Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 7, 1996, 95–113

DOMAIN VARIATION FOR CERTAIN SETS OF SOLUTIONS AND APPLICATIONS

E. N. Dancer

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

The purpose of this paper is three-fold. We generalize work of our earlier papers [8]–[10] to show that certain solutions or sets of solutions of

$$-\Delta u = f(u) \quad \text{in } \Omega \tag{1}$$

(or systems of equations) with either Dirichlet or Neumann boundary conditions continue if Ω is perturbed in quite a general way. More precisely, in the earlier work, we showed that if the set of solutions has non-zero Leray–Schauder degree, then it does continue if Ω is perturbed. Here we prove similar results when we consider sets of solutions of non-zero homotopy index (or Morse numbers), where the homotopy index is defined in Rybakowski [22]. The proof of this is much more delicate than the earlier case since we need to retain the variational structure.

We become interested in this problem for two reasons. Firstly, if Ω is invariant under the orthogonal action of a compact Lie group G, then the set of solutions of (1) is invariant ynder the natural action of the symmetry group G. Thus the solutions of (1) are usually orbits under this group action rather than isolated points. Then a theorem of Sylvester [25] implies that these orbits frequently have Leray–Schauder degree zero (for example if $G = S^1$ and the orbit consists of more than one point). Thus the old arguments do not apply but the new result does apply. Note that one cannot always avoid the problem by using subspaces

1991 *Mathematics Subject Classification.* 35J65, 35B32, 58E30.
Research partially supported by the Australian Research Council.

fixed by a subgroup of G. We will use this result and some additional arguments to answer a question of Jimbo and Morita [17]. More precisely, we construct a contractible domain D in $\mathbb{R}^m$ where $m \geq 3$ such that the equation

$$-\Delta u = \lambda u(1-|u|^2) \quad \text{in } D, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D,$$

has a stable non-constant solution. Here u is *complex-valued*. Note that, by stability we mean orbital stability for the natural corresponding parabolic equation. This equation is known as the Ginzburg–Landau equation and has been studied under various boundary conditions extensively. See [2], [4] and [17] where many further references can be found. Note that our example contradicts their original conjecture. Subsequent to my outlining the construction here, Jimbo and Morita [18] have found a different example by a different proof. Their method seems to depend more strongly on λ being large. Our method has the advantage that it can also be used to construct various types of unstable solutions. We do not know if there are similar examples with $m = 2$.

As a second application, we use similar ideas to prove some results on the existence of positive solutions of the exterior problem

$$(2) \qquad -\Delta u = u^p \quad \text{on } \mathbb{R}^n \backslash D, \quad u = 0 \quad \text{on } \partial D,$$

for p close to but less than the critical exponent p^*. We do not solve the conjecture in [11] but give partial results which strongly support the conjecture. Note that (2) was studied in [11] because it was of importance for problems on bounded domains with a small hole.

In §1, we prove our main perturbation results while in §2 we construct our counterexample for the Jimbo–Morita problem. Finally, in §3, we consider the exterior problem.

1. Perturbation theorems

Assume $g : \mathbb{R} \to \mathbb{R}$ is C^1 and Ω is a domain in $\mathbb{R}^m$. We consider the equation

$$(3) \qquad -\Delta u = g(u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where we consider the equation in the weak ($W^{1,2}$) sense. We will prove that, for suitable sublinear g, if T is a compact connected set of solutions on which the energy is constant and which has non-trivial homotopy index in the sense of [22], then some of these solutions persist when we perturb Ω in quite a general way. We now make this more precise.

We assume g is C^1 and bounded and there exist $\alpha < \beta$ such that $g(y) < 0$ if $y \geq \beta$ and $g(y) > 0$ if $y < \alpha$. By the weak maximum principle (or test function arguments) we see that any solution of (3) in $W^{1,2}(\Omega)$ satisfies $\alpha \leq u(x) \leq \beta$ on Ω. (Note that if g were not bounded, we could truncate g to be bounded without affecting the bounded solutions). Now choose a small and positive such that $g(y) + ay$ is negative at β and positive at α. We then define $\widetilde{g}$ C^1 and bounded (and $\widetilde{g}'$ bounded) so that $\widetilde{g}(y) = g(y) + ay$ on $[\alpha, \beta]$, $\widetilde{g}(y) < 0$ on $[\beta, \infty)$ and $\widetilde{g}(y) > 0$ on $(-\infty, \alpha]$. By similar arguments to above, we see that (3) is equivalent to the problem

$$\Delta u + au = \widetilde{g}(u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Now if Ω is a bounded open set in $\mathbb{R}^m, -\Delta + aI$ is a positive self-adjoint operator on $L^2(\Omega)$ and hence it has a positive self-adjoint square root H_Ω which has compact resolvent if $-\Delta + aI$ does. We drop the Ω when the meaning is clear. It is well known (cp. [20]) that H_Ω has domain $W^{1,2}(\Omega)$. Let $S_\Omega = H_\Omega^{-1}$. Then our problem is equivalent to the problem

$$v = S_\Omega \widetilde{g}(S_\Omega v) \tag{4}$$

where $u = S_\Omega v$. (Remember that H_Ω is a bijection of $W^{1,2}(\Omega)$ onto $L^2(\Omega)$ and H_Ω is injective because $-\Delta + aI$ is.) We work with this equation henceforth. Note that the solutions of (4) correspond to the critical points of $\widehat{E}(v) = \frac{1}{2}\|v\|^2 - \widetilde{G}(S_\Omega v)$ where $\widetilde{G} : \mathbb{R} \to \mathbb{R}$ and $\widetilde{G}' = \widetilde{g}$.

We now consider domains Ω_n approaching Ω. More precisely, as in [10] we write $\Omega_n \to_n \Omega$ as $n \to \infty$ if Ω and Ω_n are bounded open sets such that the following properties hold:

(i) $\Omega_n \supseteq \Omega$ for each n and each Ω_n has Lipschitz boundary,
(ii) $m(\Omega_n \backslash \Omega) \to 0$ as $n \to \infty$ where m denotes Lebesgue measure,
(iii) the natural inclusion $i : W^{1,2}(\Omega) \to L^2(\Omega)$ is compact and
(iv) $\{u|_\Omega : u \in C_0^\infty(\mathbb{R}^m)\}$ is dense in $W^{1,2}(\Omega)$ in the $W^{1,2}(\Omega)$ norm.

As discussed in [10], these are rather weak assumptions that are satisfied very generally. Note that we do not assume that Ω is connected.

Next we assume that T is a component in $W^{1,2}(\Omega)$ of the weak solutions of (3) such that any two points of T can be joined by a curve in T which is continuous and piecewise differentiable (in the $W^{1,2}$ norm). Note that this assumption is satisfied in many cases. For example, it is easy to check that it is satisfied if T is an orbit under the continuous linear action of a compact connected Lie group (by [13]) or if g is real analytic (because our solutions are then the zeros of a real analytic Fredholm map in an appropriate space and T is locally diffeomorphic to a real analytic variety in finite dimensions). The reason for the interest in this assumption is that it implies that the energy $E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \widetilde{G}(u)$ is constant on T. This follows simply by differentiating E on such a piecewise differentiable curve. Note that by a simple calculation $E(u) = \widehat{E}(v)$ where $u = S_\Omega v$. Thus our comments also apply to (4). We define the homotopy index $h(T)$ to be the homotopy index in the sense of [22] of the flow of the differential equation

$\dot{u} = -u + S_\Omega \widetilde{g}(S_\Omega u)$ on $L^2(\Omega)$ on a suitable small neighbourhood of $H_\Omega T$. We will prove a little later that this is well defined provided that there are no other solutions of (4) close to $H_\Omega T$.

We now state our main result for Neumann problems.

THEOREM 1. *Assume that the above conditions on g are satisfied, T is a component of the set of solutions of* (3) *satisfying our earlier assumptions, no other solutions of* (3) *are close to T in $W^{1,2}(\Omega)$, $\Omega_n \to_n \Omega$ as $n \to \infty$ and $h(T)$ is non-trivial. Fix $p > 1$. For large n, there is a solution u_n of* (3) *on Ω_n such that $\|u_n - w\|_p$ is small for some $w \in T$.* (*Here we compare solutions on different sets by extending them to be zero outside their domain.*)

Before proving this result, we need a technical lemma on square roots. Let $P_n : L_{2,\mathrm{loc}}(\mathbb{R}^m) \to L_2(\Omega_n)$ be the natural restriction operator. P is defined analogously (for Ω_n replaced by Ω).

LEMMA 1. *Assume $\Omega_n \to_n \Omega$ as $n \to \infty$. Then S_{Ω_n} are uniformly bounded as maps on $L^\infty(\Omega_n)$ and $L^2(\Omega_n)$. If $f \in L^2(\mathbb{R}^m), S_{\Omega_n} P_n f \to S_\Omega P f$ in $L^2(\mathbb{R}^m)$ as $n \to \infty$. Moreover, S_{Ω_n} is compact on $L^2(\Omega_n)$. If $\Omega \cup \bigcup_{n=1}^\infty \Omega_n \subseteq B$ and W is bounded in $L^\infty(B)$, then $\bigcup_{n=1}^\infty S_{\Omega_n} P_n W$ lies in a compact subset of $L^2(B)$* (*where we extend functions on Ω_n to B by defining them to be zero on $B \backslash \Omega_n$*).

PROOF. It is well known and easy to prove that $-\Delta + aI$ with Neumann boundary conditions has inverse with norm bounded by a^{-1} both on $L^2(\Omega)$ and $L^\infty(\Omega)$. (For the case of $L^\infty(\mathbb{R}^m)$, we use the fact that the inverse is positivity preserving and $(-\Delta + aI)^{-1}(1) = a^{-1}$.) Since $(-\Delta + aI)^{-1}$ is self-adjoint, the bound for S_Ω on $L^2(\Omega)$ follows by standard results. To obtain the bound on $L^\infty(\mathbb{R}^m)$, we use the formula for S_Ω (cp. Kato [20], p. 282). If $f \in L^2(\Omega)$, then

$$S_\Omega f = \int_0^\infty \lambda^{-1/2} (-\Delta + (a + \lambda) I)^{-1} f \, d\lambda. \tag{5}$$

Here Δ means the Laplacian on Ω with Neumann boundary conditions. If $f \in L^\infty(\Omega)$, by our estimate above,

$$\|(-\Delta + (a + \lambda) I)^{-1} f\|_\infty \le (a + \lambda)^{-1} \|f\|_\infty$$

for $\lambda \ge 0$ and hence, by (5),

$$\|S_\Omega f\|_\infty \le \|f\|_\infty \int_0^\infty \lambda^{-1/2} (a + \lambda)^{-1} d\lambda \le K \|f\|_\infty$$

where K is independent of Ω. To see that $S_{\Omega_n} P_n f$ converges to $S_\Omega P f$, first note that the arguments in [9] and [10] imply that $(-\Delta + (a + \lambda) I)_{\Omega_n}^{-1} P_n f \to (-\Delta + (a + \lambda) I)_\Omega^{-1} P f$ in $L^2(\mathbb{R}^m)$ for each $f \in L^2(\mathbb{R}^m)$ and each $\lambda \ge 0$. Indeed, since the resolvents are uniformly bounded, it follows easily from the resolvent equation that the convergence is locally uniform in λ for fixed f. The convergence of the square root follows easily from this and the integral formula (5) if we note that, in the integral, we can uniformly estimate the parts for λ small and λ large. The compactness statement follows from standard results.

To prove the last statement, we first note that S_{Ω_n} are uniformly bounded as maps of $L^2(\Omega_n)$ to $W^{1,2}(\Omega_n)$. It is convenient to estimate instead H_{Ω_n}. The estimate follows, because if $u \in \mathcal{D}((-\Delta + aI)_{\Omega_n})$ then

$$\|S_{\Omega_n} u\|_2^2 = (S_{\Omega_n} u, S_{\Omega_n} u) = ((-\Delta + aI)u, u) = \int_{\Omega_n} (|\nabla u|^2 + au^2).$$

This establishes the bound for $u \in \mathcal{D}((-\Delta + aI)_{\Omega_n})$ and it extends to $W^{1,2}(\Omega_n)$ by density.

Hence we see from the compactness of the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ that $\bigcup_{n=1}^{\infty} S_{\Omega_n} P_n W|_\Omega$ lies in a compact subset of $L^2(\Omega)$. Moreover, by our earlier estimates, $\bigcup_{n=1}^{\infty} S_{\Omega_n} P_n W$ is bounded in $L^\infty(B)$ and hence we easily see that $\|S_{\Omega_n} P_n w\|_{2,\Omega_n \setminus \Omega}$ is small if n is large uniformly for $w \in W$. The required compactness follows easily from this and the precompactness of $\bigcup_{n=1}^{\infty} S_{\Omega_n} P_n W|_\Omega$. (The precompactness of $S_{\Omega_n} P_n W$ for fixed n is similar but much easier.)

Proof of Theorem 1. Choose a ball B such that $\Omega \cup \bigcup_{n=1}^{\infty} \Omega_n \subseteq B$. We consider the maps

$$F_n(v) = v - i_n S_{\Omega_n} \widetilde{g}(S_{\Omega_n} P_n v)$$

on $L^2(B)$. Here i_n is the natural inclusion of $L^2(\Omega_n)$ into $L^2(B)$ and i is defined analogously (with Ω_n replaced by Ω). Note that F_n is the gradient of f_n where $f_n(v) = \frac{1}{2}\|v\|^2 - \widetilde{G}(S_{\Omega_n} P_n v)$, and F and f are defined analogously. Note also that the zeros of F_n are in $R(P_n)$ and hence are solutions of $v = S_{\Omega_n} \widetilde{g}(S_{\Omega_n} v)$ on Ω_n extended to be zero on $B \setminus \Omega_n$.

Choose a neighbourhood W of $H_\Omega T$ in $L^2(\Omega)$ such that no other solution of (4) lies in $\overline{W}$. We prove that no solution of the equation

$$\dot{u} = -u + S_\Omega \widetilde{g}(S_\Omega u) \quad \text{on } L^2(\Omega) \tag{6}$$

can be completely contained in $\overline{W}$ (except for the constant solutions at points of $H_\Omega T$). Note that the flow π_0 for (6) is defined since the right hand side is Lipschitz and is strongly admissible in the sense of [22] by Theorem III.4.4 there. Now by a simple differentiation the energy is strictly decreasing on solutions except at zeros of $I - F$ in $\overline{W}$. It follows easily from this and the strong admissibility that any bounded solution of (6) approaches the set of stationary solutions as $t \to \pm\infty$. Thus any solution which lies on $\overline{W}$ for all t is either constant and a point of $H_\Omega T$ for all t or approaches two points of $H_\Omega T$ of different energy as t tends to $\pm\infty$. The latter case is impossible since the energy $\widehat{E}$ is constant on $H_\Omega T$. This proves our claim. Hence the homotopy index $h(T)$ is defined for the flow of (6) by using W as an isolating neighbourhood.

Next we note that we obtain the same homotopy index if we choose the flow π_0 of

$$\dot{u} = -u + iS_\Omega \widetilde{g}(S_\Omega Pu)$$

on $L^2(B)$ where i is the natural inclusion of $L^2(\Omega)$ into $L^2(B)$. One easily sees that this is still a gradient system. The flow is a product flow for the decomposition $L^2(B) = L^2(\Omega) \oplus L^2(B \setminus \Omega)$. Hence the product theorem for the homotopy index as in [22], Theorem I.10.6, implies that the homotopy index of this flow in a small neighbourhood $\widehat{N}$ of $H_\Omega T$ in $L^2(B)$ is still $h(T)$. (Note that, on $L^2(B \setminus \Omega)$, the flow is a simple stable linear flow.)

We now prove that the flow π_n for the system

$$\dot{u} = -u + i_n S_{\Omega_n} \widetilde{g}(S_{\Omega_n} P_n u)$$

on $L^2(B)$ has the property that $\{\pi_n\}$ are strongly admissible and that

$$\pi_n(u_n, t_n) \to \pi_0(u, t) \tag{7}$$

as $u_n \to u$ in $L^2(B)$ and $t_n \to t$. The strong admissibility follows from Lemma 1 above and Remark 1 on p. 167 of [22]. (Note that, as before, the flows are globally defined because the right side of the differential equation is globally Lipschitz.) Thus it remains to prove the convergence property. By the flow properties, it suffices to prove (7) for $0 \le t_n \le \widetilde{T}$ where $\widetilde{T} > 0$ is fixed. This follows easily from Theorem 3.4.8 of Henry [16] once we note that $i_n S_{\Omega_n} \widetilde{g}(S_{\Omega_n} P_n u) \to iS_\Omega \widetilde{g}(S_\Omega Pu)$ for each $u \in L^2(B)$. This follows easily from Lemma 1 and standard continuity properties of the Nemytskiĭ operator.

It now follows from (7) and Theorem I.2.3 of [22] that the homotopy index is defined for the flow π_n on $\widehat{N}$ for large n and is the same as $h(T)$, in particular, it is non-trivial. Thus π_n must have a bounded solution completely contained in $\widehat{N}$ and, since it is a gradient system, it follows that (4) on Ω_n rather than Ω has a solution in $\widehat{N}$, that is, a solution close to $H_\Omega T$ in $L^2(B)$.

Note that solutions of $F_n(u) = 0$ are uniformly bounded because $\widetilde{g}$ is bounded and by using Lemma 1. To complete the proof we need to check that if $\{u_n\}_{n=0}^\infty$ are uniformly bounded and $\|u_n - u_0\|_{2,\Omega}$ is small then $\|S_{\Omega_n} u_n - S_{\Omega_n} u\|_p$ is small where we compare functions on different sets by extending them to be zero outside. Since $\{S_{\Omega_n} u_n\}$ are uniformly bounded in L^∞ (by Lemma 1), one sees that it suffices to prove $\|S_{\Omega_n} u_n - S_\Omega u\|_{2,\Omega}$ is small. This follows easily from Lemma 1.

Remarks. 1. We do not really need that $\Omega_n \supseteq \Omega_0$. We could allow small holes much as in [9] or [10].

2. With care we could avoid the assumption that the energy is constant on T if we assume T has a suitable isolating neighbourhood in the sense of [22].

3. We could replace the homotopy index condition by the condition that some cohomology $\widetilde{H}^i(\widehat{E}^c \cap H_\Omega(T), \widehat{E}^c \cap (V \setminus H_\Omega(T)), \mathbb{Z})$ is non-zero where $c = \widehat{E}(T)$, $\widehat{E}^c = \{u \in L^2(\Omega) : \widehat{E}(u) \leq c\}$ and V is a small closed neighbourhood of $H_\Omega(T)$. To do this, we need to prove a slight generalization of Theorem III.4.8 of [22]. This is not difficult.

4. The theorem readily generalizes to systems. Assume $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 gradient mapping such that $g(y) \cdot y < 0$ for $|y| \geq r$ where $\cdot$ denotes the usual scalar product on $\mathbb{R}^2$. It follows easily from the maximum principle that any solution of the two-dimensional system $-\Delta u = g(u)$ on Ω with Neumann boundary conditions satisfies $|u| \leq r$ on Ω. We can truncate g for $|y| \geq r$ such that $g(y) \cdot y < 0$ for $|y| \geq r$ and g is bounded and C^1. It is then easy to find a small positive a and a C^1 function $\widetilde{g}$ which is bounded and C^1 such that $\widetilde{g}(y) = g(y) + ay$ for $|y| \leq r$ and $\widetilde{g}(y) \cdot y < 0$ for $|y| \geq r$. It once again is easy to check from the maximum principle that any solution of the system $-\Delta u + au = \widetilde{g}(u)$ in Ω (with Neumann boundary conditions) satisfies $|u| \leq r$ and hence is a solution of the original equation. It is now easy to generalize the proof of Theorem 1 to this case. (The square root is taken componentwise.) This can be easily generalized to systems of more than two equations.

We now consider the corresponding result for the Dirichlet problem. Here we can do somewhat better by allowing a slightly more general definition of domain convergence from [9] and more interestingly we can allow a much more general growth of g: it only needs to grow subcritically (and indeed we can usually remove this by a truncation argument). This case is easier than the Neumann case because we can take $a = 0$. We do not need to assume any sign condition on g but do assume g is bounded. Then the analogue of Theorem 1 holds with essentially the same proof. The only change that needs to be made is that we obtain the uniform (in the domain) estimates for $(-\Delta + aI)^{-1}$ on $L^\infty(\Omega_n)$ from the L^p-L^q estimates for elliptic equations as in Gilbarg and Trudinger [15]. (These are estimates for *Dirichlet* boundary conditions.) Note that, for $a \geq 0$, we can always eliminate the a term in the test function estimates in the proofs in [15]. We obtain the compactness condition on $\bigcup_{n=1}^\infty S_{\Omega_n} P_n W$ more easily because this set is bounded in $\dot{W}^{1,2}(B)$.

The other major change is in the proof of the bound for S_{Ω_n} on $L^\infty(\Omega_n)$. We still use the integral formula for the square root but to estimate $(-\Delta + \lambda I)^{-1}$ for large λ we use the fact that if $f \geq 0$, then $(-\Delta + \lambda I)^{-1} f$ for Dirichlet boundary conditions is dominated by the same expression but for Neumann boundary conditions. Hence $\|(-\Delta + \lambda I)^{-1}\| \leq \lambda^{-1}$ if $\lambda > 0$ and if we use Dirichlet boundary conditions (where the operator norm is on $L^2(\Omega)$). Otherwise the arguments are as before.

We now weaken the growth condition on g to requiring that g is C^1 and $|g(y)| \leq K|y|^s$ for $|y|$ large where $s < (m+2)(m-2)^{-1}$ ($s < \infty$ if $m = 1$ or 2). To do this, we choose g_j C^1 and bounded on $\mathbb{R}$ such that $g_j(y) = g(y)$ if $|y| \leq j$ and $|g_j(y)| \leq K|y|^s + K_2$ on $\mathbb{R}$ where K and K_2 do not depend on j. It suffices to prove that if u_n are solutions of (4) (with g replaced by g_n and Ω replaced by Ω_n) with $\|u_n - u_0\|_{2,B}$ small for some $u_0 \in H_\Omega(T)$ then $S_{\Omega_n} u_n$ are bounded in L^∞ (because $S_{\Omega_n} u_n$ are then solutions of the original equation). If $\|u_n - u_0\|_{2,B}$ is small, then $\|u_n\|_{2,\Omega_n}$ are uniformly bounded and hence it follows easily that $S_{\Omega_n} u_n$ are uniformly bounded in $W^{1,2}(\Omega_n)$. By the Sobolev embedding theorems for the space $\dot{W}^{1.2}(\Omega)$, it follows that $v_n \equiv S_{\Omega_n} u_n$ are uniformly bounded in $L^s(\Omega_n)$ where $s = (m+2)(m-2)^{-1}$. By using the L^p-L^q estimates for the Laplacian (as earlier) and a bootstrapping argument, it follows that v_n are uniformly bounded as required. It remains to prove that v_n is close to $v = S_\Omega u_0$. This follows easily from Lemma 1 since u_n is close to u_0 in $L^2(B)$. (Note that we can use a simple homotopy invariance argument similar to those in §3 to show that $h(T)$ is independent of the truncation.)

We have proved the following theorem for the problem

$$-\Delta u = g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{8}$$

THEOREM 2. *Assume that g is C^1 and $|g(y)| \leq K|y|^s$ for large $|y|$ where $s < (m+2)(m-2)^{-1}$ ($s < \infty$ if $m = 2$). Assume that T is a component of the set of solutions of* (8) *such that T satisfies the regularity condition before Theorem* 1, *no other solutions of* (8) *are close to T in $\dot{W}^{1,2}(\Omega), \Omega_n \to \Omega$ as $n \to \infty$ in the sense of* §1 *of* [9] *and $h(T)$ is non-trivial. Fix $p > 1$. For large n, there is a solution u_n of* (8) *on Ω_n such that $\|u_n - w\|_p$ is small for some $w \in T$.* (Note that in the definition of $h(T)$, we use S_Ω for the Dirichlet problem.)

REMARKS. 1. If T is bounded in $L^\infty(\Omega)$, we could use truncation arguments to obtain results in supercritical cases. We could also allow f to depend continuously on x.

2. We can prove the theorem for the Dirichlet case by working in $\dot{W}^{1,2}(B)$ rather than $L^2(B)$, looking at weak solutions and avoiding square roots. This works easily in this case because there is a natural inclusion of $\dot{W}^{1,2}(\Omega_n)$ into $\dot{W}^{1,2}(B)$.

3. In the Dirichlet case, as in [8], it is much easier to keep control of the stability properties of solutions.

In many applications, it is important to know if the solutions u_n on Ω_n are positive for large n. (In many applications, this arises naturally.) We assume Ω_n are connected. It is clearly necessary for $u \in T$ to be non-negative on Ω for the u_n to be positive. We will use Dirichlet boundary conditions. (Neumann seems much more technical.) If $u(x) > 0$ on Ω for $u \in T$ and $f(0) \geq 0$, it is not difficult to apply weak maximum principle type arguments to u_n to prove that $u_n(x) > 0$ on Ω_n for large n. We now assume $T = \{u_0\}$. If $f(0) \geq 0$, and $u_0(x) \geq 0$ on Ω, we can easily use the maximum principle type arguments to prove that $u_0(x) > 0$ on some components of Ω_0 and vanishes identically on the others. From the results in [8] and [14], it is clear that the stability of u_0 on the components of Ω where it vanishes identically plays a crucial role. In fact, it can be shown that for u_n to be positive it is necessary for u_0 to be stable on any component of Ω where it vanishes. For the remainder of this paragraph, let us assume that $f(0) = 0$. If u_0 is non-degenerate and stable on the components where u_0 vanishes and if u_0 has non-trivial homotopy index on the other components (on the space $\dot{W}^{1,2}$), then one can prove that u_n must be positive on Ω_n. The proof uses eigenvalue estimates.

2. The Jimbo–Morita problem

In this section, we use the results of §1 to construct, for each $m \geq 3$, a contractible domain Ω in $\mathbb{R}^m$ such that the equation (with u complex-valued)

$$-\Delta u = \lambda u(1 - |u|^2) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{9}$$

has a non-constant stable solution $\widetilde{u}$. Here by stability we mean stability for the natural corresponding parabolic equation and more precisely orbital stability of the orbit through u (under the action of the group S^1). Here S^1 acts by the pointwise action on the components of u, that is, $(e^{i\theta}u)(x) = e^{i\theta}u(x)$. For future reference note that this action is easily seen to be a smooth action on any of the usual function spaces ($L^2(\Omega)$ etc.). It suffices to obtain an example with $m = 3$ because if we find an example on $\Omega \subseteq \mathbb{R}^3$, it follows that we have an example on $\Omega \times B$, where B is a ball, by making u independent of the other coordinates. We could then repeat the arguments below to obtain an example on a smooth domain with u depending on all coordinates.

To construct our counterexample, we choose an annulus $D = \{\underline{x} \in \mathbb{R}^2 : a < \|\underline{x}\| < 1\}$ and a $\lambda > 0$ such that for $\Omega = D$, (9) has a non-constant stable solution $\widetilde{u}$ (cp. [17]). The construction shows that the linearization about this solution has a one-dimensional kernel (which is as small as it can be). We can think of this solution as a solution on $\widetilde{\Omega}_1 \equiv \Omega \times [-1, 1]$ (with $\widetilde{u}_1$ independent of the last coordinate). On the other hand, there is a stable constant solution $\widetilde{u}_2$ on the domain $\widetilde{\Omega}_2 = B \times [-2, -1 - \delta]$ where B is the unit ball and where we will choose a fixed small δ later. We choose smooth domains Ω_n approximating $\widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ in a natural way as in Fig. 1. (The diagram shows a cross section.) We will prove that there is a stable non-constant solution on Ω_n for n large if δ is small (and fixed).

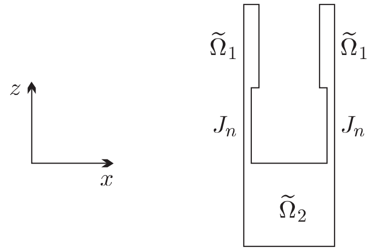

FIGURE 1

We first use Theorem 1 to show that there is a solution close in L^2 to the set $\mathcal{P}$ which is the product of the orbits of $\widetilde{u}_1$ and $\widetilde{u}_2$. Since $\widetilde{u}_1$ and $\widetilde{u}_2$ are each non-degenerate (modulo the symmetry), their orbits are isolated components of their respective solution sets (cp. [13]). A slightly technical point occurs here. The theory in [13] works more conveniently in $L^p(\Omega)$ for p large but since solutions are uniformly bounded in $L^\infty(\Omega)$, they are close in $L^p(\Omega)$ if they are close in $L^2(\Omega)$. Hence we easily see that Theorem 1 applies to the set $\mathcal{P}$ where $\Omega_n \to \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ as the thickness of the joining "cylinder" tends to zero as n tends to ∞. The only point to check is that the homotopy index is non-trivial on $\Omega_0 = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$. Now on $\Omega_0 = \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2$ our map is a product of the maps on $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ and hence its homotopy index is the smash product of the homotopy indices on $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$. Hence if we prove that each of these homotopy indices is $S^1 \cup \{*\}$ where $*$ is a base point, it will follow easily that the homotopy index of $\widehat{\mathcal{P}}$ is non-trivial. (In fact, it is $(S^1 \wedge S^1) \cup \{*\}$.) Note that the flow is the analogue of (6) and $\widehat{\mathcal{P}}$ is the product of the orbits of $H_{\widetilde{\Omega}_1}\widetilde{u}_1$ and $H_{\widetilde{\Omega}_2}\widetilde{u}_2$.

Now our assumptions imply that the orbit through $H_{\widetilde{\Omega}_1}\widetilde{u}_1$ is a strict local minimum of the energy $\widehat{E}$ (strict modulo the symmetry). Hence it is easy to construct a small neighbourhood N of the orbit of the homotopy type of S^1 such that the flow is inward on ∂N. One simply uses S, the component of $\{u \in L^2(\Omega_1) : \widehat{E}_{\Omega_1}(u) \leq \widehat{E}_{\Omega_1}(H_{\widetilde{\Omega}_1}\widetilde{u}_1) + \mu\}$ containing $\widetilde{u}_1$, where μ is small and positive and note the flow is inward across the boundary of S. (Note that one can use the non-degeneracy to check that if μ is small then S is a small neighbourhood of the orbit through $\widetilde{u}_1$.) Thus N is an isolating neighbourhood for the flow on $L^2(\Omega_1)$ with empty exit set and hence by the definition of the homotopy index $h(\mathcal{P}_1) = S^1 \cup \{*\}$ where $(*)$ is a base point and $\mathcal{P}_1$ is the orbit of $\widehat{u}_1 = H_{\widetilde{\Omega}_1}\widetilde{u}_1$. In particular, it is not trivial. Since we can use the argument on $\widetilde{\Omega}_2$, it follows from our earlier remarks that the homotopy index of $\widehat{\mathcal{P}}$ must be non-trivial. Hence all the assumptions of Theorem 1 and a remark after it are satisfied and hence there is at least one solution u_n of (9) on Ω_n for n large near the product of the orbits of $\widetilde{u}_1$ and $\widetilde{u}_2$.

We prove that if we choose things carefully, u_n is (orbitally) stable. To do this, it is convenient to first obtain a little more information from the homotopy index. By earlier, $h(\widehat{\mathcal{P}})$ is $(S^1 \wedge S^1) \cup \{*\}$ where $*$ is a base point and hence, for n large, there is a neighbourhood W_n of the set of solutions of the analogue of (4) on Ω_n near $\widehat{\mathcal{P}}$, the product of the orbits of $\widehat{u}_1$ and $\widehat{u}_2$, such that its homotopy index is $(S^1 \wedge S^1) \cup \{*\}$. This means that there is no exit set for the flow on W_n. This follows because in the construction of the homotopy index, we can choose the neighbourhood N of the connected set $\widehat{\mathcal{P}}$ to be connected and thus by the construction in [22], W_n can also be chosen connected. Hence W_n quotient by its exit set will only have a separate base point if the exit set is empty. This proves our claim.

Now let Z_n be the solution of the analogue of (6) of least energy among the compact set of solutions in W_n. It follows that Z_n must be a local minimum of $\widehat{E}_n$. Here $\widehat{E}_n$ is the energy on Ω_n. (If not, solutions starting close to Z_n with energy strictly less than Z_n would have to leave W_n because in future time such solutions have energy decreasing and less than $\widehat{E}_n(Z_n)$ and thus there is no point in W_n for these solutions to approach as $t \to \infty$. Thus some solutions leave W_n, which contradicts our earlier claim.) It follows that $S_{\Omega_n} Z_n$ is a local minimum of E on Ω_n. If the orbit containing $S_{\Omega_n} Z_n$ is isolated among the solutions of (9) (and thus $E(w) > E(S_{\Omega_n} Z_n)$ at every point close to the orbit but not on the orbit), a standard Lyapunov functional argument implies that $S_{\Omega_n} Z_n$ is stable. (We first prove a stronger coercivity inequality near the orbit.) If the orbit is not isolated and Ω_n is chosen carefully, we will prove that near $S_{\Omega_n} Z_n$ the solutions form a stable hyperbolic 2-manifold and the stability follows from Exercise 6 on p. 108 of Henry [16] (in fact stability rather than orbital stability). It remains to prove the claim above. It in fact suffices to prove that, if Ω_n is chosen suitably, then the linearization at a solution in W_n has at most 2 non-positive eigenvalues counting multiplicity. Assuming this, we show that, if the orbit through $S_{\Omega_n} Z_n$ is not isolated, the solutions nearby form a smooth 2-manifold which is easily seen to be stable and hyperbolic. To prove this, we use a tubular neighbourhood construction. It is convenient to work with (9) rather than the analogue of (4). The tubular neighbourhood construction (cp. [13]) implies that, if the orbit is not isolated, then $S_{\Omega_n} Z_n$ is not an isolated solution in $\widetilde{T} = \{w \in W^{1,2}(\Omega_n) : \langle w - S_{\Omega_n} Z_n, A S_{\Omega_n} Z_n \rangle = 0\}$ where $\langle \, , \rangle$ is the usual scalar product on $W^{1,2}(\Omega_n)$ and A is the infinitesimal generator of the group action. Let P_n be the orthogonal projection onto the subspace $\widetilde{T} + S_{\Omega_n} Z_n$ and let our equation (9) written in weak form on $W^{1,2}(\Omega_n)$ be $L(w) = 0$. Note that L is real analytic (since the nonlinear terms are polynomial). We look at the equation $P_n L(w) = 0$ as a mapping on $\widetilde{T}$ into $R(P_n)$. It is easy to see the derivative at $S_{\Omega_n} Z_n$ is Fredholm of index zero and has a one-dimensional kernel.

(The kernel cannot be trivial because $S_{\Omega_n}Z_n$ is not isolated in $\widetilde{T}$.) By a standard Lyapunov–Schmidt argument, it follows that the solutions of $P_nL(w)=0$ near $S_{\Omega_n}Z_n$ in $\widetilde{T}$ are determined by a one-dimensional bifurcation equation $h(t)=0$ where $h:\mathbb{R}\to\mathbb{R}$ is real analytic (and $t=0$ corresponds to $w=S_{\Omega_n}Z_n$). Since $S_{\Omega_n}Z_n$ is not isolated, 0 is a non-isolated zero of h and hence h vanishes identically (since h is real analytic). Thus the solutions $\widetilde{Z}$ of $P_nL(w)=0$ in T near $S_{\Omega_n}Z_n$ form a real analytic 1-manifold. If $w\in\widetilde{Z}$, then

$$L(w)=\alpha AS_{\Omega_n}Z_n \tag{10}$$

by the definition of P_n. On the other hand, by differentiating the energy along an orbit,

$$\langle L(w),Aw\rangle=0 \tag{11}$$

always. Now, by continuity, Aw is close to $AS_{\Omega_n}Z_n\neq 0$ and hence (10) and (11) imply that $\alpha=0$, i.e. points of $\widetilde{Z}$ are solutions of $L(w)=0$. Thus the solutions of $L(w)=0$ in $\widetilde{T}$ form a smooth 1-manifold and hence by the tubular neighbourhood and the group action the solutions near the orbit of Z_n form a smooth 2-manifold, as required.

It remains to prove our claim on the eigenvalues of the linearization. To do this we need to specify the joining "cylinder" J_n more carefully. We assume that the joining cylinder J_n is $\{(\underline{x},z)\in\mathbb{R}^2\times\mathbb{R}:1-n^{-1}<\|\underline{x}\|<1,\,-1-\delta<z<-1\}$. We assume that for a sequence of n's tending to infinity (which we will relabel to be the whole sequence) the linearization of (9) at a solution u_n near $\mathcal{P}$ on Ω_n has 3 eigenvalues $\lambda_n^1,\lambda_n^2,\lambda_n^3$ with $\lambda_n^i\le 0$ (not necessarily distinct) and orthogonal normalized eigenfunction v_n^i (in L^2). Without loss of generality, we may assume $u_n\to u_0$ as $n\to\infty$. It is easy to see that λ_n^i are bounded below and v_n^i are bounded in $W^{1,2}$. Fix i. Let $\mathcal{R}=\widetilde{\Omega}_2$ and $\mathcal{L}=\widetilde{\Omega}_1$. Now $v_n^i|_{\mathcal{R}}$ is bounded in $W^{1,2}(\mathcal{R})$ and hence by the Sobolev embedding $v_n^i|_{\mathcal{R}}$ has a subsequence converging weakly in $W^{1,2}(\mathcal{R})$ to w^i. Choose a further subsequence so $\lambda_n^i\to\alpha\le 0$ as $n\to\infty$. We prove that w^i is a multiple of $Au_0|_{\mathcal{R}}$. It suffices to prove that w^i is a solution of the Neumann problem for

$$-\Delta w=\lambda(1-\widetilde{h}'(u_0))w+\alpha w\quad\text{on }\mathcal{R}. \tag{12}$$

Here $\widetilde{h}:\mathcal{R}^2\to\mathcal{R}^2$ is defined by $\widetilde{h}(y)=|y|^2y$ and $u_0\in P$. This follows because we know from our construction that all eigenvalues α of (12) satisfy $\alpha\ge 0$. Hence $\alpha=0$ and our claim follows from our construction. Since Y, the set of smooth functions vanishing near the corners of $\mathcal{R}$, is dense in $W^{1,2}(\mathcal{R})$ (by using capacity ideas as in [9]), it suffices to show that the weak form of (12) holds for $\phi\in Y$. If $\phi\in Y$, we can think of $\phi\in W^{1,2}(\Omega_n)$ for large n by defining ϕ to be zero on $\Omega_n \setminus \mathcal{R}$. Substituting this in the weak form of the equation we see that

$$\int_{\mathcal{R}} \nabla w_n^i \nabla \phi + \int_{\mathcal{R}} \lambda(1 - \widetilde{h}'(u_n)) w_n^i \phi = \lambda_n^i \int_{\mathcal{R}} w_n^i \phi$$

and we obtain the required inequality by passing to the weak limit. Similarly, $v_n^i|_{\mathcal{L}}$ has a subsequence converging weakly in $W^{1,2}(\mathcal{L})$ to a multiple of $Au_0|_{\mathcal{L}}$. By choosing subsequences, we can arrange that $v_n^i|_{\mathcal{R}\cup\mathcal{L}}$ converges weakly in $W^{1,2}(\mathcal{L}\cup\mathcal{R})$ for $i = 1, 2, 3$. It follows easily that we can choose α^i, $i = 1, 2, 3$, with $\sum_{i=1}^3 \alpha_i^2 = 1$ such that $\widehat{v}_n = \sum_{i=1}^3 \alpha^i v_n^i$ converges weakly to zero in $W^{1,2}(\mathcal{R} \cup \mathcal{L})$ (and hence strongly to zero in $L^2(\mathcal{R} \cup \mathcal{L})$). Now $E_n^L(\widehat{v}_n) \leq 0$ since $\widehat{v}_n$ is a linear combination of orthogonal eigenfunctions corresponding to non-positive eigenvalues. Here E_n^L is the natural energy for the linearization at u_n. Now E_n^L is an integral over Ω_n. We decompose the integrals over Ω_n into the integrals over $\Omega_n', \Omega^{\mathcal{R}}, \Omega^{\mathcal{L}}$ where Ω_n' is the enlarged joining cylinder where we allow z to vary from $-1-2\delta$ to $-1+\delta$, $\Omega^{\mathcal{R}} = (\Omega_n \setminus \Omega_n') \cap \mathcal{R}$, and $\Omega^{\mathcal{L}} = (\Omega_n \setminus \Omega_n') \cap \mathcal{L}$.

Now $\lim_{n\to\infty}$ inf $E_{\mathcal{L}}^L(\widehat{v}_n) \geq 0$, where $E_{\mathcal{L}}^L$ denotes the contribution to $E_{\Omega_n}^L$ from the integral over $\Omega^{\mathcal{L}}$. Note that this follows easily because $\widehat{v}_n \to 0$ in $L^2(\Omega^{\mathcal{L}})$ while the term involving the gradient is non-negative. Similarly $\lim_{n\to\infty}$ inf $E_{\mathcal{R}}^L(\widehat{v}_n) \geq 0$ (with the obvious notation). Since $E^L(\widehat{v}_n) \leq 0$, it follows that

$$\lim_{n\to\infty} \sup E_{\Omega_n'}^L(\widehat{v}_n) \leq 0 \tag{13}$$

(with the obvious notation). We will prove in a moment that there is a $k > 0$ independent of n and μ_n with $\mu_n \to 0$ as $n \to \infty$ such that

$$E_{\Omega_n'}^L(\widehat{v}_n) \geq k\|\widehat{v}_n\|_{2,\Omega_n'} - \mu_n. \tag{14}$$

Hence (13) implies that $\|\widehat{v}_n\|_{2,\Omega_n'} \to 0$ as $n \to \infty$. This is impossible since $\|\widehat{v}_n\|_{2,\Omega_n} = 1$, $\|\widehat{v}_n\|_{2,\Omega^{\mathcal{L}}} \to 0$ and $\|\widehat{v}_n\|_{2,\Omega^{\mathcal{R}}} \to 0$ as $n \to \infty$.

Thus it remains to prove (14) (at least for δ small). First note that

$$\widetilde{\Omega} = \Omega_n' \cap \left\{z \leq -1 - \tfrac{3}{2}\delta \text{ or } z \geq -1 + \tfrac{1}{2}\delta\right\}$$

is in the interior of $\mathcal{R} \cup \mathcal{L}$ and hence by standard $W^{2,p}$ interior estimates we can ensure that $v_n^i|_{\widetilde{\Omega}}$ converges strongly in C^1. Hence $\widehat{v}_n \to 0$ in C^1 on $\widetilde{\Omega}$. Now if δ is small, the first eigenvalue of $-\Delta$ on Ω_n' with Dirichlet boundary conditions on $z = -1-2\delta$ or $-1+\delta$ and Neumann on the other parts of the boundary is large (greater than or equal to k). For example, we can separate variables. Choose $\phi(z)$ smooth and scalar-valued so that $\phi(z) = 1$ if $-1 - \frac{3}{2}\delta \leq z \leq -1 + \frac{1}{2}\delta$ and $\phi(z) = 0$ if $z = -1+\delta$ or $-1-2\delta$. Then $\widehat{v}_n\phi(z)$ is a suitable test function in the energy form for $-\Delta$ on Ω_n' with the above boundary conditions. Hence

$$\int_{\Omega_n'} \nabla(\widehat{v}_n\phi(z))^2 \geq k \int_{\Omega_n'} (\widehat{v}_n\phi(z))^2.$$

Since $\widehat{v}_n$ is C^1 small on $\widetilde{\Omega}$ and $\phi(z) = 1$ if $-1 - \frac{3}{2}\delta \leq z \leq -1 + \frac{1}{2}\delta$, it follows easily that

$$\int_{\Omega'_n} |\nabla \widehat{v}_n|^2 \geq k \int_{\Omega'_n} |\widehat{v}_n|^2 - \mu_n$$

where $\mu_n \to 0$ as $n \to \infty$. (14) follows easily from this if we choose δ small so that k is large. (The other term in the energy $E^L_{\Omega'_n}$ is easily seen to be dominated by $4\lambda \int_{\Omega'_n} |\widehat{v}_n|^2$.)

This completes the construction of our counterexample.

REMARKS. 1. Clearly, we could allow rather more general joining strips but our arguments need some restriction on the joining strip. We could also round off the corners to get an example which is a smooth domain.

2. The difficulties in the argument are largely caused by the fact that the product of the symmetries leave some indeterminacy on where the stable solution should lie.

3. We *suspect* that there is an alternative way of overcoming some of the difficulties. It seems very likely that we can modify the arguments of Saut and Temam [23] to prove that, for generic smooth Ω, the linearization of (9) at every solution has at most a one-dimensional kernel. This would enable one to choose Ω_n so that each of the orbits of solutions is isolated. This would greatly simplify the proof above (by choosing suitable Ω_n).

3. Exterior near critical problems

The purpose of this section is to prove some theorems on the existence of positive solutions of

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^m \setminus \Omega, \quad u = 0 \quad \text{in } \partial\Omega, \tag{15}$$

where Ω is a smooth bounded open set in $\mathbb{R}^m$ (not necessarily connected) but with $\mathbb{R}^m \setminus \Omega$ connected, $m > 2$ and p is less than but close to $p^* = (m+2)(m-2)^{-1}$. By an inversion about some point in Ω, we see as in [11] that this is equivalent to finding positive solutions of

$$-\Delta u = \|x\|^{-q(p)}|u|^{p-1}u \quad \text{in } \Omega^* \setminus \{0\}, \quad u = 0 \quad \text{on } \partial\Omega^*, \tag{16}$$

where $\Omega^* \setminus \{0\}$ is the image of $\mathbb{R}^m \setminus \Omega$ under the inversion and $q(p) = (m+2) - p(m-2)$.

We prove results which show that many positive solutions of the problem

$$-\Delta u = |u|^{p^*-1}u \quad \text{in } \Omega^*, \quad u = 0 \quad \text{on } \partial\Omega^*, \tag{17}$$

can be continued to give positive solutions of (16) and hence of (15). We use similar techniques to those of §1 except that it is more convenient to work in the space $\dot{W}^{1,2}(\Omega^*)$ rather than $L^2(\Omega^*)$.

We assume that T is a component of the positive solutions of (17) in $\dot{W}^{1,2}(\Omega^*)$ such that T is compact, is isolated and bounded in L^∞. We also assume that the natural energy is constant on T. (If T is reasonably smooth as in §2 or if $m = 3$ or 4 or 6 so that our nonlinearity is real analytic on $L^\infty(\Omega^*)$, it is not difficult to show that this last condition holds automatically).

It is easy to see that non-trivial solutions u of (17) cannot be small in the $W^{1,2}$ norm and we can use similar arguments to those in Theorem 2.3 of [3] to show that $\|u^-\|_{1,2}$ cannot be small and non-zero. This shows that components of positive solutions of (17) are also components of all solutions of (17).

Since our non-linearity is locally Lipschitz on $\dot{W}^{1,2}(\Omega^*)$ (in fact C^1), it is not difficult to deduce that there is a neighbourhood W of T in $\dot{W}^{1,2}(\Omega^*)$ such that the flow of

$$\dot{u} = -u + S(u)$$

is globally defined on W. Here S is defined by

$$\langle S(u), v\rangle = \int_{\Omega *} |u|^{p^*-1}uv$$

and we have used the Sobolev embedding theorem. It can be shown (see later) that this flow satisfies the strong admissibility condition of [22] on W so that homotopy indices can be defined. In particular, arguing as in §1 we see that there are suitable neighbourhoods of N of T (with $N \subseteq W$) such that the homotopy index $h(T)$ is defined and is independent of N.

Theorem 3. *Assume there is a component T as above such $h(T)$ is non-trivial. Then the exterior problem* (1) *has positive solutions for all p close to but less than p_*.*

We actually construct solutions of (16) which are in $\dot{W}^{1,2}(\Omega^*)$.

Before proving the theorem, we make some remarks on the assumptions. Firstly, it is well known that all solutions of (17) in $\dot{W}^{1,2}(\Omega^*)$ are smooth and that all solutions $W^{1,2}$ close to a fixed solution in the $W^{1,2}$ norm are uniformly bounded in L^∞. (To prove the latter, we use the fact that given a compact subset Y in $\dot{W}^{1,2}(\Omega^*)$, and a $c > 0$, we can choose $\lambda > 0$ and a neighbourhood Z of Y on $\dot{W}^{1,2}(\Omega^*)$ such that $\int_{|u|>\lambda} |u|^{4/(m-2)} < c$ for $u \in Z$. This follows from the Sobolev embedding theorem. We then use the proof in [3] or Kavian [21] to bound u in L^q for large q and then use the Gilbarg–Trudinger estimates.) Hence a compact set in $\dot{W}^{1,2}(\Omega^*)$ is automatically uniformly bounded in $L^\infty(\Omega^*)$ and an isolated compact component in $\dot{W}^{1,2}(\Omega^*)$ is also isolated and compact in $L^\infty(\Omega^*)$ and vice versa.

Proof of Theorem 3. Choose $K > \sup\{\|u\|_\infty : u \in T\}$. Let $f_{K,p}$ be C^1 and increasing such that $f_{K,p}(y) = |y|^p \operatorname{sign} y$ if $|y| \leq K$ and $K^p \operatorname{sign} y$ if

$|y| \geq K+1$. We will restrict K a little more later. If p is close to p^*, we will prove that the equation

$$u - \widetilde{f}_{K,p}(u) = 0 \tag{18}$$

has a solution u near T in $\dot{W}^{1,2}(\Omega^*)$ with $\|u\|_\infty < K$. This will complete the proof. Here $\widetilde{f}_{K,P}$ is the mapping of $\dot{W}^{1,2}(\Omega^*)$ into itself defined by

$$\langle \widetilde{f}_{K,p}(u), v\rangle = (\|x\|^{-q(p)} f_{K,p}(u), v).$$

It follows easily from the Sobolev embedding theorem that $\widetilde{f}_{K,p}$ is a well defined continuous map for p close to p^*.

We first consider solutions of $u = \widetilde{f}_{K,p^*}(u)$ near T. Equivalently, we consider solutions of $\Delta u = f_{K,p^*}(u)$ on Ω^* with Dirichlet boundary conditions on $\partial\Omega^*$. Let g_K by defined by $f_{K,p^*}(y) = y g_K(y)$. It is easy to see from the Sobolev embedding theorem that given $c > 0$ we can choose K_1 large enough independent of K so that if u is close to T in $W^{1,2}(\Omega^*)$, then $\|g_K(u)\chi_{(|u|>K_1)}\|_t \leq c$ where $t = 4/(m-2)$. The proof of Theorem 2.3 in Brezis and Kato [3] then implies a bound for $\|u\|_q$ for each large q. (Remember that, since u is close to T, $\|u\|_2$ is bounded.) Choosing $q > \frac{1}{2}mp^*$ and a simple bootstrapping argument implies that we have a bound independent of K for $\|u\|_\infty$ when u is close to T. We now choose K larger than this bound. This means that all solutions of (18) (for $p = p^*$) near T in $\dot{W}^{1,2}(\Omega^*)$ are bounded above by K and hence are solutions of (16). Hence T is an isolated (in $\dot{W}^{1,2}(\Omega^*)$) component of solutions of (17). We can use the same argument for the equations (for $0 \leq s \leq 1$)

$$-\Delta u = s|u|^{p^*-1}u + (1-s)f_{K,p^*}(u) \quad \text{in } \Omega^*.$$

Hence the homotopy index for the flow of

$$\dot{u}(t) = -u(t) + (1-s)\widetilde{f}_{K,p^*}(u(t)) + sS(u(t))$$

on an isolating neighbourhood of T (but contained in W) is independent of s. Here as in §1 we use Theorem I.12.2 of [22]. The only difficulty is to prove that the flow is strictly admissible on W for each s. To do this it suffices to show our map satisfies the Palais–Smale condition on W. The easiest way to see this is to use the fact that the map $u \to u - (1-s)\widetilde{f}_{K,p^*}(u) - sS(u)$ is locally proper near T. This follows easily because the map is C^1 (cp. [5]), and its derivative is Fredholm on T (cp. [24]).

Hence we see that $I - \widetilde{f}_{K,p^*}$ also has non-trivial homotopy index on some isolating neighbourhood of T. If we prove that $\widetilde{f}_{K,p}$ is uniformly close to $\widetilde{f}_{K,p^*}$ on W in the $\dot{W}^{1,2}$ norm if p is near p^*, it will follow that $I - \widetilde{f}_{K,p}$ has non-trivial homotopy index on an isolating neighbourhood of T and hence has a zero near T. Hence we will have completed the proof if we show that $\widetilde{f}_{K,p}$ is uniformly close to $\widetilde{f}_{K,p^*}$ on W in the $W^{1,2}$ norm and that solutions u of $u = \widetilde{f}_{K,p}(u)$ near T satisfy $\|u\|_\infty \leq K$.

To prove the first of these statements, note that by the Sobolev embedding theorem, it suffices to obtain an estimate of $\|\widehat{f}_{K,p}(u) - f_{K,p^*}(u)\|_s$ where $s = (m+2)(m-2)^{-1}$ and $\widehat{f}_{K,p}(u) = \|x\|^{-q(p)} f_{K,p}(u)$. Away from the origin, it is easy to see that $\widehat{f}_{K,p}(y) - f_{K,p^*}(y)$ is uniformly small for $y \in \mathbb{R}$. Thus it suffices to make the estimate near zero. The estimate near zero follows easily since $|\widehat{f}_{K,p}(y) - f_{K,p^*}(y)| \leq \|x\|^{-q(p)} K_2$ (where K_2 depends on K).

Thus it suffices to prove that solutions u of

$$-\Delta u = \widehat{f}_{K,p}(u)$$

near T in the $\dot{W}^{1,2}$ norm satisfy $\|u\|_\infty \leq K$. If $q > \frac{1}{2}m$ and fixed, a slight variant of the argument of the previous paragraph shows that $\|\widehat{f}_{K,p}(u) - f_{K,p^*}(u)\|_q$ is uniformly small on W if p is near p^*. On the other hand, since f_{K,p^*} is continuous and bounded the map $u \to f_{K,p^*}(u)$ is continuous as a mapping of $W^{1,2}(\Omega^*)$ into $L^q(\Omega^*)$. (Remember that the inclusion of $W^{1,2}(\Omega^*)$ into $L^{\widetilde{q}}(\Omega^*)$ is continuous if $\widetilde{q} = 2m(m-2)^{-1}$.) Hence, since T is compact, if u is close to T in $W^{1,2}(\Omega^*)$, then $f_{K,p^*}(u)$ is close to $f_{K,p^*}(v)$ in $L^q(\Omega^*)$ for some $v \in T$. Thus $\widehat{f}_{K,p}(u)$ is close to $f_{K,p^*}(v)$ in $L^q(\Omega^*)$ for some $v \in T$. Hence, by L^∞ regularity theory, $(-\Delta)^{-1}\widehat{f}_{K,p}(u)$ is $L^\infty(\Omega^*)$ close to $(-\Delta)^{-1} f_{K,p^*}(v) = v$ for some $v \in T$. Hence we see that any fixed point of $\widetilde{f}_{K,p}$ near T in $\dot{W}^{1,2}(\Omega^*)$ for p close to p^* is L^∞ close to some element of T. By our earlier bounds for T in L^∞, it follows that any fixed point u of $f_{K,p}$ for p near p^* satisfies $\|u\|_\infty < K$, as required. There is one minor point in the above proof. We need to check solutions we obtain are positive. This follows easily by multiplying by u^- since u is uniformly bounded and $\|u^-\|_{2m/(m-2)}$ is small. This is similar but much easier than arguments in [3]. This completes the proof.

Remarks. 1. If T is a single point (and with some work for all such T), we can apply Theorem III.4.8 of [22] to prove that the cohomology of $h(T)$ with coefficients in $\mathbb{Z}$ is simply $H^i(E^\mu \cap V, E^\mu \cap (V \setminus T), \mathbb{Z})$ where $\mu = E(T)$, $E^\mu = \{u \in \dot{W}^{1,2}(\Omega^*) : E(u) \leq \mu\}$ and V is some neighbourhood of T in $\dot{W}^{1,2}(\Omega^*)$.

2. Note that the homotopy index of T is unchanged if we replace $|u|^{p^*-1}u$ by $(u^+)^{p*}$.

3. The reason for the interest in the above theorem is the following. In [11], we conjectured that if Ω has non-trivial homology with coefficients in $\mathbb{Z}_2$, then (15) has a positive solution for p close to but less than p^*. The above result gives strong grounds for believing this conjecture. If we can find a, b positive such that the cohomology $H(E^b_+, E^a_+)$ is non-trivial, the positive solutions of (17) with energies in $[a, b]$ are isolated and if the Palais–Smale condition holds for u with

$E_+(u) \in [a, b]$, then it follows easily from Theorem 3 and Remark 1 after Theorem 3 that the conjecture is true in this case. Here E_+ is the natural energy for the non-linearity $(u^+)^{p*}$. Moreover, the proof of the Bahri–Coron theorem [2] seems to show that, under the above conditions on Ω and if all positive solutions of (17) are isolated, then there exist a, b positive such that $H(E_+^b, E_+^a)$ is non-trivial, the Palais–Smale condition holds on $[a, b]$ except possibly at the ends and there is a positive solution of (17) with non-trivial homotopy index. (If there are positive solutions with energies at the critical levels where the Palais–Smale condition fails, we also need to use the removability theorem of [12].) Thus, in a variety of cases, Theorem 3 implies that our conjecture is true and provides strong support that it is true in general. In fact it seems that the isolatedness assumptions can be greatly weakened especially in the cases where $m = 3, 4, 6$ where the non-linearity u^{p^*} is real analytic. Indeed, in these cases, the occasion where our attempted proof of our conjecture seems to have most serious difficulties is when there is a non-compact component of the set of positive solutions of (17).

REFERENCES

[1] A. BAHRI AND J. CORON, *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 253–294.

[2] F. BETHUEL, H. BREZIS, AND F. HÉLEIN, *Ginzburg–Landau Vortices*, Birkhäuser, Boston, 1994.

[3] H. BREZIS AND T. KATO, *Remarks on the Schrödinger equation with complex potentials*, J. Math. Pures Appl. **58** (1979), 137–151.

[4] K. J. BROWN, P. C. DUNNE AND R. A. GARDNER, *A semilinear parabolic system arising in the theory of superconductivity*, J. Differential Equations **40** (1981), 232–252.

[5] K. C. CHANG, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.

[6] C. CONLEY, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conf. Ser. in Math., vol. 38, Amer. Math. Soc., Providence, 1978.

[7] E. N. DANCER, *Degenerate critical points, homotopy indices and Morse inequalities*, J. Reine Angew. Math. **350** (1984), 1–22.

[8] ———, *The effect of domain shape on the number of positive solutions of certain weakly nonlinear equations*, J. Differential Equations **74** (1988), 120–156.

[9] ———, *The effect of domain shape on the number of positive solutions of certain nonlinear equations II*, J. Differential Equations **87** (1990), 316–339.

[10] ———, Addendum to [9], J. Differential Equations **97** (1992), 379–381.

[11] ———, *Superlinear problems on domains with holes of asymptotic shape and exterior problems*, preprint, University of Sydney, 1995.

[12] ———, *Degenerate critical points, homotopy indices and Morse inequalities III*, Bull. Austral. Math. Soc. **40** (1989), 97–108.

[13] ———, *The G-invariant implicit function theorem in infinite dimensions II*, Proc. Roy. Soc. Edinburgh Sect. A **102** (1986), 211–220.

[14] E. N. Dancer, *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. Appl. **91** (1983), 131–151.

[15] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[16] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin, 1981.

[17] S. Jimbo and Y. Morita, *Stability of non-constant steady state solutions to a Ginzburg–Landau equation in higher space dimensions*, Nonlinear Anal. **22** (1994), 753–770.

[18] ———, *Stable solutions with zeros to the Ginzburg–Landau equation with Neumann boundary condition*, preprint.

[19] S. Jimbo, Y. Morita and J. Zhai, *Ginzburg–Landau equation and stable steady state solutions in a non-trivial domain*, Comm. Partial Differential Equations (to appear).

[20] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.

[21] O. Kavian, *Introduction à la théorie des points critiques*, Springer-Verlag, Paris, 1993.

[22] K. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Springer-Verlag, Berlin, 1987.

[23] J. Saut and R. Temam, *Generic properties of nonlinear boundary value problems*, Comm. Partial Differential Equations **4** (1979), 293–319.

[24] S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.

[25] J. Sylvester, *A generalization of the Leray–Schauder index formula*, J. Funct. Anal. **45** (1982), 213–225.

[26] Z. Q. Wang, *Equivariant Morse theory for isolated critical orbits and its applications to nonlinear problems*, Lecture Notes in Math., vol. 1306, Springer-Verlag, Berlin, 1988, pp. 202–221.

Manuscript received May 10, 1996

E. N. Dancer
School of Mathematics and Statistics
University of Sydney
N.S.W. 2006, AUSTRALIA