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EQUIVARIANT DEGREE FOR ABELIAN ACTIONS PART II: INDEX COMPUTATIONS

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Dedicated to Louis Nirenberg on his 70th birthday

Introduction

This paper represents the second part of the study of the equivariant degree for abelian actions and constitutes another step towards the completion of our rather long journey along the paths of equivariant homotopy and equivariant degree theory initiated in [7]. A program of development of this theory was announced in [8] and followed chronologically in [9] and [10].

Here, using the results of [10], we compute the equivariant degree for abelian actions and use it in order to prove results on twisted orbits, Borsuk–Ulam type theorems, symmetry breaking problems and applications to ODE's.

Let us briefly subsume our definition of equivariant degree in the finite-dimensional setting (see [8]). Let V and W be finite-dimensional spaces and let Γ be a (not necessarily abelian) compact Lie group acting linearly (isometrically) on both V and W (with possibly different actions). Let $\Omega \subset V$ be a Γ-invariant open and bounded subset of V and let $f : \overline{\Omega} \to W$ be a continuous Γ-equivariant map such that $f(x) \neq 0$ on the boundary $\partial\Omega$ of Ω. Now, our construction is as follows. Take a sufficiently large ball $B \subset V$ centered at the origin such that $\Omega \subset B$ and let $\widehat{f} : B \to W$ be a Γ-equivariant continuous extension of f. Let N be a bounded, open and Γ-invariant neighborhood of $\partial\Omega$ such that $\widehat{f}(x) \neq 0$ for any $x \in \overline{N}$. Let $\widehat{F} : [0,1] \times B \to \mathbb{R} \times W$ be the continuous map

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defined by $\widehat{F}(t,x) = (2t + 2\phi(x) - 1, \widehat{f}(x))$, where $\phi : B \to [0,1]$ is a Γ-invariant Urysohn function such that $\phi(x) = 0$ if $x \in \overline{\Omega}$ and $\phi(x) = 1$ if $x \notin \Omega \cup N$. We assume, moreover, that Γ acts trivially on both [0,1] and $\mathbb{R}$. Clearly, $\widehat{F}(t,x) = 0$ only if $x \in \Omega$, $\widehat{f}(x) = f(x) = 0$ and $t = 1/2$. Thus, $\widehat{F}$ can be regarded as a Γ-equivariant map from $S^V \cong \partial([0,1] \times B)$ into $S^W \cong \mathbb{R} \times W \setminus \{0\}$. Following [8] we define the Γ-degree of f, denoted by $\deg_\Gamma(f;\Omega)$, as the Γ-equivariant homotopy class $[\widehat{F}]_\Gamma$ considered as an element of the Γ-equivariant homotopy group of spheres $\Pi^\Gamma_{S^V}(S^W)$. It is not hard to show that if Γ reduces to the trivial group, $\Gamma = \{e\}$, then $\deg_\Gamma(f;\Omega)$ is nothing else but the classical Brouwer topological degree of f.

The infinite-dimensional case, $\dim V = \dim W = \infty$, can be handled with appropriate Γ-equivariant suspension theorems (cf. [10, Theorem 9.1]) after imposing the usual compactness assumptions on f. Thus, for example, if $\Gamma = \{e\}$ and f is a compact perturbation of the identity, our Γ-degree reduces to the classical topological Leray–Schauder degree (see [8]).

Even though the above definition runs for any compact Lie group Γ, we shall stick in this paper, as in [10], to the case when Γ is abelian.

A description of the structure of the present paper is in order. Section 0 is essentially a collection of results from [10] (in some cases suitably reformulated) that permit us to proceed efficiently towards further investigations. It also contains the important assumption (H) that will hold true almost throughout this paper. In Section 1 we refine some results of [10] related to the action of a torus that allow us to recover some well-known results contained in [13]. In Section 2 we show that in some cases the computation of the Γ-degree may be reduced to the computation of the classical degree of the corresponding Poincaré sections. In Section 3 we compute the index of isolated orbits (see Theorem 3.2). As a consequence we obtain interesting global bifurcation results involving period doubling phenomena (see Corollary 3.1). In Section 4 we apply these degree computations to Borsuk–Ulam type theorems. Section 5 deals with the index of an isolated loop of stationary solutions and its applications to abstract Hopf bifurcation. Section 6 treats the problem of symmetry breaking, products and composition of mappings.

Finally, let us mention that [10] contains some misprints in the References. For example references [1], [2], [3] should be cyclically permuted and reference [16] should be split into two references: the one reported in [10] under [16] and another one, say [16a], which is reference [3] of the present paper.

0. Preliminaries

In this section we shall collect the results from [10] which are most frequently used in the present paper.

$\Gamma \cong T^n \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ is a compact abelian Lie group acting linearly, via isometries, on finite-dimensional spaces V and W (in the case of infinite-dimensional spaces one has to reduce the study to maps which have the right compactness properties). If B_R is the ball of radius R centered at the origin in V and t is in $I = [0, 1]$, one considers the Γ-homotopy classes of Γ-equivariant maps $F(t, X)$ with $F(t, \gamma X) = \widetilde{\gamma} F(t, X)$ from $\partial(I \times B_R)$ into $\mathbb{R} \times W \setminus \{0\}$. The resulting abelian group was called $\Pi^{\Gamma}_{S^V}(S^W)$ in [8] and if $f : V \supset \overline{\Omega} \to W$ is a Γ-equivariant map which is not zero on $\partial\Omega$, where Ω is an open, bounded, and Γ-invariant subset of V, then the Γ-degree of f is an element of $\Pi^{\Gamma}_{S^V}(S^W)$, as recalled in the introduction.

In most of this paper, unless specified otherwise, we shall assume the following standing hypothesis:

(H) $V = \mathbb{R}^k \times U$, and for any pair of isotropy subgroups H and K for U, one has $\dim U^H \cap U^K = \dim W^H \cap W^K$.

If (H) holds, then there is a "suspension" map from $(V^\Gamma)^\perp$ into $(W^\Gamma)^\perp$ given by $x_j \to x_j^{l_j}$, which is Γ-equivariant [10, Lemma 7.1]. Furthermore, $\Pi^{\Gamma}_{S^V}(S^W) \cong \Pi_{k-1} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, where Π_{k-1} corresponds to the isotropy subgroups K with $\dim \Gamma/K < k$ and there is one $\mathbb{Z}$ for each isotropy subgroup H with $\dim \Gamma/H = k$ [10, Theorem 7.1]. There are explicit generators, $[F_H]_\Gamma$, for each of the $\mathbb{Z}$-components. If $[F]_\Gamma \in \Pi(H)$, defined as the set of Γ-homotopy classes of maps such that $F^K = I \times B^K_R \to \mathbb{R} \times W^K \setminus \{0\}$ for any $K > H$, then, if $\dim \Gamma/H = k$, one has $\Pi(H) \cong \mathbb{Z}$ and $[F^H]_\Gamma$ is given by the "extension degree", $\deg_{\mathrm{E}}(F)$, of F^H, defined on the "fundamental cell" $\mathcal{C} = \{(t, x_1, \ldots, x_l) \in I \times V^H : 0 < t < 1$, $|x_j| < R$, $0 < \operatorname{Arg} x_j < 2\pi/k_j\}$ where $k_j = |\widetilde{H}_{j-1}/\widetilde{H}_j|$ and $\widetilde{H}_j = H_1 \cap \ldots \cap H_j$ with H_i being the isotropy subgroup of x_i. In this case there are exactly k variables, $z_1, \ldots, z_k$, with $k_j = \infty$ for $j = 1, \ldots, k$. Furthermore, if $B^H_k = \{(t, X) \in I \times B^H_R : z_j$ real and positive for $j = 1, \ldots, k\}$, then $\deg(F^H; B^H_k) = (\prod k_j) \deg_{\mathrm{E}}(F^H)$, where the product is taken over all finite k_j's [10, Theorem 4.1].

If $k = 0$, then $\Pi^{\Gamma}_{S^V}(S^W) \cong \Pi^{\Gamma'}_{S^{V'}}(S^{W'})$ where $\Gamma' = \Gamma/T^n$, $V' = V^{T^n}$, $W' = W^{T^n}$, and if $k = 1$, then $\Pi_0 = \Pi^{\Gamma'}_{S^{V'}}(S^{W'}) = \prod \Pi(H)$ with $|\Gamma/H| < \infty$ [10, Corollary 5.1]. Furthermore, if Γ/H is a finite group and if for each z_j with $k_j > 1$ there is another variable z'_j with the same isotropy (two variables if z_j is real and Γ acts as $\mathbb{Z}_2$, i.e. a suspension result), then $\Pi(H)$ is a finite group [10, Theorem 8.2]. In particular, if $V = \mathbb{R} \times W$ and $\Gamma/H \cong \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_m}$ then $\Pi(H) \cong \mathbb{Z}_{q_0} \times \ldots \times \mathbb{Z}_{q_m}$ with $q_0 = \text{g.c.d.}(2, p_1, \ldots, p_m)$, $q_m = \text{l.c.m.}(2, p_1, \ldots, p_m)$, $q_j = k_j/k_{j-1}$, where k_j is the largest common factor of all possible products of j of the p_i's. Hence, if any two p_i, p_j are relatively prime and odd, then $\Pi(H) \cong \mathbb{Z}_{2|\Gamma/H|}$. There are explicit generators $\eta_j, \widetilde{\eta}$ such that $2\widetilde{\eta} = 0$ and $q_j(\eta_j + \widetilde{\eta}) = 0$. For example, if $\Gamma/H \cong \mathbb{Z}_n$, then $\Pi(H) \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ if n is even and $\Pi(H) \cong \mathbb{Z}_{2n}$ if n is odd [10, Theorem 8.5].

Finally, for all the above cases any element of $\Pi(H)$ if $\dim\Gamma/H = k$, or any element of $\Pi^{\Gamma}_{SV}(S^W)$ if $k = 0$ or 1, is achieved as the Γ-degree of a map from Ω into W, provided $\Omega^H \neq \emptyset$ [10, Theorem 2.2]. We would also like to stress our results on the suspension, [10, Theorem 9.1], which will automatically hold in the present paper.

Remarks. 1) Some of the results listed above hold with weaker hypotheses, as proved in [10]. In case of need we shall recall these hypotheses in the appropriate places of the present paper.

2) In [10], Lemma 7.1, and hence Theorems 7.1, 8.2 and 9.2, were stated with the hypothesis (H3): $\dim U^H = \dim W^H$, which is incomplete, as the following example shows:

On $\mathbb{C}^2$, consider the following action of $\mathbb{Z}_{p^2q}$, where p and q are relatively prime. On (z_1, z_2) in U, Γ acts via $(e^{2\pi ik/p^2}, e^{2\pi ik/(pq)})$ for $k = 0, \ldots, p^2q - 1$, and on (ξ_1, ξ_2) in W, Γ acts via $(e^{2\pi ik/p}, e^{2\pi ik/(p^2q)})$. The isotropy subgroups for U are $H \cong \mathbb{Z}_q$ for k a multiple of p^2, and $U^H = \{(z_1, 0)\}, K \cong \mathbb{Z}_p$ for k a multiple of pq, and $U^K = \{(0, z_2)\}$, and $U^{\{e\}} = U$. One has $W^H = W^K = \{(\xi_1, 0)\}$ and (H3) holds but not (H). Also, $(U^H)^\perp = 0 \times \mathbb{C}, (W^H)^\perp = 0 \times \mathbb{C}$ and there is no non-zero equivariant map between these two last spaces, since $(U^H)^\perp \cap U^K = U^K$ and $(W^H)^\perp \cap W^K = \{0\}$. Hence, hypothesis (H2) of [10] is not met.

Consider the equivariant map $F(z_1, z_2) = (z_1^p + z_2^q, z_1^\alpha z_2^\beta)$, where $\alpha q + \beta p = 1$ (recall that a negative power is taken as a conjugate). The zeros of $F - (\varepsilon, 0)$ are at $(0, \varepsilon^{1/q} e^{2k\pi i/q})$ and $(\varepsilon^{1/p} e^{2k\pi i/p}, 0)$ with index α and β respectively. Hence the degree of F with respect to any neighborhood of $(0, 0)$ is $\alpha q + \beta p = 1$: $\deg F = 1$. Similarly $\deg F^H = p$, $\deg F^K = q$.

Note that we shall prove, in Section 4, that any equivariant map G from $I \times B$, with $B = \{(z_1, z_2) : |z_i| < 2\}$, into $\mathbb{R} \times \mathbb{C}^2 \setminus \{0\}$ is classified by $[G]_\Gamma = d_\Gamma[2t - 1, F]_\Gamma + d_H[F_H]_\Gamma + d_K[F_K]_\Gamma + d_e[F_e]_\Gamma$, where F is the above map, $F_H = (2t + 1 - 2|z_1|^2, (z_1^{p^2} - 1)z_1^p, z_1^\alpha z_2^\beta)$, $F_K = (2t + 1 - 2|z_2|^2, (z_2^{pq} - 1)z_2^q, z_2^\alpha, z_1^\alpha z_2^\beta)$ and $F_e = (2t + 1 - 2|z_1|^2|z_2|^2, (z_1^{p^2} - 1)z_1^p, z_1^\alpha z_2^\beta(\overline{z}_1^p z_2^q - 1))$.

It is then not difficult to show that

$$\begin{pmatrix} \deg G^\Gamma \\ \deg G^H \\ \deg G^K \\ \deg G \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p & p^2 & 0 & 0 \\ q & 0 & pq & 0 \\ 1 & \beta p^2 & \alpha pq & p^2 q \end{pmatrix} \begin{pmatrix} d_\Gamma \\ d_H \\ d_K \\ d_e \end{pmatrix}.$$

Lemma 7.1 of [10] is then replaced by

Lemma 0. *Hypothesis* (H) *holds if and only if*

(a) $\dim U^H = \dim W^H$,

(b) *there are integers* l_j *such that the map* $F : (z_1, \ldots, z_n) \to (z_1^{l_1}, \ldots, z_n^{l_n})$ *is* Γ-*equivariant.*

PROOF. If (H) holds then, if $H_0 = \bigcap H_i$, one has $U^{H_0} = U$ and one obtains (a). Also, as in [10, Lemma 7.1] one gets $\det\gamma\det\widetilde{\gamma} > 0$, and one obtains F^H for any maximal H (on U^Γ the identity is an appropriate map). Choose such a maximal H and let K and L be isotropy subgroups for $(U^H)^\perp$. Then

$$\dim(U^H)^\perp \cap U^L \cap (U^H)^\perp \cap U^K = \dim U^L \cap U^K - \dim U^H \cap U^L \cap U^K.$$

Let H_0 be the isotropy subgroup for $U^K \cap U^L$, i.e. H_0 is the intersection of the isotropy subgroups for all the coordinates in that subspace. Then $U^K \cap U^L \subset U^{H_0}$. Since K and L are also intersections of the corresponding subgroups, it is clear that K and L are subgroups of H_0 and thus, $U^{H_0} \subset U^K \cap U^L$, that is, $U^{H_0} = U^K \cap U^L$ while $W^{H_0} \subset W^K \cap W^L$. Since, from (H), $\dim U^{H_0} = \dim W^{H_0}$ and $\dim U^L \cap U^K = \dim W^L \cap W^K$, one gets $W^{H_0} = W^K \cap W^L$. Thus, $\dim(U^H)^\perp \cap U^K \cap U^L = \dim(W^H)^\perp \cap W^K \cap W^L$, and one may repeat the argument of Lemma 7.1 in [10] for a maximal isotropy subgroup for $(U^H)^\perp$.

Note that if $\Gamma/H \cong \mathbb{Z}_2$ and $\Gamma/K \cong \mathbb{Z}_2$ for $H \neq K$, then if x belongs to $U^H \cap U^K$, one has $\Gamma_x > H \cup K$, and since H and K are maximal among subgroups of Γ (not just among isotropy subgroups), $\Gamma_x = \Gamma$. Thus, for such subgroups, hypothesis (a) is equivalent to (H).

Conversely, if the map F exists, then it is clear that $\dim U^H \leq \dim W^H$, and it is easy to give examples with a strict inequality. While, if (a) and (b) hold, it is easy to see, by inspection, that (H) is true. □

1. Action of a torus

In [10, §1] we gave an explicit form for the action of an abelian group on an irreducible representation. In the present section we collect some further results on these actions.

Let $T^n = \{(\phi_1, \dots, \phi_n) : 0 \leq \phi_j \leq 2\pi\}$ act on $\mathbb{C}^m = \{z_1, \dots, z_m\}$ via $\exp i(\sum_{j=1}^n n_j^l \phi_j)$ for $l = 1, \dots, m$. The isotropy subgroup H_l for z_l will consist of those $(\phi_1, \dots, \phi_n)$ with $\sum n_j^l \phi_j \equiv 0 \pmod{2\pi}$. Assume that $\dim T^n/(H_1 \cap \ldots \cap H_m) = k$. Then we have seen in [10, §2] that there are exactly k coordinates $(z_1, \dots, z_k)$ such that $T^n/H_1, H_1/H_1 \cap H_2, \dots, (H_1 \cap \ldots \cap H_{k-1})/(H_1 \cap \ldots \cap H_k)$ are isomorphic to S^1 and $(H_1 \cap \ldots \cap H_{l-1})/(H_1 \cap \ldots \cap H_l)$ are finite groups for $l > k$. Note that without loss of generality we are taking $z_1, \dots, z_k$ to be the first k coordinates.

LEMMA 1.1. *Under the above circumstances there is an action of T^k on $\mathbb{C}^m$, generated by $\Phi = (\Phi_1, \dots, \Phi_k)$, such that $\sum_{j=1}^n n_j^l \phi_j = \sum_{j=1}^k N_j^l \Phi_j$ for $l = 1, \dots, m$ and for some integers N_j^l with $N_j^l = \delta_{jl} N_j$ if $j = 1, \dots, k$.*

PROOF. Let A be the $m \times n$ matrix given by (n_j^l), $l = 1, \dots, m$, $j = 1, \dots, n$. The relation $\sum_{j=1}^n n_j^l \phi_j = (A\phi)_l = 0$ gives a hyperplane in $\mathbb{R}^n$. The hypothesis on the isotropy subgroups implies that A has an $(n-k)$-dimensional kernel and that if $(A\phi)_l = 0$ for $l = 1, \ldots, k$ then $(A\phi)_j = 0$ for $j = k+1, \ldots, m$ since if not one would have an S^1-non-trivial action on the corresponding variable z_j.

Let A_0 be the matrix obtained from A by taking the first k rows. Then A_0 is onto $\mathbb{R}^k$ and as such it has a $k \times k$ non-zero minor. Assume, without loss of generality, that it corresponds to the determinant of A_1 given by (n_j^l), $l = 1, \ldots, k, j = 1, \ldots, k$. It is clear that there are positive integers $N_1, \ldots, N_k$ such that

$$\widetilde{A}_1^{-1} \equiv A_1^{-1} \begin{pmatrix} N_1 & 0 \\ 0 & N_k \end{pmatrix}$$

has integer entries. Let $\Phi_j = (A\phi)_j / N_j$ for $j = 1, \ldots, k$. Then, if $\phi^T = (\widetilde{\phi}^T, \widehat{\phi}^T)$ with $\widetilde{\phi}^T = (\phi_1, \ldots, \phi_k)$, one has $A_0\phi = A_1\widetilde{\phi} + B\widehat{\phi} = (N_1\Phi_1, \ldots, N_k\Phi_k)^T$ and $\widetilde{\phi} = \widetilde{A}_1^{-1}\Phi - A_1^{-1}B\widehat{\phi}$. Thus,

$$(A\phi)_l = \sum_{j=1}^{n} n_j^l \phi_j = \sum_{j=1}^{k} N_j^l \Phi_j + \sum_{j=k+1}^{n} n_j^l \widehat{\phi}_j - \sum_{j=1}^{k} n_j^l (A_1^{-1} B\widehat{\phi})_j.$$

The relation $(A\phi)_l = 0$ if $\Phi_1 = \ldots = \Phi_k = 0$ implies that the last two sums cancel each other. $\square$

Another simple but useful observation is the following

Lemma 1.2. *Let T^n act on V via $\exp i(\sum_{j=1}^{n} n_j^l \phi_j), l = 1, \ldots, m$. Then there is a morphism $S^1 \to T^n$ given by $\phi_j = N_j\phi, N_j$ integers, such that $\sum_{j=1}^{n} n_j^l N_j \not\equiv 0 \ (\mathrm{mod}\, 2\pi)$ unless $n_j^l = 0$ for all j's, and $V^{S^1} = V^{T^n}$.*

Proof. The congruences $\sum n_j^l \phi_j \equiv 0 \ (\mathrm{mod}\, 2\pi)$ give families of hyperplanes with normal parallel to $(n_1^l, \ldots, n_n^l)$, if this vector is non-zero. From the denseness of $\mathbb{Q}$ in $\mathbb{R}$, it is clear that one may find integers $(N_1, \ldots, N_n)$ such that the direction $\{\phi_j = N_j\phi\}$ is not in any of the hyperplanes $\sum n_j^l \phi_j = 0$ for $l = 1, \ldots, m$. Thus, $\sum n_j^l N_j \neq 0$ and, being an integer, this number cannot be another multiple of 2π, unless all n_j^l are zero and one is in V^{T^n}. $\square$

As a simple consequence of this last lemma, one may recover the following well known results (see [13, Theorem 2.2]).

Theorem 1.1. *Let T^n act on V and W such that $\dim V = \dim W$ and $\dim V^{T^n} = \dim W^{T^n}$. Let F be a T^n-equivariant map from $I \times V$ into $\mathbb{R} \times W$ which is non-zero on $\partial(I \times B)$. Then*

(a) *There is a non-zero integer β, independent of F, such that*

$$\deg(F; I \times B) = \beta \deg(F^{T^n}; I \times B^{T^n}).$$

(b) *If H is any isotropy subgroup of T^n on V and* $\deg(F^{T^n}; I \times B^{T^n}) \neq 0$, *then* $\dim V^H \leq \dim W^H$. *In this case* $\beta = \pm(\prod_{l=1}^k a'_l)/(\prod_{l=1}^k a_l)$, *where a_l is the greatest common divisor of $(n_1^l, \dots, n_n^l)$ and similarly for a'_l.*

Proof. Choose an S^1-action as in Lemma 1.2, for V and W. From [8, Theorem 4.4], one has

$$\deg(F; I \times B) = \frac{\prod_{l=1}^k(\sum_{j=1}^n n_j'^l N_j)}{\prod_{l=1}^k(\sum_{j=1}^n n_j^l N_j)} \deg(F^{T^n}; I \times B^{T^n}),$$

where $n_j'^l$ correspond to the action of T^n on W and $k = \dim V - \dim V^{T^n}$. It is clear that the quotient, β, is independent of the S^1-action chosen. Furthermore, the dimension inequality of part (b) also follows from the same reference since F^H maps $\partial(I \times B^H)$ into $\mathbb{R} \times W^H \setminus \{0\}$.

If for any F one has $\deg(F^{T^n}; I \times B^{T^n}) = 0$, then one may as well choose β to be 1. If there is an F with $\deg(F^{T^n}; I \times B^{T^n}) = 1$, then clearly β is an integer. This is the case if hypothesis (H2)$'$ of [10] is satisfied, i.e. there is an equivariant map $F^\perp : (V^{T^n})^\perp \setminus \{0\} \to (W^{T^n})^\perp \setminus \{0\}$; then one may complement $F^\perp$ by any map of degree 1 from $I \times B^{T^n}$ into itself. Note that under hypothesis (H2)$'$, from [10, Corollary 5.1(a)], $\Pi_{SV}^{T^n}(S^W) \cong \mathbb{Z}$ and $[F]_{T^n}$ is characterized by $[F^{T^n}]$, i.e. by $\deg(F^{T^n}; I \times B^{T^n})$. Since F and $(F^{T^n}; F^\perp)$ have the same degree for their invariant part, we have $[F]_{T^n} = [(F^{T^n}, F^\perp)]_{T^n}$.

If (H2)$'$ is not satisfied, let $m_l = \sum_{j=1}^n n_j^l N_j, M = \prod m_l, M' = \prod m'_l$ and assume p^α is a factor of $|M|$ with p a prime number, and $p^{\alpha'}$ the corresponding factor of $|M'|$. Take the set of $\{m_l\}$ which are multiples of p and suppose there are b_1 of them which are multiples of p^{α_1}, b_2 which are multiples of p^{α_2}, with $\alpha_2 < \alpha_1$, not including the first set, and so on up to b_k which are multiples of p^{α_k}, with $1 \leq \alpha_k < \alpha_{k-1} < \dots < \alpha_1$ and not included in the preceding sets. Let b'_1 be the number of j's such that $\alpha_1 \leq \alpha'_j$ and $p_j^{\alpha'_j}$ divides $|m'_j|$, b'_i be the number of j's such that $\alpha_i \leq \alpha'_j < \alpha_{i-1}$ for $i = 1, \dots, k$, and finally, b'_{k+1} the number of j's with $1 \leq \alpha'_j < \alpha_k$, in case $\alpha_k > 1$. Then $\alpha = \sum_{j=1}^k \alpha_j b_j$ and $\alpha' = \sum_{j=1}^{k+1} \alpha'_j \geq \sum_{j=1}^k \alpha_j b'_j + b'_{k+1}$.

Now, if $H_j = \{\phi = 2\pi e/p^{\alpha_j} : 0 \leq e < p^{\alpha_j}\}$, then the inequalities $\dim V^{H_j} \leq \dim W^{H_j}$ and $\dim V^{S^1} = \dim W^{S^1}$ imply the relations $\sum_{j=1}^i b_j \leq \sum_{j=1}^i b'_j$ for $i = 1, \dots, k+1$ (here we are taking $b_{k+1} = 0$ and $\alpha_{k+1} = 0$). From the telescoping sum, $\sum_{j=1}^k \alpha_j b_j = \sum_{j=1}^k (\alpha_j - \alpha_{j+1}) \sum_{l=1}^j b_l$, one has $\alpha \leq \alpha'$, which implies that $|M|$ divides $|M'|$ and β is an integer.

Now, under the above hypothesis, the integer β is independent of the N_j's, provided no m_l or m'_l is zero. Since the number of terms in the quotient is the same, one sees that β is the same if one takes the N_j's to be rational (provided the new m_l's and m'_l's are non-zero) and, by denseness, for N_j real, we obtain the quotient of homogeneous polynomials of degree 1. Then for each l there is a q, and conversely, such that $\sum n_j'^l N_j = c_{lq} \sum n_j^q N_j$ for all N_j in $\mathbb{R}$, where c_{lq} is a constant. Thus, $n_j'^l = c_{lq}^q n_j$ or else $c_{lq} a_q/a_l' = m_j'/m_j = m'/m$ for all $j = 1, \ldots, n$, where $|m'|$ and $|m|$ are relatively prime, $n_j^q = a_q m_j$ and $n_j'^l = a_l' m_j'$. But then $|m|$ divides all $|m_j|$'s and $|m'|$ divides $|m_j'|$, and since the $|m_j|$'s are relatively prime, we have $|m| = |m'| = 1$. Hence, $n_j'^l = \eta_{lq}(a_l'/a_q) n_j^q$ for all j's, with $|\eta_{lq}| = 1$, and $|\beta| = (\prod a_l')/(\prod a_q)$, recovering the result of [13], where one had the assumption $n_j^q, n_j'^l \geq 0$. $\square$

Note that here we are not asking for the condition $\dim V^H = \dim W^H$. In fact, one could have a strict inequality, hence a zero degree for F^H, for all H's but the smallest: take $n = 1$ and an action on W of the form $e^{iN\varphi}$ where N is a multiple of all the n_j's; then $W^H = W$.

It is easy to show, from [8, Theorem 4.4], that if $\dim V = \dim W$ but $\dim V^{T^n} \neq \dim W^{T^n}$ then the degree of F is 0.

Note also that if $R(\varphi)$ is the 2×2 real matrix corresponding to the complex action $e^{i\varphi}$ of S^1, then $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is such that $R(\varphi)A = AR(-\varphi)$, i.e. the real representations of S^1 given by $R(\varphi)$ and $R(-\varphi)$ are equivalent and A corresponds to conjugation [3, p. 110]. However, for the case of a higher dimensional torus, one may not choose the n_j's to be positive. In fact, there is no real invertible matrix A such that $R(\varphi_1 + \varphi_2)A = AR(\varphi_1 - \varphi_2)$: take $\varphi_1 = \varphi_2$ for example.

2. Poincaré sections

In some cases one may compute the Γ-degree of a map by reducing the situation to the computation of ordinary degrees on Poincaré sections. This will be the case with the "free part" of the Γ-degree when considering isolated orbits.

Recall that, under the standing hypothesis (H), one has $\Pi_{S^V}^{\Gamma}(S^W) \cong A \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, where A corresponds to the isotropy subgroups H for V such that $\dim W(H) < k$ and there is one $\mathbb{Z}$ for each isotropy subgroup for V such that $\dim W(H) = k$ (see [10, Theorem 7.1]).

In particular, any element $[F]_\Gamma$ in $\Pi_{S^V}^{\Gamma}(S^W)$ can be written as $[F]_\Gamma = \sum d_K [F_K]_\Gamma + [\widetilde{F}]_\Gamma$, where $[\widetilde{F}]_\Gamma \in A, [F_K]_\Gamma$ are the explicit generators given in [10, p. 394] and d_K are the free components in $\mathbb{Z}$.

Now, we have seen in [10, §2] that if H is such that $\dim W(H) = k$, then there are exactly k complex coordinates $z_1, \ldots, z_k$ with corresponding isotropy subgroups $H_1, \ldots, H_k$ such that $\Gamma/H_1, H_1/H_1 \cap H_2, H_1 \cap H_2/H_1 \cap H_2 \cap H_3, \ldots, H_1 \cap \ldots \cap H_{k-1}/H_1 \cap \ldots \cap H_k$ are isomorphic to S^1, i.e. these coordinates define part of the "fundamental cell" for H. Let $H_0 = H_1 \cap \ldots \cap H_k$. Then H_0 is one of the maximal isotropy subgroups for V with $\dim W(H_0) = k$. Let H_0 be such a maximal isotropy subgroup with the corresponding variables $z_1, \ldots, z_k$. Note that if $H < H_0$ is an isotropy subgroup with $\dim W(H) = k$, then H_0

acts on V^H as a finite group: in fact, $\Gamma/H \simeq (\Gamma/H_0)(H_0/H)$ and the fact that $\dim \Gamma/H = \dim \Gamma/H_0$ implies that H_0/H is finite. Then, as in [10, p. 371], the set $\{X \in V : |H_0/(H_0 \cap \Gamma_X)| < \infty\}$ is the subspace $V^{T^{n-k}}$, where T^{n-k} is the maximal torus of H_0, i.e. $H_0 = T^{n-k} \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ where $\Gamma = T^n \times \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_l}$. In particular, there is a minimal $\underline{H}$, corresponding to $V^{T^{n-k}}$, with $\underline{H} < H_0$ and $\dim \Gamma/\underline{H} = k$.

As a last preliminary step, recall that, under our standing hypothesis, for any isotropy subgroups $H_i < H_j$, there is an equivariant map $(x_1, \ldots, x_n) \to (x_1^{l_1}, \ldots, x_n^{l_n})$ from $(V^{H_j})^{\perp_{H_i}}$ into $(W^{H_j})^{\perp_{H_i}}$, the orthogonal complement of V^{H_j} (respectively W^{H_j}) in V^{H_i} (respectively in W^{H_i}), with index at zero equal to $\beta_{ij} = \prod l_k$. Hence $\beta_{ij} = 1$ if $V = \mathbb{R}^k \times W$. Let $B_k \equiv \{(t, X) : 0 < t < 1, \|X\| < R, z_1, \ldots, z_k$ real and positive$\}$. If $k = 0, B_k$ is just the ball $I \times B_R = \{(t, X) : 0 < t < 1,\ \|X\| < R\}$.

We shall consider equivariant maps $F : I \times B_R \to \mathbb{R} \times W$ which are non-zero on $\partial(I \times B_R)$ and on the sets $\{z_j = 0\}, j = 1, \ldots, k$. For such a map and for any isotropy subgroup H, F^H is non-zero from ∂B_k^H to $I \times W^H$, where B_k^H and $I \times W^H$ have the same dimension. Hence the degree of the Poincaré section $F^H|_{B_k^H}$ is well defined. On the other hand, $[F]_\Gamma = \sum d_K [F_K]_\Gamma + [\widetilde{F}]_\Gamma$ as above. One has the following result:

THEOREM 2.1. *Under the above hypothesis,* $[\widetilde{F}]_\Gamma = 0$*,* $d_K = 0$ *if* K *is not a subgroup of* H_0*, and*

$$\deg(F^{H_i}|_{B_k^{H_i}}; B_k^{H_i}) = \sum_{H_i < H_j < H_0} \beta_{ij} d_j |H_0/H_j|$$

for all $\underline{H} < H_i < H_0$ *with* $\dim \Gamma/H_i = k$*,* $k \geq 0$.

The case $k = 0$ was given in [10, Theorem 6.1]. For $k > 0$, the proof is not straightforward since a Γ-homotopy on $\partial(I \times B_R)$ does not imply, a priori, an H_0-homotopy on ∂B_k.

PROOF OF THEOREM 2.1. If K is not a subgroup of H_0, in particular if $\dim \Gamma/K < k$, then $z_j = 0$ for some $j = 1, \ldots, k$ on V^K. This implies that $F^K \neq 0$, in particular $[F]_\Gamma \in \Pi(k)$, as defined in [10, p. 381], and $[\widetilde{F}]_\Gamma = 0$. Furthermore, since $0 = [F^K]_\Gamma = \sum_{K<H_j} d_j [F_j^K]_\Gamma$, as seen in [10, p. 388], and noting that at this level the suspension is an isomorphism, one has $d_j = 0$ if $K = H_j$ is maximal, in which case d_j is the extension degree for F^K. On the other hand, if K is not maximal, then no H_j with $K < H_j$ can be a subgroup of H_0 and it is easily seen that $d_j = 0$ by solving the triangular relations $[F^{H_j}]_\Gamma = 0$. One then has

$$[F]_\Gamma = \sum_{\underline{H} < H_j < H_0} d_j [F_j]_\Gamma,$$

where F_j are the corresponding generators. Note that $[F^{H_0}]_\Gamma = d_{H_0}[F_0]_\Gamma$ with $d_{H_0} = \deg(F^{H_0}|_{B_k^{H_0}}; B_k^{H_0})$, according to [10, Theorem 4.1].

Let $H < H_0$ with $\dim \Gamma/H = k$. Then $[F^H]_\Gamma = \sum_{H<H_j<H_0} d_j[F_j^H]_\Gamma$ and on V^H there is an action of T^k such that $\sum_{j=1}^n n_j^l \varphi_j = \sum_{j=1}^k N_j^l \Phi_j$ with $N_j^l = \delta_{jl}N_j$ if $j = 1,\dots,k$, as in Lemma 1.1. Furthermore, our standing hypothesis implies that there is a Γ-equivariant map $\{x_j\} \to \{x_j^{l_j}\}$ from $(V^\Gamma)^{\perp_H}$ into $(W^\Gamma)^{\perp_H}$, the orthogonal complements in V^H and W^H respectively. It is clear that such a map implies that there is also an action of T^k on W^H, i.e. that the action of T^n can be formulated in terms of Φ.

Let $\widetilde{V}^H$ be a T^k-space of the same dimension as V^H and where the action of T^k differs only on the variables $\xi_1,\dots,\xi_k$, where it is $e^{i\Phi_j}$, $j = 1,\dots,k$, instead of $e^{iN_j\Phi_j}$. Any T^k-equivariant map $F(X_0, z_1,\dots,z_k, x_j)$ from V^H into W^H will generate a T^k-equivariant map $\widetilde{F}(X_0,\xi_1,\dots,\xi_k,x_j) = F(X_0,\xi_l^{N_1},\dots,\xi_k^{N_k},x_j)$ from $\widetilde{V}^H$ into W^H. Now, if K is an isotropy subgroup for the action of T^k on $\widetilde{V}^H$ with $K \neq \{e\}$, then on $\widetilde{V}^K$ one has $\xi_j = 0$ for some j in $\{1,\dots,k\}$, and the original map $\widetilde{F}^H$ as well as the generators $\widetilde{F}_j^H$ are non-zero on $\widetilde{V}^K$. Thus, $[\widetilde{F}^H]_{T^k}$ and $[\widetilde{F}_j^H]_{T^k}$ are elements of $\Pi(e, T^k)$, as defined in [10, Theorem 5.1].

Note that the existence of the Γ-equivariant map $\{x_j\} \to \{x_j^{l_j}\}$ implies that hypotheses (H) and (H2) of [10] are satisfied. Thus, $[\widetilde{F}^H]_{T^k}$ and $[\widetilde{F}_j^H]_{T^k}$ are uniquely determined by their extension degrees on the corresponding fundamental cell $\widetilde{\mathcal{C}}$ in $\widetilde{V}^H$, defined by $\xi_1,\dots,\xi_k$ real and positive. Furthermore, if $\widetilde{B}_k \equiv \{(t,X) : 0 < t < 1,\ \|X\| \le R,\ \xi_1,\dots,\xi_k$ real and positive$\}$ then, from [10, Theorem 4.1], these extension degrees are the usual degrees of the maps restricted to $\widetilde{B}_k$.

Finally, since $[\widetilde{F}^H]_{T^k} = \sum d_j[\widetilde{F}_j^H]_{T^k}$, as is easily seen from the corresponding equality for F^H and a Γ-action, and since the extension degree is a morphism onto $\Pi(l, T^k)$ (the sum is defined on the t variable), one has, from [10, Theorem 4.3],

$$\deg(\widetilde{F}^H|_{\widetilde{B}_k^H}; \widetilde{B}_k^H) = \sum d_j \deg(\widetilde{F}_j^H|_{\widetilde{B}_k^H}; \widetilde{B}_k^H).$$

Since $\xi_j^{N_j}$ is deformable to ξ_j on $\widetilde{B}_k$, the same relation holds on B_k^H and, from [10, p. 395], $\deg(F_j^H|_{B_k^H}; B_k^H) = (\prod l_i)(\prod k_i)$ where $\prod k_i = |H_0/H_j|$ and l_i corresponds to the "suspension map" of V^{H_j} in V^H as defined above, that is, $\prod l_i = \beta_{ij}$ for $H = H_i$. □

Remark 2.1. Let $V^* = \mathbb{R}^k \times V|_{B_k}$, where one identifies homotopically the set $\{z_i$ real and positive$\}$ with $\mathbb{R}$. Then $F|_{B_k}$ is an H_0-equivariant map and defines an element of $\Pi_{SV^*}^{H_0}(S^W)$. Now, any isotropy subgroup H of Γ with $\underline{H} < H < H_0$ gives an isotropy subgroup of H_0, since if $H = \Gamma_X = \bigcap H_j$ then $H_{0X} = \bigcap(H_j \cap H_0) = \Gamma_X \cap H_0 = H$, and conversely. Furthermore, one has

$\dim V^{*H} = \dim W^H$. From [10, Theorem 6.1],

$$[F|_{B_k}]_{H_0} = \sum_{H'_j < H_0} d'_j [F'_j]_{H_0} \quad \text{with } \dim H_0/H'_j = 0,$$

where $\{d'_j\}$ is obtained from the set of degrees $\deg(F^{H'_i}|_{B_k^{H'_i}}; B_k^{H'_i})$. By applying this argument to $[F|_{B_k}]_{H_0} - \sum_{\underline{H}<H_j<H_0} d_j [F_j|_{B_k}]_{H_0}$, one sees that the corresponding degrees are all 0, from Theorem 2.1 and the fact that the sum of the degrees is the degree of the sum. This implies that the corresponding d'_j are 0, since the triangular matrix for H_0 is invertible; thus, $[F|_{B_k}]_{H_0} = \sum_{\underline{H}<H_j<H_0} d_j [F_j|_{B_k}]_{H_0}$, since again the sum is well defined.

Hence, in this case the Γ-homotopy implies an H_0-homotopy on V^*.

Remark 2.2. Let F be as in Theorem 2.1, hence $[F]_\Gamma = \sum_{\underline{H}<H_j<H_0} d_j [F_j]_\Gamma$ with $\deg(F^{H_i}|_{B_k^{H_i}}; B_k^{H_i}) = \sum \varepsilon_{ij} \beta_{ij} d_j |H_0/H_j|$ for all $\underline{H} < H_i, H_j < H_0$ and $\varepsilon_{ij} = 1$ if $H_i < H_j$ and 0 otherwise. Now, one may also consider the map $\widetilde{F} = (F^{\underline{H}}, F^\perp_{\underline{H}})$, where $F^\perp_{\underline{H}}$ is the "suspension" map by $\{x_j^{l_j}\}$ from $(V^{\underline{H}})^\perp$ to $(W^{\underline{H}})^\perp$. It is clear that $\widetilde{F}^{H_i} = F^{H_i}$ for any $\underline{H} < H_i < H_0$ and that these two maps have the same set of degrees. Thus, the Γ-degrees of these maps are equal, i.e. the d_j's are the same, and F and $\widetilde{F}$ are Γ-homotopic on $\partial(I \times B_R)$. Furthermore, the preceding remark implies that $F|_{B_k}$ and $\widetilde{F}|_{B_k}$ are H_0-homotopic. In particular, $\deg(F|_{B_k}; B_k) = \deg(\widetilde{F}|_{B_k}; B_k) = (\prod l_j) \deg(F^{\underline{H}}|_{B_k}; B_k)$, that is,

$$\deg(F|_{B_k}; B_k) = \Big(\prod l_j\Big) \sum \beta_{ij} d_j |H_0/H_j|.$$

Remark 2.3. The relations given in Theorem 2.1 may be expressed in the form

$$\begin{pmatrix} \deg(F^{H_0}; B_k^{H_0}) \\ \deg(F^{H_i}; B_k^{H_i}) \\ \deg(F^{\underline{H}}; B_k^{\underline{H}}) \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ \beta_{i1} & |H_0/H_j| & \\ \beta_{s1} & \beta_{sj}|H_0/H_j| & |H_0/\underline{H}| \end{pmatrix} \begin{pmatrix} d_1 \\ d_j \\ d_s \end{pmatrix}.$$

Since the lower triangular matrix is invertible, the Γ-degree is completely determined by the degrees of the Poincaré sections. One may give a compact expression for the inverse by using the Möbius inversion formula, as in [11].

3. Index of an isolated orbit

Let $f : \overline{\Omega} \to W$ be a Γ-equivariant map, where Ω is an open bounded and invariant subset of $V = \mathbb{R}^k \times U$. Assume that $f^{-1}(0) = \Gamma X_0$, with $\Gamma_{X_0} \equiv H$ such that $\dim \Gamma/H = k$. Then f has a well defined Γ-degree with respect to Ω, given by the class of $F(t, X) = (2t + 2\varphi(X) - 1, \widetilde{f}(X))$ in $\Pi^\Gamma_{S^V}(S^W)$. From the excision property of the Γ-degree we may assume that Ω is a small invariant neighborhood of the orbit ΓX_0. Furthermore, X_0 has coordinates $z_1, \dots, z_k$ which are non-zero and with $H_0 = H_1 \cap \ldots \cap H_k$ such that $\dim W(H_0) = k$, as in the preceding section.

Thus, Ω can be chosen such that $z_j|_{\overline{\Omega}} \neq 0, j = 1, \ldots, k$, and $\varphi(X)$ can be constructed in such a way that $\varphi|_{\{z_j=0\}} = 1$ for $j = 1, \ldots, k$: in fact, this can be done for all the coordinates x_j in V for which X_j^0, the corresponding coordinate of X_0, is non-zero. This implies that $F|_{V^K} \neq 0$ for any K which is not a subgroup of H (and not only of H_0 as in the last section). As in the proof of Theorem 2.1, one has $\deg_\Gamma(f;\Omega) = \sum_{\underline{H}<H_j<H} d_j[F_j]_\Gamma$ and

$$\deg(f^{H_i}|_{B_k^{H_i}}; \Omega^{H_i} \cap B_k) = \sum_{H_i<H_j<H} \beta_{ij} d_j |H_0/H_j|.$$

The fact that $\deg(F^{H_i}|_{B_k^{H_i}}; B_k^{H_i})$ is the Brouwer degree of f^{H_i} on $\Omega^{H_i} \cap B_k$ follows from [8, p. 447]. Now, $|H_0/H_j| = |H_0/H| \cdot |H/H_j|$ and, as in [10, p. 377], due to the H_0-action on B_k, $f^{-1}(0) \cap B_k$ has $|H_0/H|$ points, each with the same index i_j on $V^{H_j} \cap B_k$. Hence, one may divide the above relation by $|H_0/H|$ and obtain the following result.

Theorem 3.1. *Let f be as above and let i_j be the Poincaré index of $f^{H_j}|_{B_k}$ at X_0. Then*

$$i_j = \sum_{\underline{H}<H_j<H} \varepsilon_{ij} \beta_{ij} d_j |H/H_j|.$$

Assume now that f is C^1 in a neighborhood of ΓX_0. It is easy to see that $Df(X_0)$ has a block-diagonal structure

$$Df^{H_i}(X_0) = \begin{pmatrix} Df^H(X_0) & 0 \\ 0 & Df^{\perp_i}(X_0) \end{pmatrix}$$

and that $Df^{H_i}(X_0)(X^{H_i})$ is an H-equivariant map [6, p. 412]. Suppose also that 0 is a regular value of f on Ω, that is, $Df(X_0)|_{B_k}$ is invertible. Then it follows from [6, pp. 403–404] that the H-representations $V \cap B_k$ and $W \cap B_k$ are equivalent. We shall then assume that $V = \mathbb{R}^k \times W$. This implies that $\beta_{ii} = 1$ and that $i_i = \operatorname{Sign}\det Df^{H_i}(X_0)|_{B_k} = i_H \operatorname{Sign}\det Df^{\perp_i}(X_0)$.

On $\Omega \cap B_k$, $f(X)$ is H-deformable to $(Df^H(X_0)(X^H - X_0), Df^\perp(X_0)X^\perp)$ and one may compute the H-degree of the linearization $Df(X_0)|_{B_k}$. From Remark 2.1, one has

$$[F|_{B_k}]_H = \sum_{\underline{H}<H_j<H} d_j [F_j|_{B_k}]_H$$

for the map $[2t + 2\varphi(X) - 1, f(X)]_H$. On the other hand,

$$[2t + 2\varphi(X) - 1, Df(X_0)(X - X_0)|_{B_k}]_H = \sum_{\underline{H}<H_j'<H} d_j' [F_j']_H$$

where F_j' are the generators for the action of H on B_k. Now, we have seen that H acts as a finite group on V^*. If one decomposes V^* into equivalent irreducible representations, then $Df(X_0)|_{B_k}$ has a block-diagonal structure [6, Chapter IV, Theorem 1.2, p. 407], where each block is a real matrix if H acts as $\mathbb{Z}_2$ or a complex matrix if H acts as $\mathbb{Z}_m, m \geq 3$. Furthermore, each block is H-deformable to a matrix of the form $\left(\begin{smallmatrix} \det A & 0 \\ 0 & I \end{smallmatrix}\right)$ if H acts as $\mathbb{Z}_2$ and to I if H acts as $\mathbb{Z}_m, m \geq 3$. This implies that $Df(X_0)|_{B_k}$ gives a suspension by the identity on the irreducible representations where H acts as $\mathbb{Z}_m, m \geq 3$, and $\deg_H(f|_{B_k}; B_k)$ has to take into account only those H'_j coming from coordinates where H acts trivially or as $\mathbb{Z}_2$, since the suspension is an isomorphism. In other words, one may consider H'_j such that $H/H'_j \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and $d'_j = 0$ if H'_j comes from an irreducible representation with an action of H of the form $\mathbb{Z}_m, m \geq 3$.

Now, if $H_i < H$, then, as we have already seen, H_i gives an isotropy subgroup of H on V^* and conversely. Furthermore, since $(z_1, \ldots, z_k)$ are the first variables, the fundamental cell for H_i as a subgroup of H is the restriction to B_k of the fundamental cell for H_i as a subgroup of Γ. If F_i is the generator corresponding to H_i, then it is easy to see that, by construction, $F_i|_{V^{H_j} \cap B_k} \neq 0$ for all $H_j > H_i$. Hence the relation $[F_i|_{B_k}]_H = \sum d_{ij}[F'_j]_H$ reduces to $[F_i|_{B_k}]_H = d[F'_i]_H$ where d is the extension degree of $F_i|_{B_k}$, that is, $d = \deg(F_i|_{B_k}; B_k)/\prod k_j$, as in [10, Theorem 4.1]. But then it is easy to see directly that $d = 1$, that is, $[F_i|_{B_k}]_H = [F'_i]_H$.

The above arguments imply that $d'_j = d_j$ and that $d_j = 0$ if $H_j = H \cap H_{i_1} \cap \ldots \cap H_{i_p}$ (H_{i_l} corresponding to the irreducible representation of H on V^*) and one of these is such that $H/H_{i_l} \cong \mathbb{Z}_m$, $m \geq 3$.

It remains to compute the other d_j's. It is easy to see that $d_H = i_H$ and, from $i_K = d_H + 2d_K$, that $d_K = (i_K - i_H)/2$ for any maximal K, i.e. with $H/K \cong \mathbb{Z}_2$. If K is not maximal, with $H/K \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$, then $Df(X_0)^K|_{B_k}$ has the form $\operatorname{diag}(A^H, A^{\perp K_1}, \ldots, A^{\perp K_s})$ with $i_H = \operatorname{Sign}\det A^H$ and $i_{K_j} = i_H \operatorname{Sign}\det A^{\perp K_j}$, where $H/K_j \cong \mathbb{Z}_2$. Hence, $i_K = i_H \prod_{j=1}^s (i_{K_j}/i_H)$ and Theorem 3.1 gives

$$\sum_{H_i < H_j} d_j |H/H_j| = i_H \Big[\prod (i_{K_e}/i_H) - 1 - \sum (i_{K_e}/i_H - 1) \Big]$$

where on the left side one has a sum over those H_j which are not maximal, i.e. different from $K_1, \ldots$, and on the right the product and the sum are over all maximal K_e with $H_i < K_e$.

These relations give a lower triangular matrix which is invertible (one may use the Möbius inversion formula for example) and the right hand side is completely determined by i_H and i_K for all maximal K's. We have proved the following result.

Theorem 3.2. *Let $V = \mathbb{R}^k \times W$ and 0 be a regular value of f on Ω with an isolated orbit ΓX_0 such that $\dim \Gamma/\Gamma_{X_0} = k$ and isotropy $\Gamma_{X_0} = H$. Then the Γ-index of the orbit is given by $(d_H, d_{K_1}, \ldots)$ such that $d_H = i_H, d_{K_j} =$*

$(i_{K_j} - i_H)/2$ if $H/K_j \cong \mathbb{Z}_2$, d_K is completely determined by the above integers if $H/K \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ with more than one $\mathbb{Z}_2$ factor, and $d_K = 0$ otherwise. Here i_K is the Poincaré index of f^K at X_0.

REMARK 3.1. If $\Gamma = S^1$ and $k = 1$, these index computations were given in [9, Proposition 5.2]. In this case $H \cong \mathbb{Z}_m$ and H/K cannot be a product.

REMARK 3.2. A similar result is given in [5, Proposition 4.7] and in [15].

REMARK 3.3. As an example one may consider the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}^3$ given by $(\gamma_1 x, \gamma_2 y, \gamma_1\gamma_2 z)$ with $\gamma_1^2 = \mathrm{Id}$, $\gamma_2^2 = \mathrm{Id}$ and $f(x,y,z) = -(x,y,z)$ (see [15, p. 85]). One has the following isotropy subgroups and corresponding subspaces: $H_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $(0,0,0)$, $H_1 = \mathbb{Z}_2 \times \{1\}$ and $(0,y,0)$, $H_2 = \{1\} \times \mathbb{Z}_2$ and $(x,0,0), H_3 = \{(1,1),(-1,-1)\}$ and $(0,0,z), H_4 = \{(1,1)\}$ and $\mathbb{R}^3$. By adding $2t-1$, with index i_0 equal to 1 at $t = 1/2$, one has

$$\begin{pmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

Hence, $i_0 = d_0 = 1, i_j = d_j = -1$ for $j = 1,2,3$, $i_4 = -1 = i_1 i_2 i_3$ and so $d_4 = 1$.

As an easy consequence of Theorem 3.2, one may obtain an abstract bifurcation and period doubling result of the following sort: assume $f(\lambda, X)$ is a family of Γ-equivariant functions from $\mathbb{R}^k \times W$ into W, with 0 as a regular value for $\lambda \neq \lambda_0$. If $X_0(\lambda)$ is the corresponding curve in V^H, where $H = \Gamma_{X_0(\lambda_0)}$ with $\dim \Gamma/H = k$, it is easy to see that $i_K(\lambda)$ and $d_K(\lambda)$ are well defined for $\lambda \neq \lambda_0$ and any K as above.

COROLLARY 3.1. (a) *If $i_H(\lambda)$ changes sign at λ_0, then one has a global bifurcation at λ_0 in V^H.*

(b) *If $i_H(\lambda)$ is constant and $i_K(\lambda)$ changes sign at λ_0 for some K with $H/K \cong \mathbb{Z}_2$, then there is a global bifurcation at λ_0 in V^K, i.e. with a period doubling. Topologically all bifurcations are in maximal isotropy subgroups, i.e. with $H/K \cong \mathbb{Z}_2$.*

PROOF. By global bifurcation we mean the existence of a continuum in $V \times \mathbb{R}$ going to infinity or returning to the set $(X_0(\lambda), \lambda)$, for $\lambda \neq \lambda_0$, or going to points where the hypothesis of the corollary does not apply any more (see [6]). The last sentence of the corollary means that if i_H and i_K, for all K's with $H/K \cong \mathbb{Z}_2$, do not change, then there will be no other changes for smaller isotropy subgroups. As is well known this does not hold for non-abelian actions. □

Our last result in this section relates the Γ-index to the "Floquet multipliers" in the generic case of a "hyperbolic orbit" as in [8, p. 474] and [9, p. 106].

We shall take the following setting: $V = \mathbb{R}^k \times W, F(\lambda, X) = X - f(\lambda, X)$ from V into W is C^1 and $f(\lambda, X)$ is a compact map with $F(\lambda_0, X_0) = 0$, and $\Gamma_{X_0} = H$ is such that $W(H)$ has dimension k. As before we choose the orientation of W in such a way that the first variables $z_1, \dots, z_k$ have an isotropy subgroup H_0 with $\Gamma/H_0 \cong T^k$, generated by $\Phi_1, \dots, \Phi_k$, as in Section 1, and action on z_j given by $e^{iN_j\Phi_j}$.

Since $F(\lambda_0, \gamma X_0) = 0$, one has $\frac{d}{d\Phi_j} F(\lambda_0, X_0) = 0 = F_X(\lambda_0, X_0) A_j X_0$, where A_j is the generator of the action of Φ_j. In other words, $\{A_j X_0\}$ generate the Lie algebra of Γ/H. Note that $A_j X_0$ has $iN_j z_j^0$ as its jth coordinate, hence the elements $\{A_j X_0\}$ are linearly independent. Here z_j^0 is the jth coordinate of X_0, which will be taken, without loss of generality, real and strictly positive.

DEFINITION 3.1. Let $K < H$. Then (λ_0, X_0) is said to be K-*hyperbolic* if and only if

(a) $\dim \ker(I - f_X^K(\lambda_0, X_0)) = k$,
(b) $f_\lambda(\lambda_0, X_0) : \mathbb{R}^k \to W$ is one-to-one, and
(c) $\operatorname{Range} f_\lambda(\lambda_0, X_0) \cap \operatorname{Range}(I - f_X^K(\lambda_0, X_0)) = \{0\}$.

Similarly (λ_0, X_0) is said to be K-*simply-hyperbolic* if (λ_0, X_0) is K-hyperbolic and the algebraic multiplicity of 0 as eigenvalue of $I - f_X^k(\lambda_0, X_0)$ is k.

Note that since X_0 is in V^H, it follows that $F(\lambda, X_0)$ belongs to W^H, and thus, $f_\lambda(\lambda_0, X_0)\mu$ belongs to W^H. Similarly, since $\Gamma X_0 \subset V^H$, $A_j X_0$ belongs to V^H. We have seen that $f_X^K(\lambda_0, X_0)$ has the diagonal structure

$$\begin{pmatrix} f_X^H(\lambda_0, X_0) & 0 \\ 0 & f_X^{\perp_K}(\lambda_0, X_0) \end{pmatrix},$$

hence it is easy to see that one has the following result.

PROPOSITION 3.1. *(λ_0, X_0) is K-hyperbolic if and only if (λ_0, X_0) is H-hyperbolic and $I - f_X^{\perp_K}$ is invertible.*

Note that $\operatorname{Range} f_\lambda(\lambda_0, X_0)$ has the right dimension to complement $\operatorname{Range}(I - f_X^H(\lambda, X_0))$ in W^H. Let

$$\mathcal{K}(\mu, Y) = (\mu_1 - \operatorname{Im} z_1, \dots, \mu_k - \operatorname{Im} z_k, f_\lambda(\lambda_0, X_0)\mu + f_X(\lambda_0, X_0)Y).$$

Then $\mathcal{K}$ is a compact linear operator on V and on V^K, for all $K < H$.

PROPOSITION 3.2. *(λ_0, X_0) is H-hyperbolic if and only if $I - \mathcal{K}^H$ is invertible.*

PROOF. If $(I - \mathcal{K})(\mu, Y) = 0$ then $\operatorname{Im} z_j = 0, f_\lambda \mu = 0$ and hence $\mu = 0$, Y belongs to $\ker(I - f_X^H)$, i.e. $Y = \sum \alpha_j A_j X_0$. By looking at $\operatorname{Im} z_j = \alpha_j N_j z_j^0$, one concludes that $\alpha_j = 0$ and $I - \mathcal{K}$ is one-to-one. Since $\mathcal{K}$ is compact, $I - \mathcal{K}$ is invertible.

Conversely, if $\dim\ker(I-f^H_X)>k$, let Y_0 be in this kernel and linearly independent of A_jX_0. Replacing Y_0 by $Y_0-\sum(\operatorname{Im}y_j/(N_jz^0_j))A_jX_0$ where y_j is the jth coordinate of Y_0, one may assume that $\operatorname{Im}y_j=0$ and $(0,Y_0)$ is in $\ker(I-\mathcal{K}^H)$. Similarly if $f_\lambda(\lambda_0,X_0)\mu=0$, then $(\mu,\sum(\mu_j/(N_jz^0_j))A_jX_0)$ is in $\ker(I-\mathcal{K}^H)$, i.e. f_λ must be one-to-one. Finally, if $f_\lambda\mu=-(I-f^H_X)Y$, then $(\mu,Y-\sum((y_j-\mu_j)/(N_jz^0_j))A_jX_0)$ is in $\ker(I-\mathcal{K}^H)$, where y_j is the jth coordinate of y. □

Let i_K be the index on the Poincaré section given by $\operatorname{Re}z_j>0$, $\operatorname{Im}z_j=0$, of the map $X-f^K(\lambda,X)$ at (λ_0,X_0), for X in W^K. Since the identity map $(\lambda_1,\ldots,\lambda_k,\operatorname{Re}z_1,\operatorname{Im}z_1,\ldots,\operatorname{Re}z_k,\operatorname{Im}z_k,\ldots)$, with the natural orientation on $\mathbb{R}^k\times W^K$, is homotopic to $((-1)^{k+1}\operatorname{Im}z_1,\ (-1)^{k+2}\operatorname{Im}z_2,\ldots,(-1)^{k+k}\operatorname{Im}z_k,\ \lambda_1,\ldots,\lambda_k,\operatorname{Re}z_1,\ldots,\operatorname{Re}z_k,\ldots)$ via a series of permutations, one has

$$i_K=(-1)^{k(3k+1)/2}\operatorname{Index}((\operatorname{Im}z_1,\ldots,\operatorname{Im}z_k,X-f^K(\lambda,X));(\lambda_0,X_0)),$$

where this Leray–Schauder index is with respect to the natural orientation on V^K. Now, by standard approximation arguments, this index is the index of $(0,0)$ for the operator $(I-\mathcal{K})(\mu,Y)$. Here $\mu=\lambda-\lambda_0$ and $Y=X-X_0$, since $\operatorname{Im}X^0_j=0$. From this last statement one has $i_K=i_H(-1)^{n'_K}$, where n'_K is the number, counted with multiplicity, of real eigenvalues of $f^{\perp K}_X$ which are larger than 1.

Note that n'_K is even if H/K is not a product of $\mathbb{Z}_2$'s. In fact, if H acts as S^1 or $\mathbb{Z}_m, m\ge 3$, on a set of equivalent irreducible representations, then the H-equivariant linear map $f^{\perp K}_X$ preserves these representations and can be seen as

$$(A+iB)(X+iY)\equiv\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix},$$

as a real operator. Since

$$\begin{pmatrix}A & -B\\ B & A\end{pmatrix}=P\begin{pmatrix}A+iB & B\\ 0 & A-iB\end{pmatrix}P^{-1}\quad\text{with}\quad P=\begin{pmatrix}I & I\\ -iI & iI\end{pmatrix},$$

it follows that

$$\det\begin{pmatrix}A-\lambda I & -B\\ B & A-\lambda I\end{pmatrix}=|\det(A-\lambda I+iB)|^2>0$$

and the algebraic multiplicity of any real eigenvalue is even. Similarly, if $(X,Y)^T$ is an eigenvector with real eigenvalue, then $(Y,-X)^T$ is also an eigenvector and the geometric multiplicity is even.

It is thus enough to compute i_H. Let $W^H=\ker(I-f^H_X)^m\oplus\operatorname{Range}(I-f^H_X)^m$, where the first term is the generalized eigenspace. Then $Y^H=u\oplus v$ and $I-f^H_X$ leaves each subspace invariant. Choose a basis for the first term in such a way that $I - f_X^H$ is in Jordan form on it, i.e., $u^T = (u_1, \dots, u_k)^T, u_j^T = (x_j^1, \dots, x_j^{\alpha_j})$, with $\sum \alpha_j = \alpha$, the algebraic multiplicity, and $\max \alpha_j = m$. Then

$$(I - f_X^H)u_j = J_{\alpha_j} U_j \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Since $A_j X_0$ is written in this basis as $u_j^T = (1, 0, \dots, 0), u_l = 0$ for $l \neq j$, we have $u = \sum x_j^1 A_j X_0 + w$, where w corresponds to the other variables. Then

$$\begin{aligned}&(I - \mathcal{K}^H)(\mu, Y)\\ &= (x_j^1 N_j z_j^0 + \operatorname{Im}(w_j + v_j), J_{\alpha_1} u_1 - f_1\mu, \dots, J_{\alpha_k} u_k - f_k \mu, (I - f_X^H)v - f_v \mu),\end{aligned}$$

where $(f_1\mu, \dots, f_k\mu, f_v\mu)$ are the components of $f_\lambda \mu$ in the basis. Furthermore, $(f_j\mu)^T = (f_j^1\mu, \dots, f_j^{\alpha_j}\mu)$ componentwise. Let Λ be the $k \times k$ matrix with jth row given by $f_j^{\alpha_j}$. One has the following result.

Theorem 3.3. *Let (λ_0, X_0) be K-hyperbolic. Then:*

(a) *$i_K = (-1)^{n'_K} i_H$, where n'_K is the number of eigenvalues of $f_X^{\perp K}$, counted with algebraic multiplicity, which are larger than* 1.

(b) *The matrix Λ is invertible and $i_H = (-1)^{k(k+1)/2}(-1)^{n_H} \operatorname{Sign}\det\Lambda$, where n_H is the number of eigenvalues of f_X^H, counted with algebraic multiplicity, which are larger than or equal to* 1.

Proof. If μ belongs to $\ker\Lambda$, then one obtains an element in $\ker(I - \mathcal{K}^H)$ by taking $v = (I - f_X^H)^{-1} f_v \mu, w_j^l = f_j^l \mu$ for $1 \le l \le \alpha_j - 1$ and $x_j^1 = -\operatorname{Im}(w_j + v_j)/(N_j z_j^0)$.

Thus, is order to compute the index, one may deform linearly, to 0, the terms f_v, f_j^l for $1 \le l \le \alpha_j - 1$, and then $\operatorname{Im}(v_j + w_j)$ to 0 and $N_j z_j^0$ to 1. One is left with the map

$$\begin{aligned}&(\mu_1, \dots, \mu_k, x_1^1, x_1^2, \dots, x_1^{\alpha_1}, x_2^1, \dots, x_2^{\alpha_2}, \dots, x_k^1, \dots, x_k^{\alpha_k}, v)\\ &\to (x_1^1, \dots, x_k^1, x_1^2, \dots, x_1^{\alpha_1 - 1}, -f_1^{\alpha_1}, x_2^2, \dots, -f_2^{\alpha_2}, \dots, x_k^2, \dots, -f_k^{\alpha_k}, (I - f_X^H)v).\end{aligned}$$

Via permutations, this map is homotopic to the map

$$(-(-1)^{\alpha_1} f_1^{\alpha_1}, -(-1)^{\alpha_2} f_2^{\alpha_2}, \dots, -(-1)^{\alpha_k} f_k^{\alpha_k}, x_1^1, \dots, x_1^{\alpha_1}, \dots x_k^1, x_k^{\alpha_k}, (I - f_X^H)v).$$

One may decompose $\operatorname{Range}(I - f_X^H)^m$ into $\sum \ker(I - \tau_l f_X^H)^{m_l} \oplus W$, where τ_l are the characteristic values of f_X^H with $0 < \tau_l < 1$. On each Jordan block for $I - \tau_l f_X^H$ of the form J, $I - f_X^H$ has the form $-I(1 - \tau_l)/\tau_l + J/\tau_l$, which is deformable to $-I$. On the other hand, on W, the operator $I - f_X^H$ is deformable to the identity. Hence

$$I_H = (-1)^{k(3k+1)/2}(-1)^{\alpha}(-1)^k(-1)^{\Sigma n_l} \operatorname{Sign}\det\Lambda,$$

where $n_l = \dim\ker(I - \tau_l f_X^H)$. Since $k(3k+1)/2 + k = 3k(k+1)/2$ has the parity of $k(k+1)/2$, one obtains the result. $\square$

Remark 3.4. In [8, Prop. 4.15, p. 475] and [9, Prop. 5.5, p. 112], a similar result was stated for the case $\Gamma = S^1$ and $k = 1$, where Λ was given in terms of generators of a complement of $\mathrm{Range}(I - f_X^H)$. By comparing the formulae it is easy to see that there is a difference of (-1) between the previous results and the one given here. This is due to the fact that in those papers we used the index of the Poincaré section given by (X, λ) with $\mathrm{Re}\, z_1 > 0, \mathrm{Im}\, z_1 = 0$, while here the section is given by (λ, X): the difference is an orientation factor of (-1), corresponding to the permutation of λ and $\mathrm{Re}\, z_1$.

Example 3.1 (Twisted orbits). Consider the problem of finding 2π-periodic solutions to the equation $dX/dt = f(X, \nu)$, where ν could be the frequency, X is in $\mathbb{R}^N$ and f is equivariant with respect to the abelian group $\Gamma_0 = T^n \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_m}$. This problem has been extensively studied (see e.g. [4]) but here we shall give a less algebraic presentation of it.

Let $X(t) = \sum X_n e^{int}$ be the Fourier series for $X(t)$ in V, an appropriate space of 2π-periodic functions. The action of Γ_0 on $\mathbb{R}^N$ induces a natural action on $\mathbb{C}^N$ such that one may find a basis for it such that on the jth coordinate of $\mathbb{C}^N, \Gamma_0$ acts as $\exp[2\pi i(\langle K_j/M, L\rangle + \langle N_j, \Phi\rangle)]$, where the vector K_j/M stands for $(k_j^1/m_1, \ldots, k_j^m/m_m)^T$ with $0 \le k_j^i < m_i, L$ is in $\mathbb{Z}^m, N_j$ in $\mathbb{Z}^n$ and Φ in $[0, 1]$ (see [10, Lemma 1.1]). Then $\Gamma \cong \Gamma_0 \times S^1$ acts on $X(t)$ as $\gamma X(t + \phi)$ and on the jth coordinate of X_n as $\exp[2\pi i(\langle K_j/M, L\rangle + \langle N_j, \Phi\rangle) + in\phi]$. Let H_{jn} be its isotropy subgroup, i.e. the set of $\{L, \Phi, \phi\}$ such that the above exponential is 1.

Note that, by [10, Lemma 1.1], $\Gamma/H_{jn} \cong S^1$ if $n \ne 0$ or $N_j \ne 0$. Hence, for a fixed ν, the only relevant isotropy subgroups for the equivariant degree are those for which $n = 0$ and $N_j = 0$, i.e. those for $V^{T^n \times S^1}$, which give stationary (in time and with respect to T^n) solutions. We shall leave this case to the reader and concentrate on the case of a free parameter ν.

Now, $H_{jn} = \{(L, \Phi, \phi) : n\phi/(2\pi) + \langle K_j/M, L\rangle + \langle N_j, \Phi\rangle \in \mathbb{Z}\}$, in particular $H_{j0} = H_j \times S^1$, where H_j is the isotropy subgroup of Γ_0 in $\mathbb{R}^N$. In order to apply Theorem 3.2 to H_{jn} we need to identify V^{jn}, the isotropy subspace for H_{jn}, and all isotropy subgroups K of Γ such that $H_{jn}/K \cong \mathbb{Z}_2$. Note that $K = \bigcap H_{lk}$ for l and k such that $(X_k)_l$ is in V^K and $K < H_{lk} \cap H_{jn} < H_{jn}$. Thus, either $H_{lk} \cap H_{jn} = H_{jn}$ and $H_{jn} < H_{lk}$, i.e. $(X_k)_l$ is in V^{jn}, or $H_{lk} \cap H_{jn} = K$ with $H_{jn}/K \cong \mathbb{Z}_2$, i.e. if (L, Φ, ϕ) is in H_{jn} then $(2L, 2\Phi, 2\phi)$ is in K and in H_{lk}. Then, if $H_{j0} < H_{lk}$, one requires $k\phi/(2\pi) + \langle K_l/M, L\rangle + \langle N_l, \Phi\rangle$ to be in $\mathbb{Z}$ for all (L, Φ) in H_j and all ϕ's. Taking $L = K_l M$ and $\Phi = 0$, this is impossible unless $k = 0$ and $H_j < H_l$. A similar argument with $(2L, 2\Phi, 2\phi)$ gives that the only possibility for $H_{j0}/K \cong \mathbb{Z}_2$ is for $k = 0$ and $H_j/H_j \cap H_k \cong \mathbb{Z}_2$. Thus, $V^{j0} \subset V^K \subset \mathbb{R}^N$, and we are dealing with stationary solutions.

If $n \neq 0$, then $H_{jn} < H_{lk}$ implies

$$\langle (kK_j - nK_l)/M, L\rangle + \langle kN_j - nN_l, \Phi\rangle = ak - bn$$

for integers a and b and all (L, Φ). Hence $kK_j/M - nK_l/M = kA - nB$ and $kN_j = nN_l$. For K, upon taking $(2L, 2\Phi, 2\phi)$, the coordinates in V^K have to satisfy $2kK_j/M - 2nK_l/M = 2kA - nB$ and $kN_j = nN_l$. This last equality implies that there are a finite number of modes, i.e. of k's, in V^K unless $N_j = N_l = 0$, i.e. with a trivial action of T^n. Note also that the element $(0, 0, 2\pi/n)$ in H_{jn} will be in H_{lk} only if k/n is an integer, and its double will be in H_{lk} only if $2k/n$ is an integer.

If $N_j \neq 0$, let $Y(t) = A(t)X(t)$, where $A(t) = \operatorname{diag}(e^{-ik_1 t}, \dots, e^{-ik_N t})$, with k_l such that $k_l N_l = nN_j$ and 0 otherwise, for the finite number of modes which satisfy the above relation. Here $A(t)$ is written in the representation induced by Γ_0 on $\mathbb{R}^N$, i.e. $A(t)$ is in fact a real matrix, but written this way for convenience. Then

$$Y'(t) = A'(t)A^{-1}(t)Y(t) + A(t)f(A^{-1}Y(t), \nu) = A'(0)Y(t) + f(Y(t), \nu)$$

since f is Γ_0-equivariant (take $t = -2\pi n\langle N_j, \Phi\rangle$). Thus, the equation is also Γ-equivariant and $Y_{0l} = X_{k_l l}$, i.e. one has frozen the rotating wave (see [4]), and one is back to the study of time stationary solutions. Hence, we shall assume $N_j = 0$.

Take the set of (k, l)'s such that $2k/n$ is an integer and $K_l = (2k/n)K_j/2 + (D_l/2)M + E_l M$, where D_l has components 0 or 1 and E_l is an integer-valued vector (they depend on k): these will contribute to V^K, while for V^{jn} one has (k, l) with k/n an integer and $D_l = 0$. Let $k_j^i/m_i = k_j^{i\prime}/m_i'$ with $k_j^{i\prime}$ and m_i' relatively prime (if $k_j^i = 0$ replace it by m_i and then these numbers are both 1). If $m^j = \text{l.c.m.}(m_1', \dots, m_m')$, then there is an L_j such that any L can be written as $cL_j + Q$ with $0 \le c < 2m^j$, $\langle K_j/(2M), L_j\rangle \equiv 1/(2m^j)$ and $\langle K_j/(2M), Q\rangle$ is an integer (see [10, Lemma 1.1] where those Q for which $\langle K_j/M, Q\rangle$ is odd are replaced by $Q - m^j L_j$ and we allow c to go up to $2m^j$).

Thus, $\langle K_l/M, L\rangle = c(2k/n)/(2m^j) + \langle D_l, cL_j + Q\rangle/2$. In these terms one has $H_{jn} = \{(c, Q, \phi) : 0 \le c < 2m^j,\ \langle K_j/M, Q\rangle$ is an even integer and $\phi/(2\pi) = -c/(nm^j) + 2d/n\}$ and $H_{lk} \cap H_{jn} = K = \{(c, Q, \phi) : \phi$ as above and $\langle D_l, cL_j + Q\rangle$ even$\}$. Thus, if (L, ϕ) is in K, then $\langle D_l, L\rangle$ is even for all k and l. Now, if $\langle D_{l_0}, L_j\rangle$ is odd for some (k_0, l_0), then any L can be written as $L = dL_j + L'$ with $\langle D_{l_0}, L'\rangle$ even (hence $\langle D_l, L'\rangle$ is even for all (k, l)), d is the parity of c and $\langle D_l, cL_j + Q\rangle \equiv c\langle D_l, L_j\rangle \pmod 2$. On the other hand, if for all (k, l) one has $\langle D_l, L_j\rangle$ even and not all D_l are 0, then there is an L_0 such that $L = dL_0 + L'$ with $\langle D_l, L'\rangle$ even for all (k, l) and an independent $\mathbb{Z}_2$-action on V^K. However, we shall see that this is never the case unless nm^j is odd and $\langle D_l, L_j\rangle$ is even for all l's.

Fix l and let $A_l = \{k > 0 : 2k/n \in \mathbb{Z},\ K_l = (2k/n)K_j/2 + D_lM/2 + E_lM\}$. If $K_j' = (k_j^{1\prime}, \ldots, k_j^{m\prime}), M' = (m_1', \ldots, m_m')$ and $k_l^{i\prime} = k_l^i m_i'/m^j$, then $2K_l' = (2k/n)K_j' + D_lM' + 2E_lM'$. Note that if k is in A_l, then so is $k + dnm^j$ for any integer d, with the same D_l. If $A_l = \emptyset$, then the corresponding coordinate does not enter in V^K. Furthermore, if k and k_1 are in A_l then $2(k_1 - k)/n = (2e_i + (d_i' - d_i)m_i')/k_j^{i\prime}$ for all i's. Since m_i' and $k_j^{i\prime}$ are relatively prime, one has $2(k_1 - k)/n = c_i m_i' = cm^j$. If $c = 2d$ is even, then $k_1 = k + dnm^j$ and $D_l' = D_l$, while if $c = 2d+1$, then $k_1 = k + (2d+1)nm^j/2$ (hence nm^j must be even) and

$$D_l' = D_l - (2d+1)m^jK_j'/M' + 2(E_l - E_l') = D_l + m^jK_j'/M' + 2E,$$

since m^jK_j'/M' is integer-valued. Thus, $\langle D_l', L_j\rangle = \langle D_l, L_j\rangle + 1 + 2e$.

Let $k_l = \min A_l^0$, where $A_l^0 = \{k \in A_l : \langle D_l, L_j\rangle \text{ even}\}$ and D_l be its corresponding element. Note that this subset A_l^0 of A_l is not empty, due to the alternating parity of $\langle D_l, L_j\rangle$, unless nm^j is odd. In this last case, we shall take $k_l = \min A_l$ and then $\langle D_l, L_j\rangle$ has always the same parity. Thus, any k in A_l^0 is of the form $k = k_l + dnm^j$ with the same D_l or, in the complement, of the form $k = k_l' + dnm^j$ with $D_l' = D_l + m^jK_j/M$ and $k_l' = k_l + nm^j/2$. Hence, in all cases, $K_l = (2k_l/n)K_j/2 + D_lM/2 + E_lM$ and any k in A_l is given by $k = k_l + dnm^j/2$, where the parity of d decides the class of k_j (d even if nm^j is odd).

Let $r_l = 2k_l/n + \langle D_l, L_j\rangle m^j$, that is, $r_j = 2$ and $r_l = 2k_l/n$ if the lth coordinate is in V^{jn}. Then the action of Γ on the l-coordinate of X_k is given by $\exp[2\pi i(cr_l/(2m^j) + dd_l/2 + k\phi/(2\pi))]$, where $d = 0$ unless nm^j is odd and $\langle D_l, L_j\rangle$ is even for all l, in which case $d_l = \langle D_l, L_0\rangle$. Hence, $r_l = 2k_l/m$ if nm^j is even and $r_l = 2k_l/m + \langle D_l, L_j\rangle m^j$ if nm^j is odd. Note that if, for some k, the pair (l, k) contributes to V^{jn}, then for that pair one has $2k/n$ even and $D_l = 0$, that is, $r_l = 2k_l/n$ is even. On the other hand, a coordinate will not contribute to V^{jn} if $A_l^0 \neq \emptyset, m^j$ is even and $r_l = 2k_l/n$ is odd, or if $A_l^0 = \emptyset$, hence nm^j is odd, and r_l is odd (in this case $2k_l/n$ is always even), or if nm^j is odd, $A_l^0 = A_l, r_l$ is even but $d_l = 1$.

Note that if $\Gamma_0 = \{e\}$, then $m^j = 1$ and $r_l = 2k_l/n$ is even, since the action has to be trivial. Let γ_0 be the matrix corresponding to $c = 1$ and $d = 0$, and γ_1 be the matrix corresponding to $c = 0$ and $d = 1$. Then the action of Γ_0 on $\mathbb{R}^N$ is generated by γ_0 and γ_1. Since $\gamma_0^{2m^j} = \mathrm{Id}$, one has a natural splitting of $\mathbb{R}^N$ as $\mathbb{R}^{N_0} \times \mathbb{R}^{N_1}$, where $\gamma_0^{m^j}$ acts as $(-1)^i \,\mathrm{Id}$ on $\mathbb{R}^{N_i}$, i.e. $\mathbb{R}^{N_0}$ corresponds to even r_l and $\mathbb{R}^{N_1}$ to odd ones. If nm^j is odd and r_l is even for all l's, then the splitting of $\mathbb{R}^N$ corresponds to the action of γ_1, since $\gamma_1^2 = \mathrm{Id}$. Thus, if $X(t)$ is in V^{jn}, one has

$$X(t) = \sum X_k e^{ikt} = \sum \sum_l e^{2\pi i r_l/(2m^j)}(X_k)_l e^{ik(t-2\pi/(nm^j))}$$

where the first sum is over k's with $k = k_l + dnm^j$ and even $r_l = 2k_l/n$. Hence, $X(t) = \gamma_0 X(t - 2\pi/(nm^j))$ and $X(t) = \gamma_1 X(t)$. Conversely, any $X(t)$ which satisfies these relations is in V^{jn}, where the components of $X(t)$ are restricted to those for which r_l is $2k_l/n$ and even, i.e. in $\mathbb{R}^{N_0}$. In fact, there $\gamma_0^{m^j} = \mathrm{Id}$ and $X(t)$ is $2\pi/n$-periodic and the only modes present in the Fourier expansion of $X(t)$ are those for which k is a multiple of n and $k = k_l + dnm^j$.

For V^K, the same arguments yield that, if nm^j is even,

$$X(t) = \gamma_0\Big(\sum_{\text{even}}\sum_l (X_k)_l e^{ik(t-2\pi/(nm^j))} - \sum_{\text{odd}}\sum_l (X_k)_l e^{ik(t-2\pi/(nm^j))}\Big)$$

where the first sum is over k's for which $k = k_l + 2dnm^j/2$ and the second over k's with $k = l_l + (2d+1)nm^j/2$. Then

$$X(t) = X_0(t) + X_1(t) = \gamma_0(X_0(t - 2\pi/(nm^j)) - X_1(t - 2\pi/(nm^j))).$$

Now, $X(t) = \gamma_0^2 X(t - 4\pi/(nm^j))$ and since $\gamma_0^{2m^j} = \mathrm{Id}$, one sees that $X(t)$ is $4\pi/n$-periodic. Conversely, any $X(t)$ with that periodicity will have modes k with $2k/n$ an integer and $X(t)$ can be split as above: In fact, by changing m^j to $2m^j$, one finds as above that $2k/n$ is an integer and $2k = 2k_l + dnm^j$. According to the parity, one will have X_0 or X_1. Note that $X(t), X_0$ and X_1 have a spatial splitting on the coordinates of $\mathbb{R}^N$, i.e. on $\mathbb{R}^{N_0} \times \mathbb{R}^{N_1}$, even and odd r_l's. The components of X_0 in $\mathbb{R}^{N_0}$ are $2\pi/n$-periodic, while those in $\mathbb{R}^{N_1}$ are $2\pi/n$-antiperiodic. The behavior of the components of $X_1(t)$ differs by a factor $(-1)^{m^j}$. Since we are working with 2π-periodic functions this implies that $X(t)$ is in $\mathbb{R}^{N_0}$ if n is odd and m^j even.

If nm^j is odd, then the only modes are those of the form $k = k_l + dnm^j$. One then has a spatial splitting, with $X(t) = X_+(t) + X_-(t)$, with X_+ in $\mathbb{R}^{N_0}$ and X_- in $\mathbb{R}^{N_1}$. Then $X(t) = \gamma_0(X_+(t - 2\pi/(nm^j)) - X_-(t - 2\pi/(nm^j)))$, and both $X_\pm$ are $2\pi/n$-periodic. The converse is clear. Finally, if nm^j is odd and r^l is even for all l, then $X(t) = X_+(t) + X_-(t) = \gamma_0 X(t - 2\pi/(nm^j))$, i.e. $X(t)$ is $2\pi/n$-periodic and one has $\gamma_1 X_\pm(t) = \pm X_\pm(t)$. Hence, we get

Lemma 3.1. (a) *The elements of* V^{jn} *are those* $X(t) = \gamma_0 X(t - 2\pi/(nm^j))$ *with components in the subspace* $\mathbb{R}^{N_0}$ *of* $\mathbb{R}^N$ *where* $\gamma_0^{m^j} = \mathrm{Id}$ *and* $\gamma_1 = \mathrm{Id}$.

(b) *The elements of* V^K *are such that* $X(t) = \gamma_0^2 X(t - 4\pi/(nm^j))$. *If* nm^j *is odd, then* $X(t)$ *is as above, with a spatial splitting induced by* $\gamma_0^{m^j}$ *and* γ_1.

Let then $\overline{X}(t)$ be in V^{jn} and a solution of $X'(t) = f(X(t), \nu_0)$. Let $B(t) = Df(\overline{X}(t), \nu_0)$. Then, since $\gamma Df(X, \nu) = Df(\gamma X, \nu)\gamma$ for any γ in Γ_0, for $\gamma_0^{m^j}$ and γ_1 which fix $\bar{X}(t)$, one has a structure of $B(t)$ of the form $\mathrm{diag}(B_+(t), B_-(t))$

where $B_\pm$ acts on $\mathbb{R}^{N_\pm}$ and $B_\pm(t)$ are $2\pi/n$-periodic. Let $\Phi(t)$ be the fundamental matrix, i.e. $d\Phi/dt = B(t)\Phi$ and $\Phi(0) = I$. From the flow invariance it is easy to see that $\Phi(t) = \mathrm{diag}(\Phi_+(t), \Phi_-(t))$. Note that $\overline{X}'(t)$ belongs to $\ker(d/dt - B(t))$ and that, as seen in [7, Appendix] and [9, Proposition 4.16], the eigenvalues of $\mathrm{Id} - F_X$ are related, including the algebraic multiplicities, to the $X(t)$ in V^{jn} or V^K such that $\frac{d}{dt}X(t) - B(t)X + \lambda X = 0, \lambda > 0$, which are given by $X(t) = e^{-\lambda t}\Phi(t)W$ with W in $\ker(\Phi(2\pi) - e^{2\pi\lambda}I)$, so that $X(t)$ is 2π-periodic, i.e. $e^{2\pi\lambda}$ is a Floquet multiplier for the Poincaré return map $\Phi(2\pi)$.

Now, if $X(t)$ is in V^{jn}, then $X(2\pi/(nm^j)) = \gamma_0 X(0)$ and $\gamma_0^{-1}A_+W = e^{\lambda 2\pi/(nm_j)}W$, where $\Phi(2\pi/(nm^j)) = \mathrm{diag}(A_+, A_-)$. Similarly, if $X(t)$ is in V^K then $\gamma_0^{-2}BW = e^{\lambda 4\pi/(nm^j)}W$ with $B = \Phi(4\pi/(nm^j))$. Let $\Psi(t + 2\pi/(nm^j)) = \gamma_0\Phi(t)\gamma_0^{-1}$. It is easy to see that Ψ is also a fundamental matrix, hence $\Psi(t) = \Phi(t)C$ with $C = A^{-1}$. Then $\Psi(2\pi s/(nm^j)) = \gamma_0\Phi(2\pi(s-1)/(nm^j))\gamma_0^{-1} = \Phi(2\pi s/(nm^j))A^{-1}$. Thus, $\Phi(2\pi s/(nm^j)) = \gamma_0^s(\gamma_0^{-1}A)^s$. In particular, $\Phi(2\pi) = \gamma_0^{nm^j}(\gamma_0^{-1}A)^{nm^j}$. Hence, $\Phi_+(2\pi) = (\gamma_0^{-1}A_+)^{nm^j}$, $\Phi_-(2\pi) = (-1)^{nm^j}(\gamma_0^{-1}A_-)^{nmj}$, $\gamma_0^{-2}B = (\gamma_0^{-1}A)^2$, $\Phi_\pm(2\pi/n) = \pm(\gamma_0^{-1}A_\pm)^{m^j}, \Phi_\pm(4\pi/n) = (\gamma_0^{-1}A_\pm)^{2m^j}$ and the generalized spectra of these matrices are easily related.

Since we are interested in the eigenvalues of $\gamma_0^{-1}A_+$ which are real and larger than 1, for V^{jn}, and in the eigenvalues of $(\gamma_0^{-1}A)^2$ which are real and larger than 1, for V^K, let $\sigma_+^\varepsilon =$ number of real eigenvalues λ of $\gamma_0^{-1}A_+$ with $\varepsilon\lambda > 1$ and likewise for σ_-^ε and A_-. If nm^j is odd, then $X_+(t) = \gamma_0 X_+(t - 2\pi/(nm^j))$, hence $\gamma_0^{-1}A_+X_+(0) = e^{\lambda 2\pi/(nm^j)}X_+(0)$, while $X_-(t) = -\gamma_0 X_-(t - 2\pi/(nm^j))$, hence $\gamma_0^{-1}A_-X_-(0) = -e^{\lambda 2\pi/(nm^j)}X_-(0)$. If $\gamma_1 \neq \mathrm{Id}$, then one has a splitting according to the eigenvalues of γ_1. We have proved

Proposition 3.3. *If the orbit $\overline{X}(t)$ is hyperbolic, then $i_H = (-1)^{\sigma_+^+}\varepsilon$, where ε depends on how f depends on ν ($\varepsilon = 1$ for $f(x,\nu) = f(x)/\nu$). We have $i_K = (-1)^{\sigma_+^+ + \sigma_-^+ + \sigma_-^+ + \sigma_-^-}\varepsilon$ if nm^j is even. If nm^j is odd and $\gamma_1 = \mathrm{Id}$, then $i_K = (-1)^{\sigma_+^+ + \sigma_-^-}\varepsilon$, and if nm^j is odd and $\gamma_1 \neq \mathrm{Id}$, then $i_K = (-1)^{\sigma_+^+ + \sigma_-^+}\varepsilon$.*

Remark 3.4. (a) If nm^j is even, then $i_H = i'_H$ and $i_K = i'_K$, where i'_H is given by the number of eigenvalues of $\Phi_+(2\pi)$ which are larger than 1, and i'_K by those eigenvalues of $\Phi(2\pi)$ which are of absolute value larger than 1. The same interpretation can be given in the cases of nm^j odd.

(b) One may also use the fact that for any $X(t)$ in V^K, one has $X(t) = X_0(t) + X_1(t)$ with $X_0(t) = \gamma_0 X_0(t - 2\pi/(nm^j))$ and $X_1(t) = -\gamma_0 X_1(t - 2\pi/(nm^j))$; then one may look at the Floquet multipliers, larger than 1, for the problem $X(t + 2\pi/n) = aX(t), \gamma_0 X(t) = bX(t + 2\pi/(nm^j))$, where $|a| = |b| = 1$ (see [4, Definition 6.1]). Write σ_{fb}^a for the number of these eigenvalues (counted with multiplicity). Then, if $b = 1, X(t) = X_0(t)$, if $b = -1$ then $X(t) = X_1(t)$, if $a = 1$ one has to look at positive eigenvalues of $\gamma_0^{-1}A$, and if $a = -1$ at negative eigenvalues. Hence $\sigma^+_{f+} = \sigma^+_+, \sigma^-_{f+} = \sigma^-_-$, since in the first case $X_0(t)$ is in $\mathbb{R}^{N_0}$ and in the second in $\mathbb{R}^{N_1}$. If $b = -1$ and $a = 1$, then σ^+_{f-} is σ^+_+ if m^j is even (then X_1 is in $\mathbb{R}^{N_0}$) and σ^+_- if m^j is odd (then X_1 is in $\mathbb{R}^{N_1}$). If $b = -1$ and $a = -1$, then σ^-_{f-} is σ^-_- if m^j is even (X_1 is in $\mathbb{R}^{N_1}$) and σ^-_+ if m^j is odd (X_1 is in $\mathbb{R}^{N_0}$). Thus $i_K = (-1)^{\sigma^+_{f+}+\sigma^-_{f+}+\sigma^-_+ +\sigma^+_-}$, where $\sigma^-_+ + \sigma^+_- = \sigma^+_{f-} + \sigma^-_{f-}$ if m^j is odd.

(c) As in [9, Chapter VII], one may define an orbit index as $(i_H + i_K)/2$.

Example 3.2 (Time dependent equations). Consider the problem of finding 2π-periodic solutions to the problem $dX/dt = f(X,t)$, where $f(X, t + 2\pi/p) = f(X,t)$ for some integer p. By writing $X(t) = \sum X_n e^{int}$, one has $inX_n - f_n(X) = 0$, where $f_n(X) = (2\pi)^{-1} \int_0^{2\pi} f(X(t),t)e^{-int}dt$. Replacing $X(t)$ by $X(t+\varphi)$ gives $f_n(X(t+\varphi)) = e^{in\varphi} \int_0^{2\pi} f(X(t), t-\varphi)e^{-int}dt$, where one has used the 2π-periodicity of X and $2\pi/p$-periodicity in t. Hence $f_n(X(t+\varphi)) = e^{in\varphi} f_n(X(t))$ for $\varphi = 2k\pi/p, k = 0,\dots,p-1$, giving a natural $\mathbb{Z}_p$-action on these functions. If $\overline{X}(t)$ is a $2\pi/p'$-periodic solution with p a multiple of p', then the linearization near $\overline{X}$ will solve the problem $dX/dt - Df(\overline{X}(t),t)X = 0$, where $B(t) = Df(\overline{X}(t),t)$ is $2\pi/p'$-periodic. It is then easy to compute the indices and relate them to the Poincaré index of the linearization. We leave this task to the reader.

4. Borsuk–Ulam results

In this section we shall show how many of the extension ideas given implicitly in [10] can be proved, with less stringent hypotheses, in the case when there are no extra parameters (in [10] the main interest was on the parameter case where obstructions are not primary). For the moment the only hypothesis is that V and W are representations of the compact abelian group Γ, with a special first coordinate t in V^Γ and W^Γ.

Lemma 4.1. *Let* $\mathrm{Iso}(V)$ *be the set of all isotropy subgroups of* Γ *for* V *and let* $A = \{H \in \mathrm{Iso}(V) : \exists K \in \mathrm{Iso}(V), K \le H$ *and* $\dim V^K > \dim W^K + \dim \Gamma/K\}$.

(a) *Let* F_0 *be an equivariant map from* $\bigcup_{H\in A} S^{V^H}$ *into* $\bigcup_{H\in A} W^H \setminus \{0\}$. *Then* F_0 *has an equivariant extension* F *from* S^V *into* $W \setminus \{0\}$.

(b) *Let* A' *be the subset of* $\mathrm{Iso}(V)$ *defined as* A *but with* $\dim V^K \ge \dim W^K + \dim\Gamma/K$ *instead of strict inequalities. Then if* F'_0 *and* G'_0 *are* Γ*-homotopic maps on* $\bigcup_{H\in A'} S^{V^H}$, *any two extensions* F' *and* G' *are* Γ*-homotopic on* S^V.

(c) *If* F_0 *is as in* (a) *and* $\dim V < \dim W$, *then any extension* F *is, non-equivariantly, deformable to a constant.*

Proof. For (a) it is enough to follow the arguments of [10, Theorem 3.1(a)] on the set $\mathcal{C} \cap S^V$, where $\mathcal{C}$ is the fundamental cell, which has the right dimension for the extension. Note that $\bigcup_{H\in A} S^{V^H}$ can be replaced by any invariant set which contains this union.

For (b), replace V by $I \times V$ and repeat the above argument. Or, consider $[F']_\Gamma - [G']_\Gamma$ in $\Pi^\Gamma_{S^V}(S^W)$ (recall that the addition is done on the first variable). This map is Γ-homotopically trivial on $\bigcup_{H\in A'} S^{V^H}$, i.e. it has a Γ-extension to $\bigcup_{H\in A'} B^H$. One may apply directly [10, Theorem 3.1(a)] to get a Γ-extension to B, i.e. $[F']_\Gamma - [G']_\Gamma = 0$. The same extension problem on A would meet obstructions given by the extension degree of [10, Theorem 3.1(b)].

(c) is trivial since $\Pi_{S^V}(S^W) = 0$ in this case. $\square$

Our next result will be used for the construction of the generators in the Hopf theorem.

Lemma 4.2. (a) *If H is not in A and $\dim V^H = \dim W^H + \dim \Gamma/H$ then there is an equivariant map F_H such that F_H is $(1,0)$ on any B^K when K is not a subgroup of H, and F_H has extension degree* 1.

(b) *If furthermore the following hypothesis holds:*

$$(\mathrm{H}_0) \qquad \forall \gamma \in \Gamma, \quad \operatorname{Sign}\det\gamma \operatorname{Sign}\det\widetilde{\gamma} > 0,$$

and $\dim V^H = \dim W^H$ and Γ/H is finite, then $\deg(F^H_H; B^H) = |\Gamma/H|$ and F_H can be constructed in such a way that $\deg(F^K_H; B^K) = \beta_{KH}|\Gamma/H|$ for some integer β_{KH}, for any $K < H$ with $\dim V^K = \dim W^K$ and Γ/K finite, while this degree is 0 *if K is not a subgroup of H.*

Proof. Define F_H as $(1,0)$ on all the balls B^K with K as above. Consider the fundamental cell $\mathcal{C}_H$ for B^H, as in [10, Section 3], $\mathcal{C}_H = \{x_j : 0 \le |x_j| \le R, 0 \le \operatorname{Arg} x_j < 2\pi/k_j\}$, where k_j are defined in [10]. Then $\mathcal{C}_H$ is a ball of dimension equal to dim W^H. For $k_j > 1$ and $x_j = 0$, extend F_H as $(1,0)$, as well for $\operatorname{Arg} x_j = 0$ and $2\pi/k_j$, if $2 \le k_j < \infty$, with x_j complex if $k_j = 2$. On the rest of $\partial\mathcal{C}_H$, construct a map of degree 1 with respect to $\mathcal{C}_H$ (one may always localize such a map in a neighborhood of any point of a sphere). This map is clearly equivariant with respect to the symmetries of $\partial\mathcal{C}_H$ (in fact, it is invariant). One extends this map, by the free action of Γ/H, to an equivariant map on B^H which is non-zero on ∂B^H. Since $H \notin A$, one may extend this last map to S^V, by using Lemma 4.1(a) on $A \cup K$, for all K's which are not subgroups of H. Note that this construction implies that $\Pi(H,K)$ in [10, Theorem 4.2] is $\mathbb{Z}$ in this case.

If (H_0) holds, then, from [10, Theorem 4.1], one sees that $\deg(\widetilde{F}_H; B_K) = \prod k_j \deg_{\mathrm{E}}(F)$ and, in the particular case of H with $\dim V^H = \dim W^H$ and Γ/H finite, $\deg(F^H_H; B^H) = |\Gamma/H| \deg_{\mathrm{E}}(F) = |\Gamma/H|$. In this case any element in $\Pi(H)$, as defined in [10], is uniquely determined by its extension degree.

For (b), if K is not a subgroup of H, then $F_H = (1,0)$, with zero degree, while, if $K < H$, let $x_{\perp_j}$ be the components in the orthogonal complement of V^H in V^K. Then $\mathcal{C}_K = \mathcal{C}_H \times \{x_{\perp_j} : 0 \le |x_{\perp_j}| < R,\ 0 \le \operatorname{Arg} x_{\perp_j} < 2\pi/k_j\}$ and B^K is $|\Gamma/H|$ images of $\mathcal{C}_H$ cross the ball $\{X_\perp : \|X_\perp\| \le R\} \equiv B_\perp$. Now, F^H_H

was defined as $(1,0)$ on $\partial\mathcal{C}_H \cap \{X : \|X\| < R\}$. By defining F_H^K as $(1,0)$ on this last set crossed by $B_\perp$, one obtains, from the dimension arguments of Lemma 4.1(a), an equivariant map F_H, which is $(1,0)$ on $\partial\mathcal{C}_H \cap \{X : \|X\| < R\}$. Since (H_0) holds, $\deg(F_H^K; B^K)$ is the sum of the degrees on $\mathcal{C}_H \times B_\perp$, and all of them are equal. Hence $\deg(F_H^K; B^K) = |\Gamma/H| \deg(F_H^K; \mathcal{C}_H \times B_\perp)$. □

Remark 4.1. (a) If there is an equivariant map $F_\perp^K$ from the orthogonal complement of V^H in V^K to the orthogonal complement of W^H in W^K, with zero only at 0, then one may take for F_H^K the couple $(F_H^H, F_\perp^K)$ for which $\deg(F_H^K; B^K) = |\Gamma/H| \deg(F_\perp^K; B_\perp)$. If one has the same situation for another isotropy subgroup $L < K$, then one would have

$$\deg(F_H^L; B^L) = \deg(F_H^K; B^K)\deg(F_\perp^K; B'_\perp),$$

where $B'_\perp$ is a ball in the orthogonal complement of V^K in V^L. This is the case of hypothesis (H2′) in [10] and, in particular, for hypothesis (H) of the present paper. However, this is not true in general: if one takes the example of Section 0, then any map G with $d_\Gamma = 1$ will have $\deg G^H = p(1 + pd_H)$, a multiple of p, and $\deg G = 1 + kp$, which is not a multiple of the previous degree.

(b) As pointed out above, (H_0) implies that the extension degree, if $\dim V^H = \dim W^H$, is independent of previous extensions, contrary to the case where $\dim \Gamma/H > 0$, where one has to add new hypotheses.

(c) If (H_0) holds, then if V'^H denotes the orthogonal complement of V^Γ in V^H, it is easy to see that $|\dim V'^H - \dim W'^H|$ is even (see [10, p. 376]). In particular, $|\dim V^H - \dim W^H|$ has the parity of $|\dim V^\Gamma - \dim W^\Gamma|$.

We are now ready for the Hopf classification theorem, which should be compared to [10, Theorems 5.2 and 6.1] with a different set of hypotheses, and to [12] and the references therein, in our particular case of a linear action of an abelian group.

Theorem 4.1. *Let* $\widetilde{A} = \{H \in \mathrm{Iso}(V) : \exists K \in \mathrm{Iso}(V), K \le H,$ *and* $\dim V^K > \dim W^K$ *when* $|\Gamma/K| < \infty$ *or* $\dim V^K \ge \dim W^K + \dim \Gamma/K$ *when* $|\Gamma/K| = \infty\}$, *i.e.* $A \subset \widetilde{A} \subset A'$. *Assume* (H_0) *holds. Then if* F *and* F_0 *are two equivariant maps which are* Γ*-homotopic on* $\bigcup_{H\in\widetilde{A}} S^{V^H}$, *one has integers* d_H *such that*

$$[F]_\Gamma = [F_0]_\Gamma + \sum_I d_H [F_H]_\Gamma \quad \text{in } \Pi^\Gamma_{S^V}(S^W),$$

where the sum is over the set I *of all* H*'s* not in $\widetilde{A}$ *with* $\dim V^H = \dim W^H$ *and* $|\Gamma/H| < \infty$, *and* F_H *is the generator constructed in Lemma* 4.2(b). *If* $\widetilde{A} = \emptyset$, *then* F_0 *is not present.*

Proof. Let $\Pi(\widetilde{A}) = \{[F]_\Gamma : F : \bigcup_{H\in\widetilde{A}} S^{V^H} \to \bigcup W^H \setminus \{0\}\}$ ($\Pi(\widetilde{A}) = [(1,0)]_\Gamma$ if $\widetilde{A} = \emptyset$). As in [10], it is easy to see that $\Pi(\widetilde{A})$ is a group. Let Π :

$\Pi^{\Gamma}_{S^V}(S^W) \to \Pi(\widetilde{A})$ be the map induced by restriction on the isotropy subgroups in $\widetilde{A}$. From Lemma 4.1, Π is a morphism onto $\Pi(\widetilde{A})$ and $\ker \Pi$ corresponds to those elements F which have an equivariant non-zero extension to all B^H for H in $\widetilde{A}$. Note that if $|\dim V^{\Gamma} - \dim W^{\Gamma}|$ is odd, then, from Remark 4.1(c), $|\dim V^H - \dim W^H|$ is odd for all H. In that case $\widetilde{A} = A'$ and, from Lemma 4.1(b), Π is one-to-one, i.e. $\Pi^{\Gamma}_{S^V}(S^W) = \Pi(\widetilde{A})$, and the theorem is proved.

Let H_0 be maximal among the isotropy subgroups not in $\widetilde{A}$ with finite Weyl group and equal dimensions for the corresponding isotropy subspaces, i.e. if $H > H_0$, then either H is in $\widetilde{A}$, or $\dim V^H < \dim W^H + \dim \Gamma/H$. Let F_0 be in $\ker \Pi$. Then F_0 is extendable to all B^H with $H > H_0$, i.e. $F_0^{H_0}$ belongs to $\Pi(H_0)$, as defined in [10], and its extendability to B^{H_0} will be characterized by its extension degree, given by the relation $\deg(F_0^{H_0}; B^{H_0}) = |\Gamma/H_0| \deg_{\mathrm{E}}(F_0)$. Let $d_{H_0} = \deg_{\mathrm{E}}(F_0)$ and F_{H_0} be the generator of Lemma 4.2(b). Then F_{H_0} is also in $\ker \Pi$ and $[F_0]_{\Gamma} - d_{H_0}[F_{H_0}]_{\Gamma} \equiv [F_1]_{\Gamma}$, which has zero extension degree, is in $\Pi(H)$ and is extendable to B^{H_0}. Let $A_0 = \widetilde{A} \cup \{H : H \geq H_0\}$. One may define, as before, $\Pi(A_0)$ and the projection Π_0 from $\Pi^{\Gamma}_{S^V}(S^W)$ onto $\Pi(A_0)$. It is clear that $[F_1]_{\Gamma}$ belongs to $\ker \Pi_0$ and one may repeat the construction with another maximal H_1. After a finite number of steps, one will arrive at $[F_0]_{\Gamma} - \sum_I d_H [F_H]_{\Gamma} = 0$. In general, if F and F_0 are as in the statement of the theorem, then $[F]_{\Gamma} - [F_0]_{\Gamma}$ is in $\ker \Pi$ and the result follows. □

We leave to the reader the task of verifying that the generators of the example in Section 0 are the appropriate ones. From the above theorem, one may obtain Borsuk–Ulam results.

Theorem 4.2. *Let V and W be two arbitrary representations of Γ with $\dim V = \dim W$, and let $F : V \setminus \{0\} \to W \setminus \{0\}$ be an equivariant map. Then:*

(a) $\deg(F; B) = 0$ *if* (H_0) *does not hold or if* $\dim V^{T^n} \neq \dim W^{T^n}$.

(b) *If* (H_0) *holds and the above subspaces have the same dimension, then* $\deg(F; B) = \beta \deg(F^{T^n}; B^{T^n})$, *where β is the non-zero integer given in Theorem* 1.1.

(c) *Let $\widetilde{A}' = \{H \in \mathrm{Iso}(V) : \exists K \in \mathrm{Iso}(V),\ T^n \leq K \leq H$ with $\dim V^K > \dim W^K\}$. Let F_0 be any equivariant extension of F, restricted to $\bigcup_{H \in \widetilde{A}'} S^{V^H}$, from $V^{T^n} \setminus \{0\}$ into $W^{T^n} \setminus \{0\}$. Assume the hypothesis of* (b) *holds. Let $I = \{H \in \mathrm{Iso}(V) : H \notin \widetilde{A}',\ T^n \leq H,\ \dim V^H = \dim W^H\}$. Then, for any $H_0 \in I$, one has*

$$\deg(F^{H_0}; B^{H_0}) = \deg(F_0^{H_0}; B^{H_0}) + \sum_I d_H \beta_{H_0 H} |\Gamma/H|,$$

where $\beta_{H_0 H} = 0$ if H_0 is not a subgroup of H, $\beta_{HH} = 1$, $\beta_{H_0 H}$ are integers independent of F and F_0, and d_H are integers which depend only on F and F_0.

If $\widetilde{A}' = \emptyset$, then F_0 is absent. If, furthermore, $W^\Gamma = \{0\}$, hence $V^\Gamma = \{0\}$ (from the existence of F), then one has to add, on the right, a term $\beta_{H_0\Gamma}$.

PROOF. Let us recall that $T^n \in \mathrm{Iso}(V)$, since $V^{T^n} = \{X : |\Gamma/\Gamma_X| < \infty\}$ (see [10, p. 371]). Also, if (H_0) does not hold, then $\deg(F;B) = 0$ [10, Remark 4.1]. The fact that $\deg(F;B) = 0$ if the dimensions of V^{T^n} and W^{T^n} are different was noted after the proof of Theorem 1.1. If $W^\Gamma = \{0\}$, hence $V^\Gamma = \{0\}$ since F^Γ maps the first space into the second, one may replace V and W by $\mathbb{R} \times V$ and $\mathbb{R} \times W$ and suspend the map F by the trivial map $2t-1$ with $0 \le t \le 1$, with the same degrees. This implies (b). Furthermore, in this case $\mathrm{Iso}(\mathbb{R} \times V) = \mathrm{Iso}(V) \cup \Gamma$, $\widetilde{A}'$ remains the same and, if $\widetilde{A}' = \{0\}$, the set I has to be supplemented by Γ. Then $\deg(2t-1; \mathbb{R}^\Gamma) = 1 = d_\Gamma$.

(c) follows from Theorem 4.1 applied to V^{T^n}, after noting that if $T^n \le H$, then $V^H \subset V^{T^n}$ and $|\Gamma/H| < \infty$. □

REMARK 4.2. (a) If one takes the usual decreasing order on the elements of I, then, as in [10], one has a matrix relation $(\deg(F^{H_0}; B^{H_0}) - \deg(F_0^{H_0}; B^{H_0})) = B(d)$, where B is a lower triangular matrix with $|\Gamma/H_0|$ on the diagonal. In particular, B is invertible. Thus, if F and F_0 have the same degrees on all B^{H_0} with H_0 in I, one has $[F^{T^n}]_\Gamma = [F_0^{T^n}]_\Gamma$.

(b) Note also that if $H \in \widetilde{A}'$ with $\dim V^H = \dim W^H$, then $\deg(F_K^H; B^H) = 0$ for all the generators F_K with K in I, by construction.

It would be interesting to know under what circumstances one may construct F_0 such that $\deg(F_0^{H_0}; B^{H_0}) = 0$ for all H_0 in I, or at least for T^n, besides the case where $\widetilde{A}'$ is empty, so that $\deg(F;B)$ would be a multiple of the greatest common divisor of the $|\Gamma/H|$'s for H in I.

COROLLARY 4.1. (a) *Assume that $\widetilde{A}'$ has a unique minimal element K. If K is not Γ, assume furthermore that there is an equivariant map $F_\perp$ from $(V^K)^{\perp_{T^n}} \setminus \{0\}$ into $(W^K)^{\perp_{T^n}} \setminus \{0\}$. Then one may construct F_0 such that $\deg(F_0^H; B^H) = 0$ for all H in I with $H < K$, in particular for T^n.*

(b) *If $K = \Gamma$ the last hypothesis is not necessary.*

(c) *For any minimal element K of $\widetilde{A}'$, one has*

$$\deg(F^{H_0}; B^{H_0}) = \sum_{I_K} d_H^K \beta_{H_0H}^K |K/H|$$

for all H_0 in I with $H_0 < K$, where I_K is the set of all H in I with $H < K$, and $\beta_{H_0H}^K = 0$ if H_0 is not a subgroup of H.

(d) *If for all minimal K_j in $\widetilde{A}'$, one has a complementing map $F_\perp^j$, then one may construct F_0 with $\deg(F_0; B^{T^n}) = 0$. Note that $F_\perp^j$ exists if for all $H \in \mathrm{Iso}((V^{K_j})^\perp)$ one has $\dim V^H \cap (V^{K_j})^\perp \le \dim W^H \cap (W^{K_j})^\perp$.*

Proof. If K is minimal, then $\dim V^K > \dim W^K$ and $\dim V^H \leq \dim W^H$ for all H with $T^n \leq H < K$. If K is unique, then $\bigcup_{H\in\widetilde{A}'} S^H = S^K$ and one may define F_0 as $(F^K, F_\perp)$. If H is in I with $H < K$, then, from Lemma 4.1(c), $F_\perp^H$ is deformable (non-equivariantly) to a constant and $\deg(F_0^H; B^H) = 0$. If $\widetilde{A}' = \Gamma$, then $\dim (V^\Gamma)^{\perp_H} < \dim (W^\Gamma)^{\perp_H}$ for all H with $T^n \leq H$, and one may construct $F_\perp$ as above.

For (c), for each minimal K, consider F as a K-equivariant map. The isotropy subgroups are those elements H of $\mathrm{Iso}(V)$ with $H < K$. The corresponding $\widetilde{A}'$ reduces to K and I to I_K. One then applies (b).

For (d), let K_1 be a minimal element, and let $[F_0]$ be $\Pi(F)$ in $\Pi(\widetilde{A}')$. Define F_1 in $\Pi(\widetilde{A}')$ by the relation $[F_0] = [F_0^{K_1}, F_\perp^1] + [F_1]$ with $F_1^{K_1} = (1,0)$. If K_2 is another minimal element, define F_2 in $\Pi(\widetilde{A}')$ by $[F_1] = [F_1^{K_2}, F_\perp^2] + [F_2]$ with $F_2^{K_2} = (1,0)$. Since $[F_1^{K_2}, F_\perp^2]^{K_1} = [F_1|_{V^{K_1}\cap V^{K_2}}, (F_\perp^2)^{K_1}] = [(1,0,(F_\perp^2)^{K_1})]$ is Γ-deformable to $(1,0)$ one may use the equivariant Borsuk extension theorem and assume that $F_2^{K_1} = (1,0)$. One will arrive at a final map F_s with $F_s = (1,0)$ on $\bigcup_{H\in\widetilde{A}'} S^{V^H}$. Hence,

$$[F_0] = \sum_{j=1}^{s} [F_{j-1}^{K_j}, F_\perp^j] \quad \text{in } \Pi(\widetilde{A}').$$

Since the maps on the right have obvious extensions to S^V, one may construct F_0 in this way. If H is in I and $H < K_j$, then $\dim (V^{K_j})^\perp \cap V^H < \dim (W^{K_j})^\perp \cap V^H$ and $(F_\perp^j)^H$ is a non-zero equivariant map between these spaces. Thus, $\deg([F_{j-1}^{K_j}, F_\perp^j]^H; B^H) = 0$. In general $\deg(F_0^H; B^H)$ will be the sum of the degrees of the maps on the right for those j's such that H is not a subgroup of K_j. In particular, $\deg(F_0; B^{T^n}) = 0$. Note that $\deg((F_{j-1}^{K_j}, F_\perp^j)^H; B^H) = 0$ unless $\dim V^{K_j} \cap V^H = \dim W^{K_j} \cap W^H$, in which case this degree is the product of $\deg(F_\perp^{jH}; (V^{K_j})^\perp \cap B^H)$ and $\deg((F_{j-1}^{K_j})^H; V^{K_j} \cap B^H)$. This last degree is again 0 if $V^{K_j} \cap V^H \subset V^{K_i}$ for some $i \leq j-1$, since there F_{j-1} is $(1,0)$. Otherwise, one could repeat the above argument on V^H and its corresponding $\widetilde{A}'$. □

One has the following extension of [13, Theorem 2.5].

Corollary 4.2. *Assume that Γ/T^n is a p-group, i.e. $|\Gamma/T^n| = p^k$ for some prime number p. If V and W are two arbitrary representations of Γ with $\dim V = \dim W$ and $F : V\backslash\{0\} \to W\backslash\{0\}$ is an equivariant map, then $\deg(F;B)$ is a multiple of p unless hypothesis* (H) *for V^{T^n} holds, in which case*

$$\deg(F^{H_0}; B_0^H) = \sum_{H_0 \leq H} d_H(\Pi_{H,H_0} l_i)|\Gamma/H|$$

for all H_0 in $\mathrm{Iso}(V)$ with $T^n \leq H_0$, where the l_i's are given in Lemma 0 *and the product corresponds to the variables in $(V^H)^{\perp_{H_0}}$. Here, $|\Gamma/H|$ is a multiple of p except for $H = \Gamma$, and $d_\Gamma = \deg(F^\Gamma; B^\Gamma)$.*

PROOF. If (H_0) does not hold, or if $\dim V^{T^n} \neq \dim W^{T^n}$, then $\deg(F;B) = 0$. Otherwise, if $\widetilde{A}' \neq \emptyset$, take any minimal element K. Then $T^n < K$ and for any H in $I_K, |K/H|$ is a non-zero power of p. Thus, $\deg(F;B)$ would be a multiple of p (from Corollary 4.1(c)).

Hence, if this degree is not a multiple of p, then (H_0) must hold, $\dim V^{T^n} = \dim W^{T^n}$ and $\widetilde{A}' = \emptyset$, in particular $\dim V^H \leq \dim W^H$ for all H with $T^n \leq H$. Now, if there is K such that $\dim V^K < \dim W^K$, then viewing F^{T^n} as a K-map, one should have

$$\deg(F^{T^n}; B^{T^n}) = \sum_{H<K} d_H^K \beta_{H_0 H}^K |K/H| \quad \text{for } H \text{ in } I_K,$$

hence a multiple of p. Thus, for all H in $\mathrm{Iso}(V^{T^n})$, one has $\dim V^H = \dim W^H$. Now, if K and H in $\mathrm{Iso}(V^{T^n})$ are such that $\dim V^H \cap V^K$ and $\dim W^H \cap W^K$ are different, consider F^K, from V^K into W^K, as an H-equivariant map. The fixed point subspaces for the action of H on these spaces are $V^H \cap V^K$ and $W^H \cap W^K$. From the preceding arguments, $\deg(F^K; B^K)$ is a multiple of p. Now, regarding F^{T^n} as a K-map one has, from Corollary 4.1, $\deg(F^{T^n}; B^{T^n}) = a\deg(F^K; B^K) + bp$, hence, in this case, a multiple of p. In conclusion, (H) holds for V^{T^n} and $[F^{T^n}]_\Gamma = \sum d_H [F_H]_\Gamma$, where each generator F_H can be chosen of the form $(F_H^H, x_j^{l_j})$ as in Lemma 0, with $\deg(F_H^H; B^H) = |\Gamma/H|$. $\square$

For instance an even map on $\mathbb{R}^d$ has degree 0 if d is odd ((H_0) does not hold) and has even degree if d is even ($d_\Gamma = 2^d$).

EXAMPLE 4.1. One may wonder if Corollary 4.1(d) depends really on the existence of the complementing maps. Here is an example to the contrary, which is inspired by [1, Example 3.21]. Let $\mathbb{Z}_{12}$ act on two copies of $\mathbb{C}^6$ in the following way. On the first copy, as $e^{2\pi ik/4}$ on x_1, x_2, x_3, x_4 and as $e^{2\pi ik/6}$ on y_1 and y_2. On the second copy, as $e^{2\pi ik/2}$ on ξ_1, ξ_2, ξ_3 and as $e^{2\pi ik/12}$ on η_1, η_2, η_3. The elements of $\mathrm{Iso}(V)$ are $K = \mathbb{Z}_3$ (for k a multiple of 4) with $V^K = \{x_1, x_2, x_3, x_4\}$ and $W^K = \{\xi_1, \xi_2, \xi_3\}, H = \mathbb{Z}_2$ (for k a multiple of 6) with $V^H = \{y_1, y_2\}$ and $W^H = W^K$, and $\{e\}$ (and Γ if one adds a dummy variable t). Here the set I is reduced to $\{e\}$, and $\widetilde{A}' = \{K, \Gamma\}$. Note that there is no equivariant map $F_\perp$ from $(V^K)^\perp \setminus \{0\}$ into $(W^K)^\perp \setminus \{0\}$, since any such map should map $(V^K)^\perp = V^H$ into W^H. If the conclusion of Corollary 4.1(a) still holded, any equivariant map F from $V \setminus \{0\}$ into $W \setminus \{0\}$ would have a degree which should be a multiple of 12. However, the following map has degree 6:

$$F = (x_1^2 - \overline{x}_2^2 - \overline{y}_1^3, x_3^2 - \overline{x}_4^2 - \overline{y}_2^3, \operatorname{Re} x_1 x_2 + i \operatorname{Re} x_3 x_4 + y_1^2 y_2, \overline{x}_1 y_1^2, \overline{x}_3 y_2^2, \overline{x}_2 y_1^2 + \overline{x}_4 y_2^2).$$

The equivariance of F and the fact that the only zero is at the origin are clear. In order to compute the degree, subtract $\varepsilon > 0$ from the last equation. The zeros of the perturbed map are at $A = (0, 0, 0, \varepsilon^{3/7}, 0, -\varepsilon^{2/7})$ and $B =$

$(0,\varepsilon^{3/7},0,0,-\varepsilon^{2/7},0)$ (at a zero one needs $y_1y_2=0$; if $y_1=0$, then $x_1=\pm\overline{x}_2=0$; $y_2\neq 0$ if $\varepsilon>0$, hence $x_3=0$, $\overline{x}_4y_2^2=\varepsilon$ and $\overline{x}_4^2+\overline{y}_2^3=0$, i.e. $-|y_2|^6y_2=\varepsilon^2$).

Near A one may deform linearly $\overline{x}_3y_2^2$ to $\overline{x}_3\varepsilon^{4/7}$ and to $\overline{x}_3$. Then x_3 can be deformed to 0 in the other equations. Then $y_1^2y_2$ is deformed to y_1^2 and the term $\overline{x}_2y_1^2$ to 0. One obtains a product of three maps: $\overline{x}_3$ with index -1, $(x_1^2-\overline{x}_2^2-\overline{y}_1^3, \operatorname{Re}x_1x_2+y_1^2,\overline{x}_1y_1^2)$ and $(-\overline{x}_4^2-\overline{y}_2^3,\overline{x}_4y_2^2-\varepsilon)$. In order to compute the index of the second map at its only zero, the origin, perturb the second equation by $-i\varepsilon$. The zeros of the perturbed map are for $x_1=0, y_1^2=i\varepsilon$. One may deform x_1 in the first two equations to 0 and y_1^2 to $i\varepsilon$ in the last. The degree will be $-\deg(-\overline{x}_2^2-\overline{y}_1^3,y_1^2-\varepsilon)$. Taking ε to 0 and $\overline{y}_1^3$ to 0, one obtains $-(-2)(2)=4$. For the third map, with a unique zero, one may deform ε to 0 and consider the map $(\overline{x}_4^2+\overline{y}_2^3-\varepsilon,\overline{x}_4y_2^2)$ with 3 zeros of the form $(x_4=0,|y_2|^3=\varepsilon)$, each of index $(-1)(-1)=1$, and two zeros of the form $(|x_2|^2=\varepsilon, y_2=0)$, each of index $(-1)(2)=-2$. The degree of the third map is -1. Hence, the index of F at A is 4.

For B, one follows the same steps, except that the term $y_1^2y_2$ which was deformed to y_1^2 is now deformed to y_2. The index of the second map is now 2, instead of 4, and the index of F at B is 2. Thus, $\deg(F;B)=6$.

Note that, as a K-map, any equivariant map may be written as

$$\begin{aligned}[2t-1,F]_K &= [2t-1,F^K,y_1^2,y_2^2,0]_K\\ &\quad + d[2t+1-2|y_1|^2,x_1,x_2,x_3,y_1^2(y_1^3-1),y_1^2(\overline{y}_1y_2-1),y_1^2x_4]_K,\end{aligned}$$

which shows that $\deg(F;B)=3d$. Viewing F as an H-map, using Corollary 4.1(a), one has

$$\begin{aligned}&[2t-1,F]_H\\ &\quad = e[2t+1-2|x_1|^2,y_1,y_2,x_1x_2-1,x_1(x_1^2-1),x_1(x_1x_3-1),x_1(x_1x_4-1)]_H,\end{aligned}$$

which gives $\deg(F;B)=2e$. Hence, $\deg(F;B)$ is a multiple of 6. The same result may be obtained by considering the action of $\mathbb{Z}_6$.

On the other hand, if F and F_0 coincide on V^K, then $[2t-1;F]_\Gamma=[2t-1,F_0]_\Gamma+f[F_e]_\Gamma$, where

$$\begin{aligned}F_e=(2t+1-2|x_1y_1|^2,x_1^2(x_1^4-1),x_1^2(\overline{x}_1x_2-1),x_1^2(\overline{x}_1x_3-1),\overline{x}_1y_1^2(\overline{x}_1^2y_1^3-1),&\\ \overline{x}_1y_1^2(\overline{y}_1y_2-1),\overline{x}_1y_1^2(\overline{x}_1x_4-1)).&\end{aligned}$$

Then $\deg(F;B)=\deg(F_0;B)+12f$.

By taking for F_0 the map of the example, one generates, for maps from $\mathbb{R}\times V$ into $\mathbb{R}\times W$, all odd multiples of 6 and by taking off $[F_e]_\Gamma$, all even multiples of 6. Hence, for Γ-maps from $\mathbb{R}\times V$ into $\mathbb{R}\times W$, all multiples of 6 are achieved. By replacing, in the example, the term $y_1^2y_2$ by $y_1^{2+6n}y_2$, where a negative exponent means conjugation, the index of A is changed to $2(2+6n)$, while that of B is unchanged. Hence, any odd multiple of 6 is achieved as the degree of a Γ-map from V into W.

EXAMPLE 4.2. In order to understand better the problems involved in the construction of equivariant maps with zero degree, let us study the simplest case where the coordinates of V have only two isotropy types K and H with $K \cap H = \{e\}$, $\dim V^K > \dim W^K$, $\dim V = \dim W$. If $K = \Gamma$ or $H = \{e\}$, then one may construct a complementing map and $[2t-1, F]_\Gamma = [2t-1, F^K, F_\perp]_\Gamma + d[F_e]$ and $\deg(F; B) = d|\Gamma|$ (in this case $\Gamma \cong \mathbb{Z}_n$). Thus, assume that $H \neq \{e\}$ and $K < \Gamma$. Then V^K and V^H are orthogonal, since $V^\Gamma = \{0\}$. Let $V^K = \{x_1, \ldots, x_n\}$ and $V^H = \{y, \ldots, y_m\}$. Let $\{\xi_1, \ldots, \xi_r\}$ be the coordinates of W^K and $\{\eta_1, \ldots, \eta_s\}$ the other coordinates of W.

Now, $\Gamma/K \cong \mathbb{Z}_u$ acts freely on V^K as $e^{2\pi i m_j/u}$, with m_j and u relatively prime, on x_j and as $e^{2\pi i k_l/u_l}$ on ξ_l, with k_l and u_l relatively prime and u_l a divisor of u. If one changes x_j to X_j with action $e^{2\pi i/u}$ and if $q_l k_l \equiv 1 \pmod{u_l}$, then from an equivariant map F from V to W one constructs a new equivariant map $F_l^{q_l}(X_1^{m_1}, \ldots, X_m^{m_m}, \ldots)$ with degree equal to $\prod m_j \prod q_l \deg(F; B)$. Hence, one may assume that $m_j = q_l = 1$, without affecting the congruences mod u or mod $|\Gamma|$. If V'^K is a subspace of V^K with the dimension of W^K, one obtains

$$\deg(F^K|_{V'^K}; B \cap V'^K) = 0 = \prod(u/u_l) + du,$$

from [10, Theorem 6.2]. If, for simplicity, we assume $u_l = p$ and $u = vp$, then $d = -v^{r-1}/p$. Thus, $r \geq 2$ and any prime factor of p divides v, while its square divides u. The simplest case is $u = p^2$.

REMARK 4.3. (a) At this point, there is the question of the existence of a $\mathbb{Z}_{p^2}$-equivariant map from V^K into W^K (and hence its extension to V, using Lemma 4.1). The map $(x_1^p(x_1^{p^2} - 1), x_2^p((x_1\overline{x}_2)^{p+1} - 1))$ has degree 0 (its zeros are $(0, 0)$ with index p^2, ($x_1 =$ p^2-root of unity, 0) each of index p, ($x_1 =$ one of these roots, $|x_2|^{p+1} = 1$) each of index -1, with total degree $p^2 + p^3 - p^2(p+1) = 0$). By repeating the map for (x_3, x_4) and using the homotopies to a constant map, one may obtain an equivariant map from $\mathbb{C}^5$ into $\mathbb{C}^4$, by using the homotopies on the sector $\{0 \leq |x_5| \leq R,\ 0 \leq \operatorname{Arg} x_5 < 2\pi/p^2\}$. See [1, Theorem 3.22]. See also [1, Corollary 5.9] for the conditions on the dimensions of V and W.

(b) If one is willing to use different u_l's, one may use the construction of [1, Proposition 3.8], by taking p_1 and p_2 relatively prime with $1 = \alpha p_1 + \beta p_2$. Then the map

$$f \equiv (z_1(\overline{z}_1^{\alpha p_1} - 1), z_2(\overline{z}_2^{\beta p_2} - 1)(\overline{z}_1^{\alpha p_1} z_2^{\beta p_2} - \varepsilon))$$

with $|\varepsilon| \neq 1$ is equivariant for the group $\mathbb{Z}_{p_1 p_2}$ from $\mathbb{C}^2$ into itself, with action on z_1 given by $e^{2\pi i k/p_1}$ and on z_2 by $e^{2\pi i k/p_2}$. The degree of f is 0. If the group acts on z as $e^{2\pi i k/(p_1 p_2)}$, then one may define, using the homotopy to a constant, an extension for the set $\operatorname{Arg} z = 0, 0 \leq z \leq R$ and by equivariance, on the boundary of the fundamental cell for z. By composing the map with f again, one gets an extension to the cell itself and an equivariant map F from $\mathbb{C}^3$ to $\mathbb{C}^2$. Then, if $\mathbb{Z}_{p_1p_2}$ acts on (x_1, x_2, x_3) in the standard way, one may look at $F(x_1^{p_2}, x_2^{p_1}, x_3)$.

Now, $\Gamma/H \cong \mathbb{Z}_v$ acts freely on V^H. As before one may assume that the action is by $e^{2\pi i k/v}$. If $W^K \cap W^H = \{0\}$, then one is in the situation of Corollary 4.1 and there is a complementing map, given by F^H, which as a map into $(W^K)^\perp$ is non-equivariantly trivial, i.e. the map (F^K, F^H) has degree 0. On the other hand, if $W^K \cap W^H \neq \{0\}$, then since action on all ξ's is the same, it follows that W^H contains W^K. This implies that v is a multiple of p, say $v = qp$. If $\Gamma \cong \mathbb{Z}_n$, the condition $H \cap K = \{e\}$ implies that $n = p^2q$. For simplicity, we shall assume that p and q are relatively prime, i.e. there are α and β such that $\alpha q + \beta p = 1$. Then, if the action on η_l is of the form $e^{2\pi i k u/(p^2 q)}$ and η_l is in W^H, one has $u = u_0 p$ and the map y^{u_0} may be used to build a complementing map. Thus, we shall assume that $u = 1$ and $W^H = W^K$. Then $n + m = r + s$, $n > r$ and we have the above standard actions of $\mathbb{Z}_{p^2q}$. Now, if $m > r$, then $\deg(F^H; B^H \cap \{y_{r+1} = \ldots = y_m = 0\}) = 0 = q^r + dpq$ (from [10, Theorem 6.2]), which is not possible since p and q are relatively prime. Thus, $m \leq r$.

Note that the expression $x^\alpha y^\beta$ is equivariant into $(W^H)^\perp$. In general if Γ is a finite group, $H = \bigcap_{j=1}^{s-1} H_j$ an isotropy subgroup with H_j the isotropy subgroup of the coordinate z_j, and η a coordinate in W^H, then $H < \Gamma_\eta$. If one considers the space $V^H \oplus \{\eta\}$, Lemma 7.2 of [10] gives the existence of an invariant monomial $z_1^{\alpha_1} \ldots z_{s-1}^{\alpha_{s-1}} \eta$, since $k_s = |H/H \cap \Gamma_\eta| = 1$. By taking $\eta = 1$ and changing α_j into $-\alpha_j$ (i.e. conjugates) one obtains a similar equivariant monomial. In order to complete the example, one has the following:

Proposition 4.1. *For the above situation, one has:*

(a) *If* $m < r$, *then* $\deg(F; B) = apq$.

(b) *If* $m = r$, *then* $\deg(F; B) = \alpha q + apq$, *hence not a multiple of* pq, *in particular not* 0.

(c) *If* $m \leq r - 2$, *then for any* F^K, *one has an extension* $\widetilde{F}_0$ *of* $(2t - 1, F^K)$ *with* $\deg(\widetilde{F}_0; I \times B) = 0$ *or equivalently* $\deg(F; B) = ap^2q$.

(d) *If* $m = r-1$, *then for any* F^K, *there is an extension* F_0 *with* $\deg(F_0; B) = \alpha^{n-m}pq + ap^2q$, *in particular non-zero and not a multiple of* p^2q.

Proof. Viewing F as a K-map, one may use Corollary 4.1(b) to prove that $\deg(F; B)$ is a multiple of q. If we view it as an H-map, the corresponding $\widetilde{A}'$ is empty and the degree is a multiple of p if $m < r$, while if $m = r$, one has

$$\deg(F; B) = \deg((F^H, x_1^\alpha, \ldots, x_m^\alpha); B) + dp = \alpha^m \deg(F^H; B^H) + dp.$$

But viewing F as a Γ-map yields $\deg(F^H;B^H)=\deg(2t-1,y_1^q,\dots,y_m^q)+epq=q^m+epq$. Hence, $\deg(F;B)=(\alpha q)^m+d_1p=1+d_2p$ (by using $\alpha q=1-\beta p$). Since $\deg(F;B)=cq=1+d_2p=\alpha q+(d_2+\beta)p$, it follows that $c=\alpha+ap$ and one obtains (b).

Note that if $m=r$, then $[2t-1,F]_\Gamma=[2t-1,F_0]_\Gamma+d_H[F_H]_\Gamma+d_e[F_e]_\Gamma$, where

$$\begin{aligned}F_H&=(2t+1-2|y_1|^2,y_1^q(y_1^{pq}-1),y_1^q(\overline{y}_1y_2-1),\dots,y_1^q(\overline{y}_1y_m-1),\\&\qquad\qquad x_1^\alpha y_1^\beta,\dots,x_n^\alpha y_1^\beta),\\F_e&=(2t+1-2|x_1y_1|^2,y_1^q(y_1^{pq}-1),\dots,y_1^q(\overline{y}_1y_m-1),\\&\qquad\qquad x_1^\alpha y_1^\beta(x_1^p\overline{y}_1^q-1),x_1^\alpha y_1^\beta(\overline{x}_1x_2-1),x_1^\alpha y_1^\beta(\overline{x}_1x_n-1)).\end{aligned}$$

It is easy to see that $\deg(F_H^H;B^H)=pq$, $\deg(F_H;B)=\alpha^npq$, $\deg(F_e^H;B^H)=0$ and $\deg(F_e;B)=p^2q$.

For (c), assume first that $m=1$ and let $V'^k=\{x_1,\dots,x_r\}$, which has the dimension of W^K. Now, from Lemma 4.1, $F|_{V'^K}$ has an equivariant extension $G(x_1,\dots,x_r,y_1)$ from $V'^K\times\mathbb{C}\setminus\{0\}$ into $W^K\times\mathbb{C}\setminus\{0\}$, with this last $\mathbb{C}$ corresponding to η_1. Let $\widetilde{G}(x_1,\dots,x_n,y_1)$ be an equivariant extension of F^K and G, which will have zeros. Let $F_0=(\widetilde{G},x_{r+1}^\alpha y_1^\beta,\dots,x_n^\alpha y_1^\beta)$, which has no zeros but the origin. In order to compute the degree of F_0, perturb the last component of $\widetilde{G}$ (on η_1) by $-\varepsilon$. Since this last component must be 0 for $y_1=0$, the zeros of the perturbed map are those of $G(x_1,\dots,x_r,y_1)-(0,\dots,0,\varepsilon)=G_\varepsilon$. For the computation of the degree of F_0, one may deform x_j to 0 in $\widetilde{G}_\varepsilon$, for $j=r+1,\dots,n$. Then $\deg(F_0;B)=\deg((G_\varepsilon,x_{r+1}^\alpha y_1^\beta,\dots,x_n^\alpha y_1^\beta);B\cap\{|y_1|>\eta\})$ for some η small enough. One may perturb G_ε to a regular map on the above set. Near each zero $(x_1,\dots,x_r,y_1\neq 0)$, one may deform $x_j^\alpha y_1^\beta$ to x_j^α and get an index equal to α^{n-r} times the index of the zero of G_ε. Thus,

$$\deg(F_0;B)=\alpha^{n-r}\deg(G_\varepsilon;B'^K\times\{|y_1|>\eta\})=\alpha^{n-r}\deg(G;B'),$$

where B' is the ball in $V'^H\times\mathbb{C}$.

Now $[2t-1,G]_\Gamma=[2t-1,G_\perp]_\Gamma+d_K[F'_K]_\Gamma+d_e[F'_e]_\Gamma$ where

$$\begin{aligned}G_\perp&=(x_1^p+y_1^q,x_2^p,\dots,x_r^p,x_1^\alpha y_1^\beta),\\F'_K&=(2t+1-2|x_1|^2,x_1^p(x_1^{p^2}-1),x_1^p(\overline{x}_1x_2-1),\dots,x_1^p(\overline{x}_1x_r-1),x_1^\alpha y_1^\beta),\\F'_e&=(2t+1-2|x_1y_1|^2,x_1^p(x_1^{p^2}-1),\dots,x_1^p(\overline{x}_1x_r-1),x_1^\alpha y_1^\beta(\overline{x}_1^py^q-1)).\end{aligned}$$

One has $\deg(G_\perp^K;B'^K)=p^r$, $\deg(G_\perp;B')=p^{r-1}$, $\deg(F'^K_K;B'^K)=p^2$, $\deg(F'_K,B')=\beta p^2$, $\deg(F'^K_e;B'^K)=0$ and $\deg(F'_e;B')=p^2q$. Hence, $\deg(G^K;B'^K)=p^r+d_Kp^2$, $\deg(G;B')=p^{r-1}+d_K\beta p^2+d_ep^2q$. Now, G^K extends to F^K, hence the first degree is 0. Thus,

$$\deg(F_0;B)=\alpha^{n-r}(p^{r-1}-\beta p^r+d_ep^2q)=\alpha^{n-r}qp^2(\alpha p^{r-3}+d_e).$$

If $r \geq 3$, choose d_e such that this last term is 0. Then $[2t-1, G_\perp]_\Gamma + d_K[F'_K] + d_e[F'_e] = [\widehat{F}_0]$ has degree 0 and $[\widehat{F}_0^K] = [2t-1, G^K] = [2-1, F|_{V'^K}]$. (Note that $\widehat{F}_0$ is not necessarily of the form $[2t-1, G]$, but, from the Borsuk extension theorem, one may assume that $\widehat{F}_0^K = (2t-1, F|_{V'^K})$.) Extend $\widehat{F}_0$ and F^K, as was done for G, to a map $F_1(t, x_1, \ldots, x_n, y_1)$ and define $\widetilde{F}_0 = (F_1, x_{r+1}^\alpha y_1^\beta, \ldots, x_n^\alpha y_1^\beta)$. Then $\widetilde{F}_e^K = (2t-1, F^K)$ and $[2t-1, F]_\Gamma = [\widetilde{F}_0]_\Gamma + d[F_e]_\Gamma$ with $\deg(\widetilde{F}_0; I \times B) = 0$ and $F_e = (F'_e, x_1^\alpha y_1^\beta(\overline{x}_1 x_{r+1} - 1), \ldots, x_1^\alpha y_1^\beta(\overline{x}_1 x_n - 1))$ with $\deg(F_e, I \times B) = pq$. If $r = 2$, then $\deg(F_0; B) = \alpha^{n-1} pq + ap^2 q$.

For $m \geq 1$, we shall use the following induction argument:

For all k with $0 \leq k \leq n - r$, there is an equivariant map with a unique zero at the origin and $t = 1/2$:

$$\widetilde{F}_{k,m} : \{t, x_1, \ldots, x_{r+k}, y_1, \ldots, y_m\} \equiv V'_{k,m} \to \mathbb{R} \times \{\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_{k+m}\} \equiv W'_{k,m}$$

such that $\widetilde{F}_{k,m}^K = (2t-1, F^K|_{V'_{k,m}})$ and $\deg(\widetilde{F}_{k,m}; B'_{k,m})$ is zero if $m \leq r-2$ and $\alpha^{k+1} pq + d_{k,m} p^2 q$ if $m = r-1$. For $m = 1$, one may take $\widetilde{F}_{k,1} = \widetilde{F}_0|_{V'_{k,1}}$, which is an extension of $(2t-1, F^K|_{V'_{k,0}})$. Furthermore, as we have seen above, $\deg(\widetilde{F}_{k,1}; B'_{k,1}) = \alpha^k \deg(F_1; B'_{k,1}) = \alpha^k p^2 q(\alpha p^{r-3} + d_e)$, with the required properties.

If we assume the induction hypothesis for $m-1$, take any equivariant extension of $\widetilde{F}_{1,m-1}|_{V'_{0,m}}$ to a map $\widetilde{F}_{0,m} : V'_{0,m} \setminus \{0\} \to W'_{0,m} \setminus \{0\}$, which exists by Lemma 4.1. Then $[\widetilde{F}_{0,m}] = [2t-1, G_\perp^{0,m}] + d_K[F_K'^{0,m}] + d_e[F_e'^{0,m}]$ where

$$\begin{aligned}
G_\perp^{0,m} &= (x_1^p + y_1^q, \ldots, x_m^p + y_m^q, x_{m+1}^p, \ldots, x_r^p, x_1^\alpha y_1^\beta, \ldots, x_m^\alpha y_m^\beta),\\
F_K'^{0,m} &= (2t + 1 - 2|x_1|^2, x_1^p(x_1^{p^2} - 1), x_1^p(\overline{x}_1 x_2 - 1), \ldots, x_1^p(\overline{x}_1 x_r - 1),\\
&\qquad x_1^\alpha y_1^\beta, \ldots, x_1^\alpha y_m^\beta),\\
F_e'^{0,m} &= (2t + 1 - 2|x_1 y_1|^2, x_1^p(x_1^{p^2} - 1), \ldots, x_1^p(\overline{x}_1 x_r - 1), x_1^\alpha y_1^\beta(\overline{x}_1^p y_1^q - 1),\\
&\qquad x_1^\alpha y_1^\beta(\overline{y}_1 y_2 - 1), \ldots, x_1^\alpha y_1^\beta(\overline{y}_1 y_m - 1)).
\end{aligned}$$

As before, $\deg(G_\perp^{0,mK}) = p^r$, $\deg(G_\perp^{0,m}) = p^{r-m}$, $\deg(F_K'^{0,mK}) = p^2$, $\deg(F_K'^{0,m}) = \beta^m p^2$ and $\deg(F_e'^{0,m}) = p^2 q$. Hence, since $\widetilde{F}_{0,m}^K = F_{V'_{0,m}}^K$ with extension F^K, one has $0 = \deg(\widetilde{F}_{0,m}^K) = p^r + d_K p^2$, $\deg(\widetilde{F}_{0,m}) = p^{r-m} + d_K \beta^m p^2 + d_e p^2 q = p^{r-m}(1 - (\beta p)^m) + d_e p^2 q$. Thus,

$$\deg(\widetilde{F}_{0,m}) = \alpha q p^{r-m}(1 + \beta p + \ldots + (\beta p)^{m-1}) + d_e p^2 q.$$

Then, if $r - m \geq 2$, one may choose d_e such that this degree is 0, while if $r = m+1$, this degree is $\alpha pq + d_{0,m} p^2 q$. For any choice of d_e, the right hand side, when restricted to V^K, is Γ-homotopic to $\widetilde{F}_{0,m}^K$, i.e. to $(2t-1, F^K|_{V'_{0,m}})$, and when restricted to $V'_{0,m-1}$, it is Γ-homotopic to any extension of $\widetilde{F}_{0,m}^K$, from the dimension condition and Lemma 4.1(b). Thus, as above one may still assume that the right hand side extends $F_{1,m-1}|_{V'_{0,m}}$.

Take now any equivariant extension F'_1, with possibly non-trivial zeros, of $\widetilde{F}_{1,m-1}$ and $\widetilde{F}_{0,m}$, from $V'_{1,m}$ into $W'_{0,m} = W'_{1,m-1}$. Define $\widetilde{F}_{1,m} = (F'_1, x^\alpha_{r+1}y^\beta_m)$. Now, since $\deg(\widetilde{F}_{1,m-1}) = 0$ (one has $m-1 < r-1$), one may extend, non-equivariantly, $\widetilde{F}_{1,m-1}$ from the sphere in $V'_{1,m-1}$ into the ball, without zeros. Hence, one may assume that $\widetilde{F}_{1,m}$ has an extension without zeros for $y_m = 0$. By perturbing this map on $|y_m| \geq \eta$ to a regular map, one shows as before that $\deg(\widetilde{F}_{1,m}) = \alpha \deg(\widetilde{F}_{0,m})$, proving the induction hypothesis for $k = 1$.

For a general k, take $\widetilde{F}_{k,m-1}$ and construct an equivariant extension $\widetilde{F}_{k-1,m}$ from $V'_{k-1,m}$ into $W'_{k-1,m} = W'_{k,m-1}$ by using Lemma 4.1. Then define F'_k as any equivariant extension, with possibly non-trivial zeros, of $\widetilde{F}_{k,m-1}$ and $\widetilde{F}_{k-1,m}$ from $V'_{k,m}$ into $W'_{k-1,m}$. Define $\widetilde{F}_{k,m} = (F'_k, x^\alpha_{r+k}y^\beta_m)$. Since $\deg(\widetilde{F}_{k,m-1}) = 0$, one may perturb F'_k as above and prove that $\deg(\widetilde{F}_{k,m}) = \alpha \deg(\widetilde{F}_{k-1,m})$ and one gets the result by induction on k. By taking $k = n-r, \widetilde{F}_0 = \widetilde{F}_{n-r,m}$, one has completed the proof. $\square$

REMARK 4.4. (a) One has $[2t-1, F]_\Gamma = [\widetilde{F}_0]_\Gamma + d[F_e]_\Gamma$, where

$$F_e = (F_e'^{0,m}, x^\alpha_1 y^\beta_1(\overline{x}_1 x_{r+1} - 1), \dots, x^\alpha_1 y^\beta_1(\overline{x}_1 x_n - 1)).$$

(b) In order to avoid the case $m = r-1$, one could suspend the map F by x^p_{n+1}, increasing r to $r+1$. Then $[2t-1, F, x^p_{n+1}]_\Gamma = [\widehat{F}_0]_\Gamma + \widehat{d}[\widehat{F}_e]$ with $\deg(\widehat{F}_0) = 0$. Thus, $\deg(F) = p^{-1}\widehat{d}p^2 q = \widehat{d}pq$, recovering Proposition 4.1(a). This suspension argument could be used to study the general case but it is not clear that it could be useful.

We conclude this section by having a closer look at the case where the standing hypothesis (H) holds on V^{T^n}, that is, there are complementing maps of the form $x_j^{l_j}$ for all isotropy subgroups. Then

$$\deg(F^H) = \beta_{H\Gamma} \deg(F^\Gamma) + \sum_{T^n \leq H \leq K < \Gamma} d_K \beta_{HK} |\Gamma/K|$$

for any H in $\mathrm{Iso}(V^{T^n})$, and $\beta_{HK} = \prod l_j$ for x_j in $(V^K)^\perp \cap V^H, \beta_{HH} = 1$.

By reducing Γ to Γ/T^n and V to V^{T^n}, we may assume that Γ is a finite group (see Theorems 4.1 and 4.2). Define $m = \text{g.c.d.}(|\beta_{T^n K}| \cdot |\Gamma/K|$ for $K < \Gamma)$, where $\beta_{T^n\Gamma}$ will be denoted by β and $\beta_{T^n K}$ by β_K. Then, from the Darboux theorem, since the d_K are arbitrary, one gets

PROPOSITION 4.2. *$\deg(F) = \beta \deg(F^\Gamma) + dm$ and any integer d is achieved. The term $\deg(F^\Gamma)$ is replaced by 1 if $V^\Gamma = \{0\}$.*

Let $m_0 = \text{g.c.d.}(|\Gamma/K|$ for $K < \Gamma)$. Then clearly m_0 divides m. Since any isotropy group H is of the form $H = \bigcap H_j$ where H_j is the isotropy of the coordinate x_j in V^H, one sees that $|\Gamma/H|$ is a multiple of $\widetilde{m}^j \equiv |\Gamma/H_j|$ for all such j's, and of course a multiple of the g.c.d.$(\widetilde{m}^j, \forall j)$ (i.e. including all the coordinates of $(V^\Gamma)^\perp$). Thus, this last greatest common divisor divides m_0. On the other hand, $H_j \in \mathrm{Iso}((V^\Gamma)^\perp)$, hence m_0 divides this g.c.d. and $m_0 = \text{g.c.d.}(\widetilde{m}^j = |\Gamma/H_j|$, H_j the isotropy group of x_j in $(V^\Gamma)^\perp)$.

Now, the action of $\Gamma = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_n}$ on a coordinate is of the form $\exp(2\pi i a)$ with $a = \sum k_j r_j/m_j = \sum \widetilde{k}_j r_j/\widetilde{m}_j$, where $0 \le r_j < m_j$, and $\widetilde{k}_j$ and $\widetilde{m}_j$ are relatively prime. If $\widetilde{m} = \text{l.c.m.}(\widetilde{m}_j : j$ such that $k_j \neq 0)$ and $R = (r_1, \ldots, r_n)$, then $a = \langle K, R\rangle/\widetilde{m}$ and, as seen in [10, Lemma 1.1], there is an R_0 such that $\langle K, R_0\rangle \equiv 1 \pmod{\widetilde{m}}$. For any vector R, let $k \equiv \langle K, R\rangle \pmod{\widetilde{m}}$ with $0 \le k < \widetilde{m}$ and $Q = R - kR_0$. Then $\langle K, Q\rangle \equiv 0 \pmod{\widetilde{m}}$. The isotropy subgroup H for the variable corresponds to these Q's and $|\Gamma/H| = \widetilde{m}$. Now, if ξ is a coordinate of W^H, with the corresponding action $a' = \sum k'_j r_j/m_j = \langle K', R\rangle/\widetilde{m}'$, with $\widetilde{m}' = \text{l.c.m.}(\widetilde{m}'_j$ with $k'_j \neq 0)$, $k'_j/m_j = \widetilde{k}'_j/\widetilde{m}'_j$, the last pair relatively prime, then by taking r_j a multiple of $\widetilde{m}_j$, and the others to be 0, one concludes that $\widetilde{m}'_j$ divides $\widetilde{m}_j$ or $k'_j = 0$. Hence, $\widetilde{m}'$ divides $\widetilde{m}$ since if $k_j = 0$ one needs $k'_j = 0$. Of course $\widetilde{m}$ divides $M = \text{l.c.m.}(m_1, \ldots, m_n)$, which in turn divides $|\Gamma|$.

Note that the action on ξ is given by $a' = k\langle K', R_0\rangle/\widetilde{m}' = kl/\widetilde{m}$, where $l = \langle K', R_0\rangle\widetilde{m}/\widetilde{m}'$ is the integer given in Lemma 0 (the term $\langle K', Q\rangle/\widetilde{m}'$ is an integer for Q in H, since ξ is in W^H). For a general R, $\langle K, R\rangle = k + d\widetilde{m}$ and $\widetilde{m}\langle K', R\rangle/\widetilde{m}' = kl + d'\widetilde{m}$. Hence, $l = (\widetilde{m}\langle K', R\rangle/\widetilde{m}' + d_0\widetilde{m})/\langle K, R\rangle$ for $d_0 = ld - d'$. If one has a right hand side with d_0 replaced by d_1 and giving an integer l_1, then, if k and $\widetilde{m}$ are relatively prime, one has $l_1 \equiv l \pmod{\widetilde{m}}$. Note that $\langle K', R_0\rangle$ and $\widetilde{m}'$ are relatively prime, since $\Gamma/\Gamma_\xi \cong \mathbb{Z}_{\widetilde{m}'}$. (Again, a negative power means conjugation.)

Now, let n be the least common multiple of some of the $\widetilde{m}^j$'s (say k of them) and let $H = \bigcap H_j$, where j is taken over all indices for which the coordinate x_j has isotropy H_j and $|\Gamma/H_j| = \widetilde{m}^j$ divides n. Then H is an isotropy subgroup and $|\Gamma/H|$ is a multiple of n (equal to n if Γ is a cyclic group). Furthermore, if V^H is larger than the space generated by the x_j's, then there is a coordinate x with $|\Gamma/\Gamma_x| = \widetilde{m}$ which does not divide n and $\Gamma_x > \bigcap H_j$. As above, let the action on x be given by $\langle K, R\rangle/\widetilde{m}$, with $R = kR_0 + Q$. Take $R = nR_0$: since $\widetilde{m}$ does not divide n, R does not belong to Γ_x. However, on x_j, $\langle K_j, R\rangle/\widetilde{m}^j$ is an integer, i.e. R is in $\bigcap H_j$ and $V^H = \{x_j\text{'s}\}$.

Conversely, if H_0 is an isotropy subgroup, let $n = \text{l.c.m.}(\widetilde{m}^j\text{'s} : x_j$ coordinate in $V^{H_0})$ and H be constructed as above. Then $H < H_0$, $V^{H_0} \subset V^H$ and $\beta_{H_0}|\Gamma/H_0|$ and $\beta_H|\Gamma/H|$ have the common factor $(\prod_{I_n} l_j)n$, where j in I_n corresponds to x_j such that $\widetilde{m}^j$ does not divide n. Now, this factor would be the factor one would obtain by considering a cyclic group $\mathbb{Z}_{\widetilde{M}}$ with $\widetilde{M} = \text{l.c.m.}(\widetilde{m}^j)$ (or a non-effective action of $\mathbb{Z}_M$) given by $\exp(2\pi i/\widetilde{m}^j)$ on x_j and $\exp(2\pi i k_j/\widetilde{m}'^j)$ on ξ_j, with k_j and $\widetilde{m}'^j$ relatively prime, $\widetilde{m}^j = s_j\widetilde{m}'^j$ and $l_j = k_j s_j$.

Proposition 4.3. *Let* $\widehat{m} = \text{g.c.d.}((\prod_{I_n} l_j)n$ *for all l.c.m.'s* n *of the* $\widetilde{m}^j$*'s). Then:*

(a) m_0 *divides* $\widehat{m}$ *which divides* m.

(b) *If* $s_j = 1$ *for all* j*'s, then* $m_0 = \widehat{m}$ *and* β *and* m_0 *are relatively prime, in particular, if* $m_0 > 1$ *then* $\beta \not\equiv 0 \pmod{m_0}$ *and if* $\deg F^\Gamma = 1$, *then* $\deg F \neq 0$.

Proof. Since $m_0 = \text{g.c.d.}(\widetilde{m}^j\text{'s}), m_0$ divides all n's and hence $\widehat{m}$. Furthermore, any term $\beta_H|\Gamma/H|$, for m, has a factor $(\prod_{I_n} l_j)n$ for some of the n's, and, as such, is a multiple of $\widehat{m}$.

For (b), let $\widehat{m} = m_0 A$ and let p be a prime factor of A. Let $I_p = \{j : m_0 p$ divides $\widetilde{m}^j\}$. From the definition of m_0, the complement of I_p is non-empty. Let $m_0 N = \text{l.c.m.}(\widetilde{m}^j\colon j \text{ not in } I_p)$. Then p does not divide N (if not, p would divide at least one $\widetilde{m}^j/m_0$). Now, in $\widehat{m}$, the term $(\prod l_j)n$ for $n = m_0 N$ is $(\prod_{I_p} k_j)m_0 N$. But, for j in $I_p, m_0 p$ divides $\widetilde{m}^j$, hence p cannot divide k_j, since k_j and $\widetilde{m}'^j = \widetilde{m}^j$ are relatively prime. Thus, the only possibility is $p = 1$ and $m_0 = \widehat{m}$. Finally, if p is a prime factor of m_0, then p divides $\widetilde{m}^j$ for all j's, and hence does not divide any k_j nor β. The rest of (b) is clear. $\square$

It is not difficult to construct examples where one has strict inequalities in (a). We leave to the reader the task of comparing the above results to the vast literature on the subject (some of which is incorrect).

Remark 4.5. A curious application of Theorem 4.2 and Corollary 4.2 is the following classical result of Jane Cronin: let $f : \mathbb{C}^n \to \mathbb{C}^n$, or $\mathbb{R}^n \to \mathbb{R}^n$, be such that $f(x) = P(x) + g(x)$, where $P_j(x)$ is a homogeneous polynomial of degree $k_j, P(x)$ has an isolated zero at the origin and $g(x)$ is small with respect to P near the origin. Then $\text{Index}(f) = \text{Index}(P) = \prod k_j$ in the complex case and modulo 2 in the real case. The first equality is clear. For the second put the standard S^1-action on the first copy of $\mathbb{C}^n$ and the action given by $e^{ik_j\varphi}$ on the second copy (in the real case replace S^1 by $\mathbb{Z}_2$). The map $P(x)$ is clearly equivariant. In the first case, from Theorem 4.2, $\text{Index}(P) = \beta$, independently of P, and $\prod k_j$ for $\widetilde{P}_j(x) = x_j^{k_j}$. In the second case, either all k_j are odd and $\text{Index}(P)$ is odd, or otherwise, from Corollary 4.2, this index is a multiple of 2.

5. Index of a loop of stationary points

Let $F : \mathbb{R} \times U \to W$ be an equivariant map such that F has a simple loop P of zeros in $\mathbb{R} \times U^\Gamma$ on which F is regular, with the usual compactness if U is infinite-dimensional. Hence DF has a one-dimensional kernel, at each point of P, generated by the tangent vector to P. This situation forces U and

W to be equivalent representations (see [6, Chapter IV]). Furthermore, if Ω is a small invariant neighborhood of P such that F^Γ has only P as zeros in Ω^Γ and $D_{X_\perp}F^\perp$ is invertible, where X is written as $X^\Gamma \oplus X_\perp, F = (F^\Gamma, F^\perp)$, then $\deg_\Gamma(F;\Omega) = \deg_\Gamma(F^\Gamma(X^\Gamma), D_{X_\perp}F^\perp(X^\Gamma)X_\perp;\Omega)$, as is now standard.

For a general P one follows the steps of [9, Proposition 6.1]. In order to stress the main point of this index computation, we shall avoid repeating the arguments of the previous reference and study the case of the Hopf bifurcation where $P = \{(\mu,\nu) : \mu^2+\nu^2 = \varrho^2\}$ and $F^\Gamma(X^\Gamma) = (\mu^2+\nu^2-\varrho^2, F_0(\mu,\nu,X_0))$, with $F(\mu,\nu,0) = 0$. Then $D_{X_0}F_0$ has to be invertible on the loop and

$$\begin{aligned}\deg_\Gamma(F;\Omega) &= \deg_\Gamma(\mu^2+\nu^2-\varrho^2, D_{X_0}F_0(\mu,\nu)X_0, D_{X_\perp}F^\perp(\mu,\nu)X_\perp);\Omega)\\ &= \Sigma_0 J^\Gamma(D_{X_0}F_0, D_{X_\perp}F^\perp),\end{aligned}$$

where Σ_0 is the suspension by $2t-1$, which is an isomorphism, and J^Γ is the J^Γ-homomorphism from the set $[S^1 \to GL^\Gamma(V)]_\Gamma$ of all Γ-homotopy classes from S^1 into $\mathrm{GL}^\Gamma(V)$, where V corresponds to $(X_0, X_\perp)$ (see [6, Chapter 2, Remark 4.2]).

Again, by standard arguments, one may assume that V is finite-dimensional and DF has a diagonal structure $\mathrm{diag}(D_{X_0}F_0,\ D_{Y_j}F_j,\dots,D_{Z_l}F_l,\dots,D_{Z_k}F_k)$, where Y_j are made of real coordinates where $\Gamma/\Gamma_{y_{j,s}} \cong \mathbb{Z}_2$ for all s, Z_l are made of complex coordinates with Weyl group of the form $\mathbb{Z}_{\widetilde{m}^l}$, and $\mathbb{Z}_k$ corresponding to coordinates where the action of Γ is that of S^1 (see [6, Chapter VI, Theorem 1.2]).

In [6, Chapter VI, in particular Theorem 6.1], one has a complete study of J^Γ as a morphism from $\Pi_1(\mathrm{GL}^\Gamma_+(V))$ into $\Pi^\Gamma_{S^{\mathbb{R}^2\times V}}(S^{\mathbb{R}\times V})$, where $\Pi_1(\mathrm{GL}^\Gamma_+(V))$ is the subset of the previous set of Γ-homotopic maps where $\det(D_{X_0}F_0)$ and $\det(D_{Y_j}F_j)$ are positive. It is clear that one may change the sign of such a determinant by multiplying one equation by -1, but, in order to be able to compare the indices, we shall give the full Γ-index of the loop.

Let I_0 be the linear map which changes the first component of X_0 into its opposite and I_j the similar map for Y_j. Since the addition in $\Pi^\Gamma_{S^{\mathbb{R}^2\times V}}(S^{\mathbb{R}\times V})$ is defined on t, the map I_j induces two morphisms on this group by $[f(I_jX)]_\Gamma = I_j^*[f(X)]_\Gamma$ and $[I_jf(X)]_\Gamma = I_j'^*[f(X)]_\Gamma$.

Since $I_j^2 = I$, one has $I_j^{*2} = I_j'^{*2} = I$ and it is easy to see that the I_j^*'s and $I_k'^*$'s all commute. It is easy to see (since the addition is defined on the first variable) that $I_0^*[f]_\Gamma = I_0'^*[f]_\Gamma = -[f]_\Gamma$.

Also, if H_j is such that $\Gamma/H_j \cong Z_2$, then $I_j^*[J^\Gamma A(\mu,\nu)]_\Gamma = I_j'^*[J^\Gamma A(\mu,\nu)]_\Gamma$ for $\dim V^{H_j} - \dim V^\Gamma \neq 2$: in fact $[\mu^2+\nu^2-\varrho^2, AI_jY_j]_\Gamma = [\mu^2+\nu^2-\varrho^2, I_jAY_j]_\Gamma$, since this is clearly true if Y_j reduces to one dimension, while if A is an $n\times n$ matrix, then A is homotopic to $\mathrm{diag}(I,\widetilde{A})$ with $\widetilde{A}$ an $(n-1)\times(n-1)$ matrix. If $n=2$, then it is easy to see that $I_j^*[J^\Gamma(A)]_\Gamma = -[J^\Gamma(A)]_\Gamma = -I_j'^*[J^\Gamma(A)]_\Gamma$, by looking at AY given by λz with $\lambda = \mu + i\nu, z = z_1 + iz_2$, which generates $\Pi_1(SO(2))$ and $J(A)$ is the Hopf map. (Note that Theorem 8.5 of [10] asks for $n \geq 3$.)

Now for any $A = A(\mu,\nu) = \operatorname{diag}(A_0, A_j, B_l, C_k)$, where A_0 corresponds to X_0, A_j to Y_j, B_l to Z_l and C_k to Z_k, let $\varepsilon_j = \operatorname{Sign}\det A_j$ for $j = 0,1,\dots,r$ (if there are r different isotropy subgroups H_j with $\Gamma/H_j \cong \mathbb{Z}_2$). Let $A_j^* = A_j I_j^{\alpha_j}$ with $\alpha_j = (1-\varepsilon_j)/2$ (i.e. $A_j^* = A_j$ if $\varepsilon_j = 1$ and $A_j^* = A_j I_j$ if $\varepsilon_j = -1$) and let $A^* = \operatorname{diag}(A_0^*, A_j^*, B_l, C_k)$. Then $A^*(\mu,\nu)$ belongs to $\Pi_1(\mathrm{GL}_+^\Gamma(V))$. Now, A^* can be written as a product of matrices of the form $\operatorname{diag}(I, A_j, I, I)$ (similarly for B_l and C_k) and, since J^Γ is a morphism on the fundamental group of $\mathrm{GL}_+^\Gamma(V)$,

$$J^\Gamma[A^*] = \Sigma_\Gamma J^\Gamma[A_0^*] + \sum_j \Sigma_\Gamma J^\Gamma[A_j^*] + \sum_l \Sigma_\Gamma J^\Gamma[B_l] + \sum_k \Sigma_\Gamma J^\Gamma[C_k],$$

where Σ_Γ is the suspension by the corresponding identity. The above argument was used in [6, Chapter VI, Proposition 5.3] to study $\ker J^\Gamma$. Here one has

$$\begin{aligned} J^\Gamma[A] = \Big(\prod_{j=0}^r I_j^{*\alpha_j}\Big)\Big[&I_0^{*\alpha_0}\Sigma_\Gamma J^\Gamma[A_0] + \sum_j I_j^{*\alpha_j}\Sigma_\Gamma J^\Gamma(A_j) \\ &+ \sum_l \Sigma_\Gamma J^\Gamma[B_l] + \sum_k \Sigma_\Gamma J^\Gamma[C_k]\Big]. \end{aligned}$$

It remains to identify I_j^* on each term and to compute $J^\Gamma[A_j], J^\Gamma[B_l], J^\Gamma[C_k]$ in terms of the generators of Π^Γ, as given in [10], in order to prove the following

Theorem 5.1. *Assume for simplicity* $\dim V^\Gamma, \dim V^{H_j} - \dim V^\Gamma \geq 3$, $\dim_{\mathbb{C}} V_l \geq 2$, *for* V_l *generated by the variables* z_s *with Weyl group of the form* $\mathbb{Z}_p$. *Then*

$$\begin{aligned} &\deg_\Gamma((|\lambda|^2 - \varrho^2, F(\mu,\nu,X_0,Y_j,Z_l,Z_k);\Omega)) \\ &= \Big(\prod_{j=0}^r I_j^{*\alpha_j}\Big)\Big[d_0[F_0]_\Gamma + \sum_j d_j I_j^{*\alpha_j}[F_j]_\Gamma + \sum_l \Big(\sum_s n_s d_s\Big)[F_l]_\Gamma + \sum_k d_k[F_k]_\Gamma\Big] \end{aligned}$$

where $d_0\eta$ *is the class of* $D_{X_0}F_0$ *in* $\Pi_1(\mathrm{GL}(V^\Gamma))$ *and* η *is the Hopf map* (d_0 *is an element of* $\mathbb{Z}_2$), $d_j\eta$ *is the class of* $D_{Y_j}F_j$ *in* $\Pi_1(\mathrm{GL}(V^\Gamma)^{\perp H_j})$ (d_j *is in* $\mathbb{Z}_2$). *If* Γ/H_l *acts as* $\mathbb{Z}_p$ (*p not necessarily prime*) *on* $Z = (Z_1,\dots,Z_l)$ *in the following form: on the coordinate* Z_s *as* $\exp(2\pi i m_s/p)$ *with* m_s *and* p *relatively prime, then* d_s *is the winding number of* $\det(D_{Z_s}F_s)$ *as a mapping from* S^1 *into* $\mathbb{C}\setminus\{0\}$. *The number* $|n_s|$ *is an* odd *integer such that* $n_s m_s \equiv 1 \pmod p$. *Finally,* d_k *is the winding number of* $\det(D_{Z_k}F_k)$, *where* Γ/H_k *acts as* $\exp(2\pi i m_k\varphi)$.

The maps $[F_u]_\Gamma, u = 0,j,l,k$, *are independent generators of* $\Pi^\Gamma_{S^{\mathbb{R}^2\times V}}(S^{\mathbb{R}\times V})$ *of the form* $\Sigma_\Gamma(1-|z|^2,\lambda z)$, *where* $\lambda = \mu + i\nu$ *and* z *is a complex coordinate with isotropy* H (*equal to* Γ, H_j, H_l, H_k) *and* z *is taken as* $x_1 + ix_2$ *for* $H = \Gamma$ *and as* $y_1 + iy_2$ *for one of the* H_j*'s.*

Furthermore, $I_0^*[F_u]_\Gamma = -[F_u]_\Gamma$ *and* $I_j^*[F_u]_\Gamma = [F_u]_\Gamma - [F_{uj}]_\Gamma$, *where*

$$F_{uj} = (1 - |y_j| \cdot |z|, 2t-1, X_0', Y_i, (y_j^2 - 1)y_j, \lambda z, \ldots, z_s, \ldots).$$

If $j \neq k$, *then* $I_k^*[F_{uj}]_\Gamma = [F_{uj}]_\Gamma - [F_{ujk}]_\Gamma$ *with*

$$F_{ujk} = (1 - |y_j| \cdot |y_k| \cdot |z|, 2t-1, X_0', Y_i, (Y_j^2 - 1)y_j, (y_k^2 - 1)y_k, \lambda z, \ldots, z_s, \ldots),$$

while $I_j^*[F_{uj}]_\Gamma = -[F_{uj}]_\Gamma$.

If H *is not a subgroup of* H_j *(always if* Γ/H *is not finite), then* $[F_{uj}]$ *is a generator for the part of the degree corresponding to* $H \cap H_j$ *and one has* $p([F_{uj}]_\Gamma + [\widetilde{F}_{uj}]_\Gamma) = 0$ *with* $2[\widetilde{F}_{uj}]_\Gamma = 0$. *If* H *is not a subgroup of* H_j *and* H_k, *then* $[F_{ujk}]_\Gamma$ *is a generator for the part corresponding to* $H \cap H_j \cap H_k$ *with* $p([F_{ujk}]_\Gamma + [\widetilde{F}_{ujk}]_\Gamma) = 0$ *and* $2[\widetilde{F}_{ujk}]_\Gamma = 0$. *If* $\Gamma/H \cong S^1$, *then* $[F_{uj}]_\Gamma$ *is the generator corresponding to* $H \cap H_j$ *and* $[F_{ujk}]_\Gamma$ *the one for* $H \cap H_j \cap H_k$. *If* $H = H_j$, *then* $[F_{uj}]_\Gamma$ *is the second generator for* H_j *with* $2[F_{uj}]_\Gamma = 0$. *Finally, if* $H < H_j$ *(hence* $\Gamma/H \cong \mathbb{Z}_p$ *with* p *even), then* $[F_{uj}]_\Gamma = 2[F_u]_\Gamma + d[\widetilde{F}_u]_\Gamma$ *with* $d = 1$ *if* $p = 2k$ *with* k *odd and* $2[\widetilde{F}_u]_\Gamma = 0$, $\widetilde{F}_u = (\varepsilon - |z^p - 1|, 2t-1, X_0', y_j, \lambda(z^p - 1)z)$ *with* $0 < \varepsilon < 1$. *We have* $I_j^*[\widetilde{F}_u]_\Gamma = [\widetilde{F}_u]_\Gamma$. *The action of* I_k^* *follows from the above.*

Proof. It is known that $\Sigma_\Gamma J^\Gamma[D_{X_0}F_0] = d_0[1-|z|^2, 2t-1, \lambda z, X_0', Y_j, Z_l, Z_k]$ is the suspension of the Hopf map η (the change from $|\lambda|^2 - \varrho^2$ to $1 - |z|^2$ is a linear deformation). Since $(\Sigma\eta)^2 = 0$, the action of I_0^* on it is the identity. The same happens for $[F_j] = [1 - |z|^2, 2t-1, X_0, \lambda z, Y_j', \ldots]$: since one is in $\mathbb{Z}_2$, the orientations play no role.

For H_l, it was proved in [6, Chapter VI, Theorem 6.1 and Remark 6.9] that each $D_{Z_s}F_s$ gives $d_s[F_s]$, where F_s is built on the same model. Furthermore, it was proved in the above reference, p. 447, that $[F_s] = n_s[F_l] + (n_s - 1)[\widetilde{F}_l]$, where $[F_l]$ and $[\widetilde{F}_l]$ generate this part of the group which is, from [10, Theorem 8.3], $\mathbb{Z}_p \times \mathbb{Z}_2$ if p is even and $\mathbb{Z}_{2p}$ if p is odd, with the relations $p([F_l] + [\widetilde{F}_l]) = 0$ and $2[\widetilde{F}_l] = 0$ (the action for F_l is taken as $\exp(2\pi i/p)$). Hence the contribution of all Z_s's is $(\sum d_s n_s)[F_l] + \sum d_s(n_s - 1)[\widetilde{F}_l]$. However, if p is even, then m_s and n_s are odd and the last term is 0 (in $\mathbb{Z}_2$). While, if p is odd and n_s is even, then $(n_s - p)m_s \equiv n_s m_s \equiv 1 \pmod p$ with $n_s - p$ odd. For H_k, we refer to [8] and [6, Chapter VI].

It remains to study the effect of the isomorphisms I_j^* on each of the above generators. If $F_u = (1 - |z|, 2t-1, X_0', Y_i, y_j, \lambda z, z_k)$ with $\lambda = \mu + i\nu$, on the ball $B = \{0 \leq t \leq 1, |z_i| \leq 2, |X_0| \leq 2, |Y_i| \leq 2, |y_j| \leq 2\}$ one may use the deformation $y_j(1 - \tau + \tau(y_j^2 - 1))$ in the computation of $\deg_\Gamma(F_u; B) = [F_u]_\Gamma$, since the suspension is an isomorphism. But

$$\deg_\Gamma(F_u; B) = \deg_\Gamma(F_u; B \cap \{|y_j| < 1/2\}) + \deg_\Gamma(F_u; B \cap \{|y_j| > 1/2\}).$$

For the first degree, one may deform y_j^2 to 0 and obtain $I_j^*[F_u]$. For the second, one may use the deformation $1-(1-\tau+\tau|y_j|)|z|$ on the set $\{|y_j|>1/2\}$, and then the second degree is $[F_{uj}]_\Gamma$, where $F_{uj}=(1-|y_j|\cdot|z|, 2t-1, X_0', Y_i, (y_j^2-1)y_j, \lambda z, z_k)$. Thus, $I_j^*[F_u]_\Gamma=[F_u]_\Gamma-[F_{uj}]_\Gamma$.

By using $I_j^{*2}=I$, it is then easy to see that $I_j^*[F_{uj}]_\Gamma=-[F_{uj}]_\Gamma$. Furthermore, by repeating the above argument, one has $[F_{uj}]_\Gamma=I_k^*[F_{uj}]_\Gamma+[F_{ujk}]_\Gamma$, as stated in the theorem. Further applications of I_l^* are built on the same scheme.

Let $H=\Gamma_z$ and $H_j=\Gamma_{y_j}$ with $\Gamma/H_j\cong\mathbb{Z}_2$ and $\Gamma/H\cong\mathbb{Z}_p$ or S^1. Now, either $H<H_j$ or there is h in H such that $hy_j=-y_j$, in which case $\Gamma/H\cap H_j=\Gamma/H\times\mathbb{Z}_2$, since h^2 is in H and acts as the identity on y_j. If $H<H_j$ and $\Gamma/H\cong S^1$, then the action of Γ on z is $\exp[2\pi i(\langle N,\Phi\rangle+\langle K,L\rangle/\widetilde{m})]$ (see [10, Lemma 1.1]). Hence, for any L there is a Φ_0 such that the exponential is 1. On y_j, the action is of the form $\exp(2\pi i\langle K_j,L\rangle/2)$. Thus, if $H<H_j$, this last expression should be 1 for any L, which is impossible since $\Gamma/H_j\cong\mathbb{Z}_2$. Since H_j is maximal, the only case where H is a subgroup of H_j is for $\Gamma/H\cong\mathbb{Z}_p$, with p even, with a generator γ such that $\gamma z=\exp(2\pi i/p), \gamma y_j=-y_j$.

Now, if H is not a subgroup of H_j and Γ/H is finite, then F_{uj} is one of the generators for $H\cap H_j$ with $p([F_{uj}]_\Gamma+[\widetilde{F}]_\Gamma)=0$ (see [10, Theorem 8.4; there are two other generators in this case, $[F_{ju}]_\Gamma$ and $[\widetilde{F}]_\Gamma$, both of order 2). Similarly if H_k and H_j do not contain H, then $[F_{ujk}]_\Gamma$ is a generator for $H\cap H_j\cap H_k$. The congruences are given in [10, Theorem 8.4]. If $\Gamma/H\cong S^1$, then the usual degree of F_{uj} on the fundamental cell for $H\cap H_j$, i.e. for $0<y_j<2, z$ in $\mathbb{R}^+$, is -1 (because the equation for $2t-1$ is in the second place). If $H=H_j$ then the fundamental cell reduces to $0<y_j<2$ and it is easy to see that F_{uj}, on that cell, is the suspension of the Hopf map, hence the second generator for the group [10, Theorem 8.1].

Finally, if $H<H_j$, then one may construct a fundamental cell for H in two ways. The first, as the set characterized by $\{0\le|z|\le 2,\ 0\le\operatorname{Arg} z<2\pi/p\}$ with the generators $[F_u]_\Gamma$ and $[\widetilde{F}_u]_\Gamma$ with $p[F_u]_\Gamma=0$, $2[\widetilde{F}_u]_\Gamma=0$, from [10, Theorem 8.4] (here p is even). The second, with $p=2k$, a fundamental cell of the form $\{0\le y_j<2, 0\le|z|\le 2, 0\le\operatorname{Arg} z<2\pi/k\}$, with the generators $\eta_1=(1-|Y|\cdot|z|, 2t-1, X_0', \lambda Y, (\overline{Y}z^k-|Y|)z)$ with $Y=y_1+iy_2$ and $\Gamma_Y=H_j, \eta_2=[F_{uj}]_\Gamma$ and $\widetilde{\eta}=[\widetilde{F}_u]$ and the relations $2\eta_1+d_2\eta_2+d\widetilde{\eta}=0, k(\eta_2+\widetilde{\eta})=0, 2\widetilde{\eta}=0$ (see [10, Theorem 8.2]). Now, since $\deg(I_j^*[F_u];\, z \text{ in } \mathbb{R}^+)=-\deg([F_u];\, z \text{ in } \mathbb{R}^+)$ (here these are ordinary degrees) and $\deg([F_{uj}]; z \text{ in } \mathbb{R}^+)=2\deg([F_u]; z \text{ in } \mathbb{R}^+)$, one has $I_j^*[F_u]_\Gamma=-[F_u]_\Gamma+d\widetilde{\eta}$ and $[F_{uj}]_\Gamma=2[F_u]_\Gamma+d\widetilde{\eta}$ (the same d because of the relations between the three maps). Furthermore, in η_1 one may perform the rotation $((\tau\lambda-(1-\tau)(\overline{Y}z^k-|Y|))Y, ((1-\tau)\lambda+\tau(\overline{Y}z^k-|Y|))z)$. The term $|Y|Y-|Y|^2z^k$ is deformed linearly to $|z|Y-z^k$, then $1-|Y|\cdot|z|$ is deformed linearly to $1-|z|^k$ and finally $Y|z|-z^k$ to Y. Thus, $\eta_1=[F_u]_\Gamma$, $d_2=-1$ and

$d = \widetilde{d}$. If k is odd, the relation $k(\eta_2 + \widetilde{\eta}) = 0$ implies that $d = 1$. Since $\widetilde{\eta}$ has the class of the Hopf map on the fundamental cell, we have $I_j^* \widetilde{\eta} = \widetilde{\eta}$. One may apply I_k^* to the previous case and study the case where H is a subgroup of $H_j \cap H_k$ or not. We leave out the determination of d when k is even. $\square$

Remark 5.1. One could have proved Theorem 3.2 by using generators as above. Note also that the easier part of the above theorem, i.e. for $\Gamma/H_k \cong S^1$, has been proved in various papers, as [6] or [15, Theorem 2.1.1].

Example 5.1. Consider the Hopf bifurcation problem for the equation

$$(\nu + \nu_0)\frac{dX}{dt} = L(\mu)X + g(X, \mu, \nu), \quad X \text{ in } \mathbb{R}^N,$$

where $X(t)$ is 2π-periodic, (μ, ν) is close to $(0, 0)$ and $g(X, \mu, \nu) = o(|X|)$, and $L(\mu)$ and $g(X, \mu, \nu)$ are Γ_0-equivariant. Then the problem is equivalent to $in(\nu + \nu_0)X_n - L(\mu)X_n - g_n(X) = 0$, where $(X_n)_j$ has isotropy H_{jn}, as in Example 3.1. The representations of Γ on $(X_n)_j$ and $(X_k)_l$ are equivalent only if $k = n, N_j = N_l$ and $K_j/M \equiv K_l/M$, where the action of Γ is given by $\exp(2\pi i(\langle K_j/M, L\rangle + \langle N_j, \Phi\rangle + n\varphi/(2\pi))$. Since we need that $in(\nu+\nu_0)I - L(\mu)$ is invertible for $\mu^2 + \nu^2 = \varrho^2$, this implies that $L(\mu)$ is invertible for $|\mu| \leq \varrho$. This fact implies that if Γ/H_{jn} is finite, then $n = 0, N_j = 0$ and the corresponding $d_{j0} = 0$. If $\Gamma/H_{j0} \cong S^1$, then again one has $d_{j0} = 0$. Thus, the bifurcation degree $\deg_\Gamma(\nu^2 + \mu^2 - \varrho^2, X - F(\nu, \mu, X))$ is $\prod I_k^{*\alpha_k}(\sum_{n \geq 1} d_{jn}[F_{jn}]_\Gamma)$, where $[F_{jn}] = \Sigma(\varepsilon^2 - |z_{jn}|^2, \lambda z_{jn})$.

Now, $L(\mu) = \operatorname{diag}(L_0(\mu), L_k(\mu), \ldots, L_l(\mu))$, since it is Γ_0-equivariant, where Γ_0 acts trivially on L_0, as $-\operatorname{Id}$ on L_k and as $\mathbb{Z}_m$ or S^1 on $L_k(\mu)$. Since d_{jn} is given by the winding number of $in(\nu+\nu_0)I - L(\mu)|_{V^{H_j}}$, where H_j is the isotropy of the variables in V^{H_j}, it follows that $L(\mu)$ is one of the above matrices. It is well known that the winding number d_{jn} is the net crossing number of eigenvalues, counted with multiplicity, of $L_j(\mu)$ at $in\nu_0$ (see [9] for instance). Note that if $d_{jn} \neq 0$, one has a Hopf bifurcation in V^{jn}, as defined in Example 3.1, i.e. with $X(t) = \gamma_0 X(t - 2\pi/(nm^j))$: see [6] or restrict the bifurcation problem to that invariant space where the Γ-degree keeps all d_{kl} with $H_{jn} < H_{kl}$. In order to determine α_k, it is enough to see which subgroups H of Γ give $\Gamma/H \cong \mathbb{Z}_2$: this is possible only if $n = 0, N_j = 0$ and Γ_0 acts as $-\operatorname{Id}$. Hence $\alpha_k = (1 - \operatorname{Sign}\det L_k)/2$ while $\alpha_0 = (1 - \operatorname{Sign}\det L_0)/2$. Then $I_k^*[F_{jn}]_\Gamma = [F_{jn}] - [F_{kjn}]$, where F_{kjn} represents the resonance of the stationary part $L_k(\mu)$, with action of Γ_0 as $-\operatorname{Id}$, on the nth mode with component z_{jn}. Note that there are at most $N/2$ d_{jn}'s which are non-zero. Compare with [4] and [15].

Example 5.2 (Hopf bifurcation for time-dependent differential equations). Consider the problem of Hopf bifurcation for the equation

$$(\nu + \nu_0)\frac{dX}{dt} = L(\mu)X + g(X, \mu, \nu) + \varepsilon h(X, \mu, \nu, t), \quad X \text{ in } \mathbb{R}^N,$$

where $X(t)$ is 2π-periodic, (ν, μ) is close to $(0,0)$, $g(X,\mu,\nu) = o(|X|)$ and $h(0,\mu,\nu,t) = 0$. If h has a linear part in X, then ε is chosen so small that, assuming $L(\mu)$ invertible for $|\mu| \leq \varrho$ and without pure imaginary eigenvalues for $\mu \neq 0$ close to a multiple of ν_0 (that is, $in(\nu+\nu_0)I - L(\mu)$ is invertible on $\mu^2+\nu^2 = \varrho^2$) then the Fredholm operator $(\nu+\nu_0)d/dt - L(\mu) - \varepsilon Dh$ is invertible, for $\mu^2+\nu^2 = \varrho^2$, on the space of 2π-periodic C^1-functions into the corresponding space of C^0-functions.

Thus, for $\varepsilon = 0$, one has an S^1-action, while for $\varepsilon \neq 0$ the action is reduced, as seen in Example 3.2, to a $\mathbb{Z}_p$-action. The hypothesis on ε implies that, for $\mu^2+\nu^2 = \varrho^2$, one may $\mathbb{Z}_p$-deform the equation to $(\nu+\nu_0)dX/dt - L(\mu)X$, considered, when $\varepsilon \neq 0$, as a $\mathbb{Z}_p$-equivariant linear map. While for $\varepsilon = 0$, any non-zero winding number d_n of $in(\nu+\nu_0)I - L(\mu)$ will give rise to a Hopf bifurcation of 2π-periodic solutions (not necessarily least periodic: see [9]), for $\varepsilon \neq 0$ we have to study the isotropy subgroups H of $\mathbb{Z}_p$ for its action on Fourier series, that is, on X_m, as $\exp(2\pi imk/p), 0 \leq k < p$, and H is the isotropy of X_n. Now, two representations of $\mathbb{Z}_p$ will be equivalent (i.e. on X_n and X_m) if and only if $m \equiv n \pmod{p}$. Furthermore, if $n/p = n'/p'$ with n' and p' relatively prime, then $H = \{k = 0, p', 2p', \dots, (p/p'-1)p'\}$, i.e. $H \cong \mathbb{Z}_{p/p'}$ and $\Gamma/H \cong \mathbb{Z}_{p'}$.

In order to apply Theorem 5.1, we need to identify the modes X_m for which the isotropy is exactly H, i.e. the action of Γ is of the form $\exp(2\pi i m_s k/p')$ for $k = 0, \dots, p'-1$, with m_s and p' relatively prime. Then $m_s = m_j + ap'$ and $m = m_j p/p' + ap$ where $1 \leq m_j < p'$ is relatively prime to p' (this has to happen for $m_s = n'$ and $m = n$). If p' is prime, then any integer between 1 and $p'-1$ is allowed. Clearly, if n_j, with $|n_j|$ odd, is such that $m_j n_j \equiv 1 \pmod{p'}$, then $m_s n_j \equiv 1 \pmod{p'}$. If $H = \Gamma$, then $m = kp$ and $m_j = n_j = p' = 1$. Finally, since Γ acts only on the non-trivial modes, I_k^* is not present, except for I_0^* where it is $\varepsilon = \operatorname{Sign}\det L(0)$. We have proved the following:

Proposition 5.1. *Under the above assumptions, the bifurcation degree has the following components:*

(a) $d_\Gamma \equiv \varepsilon \sum_{k=1}^{\infty} d_{kp} \pmod{2}$,

(b) $d_H \equiv \varepsilon \sum_j n_j \sum_{k=1}^{\infty} d_{m_j p/p' + kp} \pmod{2p'}$ *if* p' *is odd and* $\pmod{p'}$ *if* p' *is even, where* $\varepsilon = \operatorname{Sign}\det L(0), d_m$ *is the winding number of* $im(\nu+\nu_0)I - L(\mu)$ *for* $\mu^2+\nu^2 = \varrho^2, |\Gamma/H| = p'$ *and* $1 \leq m_j < p'$ *is relatively prime to* p' *and* $|n_j|$ *is odd such that* $n_j m_j \equiv 1 \pmod{p'}$.

If d_Γ *is odd, then one has Hopf bifurcation of* $2\pi/p$*-periodic solutions, while if* d_H *is not congruent to* 0*, one has Hopf bifurcation of* $2\pi p'/p$*-periodic solutions.*

Remark 5.2. Note that a mode m belongs to just one p', since if $m_1 p/p_1 + k_1 p = m_2 p/p_2 + k_2 p$, then $m_1 p_2 - m_2 p_1 = k p_1 p_2$, where m_j and p_j are relatively prime. But then $p_2 = p_1$. Thus, it is convenient to list the divisors of p in increasing order and begin with the smallest (1 corresponds to d_Γ). Then, for a given integer $j < p'$, either j is relatively prime to p' or the corresponding modes $jp/p' + kp$ have already been assigned to a smaller divisor of p. Note also that if $m_j n_j \equiv 1 \pmod{p'}$, with m_j and p' relatively prime, then this is also true for $m_1 = p' - m_j$ and $n_j = -n_j$: that is, there is a natural pairing in the congruence classes of the modes. Finally, note that if p' is an odd prime (if $p' = 2$ then $m_j = 1 = n_j$), then, due to the pairing, one has to consider all integers between 1 and $(p'-1)/2$, with $n_1 = 1, n_2 = (1+p')/2$ if this number is odd or $(1-p')/2$ if the first number is even and n, for $(p'-1)/2$, can be taken to be $p'-2$.

For instance, for $p = 2$, the components of the bifurcation index will be

$$d_\Gamma = \sum d_{2k} \pmod 2, \quad d_{\{e\}} = \sum d_{2k+1} \pmod 2.$$

For $p = 3$, one has

$$d_\Gamma = \sum d_{3k} \pmod 2, \quad d_{\{e\}} = \sum (d_{3k+1} - d_{3k+2}) \pmod 6.$$

For $p = 4$, one has

$$\begin{aligned} d_\Gamma &= \sum d_{4k} \pmod 2, \quad d_2 = \sum d_{4k+2} \pmod 2, \\ d_{\{e\}} &= \sum (d_{4k+1} - d_{4k+3}) \pmod 4. \end{aligned}$$

For $p = 5$, one has

$$\begin{aligned} d_\Gamma &= \sum d_{5k} \pmod 2, \\ d_{\{e\}} &= \sum (d_{5k+1} - d_{5k+4}) + 3 \sum (d_{5k+2} - d_{5k+3}) \pmod{10}. \end{aligned}$$

For $p = 6$, one has

$$\begin{aligned} d_\Gamma &= \sum d_{6k} \pmod 2, \quad d_3 = \sum d_{6k+3} \pmod 2 \quad \text{for } p' = 2, \\ d_2 &= \sum (d_{6k+2} - d_{6k+4}) \pmod 6 \quad \text{for } p' = 3, \\ d_{\{e\}} &= \sum (d_{6k+1} - d_{6k+5}) \pmod 6. \end{aligned}$$

Finally, for $p = 7$, one has

$$\begin{aligned} d_\Gamma &= \sum d_{7k} \pmod 2, \\ d_{\{e\}} &= \sum (d_{7k+1} - d_{7k+6}) - 3 \sum (d_{7k+2} - d_{7k+5}) \\ &\quad + 5 \sum (d_{7k+3} - d_{7k+4}) \pmod{14}. \end{aligned}$$

Recall that if the bifurcation index is 0, then, given a linear part, there is a non-linear part at the level of Fourier series (not necessarily coming from a differential equation) such that there is no bifurcation (see [6, Chapter VI, Theorem 6.1]). Here we shall give an example, which generalizes the examples of [9,

p. 156], showing how one may force a linear system which has a Hopf bifurcation with a linear time-periodic perturbation which destroys the bifurcation.

Take p any integer larger than 1 and consider the following system for 2π-periodic functions:

$$x'' - \lambda x' + \mu x + \varepsilon((p+1)\cos pty + \sin pty') = 0,$$
$$y'' - (p-1)^2\lambda y' + (p-1)^2\mu y - (p-1)\varepsilon((2p-1)\cos ptx + \sin ptx') = 0.$$

For $\varepsilon = 0$, λ close to 0 and μ close to 1, one has a vertical Hopf bifurcation for $(x, 0)$ with $n = 1$ and for $(0, y)$ with $n = p - 1$. The winding numbers are all 0 except $d_1 = d_{p-1} = 1$.

For $\varepsilon \neq 0$, the system is equivalent to

$$(-n^2 - in\lambda + \mu)x_n + (\varepsilon/2)((n+1)y_{n-p} - (n-1)y_{n+p}) = 0,$$
$$(-n^2 - i(p-1)^2\lambda + (p-1)^2\mu)y_n - (\varepsilon/2)(p-1)((p+n-1)x_{n-p} - (n-p+1)x_{n+p}) = 0.$$

Taking the first equation for $n = 1$ and the second for $n = p-1$, one has the pair $((\mu-1-i\lambda)x_1 + \varepsilon\overline{y}_{p-1}, (p-1)^2[(\mu-1-i\lambda)y_{p-1} - \varepsilon\overline{x}_1])$ with only zero giving $x_1 = y_{p-1} = 0$, except if $\mu = 1, \lambda = 0, \varepsilon = 0$. For $\varepsilon \neq 0$, the remaining equations form a closed system with invertible diagonal, that is, the only solution for $\varepsilon \neq 0$ and (λ, μ) close to $(0, 1)$ is $x = y = 0$. For $p = 1$, one takes out the factors $p - 1$ in the second equation and one has $d_1 = 2$ and the same result holds.

It would be interesting to have similar simple examples for, say, $p = 3, d_1 = 6, d_j = 0$ for $j > 1$ or $p = 5, d_1 = 3, d_2 = -1$ and $d_j = 0$ otherwise. See [6, Chapter VI, Theorem 6.1 and Remark 6.8]. For obvious reasons of space, we leave to another paper the study of forcing a loop of non-stationary solutions by a $\mathbb{Z}_p$-action (see [9, p. 122]).

6. Operations

In this last section we study the basic properties of the following operations for the Γ-degree: reduction of the group, or symmetry breaking, products and composition of maps. We leave applications of these results to subsequent papers.

(A) Symmetry breaking. Let Γ_0 be a subgroup of Γ. If a map is Γ-equivariant it is also Γ_0-equivariant and one has a morphism $P_* : \Pi^{\Gamma}_{S^V}(S^W) \to \Pi^{\Gamma_0}_{S^V}(S^W)$. Under hypothesis (H2)$'$ for Γ and Γ_0 in [10, Theorem 5.3] or under hypothesis (H) of the present paper, these groups are of the form $\prod \widetilde{\Pi}(H)$ for all isotropy subgroups H and $\widetilde{\Pi}(H)$ is the suspension by $F_H^{\perp}$ of the group $\Pi(H)$ given by the equivariant homotopy classes of maps from S^{V^H} into S^{W^H} which have equivariant extensions from B^K into $W^K \setminus \{0\}$ for all $K > H$. It is thus important to determine the relation between the isotropy subgroups for Γ and $\Gamma_0, \Pi(H)$ for Γ, and $\Pi_0(H_0)$ for Γ_0.

Lemma 6.1. (a) *Any isotropy subgroup H_0 for Γ_0 is of the form $H \cap \Gamma_0$ with H an isotropy subgroup for Γ. For a given H_0, there may be several H's. Let $\underline{H}$ be the minimal one. Then $V^{\underline{H}} = V^{H_0}$, $\dim \Gamma_0/H_0 \leq \dim \Gamma/H$, and if one has equality of the above dimensions, then $|\widetilde{H}_0^0/H_0|$ divides $|\widetilde{H}_0/H|$, where $\widetilde{H}_0$ is the maximal isotropy subgroup with Weyl group of the same dimension as $\Gamma/\underline{H}$, given in Theorem* 2.1, *i.e. $\Gamma/\widetilde{H}_0 \cong T^k$. In this case, if hypothesis* (H_0) *of Section* 4 *holds for F in $\Pi(\underline{H})$, then F belongs to $\Pi_0(H_0)$ and $\deg_{\mathrm{E}}^{\Gamma_0}(F) = (|\widetilde{H}_0/\underline{H}|/|\widetilde{H}_0^0/H_0|)\deg_{\mathrm{E}}^{\Gamma}(F)$ if $W^{\underline{H}} = W^{H_0}$ and* 0 *otherwise.*

(b) *If there is a complementing map $F_H^{\perp}$ for all H's, then this is also true for all H_0's. In this case P_* maps $\widetilde{\Pi}(H)$ into $\widetilde{\Pi}_0(H_0)$.*

(c) *If hypothesis* (H$'$) *holds for* Γ, *it will hold for* Γ_0, *where* (H$'$) *is* (H) *together with the condition $W^{\underline{H}} = W^{H_0}$ for all H_0's, which holds if $V = \mathbb{R}^k \times W$.*

Proof. If $H_0 = \Gamma_{0X} = \{\gamma \in \Gamma_0 : \gamma X = X\}$, then clearly $H_0 = \Gamma_X \cap \Gamma_0$. Hence $\underline{H}$ is the intersection of all such H's and the isotropy subgroup for V^{H_0}. If z_i is a coordinate in this space with the subgroups $\widetilde{H}_{i-1} = H_1 \cap \ldots \cap H_{i-1}$ and $\widetilde{H}_i = \widetilde{H}_{i-1} \cap H_i$, as in Section 0, and the corresponding subgroups $\widetilde{H}_{i-1}^0 = \widetilde{H}_{i-1} \cap \Gamma_0$, then either $k_i = |\widetilde{H}_{i-1}/\widetilde{H}_i|$ is infinite and the corresponding k_i^0 is infinite or not, or k_i is finite. In this case, any γ in $\widetilde{H}_{i-1}$ can be written as $\gamma = \gamma_i^{\alpha}\gamma_{H_i}$ with $0 \leq \alpha \leq k_i$, and $\gamma_i^{k_i}$ is in $\widetilde{H}_i$ as is γ_{H_i}. For γ in Γ_0, γ^{k_i} is in $\widetilde{H}_i \cap \Gamma_0$, that is, k_i^0 is finite and divides k_i. If x_l is the last coordinate in V^H, then $\widetilde{H}_l = H$. Thus, $\widetilde{H}_l^0 = H_0$ and $k_i^0 = 1$ for $i > l$. Since there are at most $k = \dim \Gamma/H$ coordinates with k_i^0 infinite, $\dim \Gamma_0/H_0 \leq \dim \Gamma/H$ and, in case of equality, $|\widetilde{H}_0/H_0| = \prod k_i^0$ divides $|\widetilde{H}_0/H| = \prod k_i$. Thus, the fundamental cell $\mathcal{C}_0$ for H_0 is made of $\prod k_i/\prod k_i^0$ copies of the fundamental cell $\mathcal{C}$ for $\underline{H}$. Thus, if $K > \underline{H}$ and $F^K \neq 0$, one has $F^{K_0} \neq 0$ for $K_0 = K \cap \Gamma_0 \geq H_0$. Conversely, if $K_0 > H_0$, then as above $K_0 = \underline{K} \cap \Gamma_0$ with $\underline{K}$ minimal, i.e. $K_0 = \bigcap H_i \cap \Gamma_0$, for H_i the isotropy subgroup of the coordinate x_i in V^{K_0}, and $\underline{K} = \bigcap H_i$. Thus, $\underline{K} > H$ and $F^{K_0} \neq 0$. In other words, the extension degree is defined for $\underline{H}$ and H_0 and the equality of the lemma comes from [10, Theorem 4.1], by computing $\deg(F^{\underline{H}}; B_k)$: in fact, since $\dim V^{\underline{H}} = \dim W^{\underline{H}} + \dim \Gamma/\underline{H}$ and, from $H_0 = \underline{H} \cap \Gamma_0 < \underline{H}$, one has $W^{\underline{H}} \subset W^{H_0}$, it follows that, if $W^{\underline{H}} = W^{H_0}$, one has the same equality of the dimensions for H_0, while if one has a strict inclusion, then any map in $\Pi(H_0)$ has a non-trivial extension.

For (b), any complementing map for $\underline{H}$ will also work for H_0. Thus, if $\underline{H} < H$, the map $(F^H, F_{\perp}^H)^{\underline{H}}$, which does not belong to $\Pi(\underline{H})$ if $\underline{H}$ is a strict subgroup of H, has the following property: if $K_0 > H_0$, hence as above, $K_0 = \underline{K} \cap \Gamma_0$ with $\underline{K} > \underline{H}$, then if $(F^H, F_{\perp}^H)^{K_0}(X) = 0$, then X is in V^H and $F^H(X) = 0$. But

$X \in V^{\underline{K}}$, thus $\underline{K}$ and H are subgroups of Γ_X. Hence, if F^H is in $\Pi(H)$ and H is a strict subgroup of Γ_X, one has $F^H(X) \neq 0$. That is, $\Gamma_X = H$ and $\underline{K} \leq H$. But the relation $\underline{H} < \underline{K}$ would imply $H_0 = K_0$, which is a contradiction. That is, $(F^H, F^H_\perp)^{K_0} \neq 0$ if $K_0 > H_0$ and the pair $(F^H, F^H_\perp)$ belongs to $\widetilde{\Pi}_0(H_0)$.

(c) is clear since $V^{\underline{H}} = V^{H_0}$ and $V^{\underline{K}} = V^{K_0}$. □

Proposition 6.1. (a) *If* (H′) *holds and* $\dim \Gamma_0/H_0 = \dim \Gamma/H = k$, *then*

$$P_*[F^H, F^H_\perp]^d_\Gamma = d\beta_{\underline{H}H} \frac{|\widetilde{H}_0/H|}{|\widetilde{H}^0_0/H_0|}[F_0^{H_0}, F^{\underline{H}\perp}]_{\Gamma_0}$$

for the generators of $\widetilde{\Pi}(H)$ *and* $\widetilde{\Pi}_0(H_0)$, *where* $\beta_{\underline{H}H} = \deg((F^H_\perp)^{\underline{H}})$.

(b) *If furthermore* $k = 0$ *and* Γ_0/H_0 *or* Γ/H *is not finite, then* $P_* = 0$.

Proof. From Theorem 2.1, since by construction $F^H \neq 0$ on the set $z_j = 0$ for any $j = 1, \ldots, k$, one has

$$\deg(F^{H_i}|_{B_k^{H_i}}; B_k^{H_i}) = \sum_{H_i < H_j < \widetilde{H}_0} \beta_{ij} d_j |\widetilde{H}_0/H_j|$$

for all H_i's and H_j's with $\dim \Gamma/H_i = k$. Hence, if $F = (F^H, F^H_\perp)$, the degree on the left will be 0, since it is a product and $F^H|_{B_k^{H_i}}$ corresponds to $V^H \cap V^{H_i}$ with isotropy larger than H, i.e. there $F^K \neq 0$, unless $H_i < H$, in which case the degree for the generator is $\beta_{H_iH}|\widetilde{H}_0/H|$. On the right hand side, one has $d_j = 0$ except for $d_H = 1$. In particular, $\deg((F^H, F^H_\perp)^{\underline{H}}; B_k^{\underline{H}}) = \beta_{\underline{H}H}|\widetilde{H}_0/H|$. Now, as a Γ_0-map, $P_*[F^H, F^H_\perp] = a[F_0^{H_0}, F_\perp^{\underline{H}}]$ for some integer a (recall that we are complementing with the same maps $F_\perp^{\underline{H}}$), and $\deg(F_0^{H_0}; B_k^{H_0}) = |\widetilde{H}^0_0/H_0|$.

For (b), it is enough to recall that $\Pi(H) = 0$ if $\dim \Gamma/H > 0$. □

Remark 6.1. In [5] the authors consider the case where $V = \mathbb{R}^k \times W, |\Gamma/\Gamma_0| < \infty$ and there is an open, bounded, Γ_0-invariant set Ω_0 such that $\gamma\overline{\Omega}_0 \cap \overline{\Omega}_0 = \emptyset$ for all γ in $\Gamma \setminus \Gamma_0$. Then they compute the free part of the Γ-degree of a map with respect to $\Omega = \Gamma\Omega_0$, i.e. the one corresponding to isotropy subgroups with Weyl group of dimension k. If Γ is abelian and x is in Ω_0, then Γ_x is a subgroup of Γ_0, due to the condition $\gamma\widetilde{\Omega}_0 \cap \Omega_0 = \emptyset$. While, if x is in Ω, i.e. $x = \gamma_0 x_0$ with x_0 in Ω_0, then if γ is in Γ_x, one has $\gamma_x = x = \gamma\gamma_0 x = \gamma_0 x$, hence γ is in $\Gamma_{x_0} < \Gamma_0$. Thus, all isotropy subgroups for Γ are isotropy subgroups for Γ_0. Then, since $\dim \Gamma = \dim \Gamma_0$, one has $\dim \Gamma/H_0 = \dim \Gamma_0/H_0$ and one has the same set of variables with $k_i = \infty$. From $\Gamma/H_0 = (\Gamma/\Gamma_0)(\Gamma_0/H_0)$, one has $|\widetilde{H}_0/H|/|\widetilde{H}^0_0/H_0| = |\Gamma/\Gamma_0|$. Hence, for these subgroups, one sees, from Lemma 6.1(a) and the previous proposition, that $P_*[F^H]_\Gamma = |\Gamma/\Gamma_0|[F^H]_{\Gamma_0}$ and the assignment $[F^H]_{\Gamma_0} \to [F^H]_\Gamma$ is an isomorphism: there are $|\Gamma/\Gamma_0|$ disjoint copies of Ω_0 in Ω.

Let $V = \mathbb{R} \times W$. Then $\Pi^\Gamma_{S^V}(S^W)$ is generated, in its free part, by $[F^H]_\Gamma$ as above for $\dim \Gamma/H = 1$, and for H with $\Gamma/H = A = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_m}$ by η'_j and $\widetilde{\eta}'$, $j = 1, \ldots, m$, given in term of an auxiliary space $X = (Z_1, \ldots, Z_m)$ with action of Γ/H on Z_j given by $\exp(2\pi i/p_j)$. Then

$$\eta'_j = \Big(1 - \prod |Z_i|^2, X_0, X_i, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i\neq j}, \lambda Z_j\Big),$$
$$\widetilde{\eta}' = \Big(\varepsilon^2 - \prod_{i<m} |Z_i|^2 |Z_m^{p_m} - \varepsilon_m|^2, X_0, X_i, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i\neq j}, \lambda(Z_m^{p_m} - \varepsilon_m)Z_m\Big),$$

with $\varepsilon_i, \varepsilon_m$ of modulus one, such that $Z_i^{p_i} - \varepsilon_i = 0$ has no real root (see [9, Theorem 8.4]). One has $p_j(\eta'_j + \widetilde{\eta}') = 0$ and $2\widetilde{\eta}' = 0$. Similar definitions hold for $\Gamma_0/H_0 = A_0$.

Proposition 6.2. (a) *If* $\dim \Gamma/H = \dim \Gamma_0/H_0 = 1$, *then*

$$P_*[F^H]_\Gamma = \frac{|\widetilde{H}_0/H|}{|\widetilde{H}_0^0/H_0|}[F^{H_0}]_{\Gamma_0}.$$

(b) *If* $\dim \Gamma/H = \dim \Gamma_0/H_0 = 0$, *then*

$$P_*[\eta'_j]_\Gamma = (|A|/|A_0|)(p_{0j}/p_j)[\eta'_{0j}]_{\Gamma_0} + \widetilde{d}_j[\widetilde{\eta}'_0]_{\Gamma_0},$$

where $\widetilde{d}_j$ *is* 0 *or* 1 *and* $\widetilde{d}_j = 0$ *if* $|A_0|$ *or* p_j *are odd. Moreover,* $P_*[\widetilde{\eta}']_\Gamma = (|A|/|A_0|)[\widetilde{\eta}'_0]_{\Gamma_0}$ *for* $j = 1, \ldots, m$.

(c) *If* $\dim \Gamma/H = 1$ *and* $\dim \Gamma_0/H_0 = 0$, *then*

$$P_*[F^H]_\Gamma = (|\widetilde{H}_0/H|/|A_0|)p_{01}[\eta'_{01}]_{\Gamma_0} + \widetilde{d}[\widetilde{\eta}'_0]_{\Gamma_0}$$

with $\widetilde{d} = 0$ *if* $|A_0|$ *is odd.*

Proof. (a) was already proved in the previous proposition. For (b), notice that if Γ acts as $\exp(2\pi i/p_j)$ on Z_j, then Γ_0 has to act as $\exp(2\pi i/p_{0j})$, where p_{0j} divides p_j. Hence $|A_0|$ divides $|A|$. As before, for the minimal $\underline{H}$, one has $\Gamma/\underline{H} = (\Gamma/H)(H/\underline{H})$ and Γ_0 acts trivially on the variables in $V^{\underline{H}} \cap (V^H)^\perp$. Now, as seen in [10, Theorem 8.4], the components of $P_*[\eta'_j]_\Gamma$ on η'_{0i} can be computed via $\deg(\eta'_j; B^{H_0} \cap (\operatorname{Arg} Z_i = 0))/\prod_{k\neq i} p_{0k}$. It is then clear that this number is 0 if $i \neq j$ and $\prod_{k\neq j} p_k/\prod_{k\neq j} p_{0k}$ if $i = j$. Thus,

$$P_*[\eta'_j]_\Gamma = \Big(\prod_{k\neq j} p_k/\prod_{k\neq j} p_{0k}\Big)[\eta'_{0j}]_{\Gamma_0} + \widetilde{d}_j[\widetilde{\eta}'_0]_{\Gamma_0}.$$

Now, if one computes the ordinary class of both sides in $\Pi_{n+1}(S^n)$, then $[P_*[\eta'_j]] = [\eta'_j] = \prod_{i\neq j} p_i\eta$, where η is the Hopf map, while on the right hand side one has the same quantity plus $\widetilde{d}_j|A_0|\eta$. Hence, if $|A_0|$ is odd, one has $\widetilde{d}_j = 0$. Since $\widetilde{\eta}$ is the Hopf map based on the fundamental cell for Γ/H and the fundamental cell for Γ_0/H_0 is generated by $|A|/|A_0|$ copies of the first one, with a suspension on the variables on $X^{\underline{H}} \cap (X^H)^\perp$, one has $P_*[\widetilde{\eta}']_\Gamma = (|A|/|A_0|)[\widetilde{\eta}'_0]_{\Gamma_0}$. Then, from the relation $p_j(\eta'_j + \widetilde{\eta}') = 0$, one has $p_j\widetilde{d}_j \equiv (|A|/|A_0|)(p_j - p_{0j}) \pmod 2$. If p_j is odd, hence p_{0j} which divides p_j is also odd, then $\widetilde{d}_j$ is even.

For (c), one has $\Gamma/H \cong S^1 \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_m}$ and, using the auxiliary space X, one may take $F^H = (1 - \prod |Z_j|^2, X_0, X_i, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i \neq j}, \lambda Z_1)$, where Γ acts as $\exp(i\varphi)$ on Z_1 and Γ_0 as $\exp(2\pi i/p_{01})$. Again, the components of $P_*[F^H]_\Gamma$ on η'_{0j} are given by $\deg(F^H; B^{H_0} \cap \{\operatorname{Arg} Z_j = 0\})/\prod_{k\neq j} p_{0k}$, i.e. 0 if $j \neq 1$ and $\prod_{k\geq 2} p_k/\prod_{k\geq 2} p_{0k}$ if $j = 1$. The fact that $\widetilde{d}$ is 0 if $|A_0|$ is odd is proved as above. □

(B) Products. Consider the classical problem of a product of maps $(f_1(X_1), f_2(X_2))$ defined on a product $\Omega = \Omega_1 \times \Omega_2$ from $V_1 \times V_2$ into $W_1 \times W_2$, where f_1 and f_2 are Γ-equivariant and Ω_i are Γ-invariant, open and bounded. The associated maps, which define the Γ-degree, are $F_i(t_i, X_i) = (2t_i + 2\varphi_i(X_i) - 1, f_i(X_i))$ and one may consider the pair $(F_1(t_1, X_1), F_2(t_2, X_2))$ from $\mathbb{R} \times V_1 \times \mathbb{R} \times V_2$ into $\mathbb{R} \times W_1 \times \mathbb{R} \times W_2$. Let $\Phi(X_1, X_2) = \varphi_1(X_1) + \varphi_2(X_2) - \varphi_1(X_1)\varphi_2(X_2) = \varphi_2(1 - \varphi_1) + \varphi_1$. Then clearly $0 \leq \Phi \leq 1$ and $\Phi = 0$ on $\Omega_1 \times \Omega_2$ and $\Phi = 1$ on the complement of $(\Omega_1 \cup N_1) \times (\Omega_1 \cup N_2)$. Furthermore, (F_1, F_2) is linearly deformable to $(2t_1 + 2\Phi - 1, \widetilde{f}_1, F_2)$, since $\widetilde{f}_i(X_i) \neq 0$ on N_i, and then to $(2t_1 + 2\Phi - 1, \widetilde{f}_1, 2t_2 - 1, \widetilde{f}_2)$.

LEMMA 6.2. *One has* $[F_1, F_2] = \Sigma_0 \deg_\Gamma((f_1, f_2); \Omega_1 \times \Omega_2)$, *where* Σ_0 *is the suspension by* $2t_2 - 1$.

Note that if $[F_i]$ is in $\Pi^\Gamma_{S^{V_i}}(S^{W_i})$, then $[F_1, F_2]$ is in $\Pi^\Gamma_{S^{V_1 \times \mathbb{R} \times V_2}}(S^{W_1 \times \mathbb{R} \times W_2})$, which defines a morphism of groups, i.e. $[F_1 + G_1, F_2] = [F_1, F_2] + [G_1, F_2]$ and $[F_1, F_2 + G_2] = [F_1, F_2] + [F_1, G_2]$. (For this last operation, with the sum defined on t_2, one has to translate this sum on t_1. This is done as in any text on homotopy.) Hence, if $[F_1]$ and $[F_2]$ are expressed as sums, as in several cases in [10] and above, one may expand $[F_1, F_2]$ in terms of elementary products. Let $V = V_1 \times \mathbb{R} \times V_2$ and $W = W_1 \times \mathbb{R} \times W_2$. We shall incorporate t_2 in V_2.

LEMMA 6.3. (a) *Any isotropy subgroup* H *for* V *is of the form* $H_1 \cap H_2$ *with* H_i *in* $\operatorname{Iso}(V_i)$. *There are minimal isotropy subgroups* $\underline{H}_i$ *with* $H = \underline{H}_1 \cap \underline{H}_2, V_i^{\underline{H}_i} = V_i^H$ *and* $\dim \Gamma/\underline{H}_i \leq \dim \Gamma/H \leq \dim \Gamma/\underline{H}_1 + \dim \Gamma/\underline{H}_2$.

(b) *If* $[F_i]$ *is in* $\Pi(\underline{H}_i)$, *then* $[F_1, F_2]$ *is in* $\Pi(H)$. *If for any* H_i, *there are complementing maps, then if* $[F_i, F_\perp^i]$ *is in* $\widetilde{\Pi}(H_i)$, *then* $[F_1, F_\perp^1, F_2, F_\perp^2]$ *is in* $\widetilde{\Pi}(H)$.

(c) *If hypothesis* ($\widetilde{\mathrm{H}}$) *holds for* V_1 *and* V_2, *it also holds for* V, *where* ($\widetilde{\mathrm{H}}$) *is* (H) *together with the condition* $W_i^{\underline{H}_i} = W_i^H$, *which is true if* $V_i = \mathbb{R}^{k_i} \times W_i$.

PROOF. If $H = \Gamma_{(X_1, X_2)}$, then $H = \Gamma_{X_1} \cap \Gamma_{X_2} = H_1 \cap H_2$, by recalling that $\Gamma_X = \bigcap H_j$ over the isotropy subgroups of the non-zero variables x_j's in X. Then $V^H = V_1^H \times \mathbb{R} \times V_2^H$. If $\underline{H}_i = \bigcap H_j$ for coordinates x_j in V_i^H, then

$H < \underline{H}_i$ and $V_i^{\underline{H}_i} = V_i^H$. Since $H = H_1 \cap H_2$, one has $\dim \Gamma/H_i \leq \dim \Gamma/H$. In the decomposition of Γ/H over the coordinates of V, one obtains the groups $\widetilde{H}^1_{i-1}/\widetilde{H}^1_i$ for the first coordinates, corresponding to $V_1^{\underline{H}_1}$ with order k_i^1, and then $H_1 \cap \widetilde{H}^2_{i-1}/H_1 \cap \widetilde{H}^2_i$, with order $\widetilde{k}^2_i$. We shall denote by k_i^2 the order of $\widetilde{H}^2_{i-1}/\widetilde{H}^2_i$ corresponding to $V_2^{\underline{H}_2}$. If k_i^2 is finite, then any γ in $\widetilde{H}^2_{i-1}$ can be written as $\gamma_i^\alpha \gamma_{H_i}$ with $0 \leq \alpha \leq k_i^2$ and γ_{H_i} in $\widetilde{H}^2_i$. In particular, for γ in $H_1 \cap \widetilde{H}^2_{i-1}$, $\gamma^{k_i^2}$ is in $H_1 \cap \widetilde{H}^2_i$, that is, $\widetilde{k}^2_i$ divides k_i^2. Thus, the number of k_i's infinite for V^H is the sum of the number of those for $V_1^{\underline{H}_1}$ and a quantity less than or equal to the number of those for $V_2^{\underline{H}_2}$. Note that when $H_1 \cap \widetilde{H}^2_{i-1} = H$, then $\widetilde{k}^2_j = 1$ for $j \geq i$.

For (b), if $K = \underline{K}_1 \cap \underline{K}_2 > H_1 \cap H_2$, then $V^K = V_1^{\underline{K}_1} \times \mathbb{R} \times V_2^{\underline{K}_2}$ is strictly contained in $V^H = V_1^{\underline{H}_1} \times \mathbb{R} \times V_2^{\underline{H}_2}$. Then either $\underline{K}_1 > \underline{H}_1$ or $\underline{K}_2 > \underline{H}_2$ and the corresponding $F_i^{\underline{H}_i} \neq 0$, i.e. $[F_1, F_2]$ is in $\Pi(H)$. Also, if $(F_1, F_\perp^1, F_2, F_\perp^2)$ has a zero at (X_1, X_2) in V^K for $K > H$, then since $F_\perp^i$ is zero only at the origin, (X_1, X_2) must be in $V_1^{H_1} \times V_2^{H_2}$ with $\Gamma_{(X_1,X_2)} \leq H_1 \cap H_2 = H$, leading to a contradiction. Thus, the above map is in $\widetilde{\Pi}(H)$.

Finally, if $(\widetilde{\mathrm{H}})$ holds for $V_i = \mathbb{R}^{k_i} \times U_i$, let $K = \underline{K}_1 \cap \underline{K}_2$ and $H = \underline{H}_1 \cap \underline{H}_2$. It is then clear that $\dim U^H \cap U^K = \dim W^H \cap W^K$, since $U^H = U_1^{\underline{H}_1} \times U_2^{\underline{H}_2}$ and likewise for K and one has $W_i^{\underline{H}_i} = W_i^H$. Note that in general $W_i^{\underline{H}_i} \subset W_i^H$. $\square$

Proposition 6.3. (a) *If* $\dim V_i^{\underline{H}_i} = \dim W_i^{\underline{H}_i} + \dim \Gamma/\underline{H}_i, i = 1, 2,$ *and* $\dim \Gamma/H = \dim \Gamma/H_1 + \dim \Gamma/\underline{H}_2$*, then, for* $[F_i]$ *in* $\Pi(\underline{H}_i)$*, one has*

$$\deg_{\mathrm{E}}(F_1, F_2) = \deg_{\mathrm{E}}(F_1) \deg_{\mathrm{E}}(F_2) \prod (k_i^2/\widetilde{k}_i^2)$$

if $W_i^{\underline{H}_i} = W_i^H$ *and* 0 *otherwise.*

(b) *If* $(\widetilde{\mathrm{H}})$ *holds and* $\dim \Gamma/H_i = k_i, \dim \Gamma/H = k_1 + k_2$*, then, for* $[F_i, F_\perp^i]$ *in* $\widetilde{\Pi}(H_i)$*, one has* $[F_1, F_\perp^1, F_2, F_\perp^2] = d_H[F_H]$*, where* F_H *is the generator for* $\widetilde{\Pi}(H_1 \cap H_2)$ *and*

$$d_H = \beta_{\underline{H}_1 H_1} \beta_{\underline{H}_2 H_2} |\widetilde{H}_1^0/H_1| \cdot |\widetilde{H}_2^0/H_2| / |\widetilde{H}_1^0 \cap \widetilde{H}_2^0/H_1 \cap H_2|.$$

Here $\widetilde{H}_i^0$ *is the maximal isotropy subgroup containing* H_i*,* $\Gamma/\widetilde{H}_1^0 \cong T^{k_i}$ *and* $\beta_{\underline{H}_i H_i} = \deg(F_\perp^{i\underline{H}_i})$.

(c) *Furthermore, if* $[F_i]_\Gamma = \sum d_j^i [F^i_{H_j}]_\Gamma + [\widetilde{F}_i]_\Gamma$ *with* $\dim \Gamma/H_i = k_i$ *and* $[\widetilde{F}_i]$ *in* $\widetilde{\Pi}_{k_i - 1}$*, then*

$$[F_1, F_2]_\Gamma = \sum_{jk} d_j^1 d_k^2 d_{H_j \cap H_k} [F_{H_j \cap H_k}]_\Gamma + [\widetilde{F}]_\Gamma,$$

where the sum is over all (j, k)*'s such that* $\dim \Gamma/H_j \cap H_k = k_1 + k_2, d_{H_j \cap H_k}$ *is as above and* $[\widetilde{F}]_\Gamma$ *belongs to* $\widetilde{\Pi}_{k_1 + k_2 - 1}$*, defined in* [10, *Theorem* 5.2].

Proof. It is clear that the fundamental cell for $H_1 \cap H_2$ is the product of the fundamental cell for H_1 by the fundamental cell for $H_1 \cap H_2$. The dimension conditions imply that $\widetilde{k}_2^j = \infty$ exactly when $k_2^j = \infty$, hence, from [10, Theorem 4.1], one has

$$\deg_{\mathrm{E}}(F_1, F_2) = \deg((F_1, F_2); B_{k_1} \times B_{k_2}) / \Big(\prod k_j^1 \prod \widetilde{k}_j^2 \Big)$$

if $W^H = W^{\underline{H}_1} \times \mathbb{R} \times W^{\underline{H}_2}$ and 0 otherwise. From the degree of the product, one obtains the result.

For (b), from Lemma 6.3(b), (c), one sees that it is enough to compute d_H. Now, as in Proposition 6.1, the map $[F_1, F_\perp^1, F_2, F_\perp^2]$ is non-zero if $z_j = 0$ for any j with k_j^1 or k_j^2 (i.e. $\widetilde{k}_j^2$) infinite, that is, one may apply Theorem 2.1. Then

$$\begin{aligned}\beta_{H_1}\beta_{H_2} \deg(F_1^{H_1}|_{B_{k_1}}, F_2^{H_2}|_{B_{k_2}}) &= \beta_{H_1}\beta_{H_2} \deg(F_1^{H_1}|_{B_{k_1}}) \deg(F_2^{H_2}|_{B_{k_2}}) \\ &= \beta_{H_1}\beta_{H_2} |H_1^0/H_1| / |\widetilde{H}_2^0/H_2| \\ &= \beta_H d_H |\widetilde{H}_1^0 \cap \widetilde{H}_2^0 / H_1 \cap H_2|,\end{aligned}$$

since clearly $\widetilde{H}_1^0 \cap \widetilde{H}_2^0$ is the maximal isotropy subgroup for $H_1 \cap H_2$. Here $\beta_{H_i} = \deg(F_i^\perp) = \deg(F_i^\perp|_{V_i^{\underline{H}_i}})\beta_{\underline{H}_i}$. Since one may complement F_H by $(F_1^\perp, F_2^\perp)|_{(V^H)^\perp}$, with degree $\beta_{\underline{H}_1}\beta_{\underline{H}_2}$, one obtains the result. Note that we have $\widetilde{H}_1^0 \cap \widetilde{H}_2^0 / H_1 \cap H_2 = (\widetilde{H}_1^0 \cap \widetilde{H}_2^0 / H_1 \cap \widetilde{H}_2^0)(H_1 \cap \widetilde{H}_2^0 / H_1 \cap H_2)$. The first group has order $\prod k_j^1$, since the coordinates coming from $\widetilde{H}_2^0$ have $k_j^2 = \infty$, and the second group has order $\prod \widetilde{k}_j^2$. Thus, (a) and (b) give the same result.

For (c), it is enough to note that if $[\widetilde{F}_1]_\Gamma$ belongs to $\widetilde{\Pi}_{k_1-1}$ for instance, i.e. to subgroups with $\dim \Gamma/H < k_1$, then, from Lemma 6.3(a), $[\widetilde{F}_1, F_2]$ is in $\widetilde{\Pi}_{k_1+k_2-1}$. Then one applies the bilinearity of the product. $\square$

Remark 6.2. In [5] and [15], the product is defined, also for non-abelian groups, in the Burnside ring, for the case where $V_1 = \mathbb{R}^k \times W_1$ and $k_2 = 0$.

Proposition 6.4. *If $V_1 = \mathbb{R} \times W_1$ and $V_2 = W_2$, then the only relevant subgroups are of the form (H_1, H_2) with $\dim \Gamma/H_1 \le 1$ and $\dim \Gamma/H_2 = 0$, with generators η_1, if $\dim \Gamma/H_1 = 1$, or η_j^1 and $\widetilde{\eta}_1$ if $\dim \Gamma/H_1 = 0, \eta_2$ for H_2 and η if $\dim \Gamma/H = 1$ or η_j and $\widetilde{\eta}$ if $\dim \Gamma/H = 0$.*

(a) *If $\dim \Gamma/H_1 = 1$, then*

$$[\eta_1, \eta_2]_\Gamma = \frac{|\widetilde{H}_0^1/H_1| \cdot |\Gamma/H_2|}{|\widetilde{H}_1^0/H_1 \cap H_2|} [\eta]_\Gamma.$$

(b) *If $\dim \Gamma/H_1 = 0$, then*

$$[\eta_j^1, \eta_2]_\Gamma = \alpha_j(r_j/p_j) \frac{|\Gamma/H_2|}{|H_1/H_1 \cap H_2|} [\eta_j]_\Gamma + \widetilde{d}_j[\widetilde{\eta}]_\Gamma, [\widetilde{\eta}_1, \eta_2]_\Gamma = \frac{|\Gamma/H_2|}{|H_1/H_1 \cap H_2|} [\widetilde{\eta}]_\Gamma,$$

where $p_j(\eta_j^1 + \widetilde{\eta}_1) = 0, 2\widetilde{\eta}_1 = 0, r_j(\eta_j + \widetilde{\eta}) = 0, 2\widetilde{\eta} = 0$, and $p_j\widetilde{d}_j - (\alpha_j r_j - p_j)|\Gamma/H_2|/|H_1/H|$ is even. Here $\alpha_j = 1$ if $r_j = p_j$ and if p_j divides r_j, then $\alpha_j r_j/p_j + \beta_j r_j/q_j = 1$, where Γ/H_2 has the cyclic subgroup $\mathbb{Z}_{q_j}$ and H_1/H the subgroup $\mathbb{Z}_{\widetilde{q}_j}$ with $\widetilde{q}_j = r_j/p_j$.

(c) *If* $[F_1]_\Gamma = d_1[\eta_1]_\Gamma + \sum_j d_j^1[\eta_j^1]_\Gamma + \widetilde{d}_1[\widetilde{\eta}_1]_\Gamma$ *and* $[F_2]_\Gamma = d_2[\eta_2]_\Gamma$, *then* $[F_1, F_2]_\Gamma$ *distributes according to* (a) *and* (b).

Proof. From Lemma 6.3(a), one has $\dim \Gamma/H = \dim \Gamma/H_0$ for the relevant groups, i.e. those for which the dimension of the Weyl group is less than or equal to the number of parameters, here only one. Since the β's are all 1 here, (a) is a reformulation of Proposition 6.3(b).

For (b), if $\Gamma/H_1 \cong \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_m}, \Gamma/H_2 \cong \mathbb{Z}_{q_1} \times \ldots \times \mathbb{Z}_{q_n}$ and $\Gamma/H_1 \cap H_2 \cong \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_s}$, then the action of Γ/H on $W_1 \times W_2$ is given, on the coordinate x_k, by $\exp(2\pi i \langle K_k/M, L\rangle)$, as seen previously. Here $M = (r_1, \ldots, r_s)^T$. If $a_j =$ g.c.d.$(k_j^1, \ldots, k_j^N)$ with $N = \dim W_1 + \dim W_2$, then a_j and r_j are relatively prime since the action of Γ/H is effective. If b_j and c_j are defined as a_j but on the coordinates of W_1, respectively those of W_2, then, if b_j and r_j are relatively prime, one has $p_j = r_j$, otherwise p_j divides r_j and $b_j/r_j = d_j a_j/p_j$ with $d_j a_j$ and p_j relatively prime and, likewise, $c_j/r_j = e_j a_j/q_j$. Since g.c.d.$(b_j, c_j) = a_j$, one sees that $d_j r_j/p_j$ and $e_j r_j/q_j$ are relatively prime, and so are $m_j = b_j/(d_j a_j)$ and $n_j = c_j/(e_j a_j)$, which are such that $m_j/r_j = 1/p_j$ and $n_j/r_j = 1/q_j$.

Thus, there are α_j, β_j such that $\alpha_j m_j + \beta_j n_j = 1$. For the auxiliary spaces X_1, X_2, X of [10, Theorem 8.4], with action of γ_j on X_j as $\exp(2\pi i/r_j)$ and on $X_i, i \neq j$, as the identity, and similarly for X_1 with coordinate Z_j and X_2 with coordinate Y_j, one may choose $X_j = Z_j$ if $p_j = r_j$, while if p_j divides strictly r_j, we shall keep (Z_j, Y_j, X_j) (just one Y_j from the above discussion and one may have $q_j = r_j$). Then one has an equivariant mapping between these variables given by $Z_j = X_j^{m_j}, Y_j = X_j^{n_j}$ and $X_j = Z_j^{\alpha_j} Y_j^{\beta_j}$. The generators given in [10, Theorem 8.4 and p. 394] are of the form

$$\eta_j^1 = \Big(1 - \prod |Z_i|^2, X_0^1, \{x_i\}, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i \neq j}, \lambda Z_j\Big),$$
$$\widetilde{\eta}_1 = \Big(\varepsilon^2 - \prod_{i<m} |Z_i|^2 |Z_m^{p_m} - \varepsilon_m|^2, X_0^1, \{x_i\}, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i<m,}, \lambda Z_m(Z_m^{p_m} - \varepsilon_m)\Big),$$

with $\lambda = \mu + i(2t_1 - 1)$. Also $\eta_2 = (2t_2 + 1 - 2\prod |Y_i|^2, X_0^2, \{y_i\}, (Y_i^{q_i} - \varepsilon_i)Y_i)$ and those for X are like η_j^1 and $\widetilde{\eta}_1$ but with Z_i replaced by X_i, p_i by r_i, and $(X_0^1, \{x_i\})$ by $(X_0^1, X_0^2, \{x_i\}, \{y_i\})$. Here $|\varepsilon_i| = 1$ and ε is small.

We shall make our computations, as in [10, Theorem 8.4], on $V_1 \times V_2 \times (X_1 \times X_2 \times X)^2$, where one repeats the variable X_j by X_j' and where one uses the suspension. Thus,

$$[\eta_j^1, \eta_2] = \Big(1 - \prod |Z_i|^2, X_0, \{x_i, y_i\}, \{(Z_i^{p_i} - \varepsilon_i)Z_i\}_{i \neq j}, Z_i', \lambda Z_j, Z_j', 2t_2 + 1 - 2\prod |Y_i|^2, (Y_i^{q_i} - \varepsilon_i)Y_i, Y_i', X_i, X_i'\Big),$$

where $X_j = Z_j$ if $p_j = r_j$. In this case, $[\eta_j^1, \eta_2] = d_j[\eta_j] + \widetilde{d}_j[\widetilde{\eta}]$, where $d_j = \deg((\eta_j^1, \eta_2); \operatorname{Arg} X_j = 0)/\prod_{i \neq j} r_i$. (We shall prove below that the other d_i's are 0). It is easy to see that this degree is $(\prod_{i\neq j} p_i)(\prod q_i) = p_j^{-1}|\Gamma/H_1| \cdot |\Gamma/H_2|$, giving the result (as seen in (A), $|H_1/H|$ divides $|\Gamma/H_2|$ and $|\Gamma/H| = |\Gamma/H_1| \cdot |H_1/H|$).

Now, on the space $X_1 \times X_2 \times X$ and the ball $B = \{(Z_i, Y_i, X_i) : |Z_i|, |Y_i|, |X_i| \leq 4\}$, one may take several fundamental cells for the action of Γ/H. We shall choose two of them:

$$\begin{aligned}
\mathcal{C} &= \{X_i : 0 \leq \operatorname{Arg} X_i < 2\pi/r_i\},\\
\mathcal{C}_1 &= \{Z_i, Y_i, Y_j, Z_j : 0 \leq \operatorname{Arg} Z_i < 2\pi/p_i,\ 0 \leq \operatorname{Arg} Y_i < 2\pi/\widetilde{q}_i,\\
&\qquad\qquad 0 \leq \operatorname{Arg} Y_j < 2\pi/q_j,\ 0 \leq \operatorname{Arg} Z_j < 2\pi/\widetilde{p}_j\},
\end{aligned}$$

where $i \neq j, \widetilde{q}_i = q_i$ if Z_i and Y_i are not related through X_i and $p_i/q_i = \widetilde{p}_i/\widetilde{q}_i$, with $\widetilde{p}_i$ and $\widetilde{q}_i$ relatively prime, otherwise. Note that, in this last case, $r_i = p_i\widetilde{q}_i$, and from $m_i\alpha_i + n_i\beta_i = 1$ one obtains $\alpha_i\widetilde{q}_i + \beta_i\widetilde{p}_i = 1$, $\widetilde{p}_i = n_i$ and $\widetilde{q}_i = m_i$.

Let $(\xi_i', \zeta_i', \zeta_j', \xi_j', \widetilde{\eta}')$ be the generators with respect to $\mathcal{C}_1$, given in [10, p. 399]. Then $[\eta_j^1, \eta_2] = \sum(d_i\xi_i' + e_i\zeta_i') + e_j\zeta_j' + d_j\xi_j' + d'\widetilde{\eta}'$, where (d_i, e_i) are given by the degree of the map on the section $\operatorname{Arg} Z_i = 0$ or $\operatorname{Arg} Y_i = 0$, provided the preceding $d_k = e_k = 0, k < i$ (see [10, p. 400]). Now, from the choice of ε_i in the maps η_j^1 and η_2, it is easy to take them non-real, that is, $d_i = e_i = e_j = 0$ for all i, and $d_j \prod p_i \prod \widetilde{q}_i q_j = \prod p_i \prod q_i q_j$, i.e. $d_j = \prod_{i\neq j}(q_i/\widetilde{q}_i) = |\Gamma/H_2|/(|H_1/H|q_j/\widetilde{q}_j)$. Thus, $[\eta_j^1, \eta_2]_\Gamma = d_j\xi_j' + d'\widetilde{\eta}'$, where $\widetilde{p}_j(\xi_j' + \widetilde{\eta}') = 0$ [10, Theorem 8.2 and p. 400], and

$$\begin{aligned}
\xi_j' = \Big(1 - |Y_j| \cdot |Z_j| \prod |Z_i| \prod |Y_i|, (Z_i^{p_i} - \varepsilon_i)Z_i, (Y_i^{q_i} - \varepsilon_i)Y_i, (\overline{Z}_i^{\widetilde{p}_i} Y_i^{\widetilde{q}_i} - \varepsilon_i)Y_i,&\\
(Y_j^{q_j} - \varepsilon_j)Y_j, \lambda Z_j, Y_i', Z_i', X_i, X_i'\Big)&,
\end{aligned}$$

where one has q_i if Z_i and Y_i are not related and $\widetilde{p}_i, \widetilde{q}_i$ otherwise, noting that $\overline{Z}_i^{\widetilde{p}_i} Y_i^{\widetilde{q}_i}$ is invariant.

Now $\xi_j' = \sum a_i\eta_i + a\widetilde{\eta}$ with respect to the generators given by the fundamental cell $\mathcal{C}$, where $a_i \prod_{k\neq i} r_k = \deg(\xi_j'; B \cap \operatorname{Arg} X_i = 0)$, provided one has deformed ξ_j' to a map which is non-zero for $X_i = 0$ [10, Theorem 8.4]. Perform first the linear deformation $(Z_i' - \tau X_i^{m_i}, Y_i' - \tau X_i^{n_i}, (1-\tau)X_i + \tau Z_i'^{\alpha_i} Y_i'^{\beta_i})$, on the variables which are related, with only zero at (0,0,0), since $m_i\alpha_i + n_i\beta_i = 1$. Replace then $|Z_i|, |Y_i|$ by $|Z_i^{p_i} + (Z_i' - X_i^{m_i})^{p_i}|$, including $i = j$, $|Y_i^{\widetilde{q}_i} + (Y_i' - X_i^{n_i})^{\widetilde{q}_i}|$, and $Z_i^{p_i}$, $\overline{Z}_i^{\widetilde{p}_i} Y_i^{\widetilde{q}_i}$, $Y_j^{q_j}$ by $Z_i^{p_i} + (Z_i' - X_i^{m_i})^{p_i}$, $(\overline{Z}_i^{p_i} + (\overline{Z}_i' - \overline{X}_i^{m_i})^{p_i})(Y_i^{\widetilde{q}_i} + (Y_i' - X_i^{n_i})^{\widetilde{q}_i})$, $Y_j^{q_j} + (Y_j' - X_j^{n_j})^{q_j}$ respectively. Recall that for the remaining variables one has $X_i = Z_i$ or Y_i. Then one makes rotations of the form $(A_i((1-\tau)Z_i + \tau(Z_i' - X_i^{m_i})), -\tau Z_i + (1-\tau)(Z_i' - X_i^{m_i}))$ with $A_i = Z_i^{p_i} + (Z_i' - X_i^{m_i})^{p_i} - \varepsilon_i$. If $A_i \neq 0$, then $Z_i = Z_i' - X_i^{m_i} = 0$ and the first equation is 1. If $A_i = 0$ and $\tau = 0$, then $|Z_i| = 1, Z_i' = X_i^{m_i}, Z_i'Y_i' = 0$

on a zero of the map, with $Y_i' - X_i^{n_i} = 0$, that is, the zeros are inside B. If $A_i = 0$ and $\tau \neq 0$, then $|A_i + \varepsilon_i| = 1 = |(Z_i' - X_i^{m_i})^{p_i}|(1 + ((1-\tau)/\tau)^{p_i})$ with $Y_i' = X_i^{n_i}, Z_i'Y_i' = 0$ and $|Z_i' - X_i^{m_i}| \leq 1, |Z_i| \leq 1$, hence the zero is inside B. Another rotation will bring the pair to $(Z_i, A_i(Z_i' - X_i^{m_i}))$ and one may deform Z_i to 0 in the remaining equations: one obtains a suspension by Z_i, with the same class, that is, one may replace $Z_i' - X_i^{m_i}$ by $Z_i - X_i^{m_i}$.

One performs the same deformation for Y_i, with A_i replaced by $B_i = [Z_i - X_i^{m_i}]^{\widetilde{p}_i}[Y_i^{\widetilde{q}_i} + (Y_i' - X_i^{n_i})^{\widetilde{q}_i}] - \varepsilon_i$: on a zero of the map, $B_i = 0$, $|Z_i - X_i^{m_i}| = 1$, $Z_iY_i' = 0$ with $|Y_i|, |Y_i' - X_i^{n_i}| \leq 1$, i.e. the zeros are in the ball of radius 2, inside B. One may replace Y_i by $Y_i - X_i^{n_i}$. The same steps are applied to Y_j and Y_j' with A_i replaced by $Y_j^{q_j} + (Y_j' - X_j^{n_j})^{q_j} - \varepsilon_j$, and to Z_j and Z_j' with A_i replaced by λ: on a zero of the map, one has $\lambda = 0, |Z_j^{p_j} + (Z_j' - X_j^{m_j})^{p_j}| = 1$, $|Y_j - X_j^{n_j}| = 1$, $Z_j'Y_j = 0$, $\tau Z_j = (1-\tau)(Z_j' - X_j^{m_j})$ with the results as above. Thus,

$$\begin{aligned}\xi_j' = \Big[1 - \prod |X_k| \prod_{i \neq j} (|Y_i - X_i^{n_i}||Z_i - X_i^{m_i}|)|Y_j - X_j^{n_j}||Z_j - X_j^{m_j}|,\\ (X_k^{r_k} - \varepsilon_k)X_k, ((Z_i - X_i^{m_i})^{p_i} - \varepsilon_i)(Z_i - X_i^{m_i}),\\ ((\overline{Z}_i - \overline{X}_i^{m_i})^{\widetilde{p}}(Y_i - X_i^{n_i})^{\widetilde{q}_i} - \varepsilon_i')(Y_i - X_i^{n_i}),\\ ((Y_j - X_j^{n_j})^{q_j} - \varepsilon_j)(Y_j - X_j^{n_j}), \lambda(Z_j - X_j^{m_j}), Z_i^{\alpha_i}Y_i^{\beta_i}, Z_j^{\alpha_j}Y_j^{\beta_j}\Big]_\Gamma.\end{aligned}$$

By computing the degree of the above map on the sections $\operatorname{Arg} X_k = 0$ or $\operatorname{Arg} X_i = 0$, with appropriate choices of $\varepsilon_k, \varepsilon_i, \varepsilon_i'$, the map has no zeros and a zero degree, i.e. $a_i = 0$ for $i \neq j$. For $\operatorname{Arg} X_j = 0$, choose ε_i and ε_i' such that one cannot have $(Z_i - X_i^{m_i})^{p_i} = \varepsilon_i, (\overline{Z}_i - \overline{X}_i^{m_i})^{\widetilde{p}_i}(Y_i - X_i^{n_i})^{\widetilde{q}_i} = \varepsilon_i'$ at the same time for $Z_i = Y_i = 0$. Thus, with the equation $Z_i^{\alpha_i}Y^{\beta_i}$, one has, for $Z_i = 0$, $p_im_i\widetilde{q}_i$ zeros of index α_i, and for $Y_i = 0, p_in_i\widetilde{q}_i$ zeros of index β_i, i.e. these terms make a contribution of $p_i\widetilde{q}_i(\alpha_im_i + \beta_in_i) = p_i\widetilde{q}_i = r_i$ to the degree. Choosing ε_j non-real, one sees, for (Z_j, Y_j, X_j), that the zeros are for $\lambda = 0, Z_j = 0, X_j = 1$ (X_j is real and positive) and $(Y_j - 1)^{q_j} = \varepsilon_j$, with a contribution to the degree of $q_j\alpha_j$. Hence, $a_j \prod_{i \neq j} r_i = \prod r_k \prod p_i\widetilde{q}_iq_j\alpha_j$, or else, $a_j = \alpha_jq_j$.

From [10, Theorem 8.4], one may choose

$$\begin{aligned}\widetilde{\eta} &= \Big[\varepsilon^2 - \prod |X_i| \cdot |X_j - \varepsilon_j|, (X_i^{r_i} - \varepsilon_i)X_i, \lambda(X_j^{r_j} - \varepsilon_j)X_j\Big],\\ \widetilde{\eta}_1 &= \Big[\varepsilon^2 - \prod |Z_i| \cdot |Z_j - \varepsilon_j|, (Z_i^{p_i} - \varepsilon_i)Z_i, \lambda(Z_j^{p_j} - \varepsilon_j)Z_j\Big],\\ \widetilde{\eta}' &= \Big[\varepsilon^2 - \prod |Z_i| \cdot |Y_i| \cdot |Y_j| \cdot |\overline{Y}_j^{\widetilde{q}_j}Z_j^{\widetilde{p}_j} - \varepsilon_j|, (Z_i^{p_i} - \varepsilon_i)Z_i, (Y_i^{q_i} - \varepsilon_i)Y_i,\\ &\qquad (\overline{Z}_i^{\widetilde{p}_i}Y_i^{\widetilde{q}_i} - \varepsilon_i')Y_i, (Y_j^{q_j} - \varepsilon_j)Y_j, \lambda(\overline{Y}_j^{\widetilde{q}_j}Z_j^{\widetilde{p}_j} - \varepsilon_j)Z_j\Big].\end{aligned}$$

As before, $[\widetilde{\eta}_1, \widetilde{\eta}_2] = \sum(d_i\xi_i' + e_i\zeta_i') + \widetilde{d}'\widetilde{\eta}'$. It is clear that, for $\operatorname{Arg} Z_i = 0$ or $\operatorname{Arg} Y_i = 0$, including $i = j$, the map $(\widetilde{\eta}_1, \eta_2)$ has no zeros, by taking ε_i non-real and ε so small that the circle $|Z_j - \varepsilon_j| = \varepsilon^2$ is inside the cell for X_1. Hence $d_i = e_i = 0$. Hence, on $\partial\mathcal{C}_1, (\widetilde{\eta}_1, \eta_2)$ represents $\prod(q_i/\widetilde{q}_i)$ times the Hopf map, i.e. $[\widetilde{\eta}_1, \eta_2]_\Gamma = (|\Gamma/H_2|/|H_1/H|)\widetilde{\eta}'$.

Similarly $\widetilde{\eta}' = \sum \widetilde{a}_i\eta_i + \widetilde{a}\widetilde{\eta}$. As before,

$$\begin{aligned}\widetilde{\eta}' = \Big[&\varepsilon^2 - \prod |X_k| \prod(|Z_i - X_i^{m_i}| \cdot |Y_i - X_i^{n_i}|)|Y_j - X_j^{n_j}| \\ &\times |(\overline{Y}_j - \overline{X}_j^{n_j})^{\widetilde{q}_j} + (Z_j - X_j^{m_j})^{\widetilde{p}_j} - \varepsilon_j|, (X_k^{r_k} - \varepsilon_k)X_k, \\ &((Z_i - X_i^{m_i})^{\widetilde{p}_i} - \varepsilon_i)(Z_i - X_i^{m_i}), \\ &((\overline{Z}_i - \overline{X}_i^{m_i})^{\widetilde{p}_i}(Y_i - X_i^{n_i})^{\widetilde{q}_i} - \varepsilon_i')(Y_i - X_i^{n_i}), ((Y_j - X_j^{n_j})^{q_j} - \varepsilon_j)(Y_j - X_j^{n_j}), \\ &\lambda((\overline{Y}_j - \overline{X}_j^{n_j})^{\widetilde{q}_j}(Z_j - X_j^{m_j})^{\widetilde{p}_j} - \varepsilon_j)(Z_j - X_j^{m_j}), Z_i^{\alpha_i}Y_i^{p_i}, Z_j^{\alpha_j}Y_j^{\beta_j}\Big].\end{aligned}$$

In the rotations, the only new term is the one of the form $\lambda((\overline{Y}_j - \overline{X}_j^{n_j})^{\widetilde{q}_j}(Z_j^{\widetilde{p}_j} + (Z_j' - X_j^{m_j})^{\widetilde{p}_j} - \varepsilon_j) = \lambda D_j$: since $\varepsilon \ll 1$, a zero of the map will imply $\lambda = 0$, $|D_j| = \varepsilon^2, |Y_j - X_j^{n_j}| = 1, \tau Z_j + (1-\tau)(Z_j' - X_j^{m_j}) = 0, Y_j Z_j' = 0$, which is handled as before. It is then clear that the new map is non-zero for $\operatorname{Arg} X_i = 0$, including $i = j$, by choosing ε_j such that the map is non-zero on $\partial\mathcal{C}$ and one has to compute how many times one gets the Hopf map. As before, one has contributions of r_k for $X_k = Z_k$ or $Y_k, (m_i\alpha_i + n_i\beta_i)p_i\widetilde{q}_i = p_i\widetilde{q}_i = r_i$ for the couples (Z_i, Y_i, X_i). For (Z_j, Y_j, X_j), if $Y_j = 0$, one obtains $n_jq_j\widetilde{p}_j$ points of index β_j and, for $Z_j = 0$, one has $m_jq_j\widetilde{p}_j$ points of index α_j, for a total contribution of $q_j\widetilde{p}_j = r_j$. Since there are $\prod r_i$ copies of $\mathcal{C}$ in the ball, one obtains $\widetilde{\eta}' = \widetilde{\eta}$ and $[\widetilde{\eta}_1, \eta_2]_\Gamma = (|\Gamma/H_2|/|H_1/H|)\widetilde{\eta}$.

Finally,

$$[\eta_j^1, \eta_2]_\Gamma = \prod_{i \neq j}(q_i/\widetilde{q}_i)\xi_j' + d'\widetilde{\eta}' = \alpha_jq_j\prod_{i\neq j}(q_i/\widetilde{q}_i)\eta_j + \widetilde{d}_j\widetilde{\eta} = \alpha_j\widetilde{q}_j\frac{|\Gamma/H_2|}{|H_1/H|}\eta_j + \widetilde{d}_j\widetilde{\eta}.$$

From the fact that $\widetilde{q}_j = m_j = r_j/p_j$ one obtains the result. From the relations $p_j(\eta_j^1 + \widetilde{\eta}_1) = 0$ and $r_j(\eta_j + \widetilde{\eta}) = 0$, one has

$$p_j\left[\widetilde{d}_j + \frac{|\Gamma/H_2|}{|H_1/H|}(1 - \alpha_jm_j)\right]\widetilde{\eta} = 0.$$

Note that we are not reaching η_K for k's corresponding to X_2.

Example 6.1. Note that we could have proved Theorem 5.1 by using the product instead of a direct computation for I_j^*: in fact, one had H_1 an elementary isotropy subgroup, with $\Gamma/H_1 \cong S^1$ or $\mathbb{Z}_p, [\eta_1^1]_\Gamma = \Sigma(1 - |z|^2, \lambda z)$, a suspension, H_2 with $\Gamma/H_2 \cong \mathbb{Z}_2$ and also

$$[F_2]_\Gamma = [2t_2 - 1, -y, Y]_\Gamma = [2t_2 - 1, y, Y]_\Gamma - [2t_2 + 1 - 2y^2, y(y^2 - 1), Y]_\Gamma;$$

as in Section 5, the map $[2t_2-1, y, Y]_\Gamma$ is deformed to $[2t_2-1, y^3, Y]_\Gamma$ and then to $[2t_2-1, y(y^2-1), Y]_\Gamma$, whose Γ-degree is decomposed on the set $|y|<1/2$, giving $[F_2]_\Gamma$, and on the set $|y|>1/2$, where it is $[2t_2+1-2y^2, y(y^2-1)y, Y]_\Gamma$. Hence $[F_2]_\Gamma = [\eta_0]_\Gamma - [\eta_2]_\Gamma$. For η_0, one has $H_1 = \widetilde{H}_0^1$ and $H_2 = \Gamma$. Then $[\eta_1^1, \eta_0]_\Gamma = [\eta]_\Gamma$. For η_2, one has $|\Gamma/H_2| = 2$, $|H_1/H| = 2$ if H_1 is not a subgroup of H_2 (which is always the case if $\dim \Gamma/H_1 = 1$, by the maximality of H_2) and $H_1/H = \{e\}$ if $H_1 < H_2$. In both cases, if $|\Gamma/H_1| < \infty$, one has $r_j = p_j$, hence $\alpha_j = 1$. Thus, $[\eta_1^1, \eta_2]_\Gamma = [\eta_1]_\Gamma + d\widetilde{\eta}'$ if H_1 is not a subgroup of H_2, and $[\eta_1^1, \eta_2]_\Gamma = 2[\eta]_\Gamma + d\widetilde{\eta}$ if $H_1 < H_2$. It is easy to recognize in the generators $\eta, \eta_1, \widetilde{\eta}$ and $\widetilde{\eta}'$ the maps of Theorem 5.1.

(C) Composition. Consider three representations V, W and U of the group Γ and assume $f: V \to W$ and $g: W \to U$ are equivariant maps. Then $g \circ f$ is also equivariant. Assume $f: \overline{\Omega} \to W$ is non-zero on $\partial\Omega$, where Ω is bounded, open and invariant. Let $\Omega_1 = f(\Omega)$. Assume Ω_1 is open and that g is non-zero on $\partial\Omega_1$. It is easy to see that Ω_1 is invariant and bounded (in infinite-dimensions this is due to the appropriate compactness) and that $f(\partial\Omega) \subset \partial\Omega_1$. Let B be the ball used for the definition of the Γ-degree of f, with its associated extension $\widetilde{f}$ of f. Then $\widetilde{f}(B) \subset B_1$ for some ball B_1 centered at the origin. If $\widetilde{g}$ is the extension of $g \circ f$ to B, then $\widetilde{g} \circ \widetilde{f}$ will be an equivariant extension of $g \circ f$. If N_1 is a neighborhood of $\partial\Omega_1$ where $\widetilde{g}$ is non-zero, then one may choose the neighborhood of $\partial\Omega$ contained in $\widetilde{f}^{-1}(N_1)$ with its associated φ. That is, $[2t + 2\varphi(x) - 1, \widetilde{f}(x)] = [F]_\Gamma = \deg_\Gamma(f; \Omega)$ is well defined in $\Pi^\Gamma_{S^V}(S^W)$, as are $\deg_\Gamma(g \circ f; \Omega)$ in $\Pi^\Gamma_{S^V}(S^U)$ and $\deg_\Gamma(g; \Omega_1)$ in $\Pi^\Gamma_{S^W}(S^U)$. Recall that one may normalize F by $F/\|F\|$ on S^V and, changing t to $2t-1=\tau$, one obtains a map from the cylinder into another cylinder, with similar characteristics, i.e. one has a pairing $\Pi^\Gamma_{S^V}(S^W) \times \Pi^\Gamma_{S^W}(S^U)$ into $\Pi^\Gamma_{S^V}(S^U)$ given by $([F]_\Gamma, [G]_\Gamma) \to [G \circ F]_\Gamma$, which is well defined on homotopy classes. Furthermore, since F can be taken to have value $(1,0)$ on $\tau = \pm 1$ [8, Proposition A.1], one sees from [14, p. 479] that, if $F(\tau, X)$ corresponds to $[F_1]_\Gamma + [F_2]_\Gamma$, then $[G \circ F]_\Gamma = [G \circ F_1]_\Gamma + [G \circ F_2]_\Gamma$. Also, if $F = \Sigma_0 f$, a suspension by t_1, then for

$$G_1 \oplus G_2 = \begin{cases} G_1(2t_1+1, Z) & \text{if } -1 \le t_1 \le 0, \\ G_2(2t_1-1, Z) & \text{if } 0 \le t_1 \le 1, \end{cases}$$

one has

$$(G_1 \oplus G_2) \circ (\Sigma_0 f) = \begin{cases} (2t_1+1, f(x)) & \text{if } -1 \le t_1 \le 0, \\ (2t_1-1, f(x)) & \text{if } 0 \le t_1 \le 1, \end{cases}$$

and $[(G_1 \oplus G_2) \circ \Sigma_0 f]_\Gamma = [G_1 \circ \Sigma_0 f]_\Gamma + [G_2 \circ \Sigma_0 f]_\Gamma$ (see [14, p. 479]; as usual one may perform the sum on τ or on t_1 and here we may assume that F is a suspension). In particular, if $[F]_\Gamma = \sum d_i [\widetilde{F}_i]_\Gamma + d[\widetilde{F}]_\Gamma$, with $\widetilde{F}_i$ and $\widetilde{F}$ suspensions by t_1, and $[G]_\Gamma = \sum e_i[\widetilde{G}_i]_\Gamma + e[\widetilde{G}]_\Gamma$ then

$$[G \circ F]_\Gamma = \sum d_i e_j [\widetilde{G}_j \circ \widetilde{F}_i]_\Gamma + \sum d_i e[\widetilde{G} \circ \widetilde{F}_i]_\Gamma + \sum e_j d[\widetilde{G}_j \circ \widetilde{F}]_\Gamma + de[\widetilde{G} \circ \widetilde{F}]_\Gamma,$$

and it is enough to compute each component. Note that if F is in $\Pi(H)$, i.e. F^H has a non-zero extension to $\bigcup_{K>H} V^K$ or else $F^K|_{S^K}$ is Γ-deformable to $F^K(0)$ and to $(1,0)$, then $(G \circ F)^H$ is in $\Pi(H)$. Similarly if G^K has a non-zero extension to W^K, this will also be true for $G \circ F|_{V^K}$. Here, we need to compute $\widetilde{G}_j \circ \widetilde{F}_i$.

LEMMA 6.4. *If $V = \mathbb{R}^{k_1+k_2} \times V'$, $W = \mathbb{R}^{k_2} \times W'$ and* (H) *holds for (V, W) and (W, U), and furthermore* $\dim V'^H = \dim U^H$ *for all H in* $\operatorname{Iso}(V)$, *then* (H) *holds for (V, U). If $\{x_i^{l_i}\}$ is a complementing map from $(V^H)^\perp$ onto $(W^H)^\perp$ and $\{z_j^{q_j}\}$ is a complementing map from $(W^H)^\perp$ onto $(U^H)^\perp$, then $\{x^{l_i q_i}\}$ will be a complementing map from $(V^H)^\perp$ onto $(U^H)^\perp$.*

PROOF. Let H and K be in $\operatorname{Iso}(V)$. Then $\dim W^H \cap W^K = \dim \widetilde{V}^{H_1} \cap \widetilde{V}^{H_2}$, where $\widetilde{V} = \mathbb{R}^{k_2} \times V'$. Let $\widetilde{H}$ be the isotropy of W^H, i.e. $\widetilde{H} = \bigcap_{Z \in W^H} \Gamma_Z$. Then $H < \widetilde{H}$ and $W^{\widetilde{H}} = W^H$. One has $\dim W'^{\widetilde{H}} \cap W'^{\widetilde{K}} = \dim U^{\widetilde{H}} \cap U^{\widetilde{K}}$. Now, $U^{\widetilde{H}} \subset U^H$. From (H), one has $\dim V^H = \dim W^H + k_1 = \dim U^{\widetilde{H}} + k_1 + k_2$, hence, from the extra hypothesis, one gets $\dim \widetilde{U}^H = \dim U^H$ and $U^{\widetilde{H}} = U^H$. Since the spaces $(V^H)^\perp$, $(W^H)^\perp = (W^{\widetilde{H}})^\perp$, $(U^H)^\perp = (U^{\widetilde{H}})^\perp$ have the same dimension and one has equivariant monomials between them, the composition will be a complementing map. $\square$

Note that the extra dimension condition will be met if $\operatorname{Iso}(V) \subset \operatorname{Iso}(W)$, since then $U^{\widetilde{H}} = U^H$. If $\widetilde{H}$ is in $\operatorname{Iso}(W)$, then, if H is the isotropy of $V^{\widetilde{H}}$, one has $\widetilde{H} < H, V^{\widetilde{H}} = V^H$ and $W^H \subset W^{\widetilde{H}}$. In order to compare the Γ-degrees of $\widetilde{F}_i$ and $\widetilde{G}_j$, we shall assume that $\operatorname{Iso}(V) = \operatorname{Iso}(W)$; this is the case if $V = \mathbb{R}^{k_1} \times W$ and $W = \mathbb{R}^{k_2} \times U$.

PROPOSITION 6.5. *Assume* (H) *holds for (V, W) and (W, U), and* $\operatorname{Iso}(V) = \operatorname{Iso}(W)$. *Let F^{H_1} be in $\Pi(H_1)$ and G^{H_2} be in $\Pi(H_2)$. Define $\widetilde{F} = (F^{H_1}, x_i^{l_i})$, $\widetilde{G} = (G^{H_2}, z_j^{q_j})$ and $H = H_1 \cap H_2$. Then:*

(a) $\dim \Gamma/H_i \le \dim \Gamma/H \le \dim \Gamma/H_1 + \dim \Gamma/H_2$. *The second inequality is an equality if and only if $V^{H_1} \cap V^{H_2} \subset V^{T^n}$.*

(b) $(\widetilde{G} \circ \widetilde{F})^H$ *is in* $\Pi(H)$.

(c) *If $\dim \Gamma/H_i = k_i$ and $\dim \Gamma/H = k_1 + k_2$, let $\widetilde{F}$ and $\widetilde{G}$ be the generators of $\widetilde{\Pi}(H_i)$. Then $[\widetilde{G} \circ \widetilde{F}]_\Gamma = d[\widetilde{F}_H]_\Gamma$, where $\widetilde{F}_H$ generates $\widetilde{\Pi}(H)$ and $d = \beta_{HH_1} \widetilde{\beta}_{HH_2} |H_1^0/H_1| \cdot |H_2^0/H_2| / |H_1^0 \cap H_2^0/H|$, where $\beta_{HH_1} = \prod l_i$ for x_i in $V^H \cap (V^{H_1})^\perp \cap (V^{H_2^0})^\perp, \widetilde{\beta}_{HH_2} = \prod q_j$ for z_j in $W^H \cap (W^{H_2})^\perp \cap (V^{H_1^0})^\perp, H_i^0$ is the maximal isotropy subgroup containing H_i such that $\dim \Gamma/H_i^0 = k_i$. More generally, if $F^{H_1}|_{\partial B_{k_1}} \neq 0$ and $G^{H_2}|_{\partial B_{k_2}} \neq 0$, then $(\widetilde{G} \circ \widetilde{F})^H|_{\partial B_{k_1+k_2}} \neq 0$ and*

$$\deg_{\mathrm{E}}((\widetilde{G} \circ \widetilde{F})^H) = d \deg_{\mathrm{E}}(F^{H_1}) \deg_{\mathrm{E}}(G^{H_2}).$$

Proof. Since H_2 is in $\mathrm{Iso}(V)$, H is the isotropy subgroup for the space generated by V^{H_1} and V^{H_2}. In $V^{H_1} \subset V^H$, there are $\dim \Gamma/H_1$ coordinates x_j with $\Gamma_{x_j} = H_j$ and $H_1^0 = \bigcap H_j$ maximal such that $\dim \Gamma/H_1^0 = \dim \Gamma/H_1$, and similarly for H_2 and H_2^0. Hence $\dim \Gamma/H_i \le \dim \Gamma/H$ and for V^H the maximal number of such variables will be $\dim \Gamma/H_1 + \dim \Gamma/H_2$, and strictly less if and only if one of them is in $V^{H_1} \cap V^{H_2}$.

Note that $\widetilde{G} \circ \widetilde{F} = \{x_i^{l_i q_i}\}$ on $(V^H)^\perp$ and that if $H_1 < H_2$, then $V^{H_2} \subset V^{H_1}$ and for any $K > H_1$, F^K is Γ-deformable to $(1,0)$, in which case $(G \circ F)^{H_1}$ is in $\Pi(H_1) = \Pi(H)$. A similar result holds if $H_2 < H_1$. In general,

$$V = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times V'^{H_1} \cap V'^{H_2} \times V'^{H_1} \cap (V'^{H_2})^\perp \times V'^{H_2} \cap (V'^{H_1})^\perp \times (V'^{H})^\perp$$

with $X = (\lambda_1, \lambda_2, X_0, X_1, X_2, Y)$ and $W = \mathbb{R}^{k_2} \times W_0 \times W_1 \times W_2 \times \overline{W}$, with its elements of the form $W = (\lambda_2, W_0, W_1, W_2, \overline{W})$, where these subspaces have the same meaning as for X. Hence

$$\widetilde{F}(X) = (F_\lambda(\lambda_1, \lambda_2, X_0, X_1), F_0(\lambda_1, \lambda_2, X_0, X_1), F_1(\lambda_1, \lambda_2, X_0, X_1), X_2^l, Y^l)$$

with $F_1(\lambda_1, \lambda_2, X_0, 0) = 0$ and $(F_\lambda, F_0)(\lambda_1, \lambda_2, X_0, 0) \neq 0$ since the isotropy of $V^{H_1} \cap V^{H_2}$ is strictly larger than H_1 and F^{H_1} is in $\Pi(H_1)$. Here (X_2^l, Y^l) stands for $\{x_i^{l_i}\}$ and one should normalize $\widetilde{F}$ as $\widetilde{F}/\|\widetilde{F}\|$. Similarly one has $\widetilde{G} \circ \widetilde{F}(X) = (G_0(F_\lambda, F_0, X_2^l), F_1^p(\lambda_1, \lambda_2, X_0, X_l), G_2(F_\lambda, F_0, X_2^l), Y^{pq})$, where $G_2(\lambda_2, W_0, 0) = 0$ and $G_0(\lambda_0, W_0, 0) \neq 0$, for G^{H_2} in $\Pi(H_2)$. Thus, $(\widetilde{G} \circ F)^{H_1}$ with $X_2 = Y = 0$ has $G_2(\lambda_2, W_0, 0)$ deformable to $(1,0)$. Similarly $(G \circ \widetilde{F})^{H_2}$ with $X_1 = Y = 0$ has $F_1 = 0$ and (F_0, F_λ) independent of Z_2 and Γ-deformable to $(1,0)$. Hence $(G \circ F)^{H_2}$ is Γ-deformable to $(G_0(1,0,X_2^l), 0, G_2(1,0,X_2^l), 0)$ and then to $(1,0)$. Thus, if H is a strict subgroup of H_i, $i = 1,2$, then $G \circ F$ is trivial on $V^{H_1} \cup V^{H_2}$.

Let now $K < H$ and decompose V^K as above. One has a non-zero Γ-extension of $\widetilde{G} \circ \widetilde{F}$ on $V^K \cap (V^{H_1} \cup V^{H_2})$, i.e. for $X_2 = 0$ or $X_1 = 0$. If $V^K \cap V^{H_1}$ is strictly contained in V^{H_1}, then X_1 has components $x_i = 0$ and the remaining variables, in X_1, have isotropy $\widetilde{H}_1$ containing strictly H_1 (if not, $V^K \cap V^{H_1} = V^{\widetilde{H}_1}$ would be V^{H_1}). Hence, on $V^K \cap V^{H_1}$ one may extend $F^{\widetilde{H}_1} = (F_\lambda, F_0, F_1)$ to a map trivial at the origin and of norm 1. Then for X in the unit ball of V^K one has either $\|X_2\| = 1$ and $(G_0, G_2) \neq 0$ or $\|X_2\| < 1$, in which case, from $\|F^{\widetilde{H}_1}\| = 1$, either $\|F_1\| = 1$ and $\widetilde{G} \circ \widetilde{F} \neq 0$ or $\|F_1\| < 1$ and $\|(F_\lambda, F_0)\| = 1$ with $(G_0, G_2) \neq 0$. Hence, in this case one has a non-zero Γ-extension to V^K. On the other hand, if $V^K \cap V^{H_1} = V^{H_1}$, then $V^K \cap V^{H_2}$ is strictly contained in V^{H_2} and (G_0, G_2) has a non-trivial Γ-extension to $W^K \cap W^{H_2}$. But (F_λ, F_0, F_1) has a Γ-extension to $V^{H_1} = V^K \cap V^{H_1}$ with norm one. If $F_1 \neq 0$, then $(\widetilde{G} \circ \widetilde{F})^K \neq 0$, while if $F_1 = 0$, then (F_λ, F_0) is in $V^K \cap V^{H_2}$ and (G_0, G_2) has the non-trivial extension. This proves (b).

For (c), let $z_j^i, j = 1, \dots, k_i, i = 1, 2$, be the variables in $V_i^{H_0}$. From the hypothesis on the dimensions, one sees that z_j^i are in X_i. Then, from [10, p. 394], one has

$$\widetilde{F} = \Big(\tau + 1 - \prod |x_j^1|^2, \lambda_2, X_0^0, (\lambda_1^1 + i(|z_2^1|^2 - 1))(z_1^1)^{l_1}, \dots, (\lambda_{k_1}^1 + i\tau)(z_{k_1}^1)^{l_{k_1}},$$
$$\{(P_j^1(X_0, X_1) - 1)(x_j^1)^{l_j}\}_{x_j^1 \neq z_i^1}, (Q_j(y_j^1) - 1)y_j^1, X_2^l, Y^l\Big)$$

with $\tau = t - 1/2$ and a similar expression for $\widetilde{G}$. Then

$$\begin{aligned}\widetilde{G} \circ \widetilde{F} = \Big(&\tau + 2 - \prod |x_j^1|^2 - \prod |\widetilde{x}_j^2|^2, X_0^0,\\ &(\lambda_1^1 + i(|z_2^q|^2 - 1))^{q_1}(z_1^1)^{q_1 l_1}, \dots, (\lambda_{k_1}^1 + i\tau)^{q_{k_1}}(z_{k_1}^1)^{q_{k_1} l_{k_1}},\\ &(\lambda_1^2 + i(|z_1^2| - 1))^{l_2}(z_1^2)^{q_2 l_2}, \dots,\\ &(\lambda_{k_2}^2 + i\Big(\tau + 1 - \prod |x_j^1|^2\Big)^{q_{k_2}} z^{2q_{k_2} l_{k_2}}, \dots\Big).\end{aligned}$$

Thus, if $z_j^1 = 0$, for some j, one has $\tau + 1 - \prod |x_j^1|^2 = \tau + 1 > 1/2$ and, on a zero, one would need $z_{k_2}^2 = 0$ and $\prod |\widetilde{x}_j^2| = 0$. Hence $\widetilde{G} \circ \widetilde{F}$ is non-zero. If $z_j^2 = 0$, then $\prod |\widetilde{x}_j^2| = 0$ (this is where the compositions of terms in X_0 are) and a zero of the map will give $|x_j^1| = 1$ (the terms P_j are designed this way), that is, the first component is non-zero. Thus, $(\widetilde{G} \circ \widetilde{F})^H$ is non-zero on $\partial B_{k_1+k_2} = \partial(B^H \cap \{\operatorname{Arg} z_j^i = 0\})$. In general, if F^{H_1} is non-zero on ∂B_{k_1}, then (F_λ, F_0, F_1) is normalized to 1 on this set and either $F_1 \neq 0$ or $\|(F_\lambda, F_0)\| = 1$ and (G_0, G_2) is non-zero on it. If G^{H_2} is non-zero on ∂B_{k_2}, since the z_j^2 are coordinates of X_2, one has $(G_0, G_2)(\lambda_2, W_0, W_2) \neq 0$ for such W_2 and in particular for $W_2 = X_2^l$. Note that $\widetilde{F}^H$ and $\widetilde{G}^H$ have zeros on $\partial B_{k_1+k_2} \cap V^H$ and $\partial B_{k_1+k_2} \cap W^H$, since $(z_j^2)^{l_j}$ and $(z_j^1)^{l_j p_j}$ appear as suspensions. However, for the ordinary degree, one may perturb these terms to $(z_j^i)^a - \varepsilon$ and have non-zero maps on $\partial B_{k_1+k_2}$. From the composition formula,

$$\begin{aligned}\deg((\widetilde{G} \circ \widetilde{F})^H; B_{k_1+k_2}) &= \deg_{\mathrm{E}}((\widetilde{G} \circ \widetilde{F})^H)|H_1^0 \cap H_2^0/H|\\ &= \deg(\widetilde{F}_\varepsilon^H; B_{k_1+k_2}) \deg(\widetilde{G}_\varepsilon^H; B_{k_1+k_2} \cap W^H)\end{aligned}$$

for the perturbed map. Now,

$$\deg(\widetilde{F}_\varepsilon^H; B_{k_1+k_2}) = \deg(\widetilde{F}^{H_1}; B_{k_1}) \prod l_j = |H_1^0/H_1| \deg_{\mathrm{E}}(\widetilde{F}^{H_1})\beta_{HH_1},$$
$$\deg(\widetilde{G}_\varepsilon^H; B_{k_1+k_2} \cap W^H) = \deg(\widetilde{G}^{H_2}; B_{k_2}) \prod q_j = |H_2^0/H| \deg_{\mathrm{E}}(\widetilde{G}^{H_2})\beta_{HH_2},$$

since the suspension of the form $z_j^a - \varepsilon$ for $\operatorname{Arg} z_j = 0$ has degree 1. Note that $H_0^1 \cap H_0^2/H \cong (H_0^1 \cap H_0^2/H_1 \cap H_0^2)(H_1 \cap H_0^2/H)$. The order of the first term divides $|H_0^1/H_1|$, from (A), and the order of the second divides $|H_0^2/H_2|$, i.e. d is an integer. $\square$

In general, if $[F]_\Gamma = \sum d_i[\widetilde{F}_i]_\Gamma + [\widetilde{F}]_\Gamma$ and $[G]_\Gamma = \sum e_j[G_j]_\Gamma + [\widetilde{G}]_\Gamma$ with $\dim \Gamma/H_i = k_1$, $\dim \Gamma/H_j = k_2$ and $[\widetilde{F}]_\Gamma$ in Π_{k_1-1}, $[\widetilde{G}]_\Gamma$ in Π_{k_2-1}, then $[G \circ F]_\Gamma =$

$\sum d_i e_j [\widetilde{F}_i \circ \widetilde{G}_j]_\Gamma + [\widetilde{K}]_\Gamma$ with $[\widetilde{K}]_\Gamma$ in $\Pi_{k_1+k_2-1}$ and $[\widetilde{F}_i \circ \widetilde{G}_j]_\Gamma = d_{ij}[\widetilde{K}_{ij}]_\Gamma$, where $d_{ij} = \beta_{HH_i}\widetilde{\beta}_{HH_j}|H_i^0/H_i| \cdot |H_j^0/H_j|/|H_i^0 \cap H_j^0/H|$ with $H = H_i \cap H_j$, provided $\mathrm{Iso}(V) = \mathrm{Iso}(W)$ and (H) holds for (V, W) and (W, U), for instance if $V = \mathbb{R}^{k_1} \times W$ and $W = \mathbb{R}^{k_2} \times U$.

Proposition 6.6. *Under the above hypotheses one has*

$$[G \circ F]_\Gamma = \sum f_k [\widetilde{K}_k]_\Gamma + [\widetilde{K}]_\Gamma \quad \text{with } f_k = \sum d_i e_j d_{ij},$$

where the second sum is over all (i, j) such that $H_i \cap H_j = H_k$.

Remark 6.3. One may prove the same result, either for maps which are such that $F^{H_i^0}$ is non-zero on ∂B_{k_1}, and $G^{H_j^0}$ is non-zero on ∂B_{k_2} (hence as above $(G \circ F)^K$ is non-zero on $\partial B_{k_1+k_2}$ for $K < H_i^0 \cap H_j^0$ and $\dim \Gamma/K = k_1 + k_2$), or for the generators F_j and G_j, by using Theorem 2.1: in this case, one has

$$\deg(\widetilde{G} \circ \widetilde{F})^K; B_{k_1+k_2}) = \sum_{K<H<H_i^0 \cap H_j^0} \widehat{\beta}_{KH} f_H |H_i^0 \cap H_j^0/H|,$$

where $\widehat{\beta}_{KH}$ corresponds to $\prod l_k q_k$ for the variables in $V^K \cap (V^H)^\perp$ and $K = H_i \cap H_j$ is such that $\dim \Gamma/K = k_1 + k_2$. This degree is $\beta_{KH_i} \deg(F^{H_i}; B_{k_1})\widetilde{\beta}_{KH_j} \times \deg(G^{H_j}; B_{k_2})$. From Theorem 2.1 and the fact that $\beta_{KH_i}\beta_{H_iH_l} = \beta_{KH_l}$ for $K < H_i < H_l$, this degree is

$$\begin{aligned}
&\Big(\sum_{H_i<H_l<H_i^0} \beta_{KH_l} d_{H_l} |H_i^0/H_k|\Big)\Big(\sum_{H_j<H_k<H_j^0} \widetilde{\beta}_{KH_k} e_{H_k} |H_j^0/H_k|\Big) \\
&\qquad = \Big(\sum_{K<H<H_i^0 \cap H_j^0} \sum_{H_l \cap H_k = H} \beta_{KH_l} \widetilde{\beta}_{KH_k} d_{H_l} e_{H_k} |H_i^0/H_l| \cdot |H_j^0/H_k|\Big).
\end{aligned}$$

By varying all possible K's, this will yield $f_H = \sum_{H_l \cap H_k = H} d_{H_l} e_{H_k} d_{H_l H_k}$, where $d_{H_l H_k}$ is defined in Proposition 6.5, after one recalls that $\widehat{\beta}_{KH} = \beta_{KH}\widetilde{\beta}_{KH}$ and $\beta_{KH_l}/\beta_{KH} = \beta_{HH_l}$ for $K < H < H_l$.

Our final result will concern the case where $k_1 = 1$, $k_2 = 0$, $V = \mathbb{R} \times W$, $W = U$. The case $\dim \Gamma/H_1 = \dim \Gamma/H = 1$, $\dim \Gamma/H_2 = 0$ was treated in the preceding proposition. There remains only the case $\dim \Gamma/H = \dim \Gamma/H_i = 0$, where $\Pi(H_1)$ is generated by η_j^1 and $\widetilde{\eta}_1$ with relations $p_j(\eta_j^1 + \widetilde{\eta}_1) = 0$, $2\widetilde{\eta}_1 = 0$, $\Pi(H_2)$ is generated by η_2 and $\Pi(H)$ by η_j and $\widetilde{\eta}$ with relations $r_j(\eta_j + \widetilde{\eta}) = 0$, $2\widetilde{\eta} = 0$. Taking the notations of Proposition 6.4, one has the following:

Proposition 6.7. *Under the above hypotheses one has*

$$\begin{aligned}
&[\eta_2 \circ \eta_j^1]_\Gamma = \alpha_j (r_j/p_j)(|\Gamma/H_2|/|H_1/H_1 \cap H_2|)[\eta_j]_\Gamma + \widetilde{d}_j [\widetilde{\eta}]_\Gamma, \\
&[\eta_2 \circ \widetilde{\eta}_1]_\Gamma = (|\Gamma/H_2|/|H_1/H|)[\widetilde{\eta}]_\Gamma,
\end{aligned}$$

where α_j and $\widetilde{d}_j$ are as in Proposition 6.4.

Proof. Take X_1, X_2 and X as auxiliary spaces, as in Proposition 6.4. Then, on $V \times (X_1 \times X_2 \times X)^2$, one has

$$\eta_j^1 = \Big(1 - \prod |Z_i|, X_0, x_i, (Z_i^{p_i} - \varepsilon_i)Z_i, Z_i', \lambda Z_j, Z_j', Y_i, Y_i', X_i, X_i'\Big),$$
$$\eta_2 = \Big(2\tau + 2 - 2\prod |Y_i|, X_0, x_i, Z_i, Z_i', (Y_i^{q_i} - \varepsilon_i)Y_i, Y_i', X_i, X_i'\Big).$$

In order to comply with the normalization of η_j^1 on ∂B, we shall take $\tau = t - 1/2$ in $[-1/2, 1/2]$ and $|X| \leq 3/2$. Then it is easy to see that

$$\eta_2 \circ \eta_j^1 = \Big(4 - 2\prod |Z_i| - 2\prod |Y_i|, X_0, x_i, (Z_i^{p_i} - \varepsilon_i)Z_i, Z_i', \lambda Z_j, Z_j', (Y_i^{q_i} - \varepsilon_i)Y_i, Y_i', X_i, X_i'\Big).$$

On the fundamental cell $\mathcal{C}_1$, already used in Proposition 6.4, i.e. with Z_j in the last place, one has $\eta_2 \circ \eta_j^1 = d_j \xi_j' + d' \widetilde{\eta}'$, where d_j is computed from $\deg(\eta_2 \circ \eta_j^1; \operatorname{Arg} X_j = 0)$, which can be calculated either directly or by using the formula for the ordinary composition. That is, $d_j = \prod_{i \neq j} (q_i / \widetilde{q}_i)$. Since we have already proved that $\xi_j' = \alpha_j q_j \eta_j + d\widetilde{\eta}$, we have proved the first formula. The argument for $\eta_2 \circ \widetilde{\eta}_1$ follows exactly the same lines and is left to the reader. □

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