 0$, the assertions on $\lim_{r\to 0}\varphi(r)$ are equivalent to assertions on $\lim_{r\to 0} r^{-\theta}u'(r)$ or $\lim_{r\to 0} r^{-\theta(\beta+1)}|u'(r)|^{\beta+1}$, that is,

$$\lim_{r\to 0} r^{-\theta(\beta+1)}|u'(r)|^{\beta+1} = \lim_{r\to 0} r^{-\theta(\beta+1)-\alpha}\int_0^r r^{\gamma} f(u(r))\,dr.$$

Since we wish to have $\varphi(0) \neq 0$, we see that one ought to have $\theta(\beta+1)+\alpha > 0$. So, using L'Hôspital's rule, we find that the above limit is equal to

$$c \lim_{r\to\infty} \frac{r^\gamma f(u(r))}{r^{\theta(\beta+1)+\alpha-1}},$$

which implies that $\gamma = \theta(\beta+1)+\alpha-1$. This determines the value of θ.

Remark 5.3. Condition (i) includes the case where f is supercritical, i.e. $\lim_{t\to\infty} f(t)/t^{q^*-1} = \infty$. Likewise, (ii) includes the subcritical case, namely when such a limit is zero.

Proof of Theorem 5.1. First we observe that if a solution of (5.1) is such that $u(0) > 0$, then $u'(r) < 0$ as long as $u(r)$ is still positive. [If u vanishes at a certain $r_0 > 0$, then u' could be still negative for a while.] All this can be seen by integrating (5.1) from 0 to r to obtain

$$-r^\alpha |u'(r)|^\beta u'(r) = \int_0^r s^\gamma f(u(s))\,ds,$$

and using hypothesis (F). Differentiating $\varphi(r)$ and replacing u'' that appears in this derivative by its expression obtained from (3.1), we obtain

$$\begin{aligned}\varphi'(r) = \sigma(\sigma+1) r^{-\theta-\alpha-1} u(r)^{-\sigma-2} |u'(r)|^{-\beta} \bigg\{ & r^{\alpha+1}|u(r)'|^{\beta+2} \\ & - \frac{1}{\sigma+1}\left(\theta + \frac{\alpha}{\beta+1}\right) r^\alpha u(r)|u'(r)|^{\beta+1} \\ & + \frac{1}{(\sigma+1)(\beta+1)} r^{\gamma+1} u(r) f(u(r)) \bigg\}.\end{aligned} \tag{5.6}$$

Assume now that the following lemma has been proved.

Lemma 5.1. *Under the hypotheses of Theorem* 5.1*, the function* ψ *defined*

$$\psi(r) := \{\textit{expression in braces in } (5.6)\}$$

is such that $\psi'(r) \geq 0$ *if* (i) *holds, while* $\psi'(r) \leq 0$ *if* (ii) *holds. In particular,* $\psi'(r) \equiv 0$ *if* $f(t) = k|t|^{q^*-2}t$*, where* k *is some positive constant.*

Since $\psi(0) = 0$, from the above lemma it follows that

$$\begin{aligned} &\psi(r) \geq 0,\ \forall r > 0, \quad \text{if (i) holds,} \\ &\psi(r) \leq 0,\ \forall r < 0, \quad \text{if (ii) holds.}\end{aligned} \tag{5.7}$$

Using in (5.6) the information given by (5.7) we conclude the proof of Theorem 5.1. $\square$

Proof of Lemma 5.1. Using the identity (4.1) with

$$a = \frac{\alpha-\beta-1}{\beta+2}, \quad b(r) = r, \quad f(r,u) = r^\gamma f(u),$$

we obtain

$$(5.8)\qquad \psi(r) = -\frac{\alpha-\beta-1}{\beta+1}\int_0^r s^\gamma f(u(s))\,ds - \frac{\beta+2}{\beta+1}\int_0^r s^{\gamma+1} f(u(s))u'(s)\,ds + \frac{\alpha-\beta-1}{(\beta+1)(\gamma+1)} r^{\gamma+1}u(r)f(u(r)).$$

Differentiating (5.8), we get

$$\psi'(r) = \frac{\alpha-\beta-1}{(\beta+1)(\gamma+1)} r^{\gamma+1}u'(r)\bigg\{u(r)f'(u(r)) + \bigg(1-\frac{(\gamma+1)(\beta+2)}{\alpha-\beta-1}\bigg)f(u(r))\bigg\}.$$

By taking into account that $u'(r) < 0$ for $r \in (0,R)$ the result of the lemma follows. □

Remark 5.4. By L'Hôspital's rule we have

$$(5.9)\qquad \varphi(0) = \lim_{r\to 0}\varphi(r) = cu(0)^{-(\gamma+1)/(\alpha-\beta-1)} f(u(0))^{1/(\beta+1)} =: U_0 > 0,$$

where

$$(5.10)\qquad c = \frac{2+\beta+\gamma-\alpha}{(\alpha-\beta-1)(\gamma+1)^{1/(\beta+1)}}.$$

As a consequence, it follows from (5.6) and (5.7) that for all $r \in [0,R]$ we have

$$(5.11)\qquad \begin{cases} \varphi(r) \ge U_0 & \text{if (i) holds,}\\ \varphi(r) \le U_0 & \text{if (ii) holds,}\\ \varphi(r) = U_0 & \text{if (iii) holds.}\end{cases}$$

Remark 5.5. From the proof of Lemma 5.1 we see that if there is strict inequality in (i) or (ii), then the corresponding statement for ψ' holds with strict inequality also. In this case, (5.7) and (5.11) hold true with strict inequality.

Lemma 5.2. *Let $u_0 > 0$. Then the problem*

$$(5.12)\qquad \begin{cases} -(r^\alpha|v'(r)|^\beta v'(r))' = r^\gamma|v(r)|^{q^*-2}v(r) & \textit{in } (0,\infty),\\ v(0) = u_0, \quad v'(0) = 0,\end{cases}$$

has a unique positive solution given by

$$(5.13)\qquad u_*(r) = \frac{1}{\big(u_0^{-\sigma} + \frac{1}{\theta+1}U_0 r^{\theta+1}\big)^{1/\sigma}},$$

where σ and θ are defined in (5.5) *and U_0 is given by* (5.9) *with $u(0)$ replaced by u_0.*

Proof. By Theorem 5.1, if u is a solution of (5.12) then

$$(5.14)\qquad r^{-\theta}[u(r)^{-\sigma}]' = \text{const},$$

where this constant is given by (5.9)–(5.10). Integrating (5.14) we get (5.13). □

Lemma 5.3. *Suppose that f satisfies* (F) *and* (ii). *Let u be a solution of* (5.1) *with $u(0) = u_0 > 0$. Then*

$$u(r) \geq u_*(r) \quad \textit{for } r \in [0, R],$$

where u_* is given in Lemma 5.2.

Proof. From (5.11) we have

$$r^{-\theta}(u(r)^{-\sigma})' \leq r^{-\theta}(u_*(r)^{-\sigma})'$$

which gives by integration

$$u(r)^{-\sigma} - u_0^{-\sigma} \leq u_*(r)^{-\sigma} - u_0^{-\sigma},$$

and the result follows. □

Theorem 5.2. *Suppose that f satisfies* (F) *and* (ii). *Then for every $u_0 > 0$, there exists a positive solution of*

$$\begin{cases} -(r^\alpha |u'|^\beta u')' = r^\gamma f(u(r)) & \textit{in } (0, \infty), \\ u(0) = u_0, \quad u'(0) = 0, \end{cases} \tag{5.15}$$

defined on $(0, \infty)$. Moreover, for $r \in (0, \infty)$ we have

$$u_*(r) \leq u(r) \leq c r^{-(\beta+2)\sigma}. \tag{5.16}$$

where $c > 0$.

Proof. By well known results, there exists $R > 0$ such that (5.15) has a unique positive solution defined in $[0, R)$. As seen before, such a solution is decreasing in $[0, R)$. By Lemma 5.3 it follows that $u(r) \geq u_*(r)$ for $r \in (0, R)$. In order to conclude that this solution is defined on the whole half-line it remains to prove that u' is bounded in $[0, R)$. [Indeed, this fact together with the boundedness of u in this interval gives that u can be continued for $r > R$.] From the identity

$$r^\alpha |u'(r)|^{\beta+1} = \int_0^r s^\gamma f(u(s))\, ds,$$

and the assumption $f' \geq 0$, we get

$$r^\alpha |u'(r)|^{\beta+1} \leq \frac{1}{\gamma+1} r^{\gamma+1} f(u_0),$$

or

$$|u'(r)| \leq c r^\theta. \tag{5.17}$$

The estimate from above appearing in (5.16) follows directly from Lemma 5.3, while the estimate from below follows by integrating the inequality $-(r^\alpha |u'|^\beta u')' \geq 0$. □

6. Liouville–Gelfand type problem for quasilinear equations

In this section we shall study the Liouville–Gelfand problem associated with the operator L. As is well known, the *Liouville–Gelfand problem* is the following boundary value problem:

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0), \\ u > 0 & \text{in } B_R(0), \end{cases} \tag{6.1}$$

where $B_R(0)$ is the open ball of radius R centered at the origin in $\mathbb{R}^N$ with $N \geq 1$ and λ is a real parameter. The following facts about this problem can easily be established:

(i) If $\lambda \leq 0$, then problem (6.1) has no solution.
(ii) There exists $\lambda^* > 0$ such that (6.1) has no solution if $\lambda > \lambda^*$.
(iii) All possible solutions of (6.1) are radially symmetric. This follows from the well known results of Gidas–Ni–Nirenberg [GNN].

As a consequence of (iii), (6.1) becomes an O.D.E. problem:

$$\begin{cases} -u'' - \frac{N-1}{r} u' = \lambda e^u & \text{in } (0, R), \\ u'(0) = u(R) = 0, \\ u > 0 & \text{in } [0, R). \end{cases} \tag{6.2}$$

This problem was first studied by Liouville [L] in the case of $N = 1$ and an explicit solution was found. Later, Bratu [BR], among other things, studied the case $N = 2$. In that case, problem (6.2) has two solutions if $\lambda < 2/R^2$ and in fact Bratu obtained explicit expressions for these solutions. All these results will appear here as an outcome of our Theorem 6.1.

The problem was revived by Gelfand [G], who discussed also higher dimensions $N \geq 3$. Further work appeared in Joseph–Lundgren [JL], Crandall–Rabinowitz [CR] and Mignot–Murat–Puel [MMP].

In this section we consider the problem

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = \lambda r^\gamma e^{u(r)} & \text{in } (0, R), \\ u'(0) = u(R) = 0, \\ u > 0 & \text{in } [0, R), \end{cases} \tag{6.3}$$

under the following assumption on the differential operator L:

$$\alpha - \beta - 1 = 0, \quad \beta > -1 \quad \text{and} \quad \gamma > -1. \tag{P6}$$

Observe that (6.3) includes (6.2) in the case of dimension $N = 2$, by taking $\alpha = \gamma = 1$ and $\beta = 0$. Also (6.3) is the Liouville–Gelfand problem for the p-Laplacian in $\mathbb{R}^p$, taking $\alpha = \gamma = p - 1$ and $\beta = p - 2$.

Theorem 6.1. *Assume that* (P6) *holds. Then there exists* $\lambda^* > 0$ *such that*

(i) *problem* (6.3) *has exactly two solutions if* $0 < \lambda < \lambda^*$,
(ii) *problem* (6.3) *has a unique solution if* $\lambda = \lambda^*$,
(iii) *problem* (6.3) *has no solution if* $\lambda \le 0$ *or* $\lambda > \lambda^*$.

This theorem will be proved later as a corollary to the following result, which in turn will be proved using the same technique employed in the previous section: the idea of a first integral. The result refers to the more general equation

$$\begin{cases} -(r^\alpha |u'(r)|^\beta u'(r))' = \lambda r^\gamma f(u(r)) & \text{in } (0,R), \\ u'(0) = 0. \end{cases} \tag{6.4}$$

Theorem 6.2. *Assume that* (P6) *holds. Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies condition* (F) *of the previous section and one of the following assumptions holds:*

(i) $f(t) \ge f'(t)$ *for* $t > 0$,
(ii) $f(t) \le f'(t)$ *for* $t > 0$,
(iii) $f(t) = ke^t$ *for some constant* $k > 0$.

Then the function φ *defined by*

$$\varphi(r) := r^{(\alpha-\gamma-1)/\alpha}(e^{-u(r)/(\alpha+1)})', \quad r > 0, \tag{6.5}$$

where u *is a positive solution of* (6.4), *is respectively* (i) *nondecreasing,* (ii) *nonincreasing, or* (iii) *constant.*

Remark 6.1. Theorem 6.2 cannot be extended to the case when $R = \infty$. This is due to the fact that there are no positive nonconstant solutions of

$$-(r^\alpha |u'(r)|^\beta u'(r))' \ge 0, \tag{6.6}$$

with $u'(0) = 0$, defined on the whole half-line. Here α, β satisfy condition (P6). In the case of the Laplacian, this is just the statement that there are no positive superharmonic functions defined on the whole of $\mathbb{R}^2$.

Proof of Theorem 6.2. We first prove that φ is continuous at $r = 0$ and

$$\varphi(0) = \frac{\lambda^{1/\alpha}}{(\alpha+1)(\gamma+1)^{1/\alpha}} e^{-u(0)/(\alpha+1)} f(u(0))^{1/\alpha}, \tag{6.7}$$

where $u(0)$ is the initial value of the positive solution u of (6.4). Indeed, since $u > 0$ in $[0,R)$, it follows that $u' < 0$ and $\varphi(r)$ can be written as

$$\varphi(r) = \frac{1}{\alpha+1} r^{(\alpha-\gamma-1)/\alpha} e^{-u(r)/(\alpha+1)} |u'(r)|. \tag{6.8}$$

From the integration of (6.4) we get

$$r^{(\alpha-\gamma-1)/\alpha}|u'(r)| = \lambda^{1/\alpha} r^{-(\gamma+1)/\alpha} \bigg(\int_0^r s^\gamma f(u(s))\, ds \bigg)^{1/\alpha}.$$

Applying L'Hôspital's rule to

$$\frac{\int_0^r s^\gamma f(u(s))\,ds}{r^{\gamma+1}}$$

we conclude that

$$\lim_{r\to 0} r^{(\alpha-\gamma-1)/\alpha}|u'(r)| = \lambda^{1/\alpha}\frac{1}{(\gamma+1)^{1/\alpha}} f(u(0))^{1/\alpha},$$

which together with (6.8) gives the result. Since we wish to have monotonicity properties of φ, we compute its derivative:

$$\varphi'(r) = \frac{r^{\sigma-1}}{(\alpha+1)^2} e^{-u(r)/(\alpha+1)}\{-\sigma(\alpha+1)u' + r|u'|^2 - (\alpha+1)ru''\} \tag{6.9}$$

where $\sigma = (\alpha-\gamma-1)/\alpha$. Using equation (6.4) we get

$$u''(r) = \frac{\alpha r^{\alpha-1}|u'(r)|^\alpha - \lambda r^\gamma f(u(r))}{\alpha r^\alpha |u'(r)|^{\alpha-1}}, \tag{6.10}$$

which replaced in (6.9) gives

$$\varphi'(r) = \frac{r^{\sigma-\alpha-1}}{\alpha(\alpha+1)^2}\cdot\frac{e^{-u(r)/(\alpha+1)}}{|u'(r)|^{\alpha-1}}\Psi(r),$$

where

$$\Psi(r) = -(\alpha+1)(\gamma+1)r^\alpha|u'(r)|^\alpha + \alpha r^{\alpha+1}|u'(r)|^{\alpha+1} + \lambda(\alpha+1)r^{\gamma+1}f(u(r)).$$

Now in order to show that φ is monotone it suffices to show that Ψ itself is monotone since $\Psi(0)=0$. For that, we differentiate the function Ψ, in the expression obtained we replace u'' using (6.10), and we come to

$$\Psi'(r) = \lambda(\alpha+1)r^{\gamma+1}|u'(r)|(f(u(r)) - f'(u(r))).$$

From this last expression, the result follows using the corresponding assumptions (i), (ii) or (iii). $\square$

PROOF OF THEOREM 6.1. We apply Theorem 6.2(iii) with $f(t) = e^t$ and obtain a necessary condition for the solvability of (6.3), namely: all possible solutions of (6.3) should satisfy $\varphi(r) = \varphi(0)$ for all $r \in [0,R)$. Hence

$$r^\sigma(e^{-u(r)/(\alpha+1)})' = \varphi(0) > 0,$$

which gives by integration

$$e^{-u(r)/(\alpha+1)} - e^{-u(0)/(\alpha+1)} = \frac{1}{-\sigma+1}r^{-\sigma+1}\varphi(0) \quad \text{for all } r\in[0,R]. \tag{6.11}$$

Since $u(R)=0$ and $\varphi(0)$ is given by (6.7) we obtain

$$1 - e^{-u(0)/(\alpha+1)} = \frac{\lambda^{1/\alpha}R^{-\sigma+1}}{(-\sigma+1)(\alpha+1)(\gamma+1)^{1/\alpha}} e^{-u(0)/(1+\alpha)}e^{u(0)\alpha}. \tag{6.12}$$

Putting

$$x = e^{u(0)/(\alpha+1)} \quad \text{and} \quad k = \frac{\lambda^{1/\alpha} R^{-\sigma+1}}{(-\sigma+1)(\alpha+1)(\gamma+1)^{1/\alpha}}$$

we can write (6.12) as

$$H(x) := kx^{(\alpha+1)/\alpha} - x + 1 = 0. \tag{6.13}$$

It is easily seen that H is a strictly convex function for $x \geq 0$. Since $H(0) = 1$ and $\lim_{x\to\infty} H(x) = +\infty$, we conclude that (6.13) cannot hold if the minimum of H is positive. The minimum of H occurs at $x = k^{-\alpha}(\alpha/(\alpha+1))^{\alpha}$ and the value of the minimum is $1 - k^{-\alpha}\alpha^{\alpha}/(\alpha+1)^{\alpha+1}$. This minimum will be positive if

$$\lambda > \frac{(\gamma+1)^{\alpha+1}}{(\alpha+1)R^{\gamma+1}} =: \lambda^*. \tag{6.14}$$

So the nonexistence statement will be proved when we show that there is no solution of (6.3) if $\lambda \leq 0$. But this is immediate since in this case one has $u' \geq 0$, which is not compatible with $u(R) = 0$ and $u(0) > 0$.

Now if $\lambda = \lambda^*$, then $H(x) = 0$ has only one solution, say x_0. It follows that if a solution u of (6.3) when $\lambda = \lambda^*$ exists, then $u(0)$ is necessarily given by $e^{u(0)/(\alpha+1)} = x_0$. Consequently, the corresponding value of $\varphi(0)$ is given by (6.7) with $f(t) = e^t$:

$$\varphi(0) = \frac{(\lambda^*)^{1/\alpha} x_0^{1/\alpha}}{(\alpha+1)(\gamma+1)^{1/\alpha}}. \tag{6.15}$$

And then the solution u of (6.3) with $\lambda = \lambda^*$ is obtained from (6.12), that is,

$$u(r) = -(\alpha+1)\ln\left\{x_0^{-1} + \frac{\alpha}{\gamma+1} r^{(\gamma+1)/\alpha}\varphi(0)\right\}, \tag{6.16}$$

where $\varphi(0)$ is given by (6.15). We complete the proof of part (ii) of Theorem 6.1 by verifying directly that u as given by (6.16) is a solution of (6.3) with $\lambda = \lambda^*$.

Finally, if $0 < \lambda < \lambda^*$, then $H(x) = 0$ has two solutions x_1 and x_2. As in the previous case we can prove that (6.3) has two solutions

$$u_i(r) = -(\alpha+1)\ln\left\{x_i^{-1} + \frac{\alpha}{\gamma+1} r^{(\gamma+1)/\alpha}\varphi_i(0)\right\}, \quad i = 1, 2, \tag{6.17}$$

where $\varphi_i(0)$ is given by

$$\varphi_i(0) = \frac{\lambda^{1/\alpha} x_i^{1/\alpha}}{(\alpha+1)(\gamma+1)^{1/\alpha}}. \tag{6.18}$$

□

Special case: $\alpha = \gamma = 1$ and $\beta = 0$. In this case

$$H(x) = kx^2 - x + 1, \quad k = \lambda R^2/8,$$

which gives $\lambda^* = 2/R^2$. So when $\lambda = \lambda^*$ we obtain $x_0 = 2$ and then $u(0) = 2\ln 2$. So $\varphi(0) = (\ln 2)/R^2$, and

$$u(r) = -2\ln\left\{\frac{1}{2} + \frac{2\ln 2}{R^2}r^2\right\}.$$

If $0 < \lambda < 2/R^2$, then $H(x) = 0$ has 2 solutions that can be written explicitly using the expressions (6.18)–(6.19). Those are Bratu's solutions of problem (6.2) [BR]. If $L = -(r^{N-1}|u'|^{p-2}u')'$, $1 < p = N$ (i.e. the p-Laplacian operator in radial coordinates) the solutions have a similar expression (see (6.16)). In this case, from the results of [KPA] it follows that positive solutions of the problem $-\mathrm{div}(|Du|^{p-2}Du) = e^u$ in B_R, $u = 0$ on ∂B_R are spherically symmetric. Our result gives in this case the complete description of the bifurcation diagram of the problem.

7. Existence of solutions for (Q_λ)

In this section we shall prove the following results:

THEOREM 7.1. *Assume that* (P1)–(P5) *hold and* $q = q^*$. *Then there exists* $\lambda^{**} > 0$ *such that* (Q_λ) *has a solution for* $\lambda^{**} < \lambda < \lambda_1$.

THEOREM 7.2. *Assume that* (P1), (P3), (P4), (P5) *hold and* $q = q^*$. *If*

$$(\delta+1)(\beta+1) - (\alpha-\beta-1)(\beta+2) \le 0,$$

then problem (Q_λ) *has a solution for all* $0 < \lambda < \lambda_1(R)$.

As in Remark 4.1 we use minimization to solve (Q_λ) for $q = q^*$.

For each $u \in X_R$ with $0 < R < \infty$, $q \le q^*$ and $\lambda \ge 0$ we define

$$S_\lambda(u; q, R) = \frac{\int_0^R r^\alpha |u'(r)|^{\beta+2} - \lambda \int_0^R r^\delta |u|^{\beta+2}}{(\int_0^R r^\gamma |u|^q)^{(\beta+2)/q}} \tag{7.1}$$

and

$$S_\lambda(q, R) = \inf\{S_\lambda(u; q, R) : u \in X_R \backslash \{0\}\}. \tag{7.2}$$

Also as in Remark 4.1, we see that $S_\lambda(q, R)$ for $q < q^*$ is achieved for all $\lambda \in [0, \lambda_1)$. This is a consequence of the compact imbedding $X_R \subset L^q_\gamma$, which by the way implies $S_\lambda(q, R) > 0$.

Recall that $S_0(q^*, R) = S$ is related to the best Sobolev constant of the imbedding $X_R \hookrightarrow L^{q^*}_\gamma$ (see Remark 1.2). In general, $S_\lambda(q^*, R)$ is not attained. Indeed, we have seen before that S is never attained for $0 < R < \infty$. However, one has the result of Proposition 7.2 below.

Proposition 7.1. *Under the above assumptions, the following holds:*

(i) *For fixed* R, $S_\lambda(q^*, R)$ *is a nonincreasing concave function of* λ *in* $[0, \lambda_1]$.
(ii) $S := S_0(q^*, R) > 0$.
(iii) $S_\lambda(q^*, R) \le C(\lambda_1 - \lambda)$ *with*

$$C = \frac{\int_0^R r^\delta |\varphi_1|^{\beta+2}\,dr}{[\int_0^R r^\gamma |\varphi_1|^{q^*}]^{(\beta+2)/q^*}},$$

where φ_1 *is the first eigenfunction introduced in Section* 3.
(iv) $S_\lambda(q^*, R) > 0$ *for* $\lambda \in [0, \lambda_1)$.

Proof. (i) Observe that for each $\lambda \in [0, \lambda_1]$ we have

$$S_\lambda(q^*, R) = \inf\bigg\{\int_0^R (r^\alpha |u'|^{\beta+2} - \lambda r^\delta |u|^{\beta+2}) : u \in X_R,\ \int_0^R r^\gamma |u|^{q^*} = 1\bigg\}.$$

From the definition of λ_1, it follows that

$$\xi(\lambda) := \int_0^R (r^\alpha |u'|^{\beta+2} - \lambda r^\delta |u|^{\beta+2}) \le 0.$$

Since ξ, as a function of λ, is nonincreasing and affine, the assertion (i) follows readily from the above expression of $S_\lambda(q^*, R)$.

(ii) This has already been established in Proposition 1.4.

(iii) Take $u = \varphi_1$ in the definition of $S_\lambda(q^*, R)$.

(iv) From the concavity of $S_\lambda(q^*, R)$ in $[0, \lambda_1]$ and the fact that $S_{\lambda_1}(q^*, R) = 0$, we have

$$S_\lambda(q^*, R) \ge S\lambda_1^{-1}(\lambda_1 - \lambda) > 0 \quad \text{for } \lambda \in [0, \lambda_1). \qquad \square$$

Proposition 7.2. *If* $0 < S_\lambda(q^*, R) < S$, *then there exists* $u \in X_R$ *with* $u > 0$ *in* $(0, R)$ *such that*

$$S_\lambda(u; q^*, R) = S_\lambda(q^*, R).$$

Remark 7.1. The minimizer u obtained in Proposition 7.2 is a (weak) solution of the Euler–Lagrange equation

$$\int_0^R r^\alpha |u'(r)|^\beta u'v' - \lambda \int_0^R r^\delta |u|^\beta uv = \mu \int_0^R r^\gamma |u|^{q-2} uv,$$

where μ is a Lagrange multiplier. As in Remark 4.1, $w = \mu^{1/(q^*-\beta-2)} u$ is a (weak) solution of (Q_λ) with $q = q^*$.

Remark 7.2. In order to apply Proposition 7.2, one should know that $0 < S_\lambda(q^*, R) < S$. The first inequality is always true in view of Proposition 7.1, when $\lambda \in [0, \lambda_1)$. Moreover, due to the decreasingness of $S_\lambda(q^*, R)$, either $S_\lambda(q^*, R) < S$ for all $\lambda \in (0, \lambda_1]$, or there exists $\lambda^{**} \in (0, \lambda_1)$ such that $S_\lambda(q^*, R) = S$ for

$\lambda \in [0, \lambda^{**}]$ and $S_\lambda(q^*, R) < S$ for $\lambda \in [\lambda^{**}, \lambda_1]$. From Proposition 7.1(iii), it follows that $\lambda^{**} \le \lambda_1 - S/c$.

PROOF OF PROPOSITION 7.1. Follows directly from Proposition 7.2 and Remark 7.2. □

PROOF OF PROPOSITION 7.2. We employ here a method introduced by Aubin [AU] and Trudinger [TR] and used by Guedda–Véron [GV] to treat the case of the p-Laplacian.

Let $\varepsilon > 0$ be fixed. Then for all $q^* - \varepsilon \le q < q^*$, $S_\lambda(q, R)$ is attained by some $u_q \in X_R$. We can assume that $\|u_q\|_{L^q_\gamma} = 1$ and $u_q \ge 0$ in $(0, R)$, since if u_q is a minimizer then so is $|u_q|$. On the other hand, since $u_q \in C^2(0, R)$ it follows from the existence and uniqueness theorem in ODE that $u_q > 0$ in $(0, R)$. As a consequence u_q satisfies the equation

$$Lu_q = \lambda r^\delta |u_q|^\beta u_q + S_\lambda(q, R) r^\gamma |u_q|^{q-2} u_q \tag{7.3}$$

in the weak sense.

(i) CLAIM. *The function $q \mapsto S_\lambda(q, R)$ from $[q^* - \varepsilon, q^*]$ into $(0, \infty)$ is continuous from the left.*

Indeed, first we see that the continuity of the function $q \mapsto S_\lambda(u; q, R)$ for each $u \in X_R$ implies

$$\limsup_{q \nearrow q_0} S_\lambda(q, R) \le S_\lambda(q_0, R) \tag{7.4}$$

for $q_0 \in (q^* - \varepsilon, q^*]$. (This is just the general property that the infimum of an arbitrary family of upper-semicontinuous functions is upper-semicontinuous.) Next given $\varepsilon > 0$ and $q < q_0$, we choose $u_1 \in X_R \backslash \{0\}$ such that

$$S_\lambda(u_1; q, R) < S_\lambda(q, R) + \varepsilon.$$

Since

$$\|u_1\|_{L^q_\gamma} \le \left[\frac{R^{\gamma+1}}{\gamma+1}\right]^{1/q - 1/q_0} \|u_1\|_{L^{q_0}_\gamma}$$

we obtain

$$S_\lambda(u_1; q, R) \ge \left[\frac{R^{\gamma+1}}{\gamma+1}\right]^{1/q_0 - 1/q} S_\lambda(u_1; q_0, R),$$

which implies

$$S_\lambda(q, R) + \varepsilon \ge \left[\frac{R^{\gamma+1}}{\gamma+1}\right]^{1/q_0 - 1/q} S_\lambda(q_0, R).$$

Passing to the limit as $q \to q_0$ we get

$$\liminf_{q \nearrow q_0} S_\lambda(q, R) + \varepsilon \ge S_\lambda(q_0, R)$$

for all $\varepsilon > 0$. This inequality together with (7.4) gives the claim.

(ii) Claim. *The set of minimizers* (u_q) *is bounded in* X_R *for all* $q \in [q^* - \varepsilon, q^*)$.

Indeed, using Lemma 3.1 we see that

$$\|u_q\|_{L^{\beta+2}_\delta} \le C\|u_q\|_{L^q_\gamma} \le \text{const.} \tag{7.5}$$

On the other hand, in view of the previous claim, there exists a constant $C > 0$ such that

$$S_\lambda(q,R) \le C \tag{7.6}$$

for $q \in [q^* - \varepsilon, q^*]$. Using (7.5) and (7.6) we conclude the proof.

(iii) Claim. *The set of minimizers* (u_q) *is bounded in* $C^{1,\theta}([0,R])$.

It follows from equation (7.3) that

$$\omega_q(r) := |u_q'(r)|^{\beta+1} = r^{-\alpha}\bigg[\lambda\int_0^r r^\delta |u_q|^{\beta+1} + S_\lambda(q,R)\int_0^r r^\gamma |u_q|^{q-1}\bigg]. \tag{7.7}$$

Now, using (7.7) and (7.5) we conclude that $|u_q'(1)| \le \text{const}$, which implies that $u_q(0) \le \text{const}$ for all $q \in [q^* - \varepsilon, q^*)$. Consequently, we obtain

$$\|u_q\|_{L^\infty} \le \text{const}, \quad \|u_q'\|_{L^\infty} \le \text{const}. \tag{7.8}$$

Next, using Hölder's inequality, we have

$$|\omega_q(r) - \omega_q(s)| \le \int_r^s |\omega_q'(t)|\,dt \le (r-s)^{1/p'}\bigg[\int_0^1 |\omega_q'(t)|^p\,dt\bigg]^{1/p} \tag{7.9}$$

for some $p > 1$, with $p' = p/(p-1)$.

Hence, in order to complete the proof of Claim (iii) we have to show that the four terms, obtained from (7.7) by differentiation, are bounded in L^p. Namely

$$\begin{aligned} &A = -\alpha r^{\alpha-1}\lambda\int_0^r r^\delta |u_q|^{\beta+1}, \quad B = \lambda r^{-\alpha+\delta}|u_q(r)|^{\beta+1}, \\ &C = -\alpha r^{\alpha-1} S_\lambda(q,R)\int_0^r r^\gamma |u_q|^{q-1}, \quad D = S_\lambda(q,R) r^{-\alpha+\gamma}|u_q(r)|^{q-1}. \end{aligned} \tag{7.10}$$

Using (7.8) we obtain the following estimates for the functions in (7.10):

$$|A|, |B| \le \text{const}\, r^{\delta-\alpha}, \quad |C|, |D| \le \text{const}\, r^{\gamma-\alpha}. \tag{7.11}$$

So, A, B, C and D are L^p-integrable for $p > 1$ such that $\delta - \alpha > -1/p$ and $\gamma - \alpha > -1/p$.

The existence of such a p follows from assumption (P4). Using then (7.11) in the inequality (7.9) we conclude that ω_q is uniformly Hölder continuous. Hence the claim is proved.

Hence it follows that there exists $u \in C^{1,\theta'}([0,R])$ with $0 < \theta' < \theta$ such $u_q \to u$ in $C^{1,\theta'}$.

(iv) CLAIM. *u is a weak solution of* (Q_λ) *and* $u \neq 0$.

We have

$$\begin{aligned} S_\lambda(q,R) &= \int_0^R r^\alpha |u_q'|^{\beta+2} - \lambda \int_0^R r^\delta |u_q|^{\beta+2} \\ &\geq S\biggl(\int_0^R r^\gamma |u_q|_q^{q^*}\biggr)^{(\beta+2)/q^*} - \lambda \int_0^R r^\delta |u_q|^{\beta+2}. \end{aligned}$$

Passing to the limit as $q \to q^*$ and using Claims (i) and (iii) above we have

$$S_\lambda(q^*,R) \geq S\biggl(\int_0^R r^\gamma |u|^{q^*}\biggr)^{(\beta+2)/q^*} - \lambda \int_0^R r^\delta |u|^{\beta+2}.$$

Since $\|u_q\|_{L^q_\gamma} = 1$ we have $\|u\|_{L^q_\gamma} = 1$. Then

$$S_\lambda(q^*,R) \geq S - \lambda \int_0^R r^\delta |u|^{\beta+2}.$$

In view of the hypothesis $S_\lambda(q^*,R) < S$ we infer that $u \neq 0$. Passing to the limit in (7.3) (understood in the weak sense) we finally conclude that u is a weak solution of (Q_λ). That is, u realizes the infimum in (7.2). $\square$

PROPOSITION 7.3. *If*

$$\delta + 1 - \frac{(\alpha-\beta-1)(\beta+2)}{\beta+1} \leq 0 \tag{7.12}$$

then $0 < S_\lambda(q^*,R) < S$ *for all* $0 < \lambda < \lambda_1(R)$, *where* $\lambda_1(R)$ *is defined in Section* 3.

REMARK 7.3. (7.12) in the case of the p-Laplacian reduces to $p^2 \leq N$. Also, in the case of the k-Hessian, (7.12) holds if $2k(k+1) \leq N$.

PROOF OF PROPOSITION 7.3. Let $\phi \in C^\infty[0,\infty)$ be such that $\phi(r) \equiv 1$ for $0 \leq r \leq r_0$ and $\phi(r) \equiv 0$ for $r \geq 2r_0$, where $2r_0 < R$. Then define for each $\varepsilon > 0$ the function

$$u_\varepsilon(r) := \phi(r)\widehat{u}_\varepsilon(r)$$

where $\widehat{u}_\varepsilon(r)$ is given in (1.13). Next we estimate the three integrals appearing in $S_\lambda(u_\varepsilon; q^*, R)$ given in (7.1).

STEP 1: *Estimating* $I_1 = \int_0^R r^\alpha |u_\varepsilon'(r)|^{\beta+2}$. Since

$$u_\varepsilon'(r) = \widehat{c}\phi'(r)\varepsilon^s(\varepsilon^n + r^n)^{-1/m} - \widehat{c}\frac{n}{m}\phi(r)\varepsilon^s r^{n-1}(\varepsilon^n + r^n)^{-(m+1)/m},$$

we have

$$\begin{aligned} I_1 = {}& \left|\widehat{c}\frac{n}{m}\right|^{\beta+2} \int_0^{r_0} r^{\alpha+(n-1)(\beta+2)} \varepsilon^{s(\beta+2)} (\varepsilon^n + r^n)^{-(m+1)(\beta+2)/m} \\ &+ \int_{r_0}^R r^\alpha |u_\varepsilon'(r)|^{\beta+2}. \end{aligned}$$

Since

$$\left|\widehat{c}\frac{n}{m}\right|^{\beta+2}\int_0^\infty r^{\alpha+(n-1)(\beta+2)}\varepsilon^{s(\beta+2)}(\varepsilon^n+r^n)^{-(m+1)(\beta+2)/m}=S^{\gamma+1/(\beta+\gamma+2-\alpha)}$$

we obtain

$$\begin{aligned}I_1=S^{(\gamma+1)/(\beta+\gamma+2-\alpha)}&+\int_{r_0}^R r^\alpha|u_\varepsilon'|^{\beta+2}\\ &-\left|\widehat{c}\frac{n}{m}\right|^{\beta+2}\int_0^\infty r^{\alpha+(n-1)(\beta+2)}\varepsilon^{s(\beta+2)}(\varepsilon^n+r^n)^{-(m+1)(\beta+2)/m}.\end{aligned}\tag{7.13}$$

The last integral in (7.13) can be estimated by

$$c\int_{r_0}^\infty r^{-\alpha/(\beta+1)}\,dr$$

with a constant c independent of ε. Observe that this integral is finite in view of the hypothesis of criticality: $\alpha-\beta-1>0$. On the other hand, since integrals of the form $\int_{r_0}^R r^a\psi(r)(\varepsilon^n+r^n)^b\,dr$ are $O(1)$ independently of $\varepsilon>0$, for any given $a,b\in\mathbb{R}$, and $\psi\in C^0[r_0,R]$, we obtain

$$I_1=S^{(\gamma+1)/(\beta+\gamma+2-\alpha)}+O(\varepsilon^{s(\beta+2)}),\quad s(\beta+2)=\frac{\alpha-\beta-1}{\beta+1}.$$

Step 2: *Estimating* $I_2=\int_0^R r^\gamma|u_\varepsilon(r)|^{q^*}$. Writing $\phi^{q^*}=1+(\phi^{q^*}-1)$ we first estimate

$$I_2=\int_0^R r^\gamma|\widehat{u}_\varepsilon|^{q^*}+O(\varepsilon^{sq^*}).$$

Since $\int_R^\infty r^\gamma|\widehat{u}_\varepsilon|^{q^*}=O(\varepsilon^{sq^*})$ we see that

$$I_2=\int_0^\infty r^\gamma|\widehat{u}_\varepsilon|^{q^*}+O(\varepsilon^{sq^*})$$

and using Proposition 1.4 we get

$$I_2=S^{(\gamma+1)/(\beta+\gamma+2-\alpha)}+O(\varepsilon^{sq^*}).$$

Step 3: *Estimating* $I_3=\int_0^R r^\delta|\widehat{u}_\varepsilon|^{\beta+2}$. As in the previous step,

$$I_3=\int_0^R r^\delta|\widehat{u}_\varepsilon|^{\beta+2}+O(\varepsilon^{s(\beta+2)}).$$

The integral above is split in two which are estimated below:

$$\begin{aligned}\int_0^{\varepsilon} r^{\delta}|\widehat{u}_\varepsilon|^{\beta+2} &= c\int_0^{\varepsilon} \varepsilon^{s(\beta+2)} r^{\delta}(\varepsilon^n + r^n)^{-(\beta+2)/m} \\ &\geq c\varepsilon^{s(\beta+2)+\delta+1-n(\beta+2)/m} = c\varepsilon^{\beta+\delta+2-\alpha}, \\ \int_{\varepsilon}^{R} r^{\delta}|\widehat{u}_\varepsilon|^{\beta+2} &= (\widehat{c}\varepsilon^s)^{\beta+2}\int_{\varepsilon}^{R} r^{\delta}(\varepsilon^n + r^n)^{-(\beta+2)/m} \\ &\geq c\varepsilon^{s(\beta+2)}\int_{\varepsilon}^{R} r^{\delta} r^{-n(\beta+2)/m}.\end{aligned}$$

Let $\eta := \delta - n(\beta+2)/m + 1$. Then we have

$$\int_{\varepsilon}^{R} r^{\delta-n(\beta+2)/m} \geq \begin{cases} c & \text{if } \eta > 0, \\ c\varepsilon^{\eta} & \text{if } \eta < 0, \\ c|ln\varepsilon| & \text{if } \eta = 0. \end{cases}$$

Thus

$$I_3 \geq c\varepsilon^{\beta+\delta+2-\alpha} + \begin{cases} O(\varepsilon^{s(\beta+2)}) & \text{if } \eta > 0, \\ c\varepsilon^{s(\beta+2)+\eta} & \text{if } \eta < 0, \\ c\varepsilon^{s(\beta+2)}|ln\varepsilon| & \text{if } \eta = 0. \end{cases}$$

Observe that $\beta + \delta + 2 - \alpha = s(\beta+2) + \eta$.

Using these estimates in the expression for $S_\lambda(u_\varepsilon; q^*, R)$ we obtain the following statements.

(i) If $\eta > 0$ we have

$$S_\lambda(u_\varepsilon; q^*, R) \leq S + O(\varepsilon^{s(\beta+2)}),$$

which is of no use.

(ii) If $\eta < 0$ we obtain

$$S_\lambda(u_\varepsilon; q^*, R) \leq S - c\lambda\varepsilon^{s(\beta+2)+\eta} + O(\varepsilon^{s(\beta+2)})$$

and consequently for $\varepsilon > 0$ sufficiently small we obtain $S_\lambda(u_\varepsilon; q^*, R) < S$, which implies $S_\lambda(q^*, R) < S$.

(iii) If $\eta = 0$ we are in a situation similar to (ii):

$$S_\lambda(u_\varepsilon; q^*, R) \leq S - c\lambda\varepsilon^{s(\beta+2)}|ln\varepsilon| + O(\varepsilon^{s(\beta+2)}),$$

and we have the same conclusion as in (ii). □

PROOF OF THEOREM 7.1. Use Propositions 7.2 and 7.3. □

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References

[AP1] F. V. Atkinson and L. A. Peletier, *Elliptic equations with nearly critical growth*, J. Differential Equations **70** (1987), 349–365.

[AP2] ———, *Emden–Fowler equations involving critical exponents*, Nonlinear Anal. **10** (1986), 755–776.

[AU] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geom. **11** (1976), 573–595.

[BL] G. Bliss, *An integral inequality*, J. London Math. Soc. **5** (1930), 40–46.

[BN] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[BR] G. Bratu, *Sur les équations intégrales non linéaires*, Bull. Soc. Math. de France **42** (1914), 113–142.

[CM] G. Caristi and E. Mitidieri, *Nonexistence of positive solutions of quasilinear equations*, preprint, 1996.

[CNS] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[CR] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Funct. Anal. **8** (1971), 321–340.

[EG1] H. Egnell, *Elliptic boundary value problems with singular coefficients and critical nonlinearities*, Indiana Univ. Math. J. **38** (1989), 235–251.

[EG2] ———, *Semilinear elliptic equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal. **104** (1988), 27–56.

[EG3] ———, *Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal. **104** (1988), 57–77.

[G] I. M. Gelfand, *Some problems in the theory of quasi-linear equations*, Amer. Math. Soc. Transl. **29** (1963), 295–381.

[GNN] B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209–243.

[GP] J. P. García Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues*, Comm. Partial Differential Equations **12** (1987), 1389–1430.

[GV] M. Guedda and L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), 879–902.

[HLP] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, 1952.

[JL] D. Joseph and T. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241–269.

[KPA] S. KESEVAN AND F. PACELLA, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities*, Appl. Anal. **34** (1994), 27–37.

[KN] M. C. KNAAP, *Nonlinear elliptic equations involving critical Sobolev exponents*, Ph.D. Thesis, University of Leiden, 1991.

[KP] M. C. KNAAP AND L. A. PELETIER, *Quasilinear elliptic equations with nearly critical growth*, Comm. Partial Differential Equations **14** (1989), 1351–1383.

[KO] A. KUFNER AND B. OPIC, *Hardy-type Inequalities*, Pitman Res. Notes in Math., vol. 219, Longman Scientific and Technical, Harlow, 1990.

[LI] J. LIOUVILLE, *Sur l'équation aux différences partielles* $\frac{d^2 \log \lambda}{du dv} \pm \frac{\lambda}{a^2} = 0$, J. Math. Pures Appl. **18** (1853), 71–72.

[LP] J. MCLEOD AND L. PELETIER, *Elliptic equations with critical growth and Moser's inequality*, Variational Methods, Progress in Nonlinear Differential Equations, vol. 4, Birkhäuser, 1990, pp. 185–196.

[MMP] F. MIGNOT, F. MURAT ET J. P. PUEL, *Variation d'un point de retournement par rapport au domaine*, Comm. Partial Differential Equations **4** (1979), 1263–1297.

[M] E. MITIDIERI, *On the definition of critical dimension*, in preparation.

[M1] J. MOSER, *On a nonlinear problem in differential geometry*, Dynamical Systems, Academic Press, New York, 1973, pp. 273–280.

[M2] ———, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[NS] K. NAGASAKI AND T. SUZUKI, *Radial and nonradial solutions for the nonlinear eigenvalue problem* $\Delta u + \lambda e^u = 0$ *on annuli in* $\mathbb{R}^2$, J. Differential Equations **87** (1990), 144–168.

[PO1] S. L. POKHOZHAEV, *Eigenfunctions for the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408–1411.

[PO2] ———, *On the Sobolev imbedding theorem for* $pl = n$, Proc. Conf. Section Math., Moscow Power Inst., 1965, pp. 158–170. (Russian)

[PO3] ———, *On eigenfunctions of quasilinear elliptic problems*, Mat. Sb. **82** (1970), 192–212. (Russian)

[PS1] P. PUCCI AND J. SERRIN, *A general variational inequality*, Indiana Univ. Math. J. **35** (1986), 681–703.

[PS2] ———, *Continuation and limit properties for solutions of strongly nonlinear second order differential equations*, Asymptotic Anal. **4** (1991), 97–160.

[PS3] ———, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. **69** (1990), 55–83.

[R] R. C. REILLY, *On the Hessian of a function and the curvature of its graph*, Michigan Math. J. **20** (1973–1974), 373–383.

[RE] F. RELLICH, *Darstellung der Eigenwerte von* $\Delta u + \lambda u = 0$ *durch ein Randintegral*, Math. Z. **46** (1940), 635–636.

[RO] E. RODEMICH, *The Sobolev inequalities with best possible constants*, Analysis Seminar at California Institute of Technology, 1966.

[TA] G. TALENTI, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

[TO] P. TOLKSDORF, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations 51 (1984), 126–150.

[TR] N. S. TRUDINGER, *On imbeddings into Orliz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[TS1] K. Tso, *Remarks on the critical exponents for Hessian operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 113–122.

[TS2] ———, *On symmetrizations and Hessian operators*, J. Analyse Math. **52** (1989), 94–106.

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Philippe Clément
Department of Technical Mathematics and Informatics
University of Technology, Delft
THE NETHERLANDS

Djairo Guedes de Figueiredo
IMECC - UNICAMP
Caixa Postal 6065
13081-970 Campinas, S. P. BRAZIL

Enzo Mitidieri
Dipartimento di Scienze Matematiche
Università degli Studi di Trieste
Piazzale Europa 1
34100 Trieste, ITALY