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THE CONLEY INDEX AND THE CRITICAL GROUPS VIA AN EXTENSION OF GROMOLL–MEYER THEORY

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

We investigate, in a variational setting, the relationship between the Gromoll–Meyer pairs of *a dynamically isolated critical set* and the Conley index pairs of its *isolating invariant neighbourhoods*. We show that the information given by the critical groups of such a set is equivalent to that given by the Conley index. This allows us to derive—in a non-compact setting—various invariance properties for the Conley index from those of the critical groups, as well as a formula relating the degree of a gradient vector field in an isolating neighbourhood to the Conley index pair associated with it.

0. Introduction

The *Conley index* provides an algebraic-topological measure of an isolated invariant subset of a compact space on which a two-sided flow is acting. In a variational context, i.e., when a gradient flow on a compact manifold is considered, such an index generalizes the *Morse index* of a non-degenerate critical point. In the last decade, Conley's theory have been actively refined and extended to the non-compact case in order to overcome the limitations to its applicability in the theory of partial differential equations. See, for example, Rybakowski [R], Benci [B], Salamon [S].

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At the same time, Morse theory was also being extended and developed to overcome the non-compactness, regularity and degeneracy problems that occur in many important variational problems. Some compactness is usually restored by imposing the *Palais–Smale condition* on the functional f under study, while the problems of regularity (i.e., when f is not C^2) have been handled by the introduction of the *critical groups* for an isolated critical point (Rothe [Ro], Chang [Ch]). An alternative definition of the critical groups—via the concept of a (G-M)-pair—was also given by Gromoll–Meyer [G-M] in a seminal paper where they also deal with possibly degenerate but always isolated critical points. In his 1983 lecture notes at Montreal, the first-named author showed—among other things—that the two notions of critical groups coincide and established the homotopy invariance of (G-M)-pairs. He also indicated how the whole Gromoll–Meyer theory can be naturally extended to *isolated critical sets*.

In this paper, we investigate the connection between these theories and more precisely we establish the close relationship between the (G-M)-pairs of an isolated critical set and the *Conley index pairs* for its isolating invariant neighbourhoods. This correspondence allows us to deduce various stability properties of the Conley index from their counterparts in the Gromoll–Meyer theory extended to isolated critical sets. Now, many of these properties (of the Conley index) have been extended to the non-locally compact case by Benci [B], Rybakowski [R], Salamon [S] and others under various hypotheses, but even though we only deal here with a variational setting, the advantages of our approach are three-fold:

First, the proofs of the various properties of the critical groups (which are ultimately transferable to the Conley index!) are much simpler than in Conley's theory and its non-compact extensions. Secondly, it eliminates some of the unnatural hypotheses that are usually imposed on the critical set like connectedness and the likes and thirdly, it allows us to establish a relationship between *the degree* of a gradient vector field in an isolating neighbourhood and the Conley index pair associated with it, a fact that is not usually dealt with in classical Conley theory.

In this paper, we shall restrict ourselves to the study of variational structures. For the convenience of the reader, the notions of isolated critical sets and their critical groups will be reviewed and studied in detail. In this paper, we shall call them *dynamically isolated critical sets* to distinguish them from the "topologically isolated" critical sets. We also recall the notions of isolated invariant set and their isolating neighbourhoods. We mainly prove that the (G-M)-pairs are Conley index pairs and that the Conley index coincides with the critical groups. Since the homotopy invariance for critical groups and other properties have been proved very simply in [Ch], many of them will automatically hold for the Conley index.

I. Dynamically isolated critical sets and their isolating neighbourhoods

A *flow* on a metric space M is a continuous map $\eta : \mathbb{R} \times M \to M$ that has the following properties:

(i) $\eta(0, x) = x$ for any $x \in M$, and

(ii) $\eta(t_1, \eta(t_2, x)) = \eta(t_1 + t_2, x)$ for any $t_1, t_2 \in \mathbb{R}$ and $x \in M$.

For a subset B of M and $\delta > 0$, we denote by $N_\delta(B)$ the δ-neighbourhood of B, i.e., the set $\{x \in M : \operatorname{dist}(x, B) < \delta\}$.

We shall use the following set of notation: For $T > 0$, let

$$B^T = \bigcup_{|t| \le T} \eta(B, t), \quad \text{where } \eta(B, t) = \{y = \eta(x, t) : x \in B\}$$

$$\widetilde{B} = B^\infty = \bigcup_{t \in \mathbb{R}} \eta(B, t) \quad \text{and} \quad \widetilde{B}^+ = \bigcup_{t \ge 0} \eta(B, t).$$

If $T_1 \le 0 \le T_2$, we shall write

$$G_{T_2}^{T_1}(B) = \{x \in \operatorname{cl}(B) : \eta(x, [T_1, T_2]) \subset \operatorname{cl}(B)\} = \bigcap_{T_1 \le t \le T_2} \eta(\operatorname{cl}(B), t),$$

$$G^T(B) = G_T^{-T}(B) \quad \text{and} \quad I(B) = G^\infty(B) = \bigcap_{t \in \mathbb{R}} \eta(\operatorname{cl}(B), t).$$

In addition, for any $x \in M$, we consider

$$\omega(x) = \bigcap_{t > 0} \operatorname{cl}(\eta(x, [t, \infty)) \quad \text{(the } \omega\text{-}\textit{limit set} \text{ of } x) \quad \text{and}$$

$$\omega^*(x) = \bigcap_{t > 0} \operatorname{cl}(\eta(x, [-\infty, -t]) \quad \text{(the } \omega^*\text{-}\textit{limit set} \text{ of } x).$$

We now recall a few definitions.

DEFINITION I.1 (invariant subset). A subset B of M is called *invariant* (with respect to η) if for all $x \in B$ and for all $t \in \mathbb{R}$, $\eta(x, t) \in B$.

It is easy to see that, given a closed set $A \subset M$, the (possibly empty) set $I(A) = \bigcap_{t \in \mathbb{R}} \eta(A, t)$ is a maximal closed invariant subset in A.

Now, we come to define the important notions of an isolated invariant set and an isolating neighbourhood.

DEFINITION I.2 (isolated invariant set). An invariant set A is called *isolated* if there exists a closed neighbourhood U and $T_1 < 0 < T_2$ such that

$$A = I(U) \subset G_{T_2}^{T_1}(U) \subset \operatorname{int}(U). \tag{I.1}$$

In this case, U is called an *isolating neighbourhood* of A.

REMARK I.3. Conley's definition of an isolated invariant set says that $A = I(U) \subset \operatorname{int}(U)$. This is equivalent to the above definition in the compact case but not in general. It was Benci [B] who came up with this stronger notion that seems to be suitable for the non-locally compact setting.

The following notion is crucial to what follows.

Definition I.4 (Mean Value Property). A subset W of M is said to have the *Mean Value Property* (for short (MVP)) if for any $x \in M$ and any $t_0 < t_1$, we have

$$\eta(x, [t_0, t_1]) \subset W \quad \text{whenever} \quad \eta(x, t_i) \in W \text{ for } i = 0, 1. \tag{I.2}$$

In the sequel, we shall restrict ourselves to a variational context that we now describe.

Suppose M is a C^1-Finsler manifold and let $f \in C^1(M, \mathbb{R})$ be a function satisfying the *Palais–Smale condition* (for short (P-S)): a sequence $(x_n)_n$ in M is relatively compact whenever $(f(x_n))_n$ is bounded and $\|df(x_n)\| \to 0$.

We denote by $K = K_f$ the critical set of f. A *pseudo-gradient vector field* for f (for short p.g.v.f.) is defined to be a section V of the tangent bundle $T(M)$ satisfying, for all $x \in M$,

$$\langle df(x), V(x)\rangle \geq \|df(x)\|^2, \tag{I.3}$$

and

$$\|V(x)\| \leq A\|df(x)\| \tag{I.4}$$

for some constant $A > 0$. With such a vector field, we may associate a flow η on M as a solution to the Cauchy problem

$$\begin{cases} \dot{\eta}(x,t) = V_1(\eta(x,t)), \\ \eta(x,0) = x, \end{cases} \tag{I.5}$$

where

$$V_1(x) = g(x)\frac{V(x)}{\|V(x)\|} \quad \text{and} \quad g(x) = \min\{\operatorname{dist}(x, K), 1\}. \tag{I.6}$$

Thus, $\|V_1(x)\| \leq 1$ for any $x \in M$ and the flow η is well defined on $M \times \mathbb{R}$.

The following notion is also useful.

Definition I.5 (invariant hull). Given a subset $S \subset M$, we define the *invariant hull* of S to be

$$[S] = \{x \in M : \omega(x) \cup \omega^*(x) \subset S\}. \tag{I.7}$$

Note that if $S \subset K_f$, then $\omega(x) = \omega(\eta(x,t))$ and $\omega^*(x) = \omega^*(\eta(x,t))$ for all t and all $x \in S$. Therefore $[S]$ is a minimal invariant set containing S.

THEOREM I.6. *Let f be a C^1-functional satisfying the (P-S) condition and suppose that W is a closed (MVP) neighbourhood of a subset S of K_f satisfying $W \cap K_f = S$ and $W \subset f^{-1}[\alpha, \beta]$, where α, β are regular values of f. Then $[S]$ is an isolated invariant set. Moreover, any closed neighbourhood U of $[S]$ with $U \subset W$ is an isolating neighbourhood for $[S]$.*

We shall split the proof into several lemmas of independent interest. First, we exploit the (P-S) condition.

LEMMA I.7. *Suppose f is a C^1-functional satisfying the (P-S) condition. Then, for any $x \in M$, the set $\omega(x)$ is compact and is a subset of K_c for some critical value c, where $K_c = K \cap f^{-1}(c)$. The same holds true for the sets $\omega^*(x)$.*

PROOF. First, we show that $\omega(x)$ is on one level, say $\omega(x) \subset f^{-1}(c)$ for some $c \in \mathbb{R}$. Indeed, if not, then there exist $t_n, t'_n \uparrow \infty$ such that $\eta(x, t_n) \to y$ and $\eta(x, t'_n) \to y'$ with $f(y) < f(y')$. We may always assume that $t'_n > t_n$, which means that

$$f(y') = \lim_n f(\eta(x, t'_n)) \le \lim_n f(\eta(x, t_n)) = f(y),$$

which is a contradiction.

Next, we prove that $\omega(x) \subset K$. Indeed, if $y \in \omega(x) \setminus K$, choose regular values $a < b$ such that both $f(x)$ and $f(y)$ are in (a, b). Since $K_a^b = K \cap f^{-1}[a, b]$ is compact, there exists $r > 0$ such that $B_r(y) \cap N_r(K_a^b) = \emptyset$. By definition, there exists $t_n \to \infty$ such that $x_n = \eta(y, t_n) \in B_r(y)$. However, we claim that there exists $t'_n \to \infty$ such that $x'_n = \eta(y, t'_n) \in \partial N_r(K_a^b)$, because if not, there must be $T > 0$ such that $\eta(y, [T, \infty)) \cap N_r(K_a^b) = \emptyset$. Since there exists $\delta > 0$ such that $\|f'(x)\| \ge \delta$ for all $x \in f^{-1}[a, b] \setminus N_r(K_a^b)$, we have

$$f(y) = \lim_n f(x_n) \le \liminf_{t \to \infty} f(\eta(x, t)) \le a,$$

which is impossible. Now we choose t'_n, $t''_n \to \infty$ with $t'_n < t''_n$ such that

$$x''_n = \eta(x, t''_n) \in B_r(y), \quad x'_n = \eta(x, t'_n) \in \partial N_r(K_a^b), \quad \eta(x, [t'_n, t''_n]) \cap N_r(K_a^b) = \emptyset.$$

Then we have

$$t''_n - t'_n \ge |\eta(x, t''_n) - \eta(x, t'_n)| \ge \operatorname{dist}(B_r(y), N_r(K_a^b))$$

and

$$f(x'_n) - f(x''_n) \ge \delta r (t''_n - t'_n).$$

Again, this is impossible and the lemma is proved.

We are now interested to know under what condition on a neighbourhood W of a critical set S, we have $I(W) = [S]$. We shall need the following

Lemma I.8. *Let f be a C^1-functional satisfying the $(P\text{-}S)$ condition and suppose that W is a closed (MVP) neighbourhood of a critical set S satisfying $W \cap K = S$. Then*

1. $I(W) = [S] \subset \operatorname{int}(W)$,
2. *for any $T_1 < 0 < T_2$, the set $G_{T_2}^{T_1}(W)$ is again a closed (MVP) neighbourhood of $[S]$.*

Proof. (1) We first establish that:

• $[S] \subset I(W)$: Indeed, if $x \in [S]$, then by definition $\omega(x) \cup \omega^*(x) \subset S$. This implies that there are $t_n^\pm \to \pm\infty$ such that $\eta(x, t_n^\pm) \in W$. By the Mean Value Property of W, we have $\eta(x, [t_n^-, t_n^+]) \subset W$. Since n is arbitrary, it follows that $\eta(x,t) \in W$ for all t. This proves that $x \in I(W)$.

• $I(W) \subset [S]$: For all $x \in I(W)$, $\eta(x,t) \in W$ for all $t \in \mathbb{R}$. Since W is closed, $\omega(x) \cup \omega^*(x) \subset W$. According to Lemma I.7, $\omega(x) \cup \omega^*(x) \subset K$. Therefore $\omega(x) \cup \omega^*(x) \subset W \cap K = S$, that is, $x \in [S]$.

• $[S] \subset \operatorname{int}(W)$: For $x \in [S]$, there exist $t_- < 0 < t_+$ and neighbourhoods $U_\pm$ of $y_\pm = \eta(x, t_\pm)$ such that $U_\pm \subset W$. Set $V_\pm = \eta(U_\pm, \mp t_\pm)$ and $V = V_+ \cap V_-$. Then V is a neighbourhood of x satisfying $\eta(V, t_\pm) \subset U_\pm \subset W$. By the (MVP) property of W, we have $V \subset W$. This completes the proof of part (1).

To prove (2) first note that $G_{T_2}^{T_1}(W)$ is closed and has the (MVP). By (1), $[S] = I(W) \subset G_{T_2}^{T_1}(W)$. It remains to verify that $[S] \subset \operatorname{int}(G_{T_2}^{T_1}(W))$. But, if that is not the case, then there exists $x \in [S] \cap \partial G_{T_2}^{T_1}(W)$. That is, there exists a sequence $x_n \notin G_{T_2}^{T_1}(W)$ satisfying $x_n \to x$. This means that there are $t_n \in [T_1, T_2]$ such that $\eta(x_n, t_n) \notin W$. By extracting a subsequence, $t_n' \to t$, we deduce that $\eta(x,t) \notin \operatorname{int}(W)$. But $x \in [S]$, which means $\eta(x,t) \in [S] \subset \operatorname{int}(W)$ by the first part. This is a contradiction.

Lemma I.9. *Suppose W is a closed (MVP) neighbourhood of a critical set S. Assume that $W \cap K = S$ and $W \subset f^{-1}[\alpha, \beta]$, where α, β are regular values of f. Then, for any neighbourhood U of S, there exists $T > 0$ such that*

$$G^T(W) = \bigcap_{|t| \le T} \eta(W, t) \subset \operatorname{int}(U). \tag{I.11}$$

Proof. Let $x \notin \operatorname{int}(U)$. We need to show that there exists $t \in [-T, T]$ such that $\eta(x,t) \notin W$. Now, by the (P-S) condition, there exists $\delta > 0$ such that $\operatorname{dist}(x, K) \ge \delta$ and $\|f'(x)\| \ge \delta$ for all $x \in W \setminus \operatorname{int}(U)$. Set $T > \delta^{-2}(\beta - \alpha)$. We consider three cases.

(a) If $x \notin W$, then take $t = 0$ so that $\eta(t,x) = x \notin W$.

(b) If $x \in W \setminus \widetilde{\mathrm{int}(U)}$ and $\eta(x, [-T, T]) \subset W$, then $\eta(x, [-T, T]) \subset W \setminus \widetilde{\mathrm{int}(U)}$ and therefore,

$$f(\eta(x, -T)) - f(\eta(x, T)) \geq 2\delta^2 T > \beta - \alpha,$$

which is impossible. It follows that in this case $\eta(x, [-T, T]) \not\subset W$.

(c) If $x \in (\widetilde{\mathrm{int}(U)} \setminus \mathrm{int}(U)) \cap W = (\widetilde{\mathrm{int}(U)} \cap W) \setminus \mathrm{int}(U)$, then

$$\text{either} \quad x \in \bigcup_{t>0} \eta(\mathrm{int}(U), t) \quad \text{or} \quad x \in \bigcup_{t<0} \eta(\mathrm{int}(U), t).$$

In the first case, we have $t_1 \leq 0 \leq t_2$ such that

$$\eta(x, [t_1, t_2]) \subset (\widetilde{\mathrm{int}(U)} \cap W) \setminus \mathrm{int}(U) \quad \text{and} \quad \eta(x, t_1 - \varepsilon) \in U, \ \eta(x, t_2 + \varepsilon) \notin W$$

for all small enough $\varepsilon > 0$. But again, we would have $\beta - \alpha > \delta^2(t_2 - t_1)$ so that $t_2 < T$. Similarly, in the second case $t_1 > -T$.

In both cases, we conclude that $\eta(x, [-T, T]) \not\subset W$ and the proof of the lemma is complete.

Now, we come to the

Proof of Theorem I.6. Let W be a closed (MVP) neighbourhood of S such that $W \cap K_f = S$. From Lemma I.8, we obtain $[S] = I(W) \subset \mathrm{int}(W)$.

Let now U be any closed neighbourhood of $[S]$ satisfying $U \subset W$. We have

$$[S] = I([S]) \subset I(U) \subset I(W) = [S],$$

so that $I(U) = [S]$. By definition, $I(U) \subset G_{T_2}^{T_1}(U)$ for any $T_1 < 0 < T_2$ and by Lemma I.9, there exists $T > 0$ such that $G_T^{-T}(U) \subset G_T^{-T}(W) \subset \mathrm{int}(U)$. The proof of Theorem I.6 is complete.

We now recall the following key concept:

Definition I.10 (dynamically isolated critical set). A subset S of the critical set K is said to be a *dynamically isolated critical set* if there exist a closed neighbourhood $\mathcal{O}$ of S and regular values $\alpha < \beta$ of f such that

$$\mathcal{O} \subset f^{-1}[\alpha, \beta] \tag{I.8}$$

and

$$\mathrm{cl}(\widetilde{\mathcal{O}}) \cap K \cap f^{-1}[\alpha, \beta] = S. \tag{I.9}$$

We shall then say that $(\mathcal{O}, \alpha, \beta)$ is an *isolating triplet* for S.

Here are some common examples of isolated critical sets, The proofs are left to the interested readers. See also the examples in Section III.

Examples. 1) It is clear that if c is an isolated critical level, i.e., there are no critical points on the levels $[c-\varepsilon, c+\varepsilon]\setminus\{c\}$ for some $\varepsilon>0$, then the set $K_c = K\cap f^{-1}(c)$ is a dynamically isolated critical set.

2) If x_0 is a *non-degenerate critical point* for a C^2-functional f, then the singleton $S=\{x_0\}$ is a dynamically isolated critical set.

3) More generally, if x_0 is an isolated critical point, i.e., $\{x_0\}=K_f\cap U$ for some open set U, that is located on an isolated critical level, then again the singleton $S=\{x_0\}$ is a dynamically isolated critical set.

Here is the main result of this section:

Theorem I.11. *Let f be a C^1-functional satisfying the (P-S) condition on a C^1-Finsler manifold M. If S is a dynamically isolated critical set for f, then $[S]$ is an isolated invariant set. Moreover, if $(\mathcal{O},\alpha,\beta)$ is an isolating triplet for S, then any closed neighbourhood U of $[S]$ with $U\subset\mathcal{O}$ is an isolating neighbourhood for $[S]$.*

Theorem I.11 follows immediately from Theorem I.6 and the following result.

Lemma I.12. *Let $(\mathcal{O},\alpha,\beta)$ be an isolating triplet associated with a dynamically isolated critical set S. Then there exists $T>0$ such that*

$$\mathcal{O}^T\cap f^{-1}[\alpha,\beta]=\widetilde{\mathcal{O}}\cap f^{-1}[\alpha,\beta]=\mathrm{cl}(\widetilde{\mathcal{O}})\cap f^{-1}[\alpha,\beta]. \tag{I.10}$$

Moreover, the set $\{\mathcal{O}\}_\alpha^\beta=\widetilde{\mathcal{O}}\cap f^{-1}[\alpha,\beta]$ is a closed (MVP) neighbourhood of both S and $[S]$.

Proof. Set $Y=\mathrm{cl}(\widetilde{\mathcal{O}})\cap f^{-1}[\alpha,\beta]$. We need to show that $Y=\mathcal{O}^T\cap f^{-1}[\alpha,\beta]$ for some $T>0$. According to the (P-S) condition, there exists $\delta>0$ such that

$$\mathrm{dist}(x,K)\geq\delta \quad\text{and}\quad \|f'(x)\|\geq\delta \quad \text{for all } x\in Y\setminus\mathcal{O}.$$

If now $\eta(x,[0,t])\subset Y\setminus\mathcal{O}$ then

$$\beta-\alpha\geq f(x)-f(\eta(x,t))\geq-\int_0^t\langle f'(\eta(x,s)),\dot\eta(x,s)\rangle\,ds\geq\delta^2 t.$$

Let $T>\delta^{-2}(\beta-\alpha)$. If there exist $y\in Y\setminus\mathcal{O}^T$, then there exist $x\in\mathcal{O}$ and $t>T$ such that $y=\eta(x,t)$ and $\eta(x,[0,t])\cap\mathcal{O}=\emptyset$. This is clearly impossible and the lemma is proved.

II. The critical groups of an isolated critical set

In this section, we define the critical groups associated with a dynamically isolated critical set and we show that they are independent of the choice of their isolating triplet.

DEFINITION II.1 (critical groups). Let S be a dynamically isolated critical set of a C^1-functional f and let $(\mathcal{O}, \alpha, \beta)$ be any isolating triplet for S. For each integer q, we shall call the qth cohomology group

$$C_q(f, S) = H^q(f_\beta \cap \widetilde{\mathcal{O}}^+, f_\alpha \cap \widetilde{\mathcal{O}}^+) \tag{II.1}$$

the qth *critical group* for S.

Obviously, we need to show the following

PROPOSITION II.2. *The critical groups of a critical set S do not depend on the special choice of its isolating triplet $(\mathcal{O}, \alpha, \beta)$ nor on the choice of the pseudo-gradient vector field for f.*

PROOF. We verify the invariance in few steps.

1) First, assume the flow η and the neighbourhood $\mathcal{O}$ are fixed. The fact that the relative cohomology groups associated with different regular levels β and α are isomorphic is an immediate consequence of the basic deformation lemma between regular sublevels. (See for instance [Ch].)

2) Suppose now that $(\mathcal{O}, \beta, \alpha)$ and $(\mathcal{O}_1, \beta, \alpha)$ are two isolating triplets for S, with $\mathcal{O} \Supset \mathcal{O}_1 \Supset S$. We need to prove that

$$H^q(f_\beta \cap \widetilde{\mathcal{O}}^+, f_\alpha \cap \widetilde{\mathcal{O}}^+) = H^q(f_\beta \cap \widetilde{\mathcal{O}}_1^+, f_\alpha \cap \widetilde{\mathcal{O}}_1^+). \tag{II.2}$$

For that, we first show the following

CLAIM 1. *If $\mathcal{O} \Supset S$, then there exists $\delta > 0$ such that* $\operatorname{dist}(\partial(\widetilde{\mathcal{O}}^+), [S]) \geq \delta$.

Indeed, if not, there exist $x_n \in \partial(\widetilde{\mathcal{O}}^+)$ such that $x_n \to x \in [S]$. This implies that there exist $Z_n \in \partial\mathcal{O}$ and $t_n \in [0, T]$ such that $x_n = \eta(Z_n, t_n) \to x$. There exists a subsequence $t'_n \to t$ so that $Z'_n = \eta(x'_n, -t'_n) \to \eta(x, t) = Z$. Since $\partial\mathcal{O}$ is closed, $Z \in \partial\mathcal{O}$. But since $x \in [S]$, we deduce that $Z \in [S]$, which is a contradiction.

Choose now $\alpha_1 < \alpha$ such that $(\mathcal{O}_1, \beta, \alpha_1)$ is also an isolating triplet for S. Set $\mathcal{L} = \widetilde{\mathcal{O}}^+ \cap f^{-1}(\alpha_1)$, $\mathcal{L}_1 = \widetilde{\mathcal{O}}_1^+ \cap f^{-1}(\alpha_1)$ and $\mathcal{R} = \{x \in \widetilde{\mathcal{O}}^+ : \omega(x) \not\subset S\}$ and define a projection $\pi : \mathcal{R} \to \mathcal{L}$ in the following way:

For $x \in \mathcal{R}$, there exists a unique $y \in \mathcal{L}$ and a unique $t \in \mathbb{R}$ such that $y = \eta(x, t)$. Let $\pi(x) = y$ and $p(x) = t$. We need to show the following

CLAIM 2. *If we let $\mathcal{C} = \mathcal{L} \setminus \pi(\mathcal{R})$, then* $\operatorname{dist}(\mathcal{C}, \pi(\partial(\widetilde{\mathcal{O}}^+))) > 0$.

Indeed, if not, there exist $y_n \in \pi(\partial(\widetilde{\mathcal{O}}^+))$ such that $y_n \to y \in \mathcal{C}$. This yields the existence of $Z_n \in \partial\mathcal{O}$ and $t_n \in [0, T]$ such that $Z_n = \eta(y_n, -t_n)$. Thus, there exists a convergent subsequence $t_{n'} \to t$ along which $Z'_n = \eta(y_{n'}, -t_{n'}) \to \eta(y, -t) = Z$. Since $\partial\mathcal{O}$ is closed, $Z \in \partial\mathcal{O}$. This is a contradiction as $y \in \mathcal{C}$ and therefore Claim 2 is proved.

Now, setting $\mathcal{R}_1 = \{x \in \widetilde{\mathcal{O}}_1^+ : \omega(x) \not\subset S\}$ and $s_0 = \operatorname{dist}(\mathcal{C}, \pi(\partial\widetilde{\mathcal{O}}_1^+)) > 0$, we define a family of functions for $\tau \in [0,1]$:

$$\varphi_\tau(s) = \begin{cases} 1-\tau, & s \geq s_0, \\ 1 - s_0^{-1} s\tau, & 0 \leq s \leq s_0, \end{cases}$$

and a deformation $\psi : [0,1] \times (f_\beta \cap \widetilde{\mathcal{O}}^+) \to f_\beta \cap \widetilde{\mathcal{O}}^+$ in the following way:

$$\psi(\tau, x) = \begin{cases} \eta(y, -\varphi_\tau(\operatorname{dist}(y, \mathcal{C})t) & \text{if } x \in \mathcal{R} \cap f^{-1}[\alpha, \beta], \\ x & \text{if } x \in (f_\beta \cap \widetilde{\mathcal{O}}^+) \setminus \mathcal{R}, \end{cases}$$

where $y = \pi(x)$ and $t = p(x)$. We have $\psi(0,x) = x$, and

$$\psi(1, x) = \begin{cases} y, & x \notin \widetilde{\mathcal{O}}_1^+ \cap f^{-1}[\alpha_1, \beta], \\ \eta(y, -(1 - \operatorname{dist}(y, \mathcal{C})/s_0)t), & x \in \mathcal{R}_1 \cap f^{-1}[\alpha_1, \beta], \\ x, & x \in (f_\beta \cap \widetilde{\mathcal{O}}^+) \setminus \mathcal{R}. \end{cases}$$

Set $D_1 = \psi(1, f_\beta \cap \widetilde{\mathcal{O}}^+)$ and $D_2 = D_1 \setminus (\widetilde{\mathcal{L}} \setminus \mathcal{L}_1)^+$. By excision and deformation, we obtain

$$\begin{aligned} H^*(f_\beta \cap \widetilde{\mathcal{O}}^+, f_\alpha \cap \widetilde{\mathcal{O}}^+) &= H^*(D_1, \psi(1, f_\alpha \cap \widetilde{\mathcal{O}}^+)) \quad \text{(by deformation)} \\ &= H^*(D_2, \psi(1, f_\alpha \cap \widetilde{\mathcal{O}}^+) \setminus (\widetilde{\mathcal{L}} \setminus \mathcal{L}_1)^+) \quad \text{(by excision)} \\ &= H^*(D_2, \psi(1, f_{\alpha_1} \cap \widetilde{\mathcal{O}}_1^+)) \quad \text{(by deformation)} \\ &= H^*(f_\beta \cap \widetilde{\mathcal{O}}_1^+, f_{\alpha_1} \cap \widetilde{\mathcal{O}}_1^+) \quad \text{(by deformation)} \\ &= H^*(f_\beta \cap \widetilde{\mathcal{O}}_1^+, f_\alpha \cap \widetilde{\mathcal{O}}_1^+). \end{aligned}$$

By a similar technique, one may also prove the invariance of the critical groups under small perturbations of the flow.

III. Gromoll–Meyer pairs associated with a dynamically isolated critical set

First recall the following definition:

Definition III.1 (Gromoll–Meyer pairs). Let f be a C^1-functional on a Finsler manifold M and let S be a subset of the critical set K_f for f. A pair (W, W_-) of subsets is said to be a *(G-M)-pair* for S associated with a p.g.v.f. X if, for the flow η associated with X by (I.5), the following conditions hold:

(1) W is a closed (MVP) neighbourhood of S satisfying $W \cap K = S$ and $W \cap f_\alpha = \emptyset$ for some α.

(2) W_- is an *exit set* for W, i.e., for each $x_0 \in W$ and $t_1 > 0$ such that $\eta(x_0, t_1) \notin W$, there exists $t_0 \in [0, t_1)$ such that $\eta(x_0, [0, t_0]) \subset W$ and $\eta(x_0, t_0) \in W_-$.

(3) W_- is closed and is a union of a finite number of submanifolds that are transversal to the flow η.

Example. If f is a C^1-functional satisfying the (P-S) condition on a Finsler manifold M and if x_0 is an isolated critical point located on an isolated critical level, then one can associate with $\{x_0\}$ a (G-M)-pair in the following way:

For simplicity, assume that M is a Hilbert space and that $x_0 = 0$ and $f(0) = 0$. Choose $\varepsilon > 0$ and $\delta > 0$ such that 0 is the unique critical value in $[-\varepsilon, \varepsilon]$ and $x_0 = 0$ is the unique critical point in the ball B_δ centred at 0. By the (P-S) condition, we have

$$\beta = \inf_{x \in B_\delta \setminus B_{\delta/2}} \|df(x)\| > 0.$$

Take $0 < \lambda < 2\delta/\beta$ and define $g(x) = f(x) + \lambda\|x\|^2$. Choose now γ and μ in such a way that if $W = f^{-1}[-\gamma, \gamma] \cap g_\mu$ and $W_- = f^{-1}(-\gamma) \cap W$, then the following conditions hold:

$$\begin{gathered} 0 < \gamma < \min\{\varepsilon, 3\delta^2\lambda/8\} \quad \text{and} \quad \delta^2\lambda/4 + \gamma < \mu < \delta^2\lambda - \gamma, \\ B_{\delta/2} \cap f^{-1}[-\gamma, \gamma] \subset W \subset B_\delta \cap f^{-1}[-\varepsilon, \varepsilon], \\ f^{-1}[-\gamma, \gamma] \cap g^{-1}(\mu) \subset B_\delta \setminus B_{\delta/2}, \end{gathered}$$

and

$$\langle dg(x), df(x) \rangle > 0 \quad \text{for all } x \in B_\delta \setminus \operatorname{int}(B_{\delta/2}).$$

We leave it to the reader to verify that (W, W_-) is a (G-M)-pair for x_0 (see [Ch]).

More generally, if f is a C^1-functional satisfying the (P-S) condition on a Finsler manifold M and if S is a dynamically isolated critical set for f, then there are many ways to associate a (G-M)-pair (W, W_-) with S. Indeed, if $(\mathcal{O}, \beta, \alpha)$ is an isolating triplet for S, then one can easily verify that $W = \{\mathcal{O}\}_\alpha^\beta$ and $W_- = W \cap f^{-1}(\alpha')$ where $\alpha < \alpha' < \min\{f(x) : x \in S\}$ form a (G-M)-pair for S. Actually, as the following proposition shows, one can find a (G-M)-pair inside any isolating neighbourhood of S.

Proposition III.2. *Assume $(\mathcal{O}, \beta, \alpha)$ is an isolating triplet for a dynamically isolated critical set S. Then, for any neighbourhood U for $[S]$ such that $U \subset \{\mathcal{O}\}_\alpha^\beta$, there exist a (G-M)-pair (W, W_-) for S such that $W \subset U$.*

Proof. Indeed, let $\alpha < \alpha' < \min\{f(x) : x \in S\}$. By Lemma I.12, $\{\mathcal{O}\}_{\alpha'}^\beta$ is an (MVP) neighbourhood of $[S]$. By Lemma I.9, there exists $T > 0$ such that $W = G_T^{-T}(\{\mathcal{O}\}_{\alpha'}^\beta) \subset \operatorname{int}(U)$. Lemma I.8 yields that W is also a closed (MVP) neighbourhood of $[S]$. It is also clear that $W \cap K = S$ and $W \cap f_\alpha = \emptyset$.

We now look for an exit set E for W. Set $L(\alpha') = \{\mathcal{O}\}_{\alpha'}^\beta \cap f^{-1}(\alpha')$, which is a submanifold of $f^{-1}(\alpha')$. Since W is a neighbourhood of S, there is no critical point in $\{\mathcal{O}\}_{\alpha'}^\beta \setminus W$. Therefore, for all $x \in E$, there exists $t > 0$ such that $y = \eta(x, t) \in L(\alpha')$. However, by the definition of W, we have $t \equiv -T$. Thus $E = \eta(L(\alpha'), -T)$ is also a submanifold that is transversal to η. This proves that $(W, W_-) = (W, E)$ is a (G-M)-pair for S.

The following theorem is the main result of this section.

THEOREM III.3. *Let f be a C^1-functional on a C^1-Finsler manifold M and let S be a dynamically isolated critical set for f. Then, for any $(G\text{-}M)$-pair (W, W_-) for S, we have*

$$H^*(W, W_-) \cong H^*(f_\beta \cap \widetilde{W}^+, f_\alpha \cap \widetilde{W}^+) \cong C_*(f, S) \tag{III.1}$$

where $(\mathcal{O}, \alpha, \beta)$ is an isolating triplet for S.

PROOF. First we show that

$$H^*(f_\beta \cap \widetilde{W}^+, f_\alpha \cap \widetilde{W}^+) \cong H^*(\widetilde{W}^+, \widetilde{W}^+_-) \tag{III.2}$$

where

$$\widetilde{W}^+ = \bigcup_{t \geq 0} \eta(W, t) \quad \text{and} \quad \widetilde{W}^+_- = \bigcup_{t \geq 0} \eta(W_-, t).$$

For that, we define deformation retractions

$$\sigma_1 : \widetilde{W}^+_- \times [0,1] \to f_\alpha \cap \widetilde{W}_- \quad \text{and} \quad \sigma_2 = \widetilde{W}^+ \times [0,1] \to f_\beta \cap \widetilde{W}^+$$

as follows: Let $\gamma_1 : \widetilde{W}^+_- \to \mathbb{R}^1$ be the first hitting time of the level α from $\widetilde{W}^+_-$. That is,

$$\begin{aligned} \eta(x, \gamma_1(x)) \in f^{-1}(\alpha) \quad & \forall x \in \widetilde{W}^+_- \setminus f_\alpha, \\ \gamma_1(x) = 0 \quad & \forall x \in f_\alpha \cap \widetilde{W}^+_-. \end{aligned}$$

Since η is transversal to $f^{-1}(\alpha)$, γ_1 is necessarily continuous.

Similarly, let $\gamma_2 : \widetilde{W}^+ \to \mathbb{R}^1$ be the first hitting time of the level β from $\widetilde{W}^+$. Again,

$$\begin{aligned} \eta(x, \gamma_2(x)) \in f^{-1}(\beta) \quad & \forall x \in \widetilde{W}^+ \setminus f_\beta, \\ \gamma_2(x) = 0 \quad & \forall x \in f_\beta \cap \widetilde{W}^+, \end{aligned}$$

and γ_2 is continuous. Set

$$\sigma_i(x, s) = \eta(x, s\gamma_i(x)), \quad i = 1, 2.$$

Noticing that $f_\alpha \cap \widetilde{W}^+_- = f_\alpha \cap \widetilde{W}^+$ (because $W \cap f_\alpha = \emptyset$), and using the mean value property, we obtain (III.2) by simply applying the strong deformation retractions σ_1, σ_2 and by exploiting the homotopy invariance of homology groups.

Next, we show that

$$H^*(\widetilde{W}^+, \widetilde{W}^+_-) \cong H^*(W, W_-). \tag{III.3}$$

For $\delta > 0$, let $W_\delta = \bigcup_{t>\delta} \eta(W_-, t)$ and let $s : \widetilde{W}^+ \to \mathbb{R}_+$ be the first hitting time of W_-, so that

$$\begin{aligned} \eta(y, -s(y)) \in W_- \quad & \text{if } y \in \widetilde{W}^+_-, \\ s(y) = 0 \quad & \text{if } y \in \widetilde{W}^+ \setminus \widetilde{W}^+_-. \end{aligned}$$

Since η is transversal to W_-, the map s is continuous and the set W_δ which is equal to $\{y \in \widetilde{W} : s(y) > \delta\}$ is open relative to $\widetilde{W}$, and the closure of W_δ in $\widetilde{W}$ is

$$\overline{W}_\delta = \{y \in \widetilde{W} : s(y) \geq \delta\} \subset \{y \in \widetilde{W} : s(y) > 0\} = \operatorname{int}(\widetilde{W}^+_-).$$

By excision, we have

$$H^*(\widetilde{W}^+, \widetilde{W}^+_-) \cong H^*(\widetilde{W}^+ \setminus W_\delta, \widetilde{W}^+_- \setminus W_\delta). \tag{III.4}$$

Now define a strong deformation retraction by reversing the flow:

$$\sigma : (\widetilde{W} \setminus W_\delta, \widetilde{W}^+_- \setminus W_\delta) \times [0,1] \to (W, W_-), \quad \sigma(y,t) = \eta(y, -ts(y)).$$

This proves that

$$H^*(\widetilde{W}^+ \setminus W_\delta, \widetilde{W}^+_- \setminus W_\delta) \cong H^*(W, W_-). \tag{III.5}$$

Combining (III.4) and (III.5), we obtain (III.3). Apply now Proposition II.2, (III.2) and (III.3) to complete the proof of the theorem.

We now study the stability of the critical groups for isolated critical sets under perturbations of the flow.

Let η_λ be a family of flows associated with the same function f. The following "uniform continuity condition" is assumed: for all $\varepsilon > 0$ and $T > 0$, there exists $\delta = \delta(\varepsilon, T)$ such that

$$\begin{aligned} &(d(x,x') + |t - t'| + |\lambda - \lambda'| < \delta \text{ and } |t|, |t'| \leq T) \\ &\qquad\qquad \Rightarrow d(\eta_\lambda(x,t), \eta_{\lambda'}(x', t')) < \varepsilon. \end{aligned} \tag{III.6}$$

Under this assumption, if $(\mathcal{O}, \alpha, \beta)$ is an isolating triplet for S for the flow η_λ, then it will still be an isolating triplet for S for the flow $\eta_{\lambda'}$ as long as $|\lambda - \lambda'|$ is small enough.

Moreover, under the same uniform continuity assumption, the notion of isolating neighbourhood of S is also stable under small perturbations of the flow, since if $|\lambda - \lambda'| < \delta$, then clearly,

$$\operatorname{dist}(\{\mathcal{O}\}^\beta_\alpha(\lambda), \{\mathcal{O}'\}^\beta_\alpha(\lambda')) < \varepsilon, \quad \operatorname{dist}([S]_\lambda, [S]_{\lambda'}) < \varepsilon,$$

and

$$\operatorname{dist}(G^T_\lambda(\{\mathcal{O}\}^\beta_\alpha), G^T_{\lambda'}(\{\mathcal{O}\}^\beta_\alpha)) < \varepsilon,$$

where λ and λ' (in the subscripts or in the brackets) denote the dependence on the flows.

Now, we turn to the study of the effect of perturbing functionals on (G-M)-pairs of isolated critical sets.

Theorem III.4. *Let f be a C^1-functional on M that satisfies the (P-S) condition and let (W, W_-) be a (G-M)-pair for a critical set S_f. If $\mathcal{O} \subset \overline{O} \subset \operatorname{int}(O)$ is a neighbourhood of S_f, then there exists $\varepsilon > 0$ depending on f and W such that for all $g \in C^1(M, \mathbb{R})$ with $\|f - g\|_{C^1(W)} < \varepsilon$, there is a p.g.v.f η_g of g for which (W, W_-) is still a (G-M)-pair for any critical set S_g for g such that $W \cap K_g = S_g$.*

Proof. Let X be the p.g.v.f. for f with which (W, W_-) is associated. In view of the (P-S) condition, we can choose $\mathcal{O}' \subset \overline{\mathcal{O}'} \subset \mathcal{O}$ such that $S_f \subset \mathcal{O}'$ and $\beta = \inf\{\|f'(x)\| : x \in W \setminus \mathcal{O}'\} > 0$. Take $0 < \varepsilon < \frac{1}{6}\beta$ and consider any $g \in C^1(M, \mathbb{R})$ such that $\|f - g\|_{C^1(W)} < \varepsilon$. Note that S_g is necessarily included in $\mathcal{O}'$.

Consider now a p.g.v.f. Y of g that satisfies $\|X - Y\| < \varepsilon$. Define $\varrho \in C^{1-0}(M, \mathbb{R})$ such that $0 \leq \varrho \leq 1$ and

$$\varrho(x) = \begin{cases} 1, & x \in \overline{\mathcal{O}'}, \\ 0, & x \notin \mathcal{O}, \end{cases}$$

and set $V(x) = \frac{5}{4}[\varrho(x)Y(x) + (1 - \varrho(x))X(x)]$. For $x \notin \mathcal{O}'$, we have

$$\|V(x)\| \leq \tfrac{5}{4}(\|Y(x)\| + \varepsilon) \leq \tfrac{13}{4}\|g'\|,$$

and

$$\langle V(x), g'(x)\rangle \geq \tfrac{5}{4}(\|g'(x)\|^2 - \varepsilon\|g'(x)\|) \geq \tfrac{5}{4}(\|g'(x)\|^2 - \tfrac{1}{5}\|g'(x)\|^2) = \|g'(x)\|^2.$$

If $x \in \overline{\mathcal{O}'}$, then $V(x) = Y(x)$, which means that V is also a p.g.v.f. of g.

Now $V(x) = \frac{5}{4}X(x)$ outside $\mathcal{O}$. Thus, the flow η_g associated with V remains the same as the flow of X. In particular, they coincide on W_-. It is then not difficult to see that W satisfies the (MVP) with respect to the flow η_g and therefore that (W, W_-) is still a (G-M)-pair for S_g with respect to η_g.

IV. Conley's index pairs of an isolating neighbourhood

In this section, we turn to the Conley index. We first establish the relationship between the (G-M)-pairs for dynamically isolated critical sets and the *Conley index pairs* for their corresponding isolating neighbourhoods. We first recall Conley's definitions.

Definition IV.1 (index pair). A pair (N, N_0) of closed subsets of M is called an *index pair* for an isolating neighbourhood U if

(a) There exists $T > 0$ such that $G^T(\operatorname{cl}(N \setminus N_0)) \subset \operatorname{int}(N \setminus N_0)$.

(b) N_0 is *positively invariant with respect to* N, i.e., for all $t > 0$ and for all $x \in N_0$, $\eta(x,[0,t]) \subset N$ implies that $\eta(x,[0,t]) \subset N_0$.

(c) N_0 is an *exit set* for N, i.e., for all $x \in N$ and any $t_1 > 0$ such that $\eta(x,t_1) \notin N$, there exists $t_0 \in [0,t_1)$ such that $\eta(x,[0,t_0]) \subset N$ and $\eta(x,t_0) \in N_0$.

(d) $\mathrm{cl}(N \setminus N_0) \subset U$ and there exists $T > 0$ such that $G^T(U) \subset \mathrm{cl}(N \setminus N_0)$.

Definition IV.2 (Conley index). Let (N, N_0) be an index pair for an isolating neighbourhood U. The homotopy type of the pointed space $N \setminus N_0$ is called the *Conley index* of U and is denoted by $[N \setminus N_0]$.

As noted by Conley–Zehnder [C-Z], Salamon [S] and Benci [B], the Conley index is a topological invariant for isolating neighbourhoods, i.e., if (N, N_0) and $(\widetilde{N}, \widetilde{N}_0)$ are two index pairs for U, then $[N \setminus N_0] = [\widetilde{N} \setminus \widetilde{N}_0]$.

Thus, for two pairs (N, N_0), $(\widetilde{N}, \widetilde{N}_0)$ for U, we have $\overline{H}^*(N, N_0) \cong \overline{H}^*(\widetilde{N}, \widetilde{N}_0)$, where the Alexander–Spanier cohomology $\overline{H}^*(A, B, F)$ is defined on a pair of topological spaces (A, B) for some coefficient field F. If (N, N_0) is a pair of ANR's, then $\overline{H}^*(N, N_0) \cong \overline{H}^*(N/N_0, [N_0])$.

Here and in the sequel, we shall omit the coefficient field F. Thus, for two pairs (N, N_0) and $(\widetilde{N}, \widetilde{N}_0)$ associated with U, we have $\overline{H}^*(N, N_0) \cong H^*(\widetilde{N}, \widetilde{N}_0)$.

Theorem IV.3. *Let f be a C^1-functional satisfying the (P-S) condition on a C^1-Finsler manifold M and let S be a dynamically isolated critical set for f with an isolating triplet $(\mathcal{O}, \alpha, \beta)$. Then:*

(1) *If (W, W_-) is any (G-M)-pair for S with $W \subset \{\mathcal{O}\}_\alpha^\beta$, then (W, W_-) is a Conley index pair for any isolating neighbourhood U of $[S]$ satisfying $W \subset U \subset \{\mathcal{O}\}_\alpha^\beta$.*

(2) *If U is any isolating neighbourhood for $[S]$ such that $U \subset \{\mathcal{O}\}_\alpha^\beta$, then for any Conley index pair (N, N_0) of U, we have $H^*(N, N_0) \cong C_*(f, S)$.*

Proof. (1) By Theorem I.6, W is also a closed neighbourhood of $[S]$. Since W_-, the exit set of W, is transversal to η, it is necessarily positively invariant. It remains to verify conditions (a) and (d) in the definition of a Conley index pair.

First note that $\mathrm{cl}(W \setminus W_-) = W$ and $\mathrm{int}(W \setminus W_-) = \mathrm{int}(W)$. By Lemma I.9, there exists $T > 0$ such that $G^T(\mathrm{cl}(W \setminus W_-)) \subset \mathrm{int}(W) = \mathrm{int}(W \setminus W_-)$ and therefore (a) holds.

As to (d), again apply Lemma I.9 to obtain $T > 0$ such that $G^T(U) \subset G^T(\{\mathcal{O}\}_\alpha^\beta) \subset \mathrm{int}(W) \subset W$. This establishes part (1) of the theorem.

For (2), first choose a (G-M)-pair (W, W_-) such that $W \subset U$. The existence of such a pair follows from Proposition III.2. According to Theorem III.3 and part (1) above, we have

$$H^*(N, N_0) = H^*(W, W_-) = C_*(f, S),$$

which completes the proof of Theorem IV.3.

Now, we can deduce the following homotopy invariance result for Conley index pairs.

Theorem IV.4. *Suppose that $\{f_\lambda\}$, $\lambda \in [0,1]$, is a family of C^1-functions satisfying the $(P\text{-}S)$ condition, and that $\{S_\lambda\}$ is a family of dynamically isolated critical sets of f_λ with isolating neighbourhoods U_λ. Assume that (N_λ, N_λ^0) is a family of index pairs for U_λ and that all U_λ's are included in a set U on which f_λ is uniformly bounded. If the map $\lambda \to f_\lambda$ is continuous from $[0,1]$ into $C^1(U)$, then the groups $H^*(N_\lambda, N_\lambda^0)$ are independent of λ.*

Proof. This is a combination of Theorems III.3 and IV.3.

In [Ch], the first-named author establishes a relationship between the critical groups of a functional and the topological degree of its gradient vector field. Theorem IV.3 now allows us to transfer such an information to the Conley index.

Corollary IV.5. *Let H be a Hilbert space and let f be a C^2-function on H satisfying the $(P\text{-}S)$ condition. Let U be an isolating neighbourhood on which the vector field df is of the form $I - T$ where T is a compact mapping. Then, for any Conley index pair (N, N_0) for U such that $0 \notin f'(\partial N)$, we have*

$$\deg(df, N, 0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} H^q(N, N_0).$$

Proof. Take any (G-M)-pair (W, W_-) for the dynamically isolated critical set $S := I(U) \cap K_f$ such that $W \subset N$. The existence of such a pair follows from Proposition III.2. Since there are no critical points in $N \setminus W$, by excision we get

$$\deg(df, N, 0) = \deg(df, W, 0).$$

On the other hand, it has been proved in [Ch, Th. 4.2] that

$$\deg(df, W, 0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} H^q(W, W_-).$$

Our conclusion then follows from Theorem IV.3.

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