Topological **M**ethods in **N**onlinear **A**nalysis
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REMARKS ON TOPOLOGICAL SOLITONS

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. A generalization of the sine-Gordon equation

In this paper we deal with soliton solutions of Lorentz invariant equations. Roughly speaking, a soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect soliton solutions have a particle-like behaviour and they occur in many questions of mathematical physics, such as classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics (see [7], [8], [11], [13]).

In general the solitonic behaviour arises when one of the following circumstances occurs:

- existence of infinitely many first integrals of motion (e.g. KdV equation);
- existence of topological constraints which characterize the solutions.

In this paper we deal with the second case, namely we shall study topological solitons which are solutions of Lorentz invariant equations in more than one space dimensions. A classical interesting one-dimensional model is the sine-Gordon equation

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$$u_{tt} - c^2 u_{xx} + \sin u = 0, \tag{1.1}$$

$$\lim_{x\to-\infty} u(x,t) = 0, \tag{1.2}$$

$$\lim_{x\to\infty} u(x,t) = 2k\pi, \tag{1.3}$$

c being the light velocity and $k \in \mathbb{Z}$. The asymptotic conditions allow to have solutions with finite energy

$$E(u) = \int_{\mathbb{R}} \left[\tfrac{1}{2}(u_t^2 + c^2 u_x^2) + (1 - \cos u)\right] dx.$$

Any static solution $u(x,t) = \varphi(x)$ of (1.1) solves the elliptic equation

$$-c^2 \ddot{\varphi} + \sin\varphi = 0, \tag{1.4}$$

$$\lim_{x\to-\infty} \varphi(x) = 0, \tag{1.5}$$

$$\lim_{x\to\infty} \varphi(x) = 2k\pi. \tag{1.6}$$

These static solutions give rise to the travelling solutions with velocity $v \in \mathbb{R}$, $|v| < c$,

$$u(x,t) = \varphi\left(\frac{x - vt}{\sqrt{1 - |v/c|^2}}\right)$$

of (1.1). The length contraction factor $\sqrt{1 - |v/c|^2}$ is related to the Lorentz invariance of the equation itself.

Observe that the solutions of (1.4) are the critical points of the energy functional

$$E(\varphi) = \int_{\mathbb{R}} \left(\frac{c^2}{2}(\dot{\varphi}(x))^2 + V(\varphi)\right) dx$$

with $V(\varphi) = 1 - \cos\varphi$.

The function space Λ on which E is defined can be divided into infinitely many connected components according to the asymptotic conditions (1.5), (1.6):

$$\Lambda = \bigcup_{k\in\mathbb{Z}} \Lambda_k, \quad \Lambda_k = \{\varphi \mid \lim_{x\to-\infty} \varphi(x) = 0,\ \lim_{x\to\infty} \varphi(x) = 2k\pi\}.$$

The existence of local minima of E in the connected components $\Lambda_{\pm 1}$ can be proved. These solutions exhibit a solitonic behaviour.

Now consider the analogous case of a scalar equation with three space dimensions:

$$u_{tt} - c^2 \Delta u + V'(u) = 0.$$

Then the energy functional for the static solutions is

$$E(\varphi) = \int_{\mathbb{R}^3} \left(\frac{c^2}{2}|\nabla\varphi|^2 + V(\varphi)\right) dx. \tag{1.7}$$

If V is bounded from below, it is not difficult to show, by a rescaling argument, that any φ minimizing (1.7) is necessarily trivial, i.e. it takes a constant value which is a minimum point of V (Derrick's Theorem, [7]).

On the other hand, if we consider a nonpositive potential, we are forced to seek saddle points, instead of minima, and for these static solutions we have lack of stability. As an example we recall that, if we take

$$V(\xi) = \tfrac{1}{2}\xi^2 - \tfrac{1}{4}\xi^4,$$

then the critical points of the energy functional

$$E(\varphi) = \int_{\mathbb{R}^3} \left(\frac{c^2}{2} |\nabla \varphi|^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4 \right) dx$$

have been found in [6] and [10] and for more general potentials in [5], [12]; but in [1] and [4] it has been proved that these static solutions are not stable.

So we are forced to study systems of nonlinear wave equations with a suitable correction. In the following we derive, by means of some heuristic arguments, one model equation.

We are interested in maps

$$u : \mathbb{R}^{3+1} \to \mathbb{R}^n, \quad u = (u^1, \ldots, u^n).$$

We refer to the target space $\mathbb{R}^n$ as an internal parameter space. Since we require Lorentz invariance, we shall consider Lagrangian densities of the form $\mathcal{L} = \mathcal{L}(u, \varrho)$ where $\varrho = (\varrho^1, \ldots, \varrho^n)$ and

$$\varrho^j = c^2 |\nabla u^j|^2 - (u_t^j)^2.$$

We shall consider

$$\mathcal{L}(u, \varrho) = -\tfrac{1}{2}\alpha(\varrho) - V(u), \tag{1.8}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ and the potential function V is defined in an open set $\Omega \subset \mathbb{R}^n$.

The action functional related to (1.8) is

$$\mathcal{S}(u) = \int_{\mathbb{R}^{3+1}} \mathcal{L}(u, \varrho)\, dx\, dt = \int_{\mathbb{R}^{3+1}} \left(-\frac{1}{2}\alpha(\varrho) - V(u) \right) dx\, dt.$$

So the Euler–Lagrange equations are

$$\frac{\partial}{\partial t}\left(\frac{\partial \alpha}{\partial \xi_j}(\varrho) u_t^j \right) - c^2 \operatorname{div}\left(\frac{\partial \alpha}{\partial \xi_j}(\varrho) \nabla u^j \right) + \frac{\partial V}{\partial \xi_j}(u) = 0 \quad (1 \le j \le n). \tag{1.9}$$

When

$$\alpha(\xi_1, \ldots, \xi_n) = \xi_1 + \ldots + \xi_n \tag{1.10}$$

the equations (1.9) reduce to a classical system of n nonlinear wave equations

$$u_{tt}^j - c^2 \Delta u^j + \frac{\partial V}{\partial \xi_j}(u) = 0 \quad (1 \le j \le n).$$

In this paper we consider a simple correction of (1.10), namely

$$\alpha(\xi_1,\dots,\xi_n)=\sum_{j=1}^{n}\left(\xi_j+\frac{\varepsilon}{3}\xi_j^3\right),$$

where $\varepsilon>0$. Then (1.9) can be written

$$\frac{\partial}{\partial t}((1+\varepsilon(\varrho^j)^2)u_t^j)-c^2\operatorname{div}((1+\varepsilon(\varrho^j)^2)\nabla u^j)+\frac{\partial V}{\partial \xi_j}(u)=0\quad(1\le j\le n),$$

or

$$\square u^j+\varepsilon\square_6 u^j+\frac{\partial V}{\partial \xi_j}(u)=0\quad(1\le j\le n)\tag{1.11}$$

where

$$\square_6 u^j=\frac{\partial}{\partial t}[(c^2|\nabla u^j|^2-(u_t^j)^2)^2u_t^j]-c^2\operatorname{div}[(c^2|\nabla u^j|^2-(u_t^j)^2)^2\nabla u^j].$$

So the static solutions φ solve the system of equations

$$-c^2\Delta\varphi^j-\varepsilon c^6\Delta_6\varphi^j+\frac{\partial V}{\partial \xi_j}(\varphi)=0\quad(1\le j\le n)\tag{1.12}$$

where

$$\Delta_6\varphi^j=\operatorname{div}(|\nabla\varphi^j|^4\nabla\varphi^j).$$

Then the energy functional becomes

$$E(\varphi)=\int_{\mathbb{R}^3}\left(\frac{c^2}{2}|\nabla\varphi|^2+\varepsilon\frac{c^6}{6}|\nabla\varphi|^6+V(\varphi)\right)dx.\tag{1.13}$$

As to the potential V and the open set $\Omega=\mathbb{R}^n\setminus\Sigma$ where it is defined, we make the following assumptions:

- Ω is connected and $1=\min_{\xi\in\Sigma}|\xi|$;
- $V\in C^2(\Omega,\mathbb{R})$;
- $V(\xi)\ge V(0)=0$ for every $\xi\in\Omega$;
- there exist $c,r>0$ such that

$$\operatorname{dist}(\xi,\Sigma)<r\Rightarrow V(\xi)>c/\operatorname{dist}(\xi,\Sigma)^6.\tag{1.14}$$

Of course we can consider our evolution equation (1.11) as a dynamical system. The configuration space for this system is given by

$$\Lambda=\{\varphi\in H\mid\forall x\in\mathbb{R}^3:\varphi(x)\in\Omega\}$$

where H denotes the completion of $C_0^\infty(\mathbb{R}^3,\mathbb{R}^n)$ with respect to the norm

$$\|\nabla\varphi\|_{L^2}+\|\nabla\varphi\|_{L^6}.$$

Since the functions in H are continuous and decay to 0 at infinity, by means of a suitable topological invariant, the algebraic structure of $\pi_3(\Omega)$ is reflected in a decomposition of the configuration space:

$$\Lambda = \bigcup_{\alpha\in\pi_3(\Omega)} \Lambda_\alpha.$$

About the structure of $\pi_3(\Omega)$ we can say that, in many cases (e.g. when it is finitely generated and torsion free) it has the structure $\pi_3(\Omega) = \mathbb{Z}^k$ with $k \in \mathbb{N}$. For example this situation occurs when $\Omega = \mathbb{R}^4 \setminus \{\xi_1, \dots, \xi_k\}$ and it has been studied in [3] in the particular case $k = 1$.

Here we state the existence and, in some cases, multiplicity results for the static solutions of system (1.11).

Theorem 1.1. *Let*

$$A = \{\alpha \in \pi_3(\Omega) \mid E \text{ attains its minimum in } \Lambda_\alpha\}.$$

Then the subgroup of $\pi_3(\Omega)$ *generated by* A *coincides with* $\pi_3(\Omega)$ *itself.*

Since every local minimum of E gives rise to a weak solution of system (1.12), if $\pi_3(\Omega)$ is not trivial, then there exists at least one nontrivial solution of (1.12) in the class

$$\Lambda^* = \bigcup_{\alpha\neq 0} \Lambda_\alpha,$$

i.e. in the class of configurations which are not homotopic to zero.

On the other hand, in the case $\pi_3(\Omega) = \mathbb{Z}^k$, Theorem 1.1 implies that there exist at least k homotopically distinct static solutions of (1.11).

The paper is organized as follows. In Section 2 we give the topological classification of the maps $\varphi \in \Lambda$. More precisely, we introduce a homotopic invariant with suitable "localization" properties; this means, roughly speaking, that it depends on the compact regions where φ is concentrated. Such an invariant allows us to split each configuration $\varphi \in \Lambda$ in two parts which we call "particles" (the regions where the invariant is not trivial) and "radiation".

In Section 3 we give the proof of our results.

In Section 4 we recall the stability properties stated in [3].

2. Topological classification of configurations. Particles and radiation

For the sake of simplicity we consider the function space

$$\mathcal{C} = \{\varphi : \mathbb{R}^3 \to \Omega \text{ continuous} \mid \lim_{|x|\to\infty} \varphi(x) = 0\},$$

which contains our configuration space Λ.

First of all we define a “global homotopic invariant”. To this end we consider a homeomorphism $\pi : S^3 \to \mathbb{R}^3 \cup \{\infty\}$ such that $\pi(*) = \infty$, where $*$ is the base point. Now, for every function $\varphi \in \mathcal{C}$, we can define the continuous map $\varphi \circ \pi : S^3 \to \Omega$. We denote by $\varphi^\#$ the homotopy class of $\varphi \circ \pi$, that is,

$$\varphi^\# = [\varphi \circ \pi] \in \pi_3(\Omega).$$

This invariant is stable under uniform convergence.

Lemma 2.1. *For every $\varphi \in \mathcal{C}$, if $\varphi^\# \neq 0$, then there exist $\overline{x} \in \mathbb{R}^3$, $\overline{\xi} \in \Sigma$, and $\lambda \in]0,1[$ such that*

$$\lambda\varphi(\overline{x}) = \overline{\xi}.$$

It follows that $|\varphi(\overline{x})| > 1$ so we also have

$$\sup_{x \in \mathbb{R}^3} |\varphi(x)| > 1. \tag{2.1}$$

Proof. Arguing by contradiction, assume that for every $x \in \mathbb{R}^3$, $\xi \in \Sigma$, and $\lambda \in]0,1[$, we have $\lambda\varphi(x) \neq \xi$, that is, $\lambda\varphi(x) \in \Omega$. Since this fact is also true for $\lambda = 0$ and $\lambda = 1$, we deduce that φ is homotopic to 0 in Ω, and then $\varphi^\# = 0$.

The second part of the assertion follows from $\min_{\xi \in \Sigma} |\xi| = 1$. □

Definition 2.2. For every $\varphi \in \mathcal{C}$ the *support* of φ is the compact set defined as follows:

$$K(\varphi) = \overline{\{x : |\varphi(x)| > 1\}}.$$

From Lemma 2.1 it follows that

$$\varphi^\# \neq 0 \Rightarrow K(\varphi) \neq \emptyset.$$

We notice that $\varphi^\#$ is localized in $K(\varphi)$ in the following sense. If we have two functions $\varphi, \psi \in \mathcal{C}$ such that $K(\varphi) = K(\psi) = K$ and $\varphi(x) = \psi(x)$ for every $x \in K$, then $\varphi^\# = \psi^\#$; indeed, the functions φ and ψ are homotopic in Ω, so $\varphi \circ \pi$ and $\psi \circ \pi$ are homotopic too.

2.1. Definition of a local invariant. Now we want to “localize” the topological invariant in a more precise fashion, by evaluating the contribution coming from a fixed compact set in $\mathbb{R}^3$; in other words, if $\varphi \in \mathcal{C}$ and $K \subset \mathbb{R}^3$, we would like to define

$$(\varphi_{|K})^\# \in \pi_3(\Omega)$$

with suitable properties (finite additivity and so on). Having in mind, as model case, the topological degree, we notice that this kind of object can be defined only for a class of admissible subsets K.

Fix $\varphi \in \mathcal{C}$, we propose the following class of admissible sets:

$$\mathcal{K}(\varphi) = \{K \subset \mathbb{R}^3 \text{ closed} \mid |\varphi(x)| \leq 1 \text{ on } \partial K\}.$$

This set is obviously not empty; indeed, it contains $K(\varphi)$, each connected component of $K(\varphi)$ and every ball with radius sufficiently large.

Before introducing the local invariant we need some notation; let

$$\Omega_1 = \{\xi \in \Omega \mid |\xi| \le 1\}.$$

LEMMA 2.3. *For every $\xi \in \Omega_1$, there exists $\varrho > 0$ such that $s\eta \in \Omega$ for every $\eta \in B_\varrho(\xi)$ (open ball) and for every $s \in [0,1]$.*

The proof is trivial.

Now we set

$$U = \bigcup_{\xi \in \Omega_1} B_\varrho(\xi).$$

Then U is open and, by the previous lemma, it is a subset of Ω star-shaped with respect to 0.

Fix $\varphi \in \mathcal{C}$ and let $K \in \mathcal{K}(\varphi)$. Since φ is continuous, there exists an open neighbourhood N of ∂K such that $\varphi(x) \in U$ for every $x \in N$. Now consider the open neighbourhood of K

$$\widetilde{N} = K \cup N.$$

By the Urysohn lemma, there exists a continuous function $c : \mathbb{R}^3 \to [0,1]$ such that

$$c(x) = \begin{cases} 1 & \text{on } K, \\ 0 & \text{on } \mathbb{R}^3 \setminus \widetilde{N}. \end{cases}$$

Now we consider $\widetilde{\varphi} : \mathbb{R}^3 \to \mathbb{R}^n$ defined as follows:

$$\widetilde{\varphi}(x) = c(x)\varphi(x).$$

We show that $\widetilde{\varphi}(x) \in \Omega$:

- if $x \in K$, then $\widetilde{\varphi}(x) = \varphi(x) \in \Omega$;
- if $x \notin \widetilde{N}$, then $\widetilde{\varphi}(x) = 0 \in \Omega$;
- if $x \in \widetilde{N} \setminus K \subset N$, since $\varphi(x) \in U$ and $c(x) \in [0,1]$, we have $\widetilde{\varphi}(x) = c(x)\varphi(x) \in \Omega$.

DEFINITION 2.4. We set

$$(\varphi_{|K})^{\#} = \widetilde{\varphi}^{\#}.$$

We have to show that this definition is correct, i.e. it does not depend on $\widetilde{\varphi}$. If we consider analogous N_1, c_1 and $\widetilde{\varphi}_1$, then $\widetilde{\varphi}$ and $\widetilde{\varphi}_1$ are homotopic in Ω. Indeed, we set

$$H(\lambda, x) = \widetilde{\varphi}(x) + \lambda(\widetilde{\varphi}_1(x) - \widetilde{\varphi}(x)) = [c(x) + \lambda(c_1(x) - c(x))]\varphi(x).$$

We prove that $H(\lambda, x)$ takes its values in Ω. First we notice that

$$c(x) + \lambda(c_1(x) - c(x)) \in [0,1],$$

because $c(x), c_1(x) \in [0,1]$. Clearly we have

- if $x \in K$, then $H(\lambda, x) = \varphi(x) \in \Omega$;
- if $x \notin \widetilde{N} \cup \widetilde{N}_1$, then $H(\lambda, x) = 0 \in \Omega$;
- if $x \in (\widetilde{N} \cup \widetilde{N}_1) \setminus K$, then $\varphi(x) \in U$, so $H(\lambda, x) \in \Omega$.

The natural connection between the global and the local homotopic invariant is

$$(\varphi_{|K(\varphi)})^{\#} = \varphi^{\#}, \tag{2.2}$$

where on the left and the right hand sides we are using respectively the local and the global invariant. Indeed, for any choice of $\widetilde{\varphi}$ in the previous definition, since φ and $\widetilde{\varphi}$ coincide on their common support, we have $\widetilde{\varphi}^{\#} = \varphi^{\#}$.

We also notice that for every closed set K such that $K(\varphi) \subset K$, we have $K \in \mathcal{K}(\varphi)$ and $(\varphi_{|K})^{\#} = \varphi^{\#}$.

The local invariant has the finite additivity property. Let $K_1, K_2 \in \mathcal{K}(\varphi)$ with $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$. Then

$$(\varphi_{|K_1 \cup K_2})^{\#} = (\varphi_{|K_1})^{\#} + (\varphi_{|K_2})^{\#}$$

(the sum is meant, of course, in the group $\pi_3(\Omega)$).

2.2. Some suggestive terminology. In this subsection we report on some terminology (quantum numbers, particles, ...) which can be useful to interpret the notions and the model introduced. However, this terminology does not refer to any specific model in field theory.

If $\pi_3(\Omega)$ has the structure $\mathbb{Z}^k$, then, after fixing a set of generators $\{\alpha_1, \ldots, \alpha_k\}$, every configuration $\varphi \in \mathcal{C}$ is characterized by a k-tuple of integers, which we call *quantum numbers* of φ: if $\varphi^{\#} = m_1\alpha_1 + \ldots + m_k\alpha_k$, then we set

$$\nu(\varphi) = (m_1, \ldots, m_k).$$

In our model, for a fixed configuration $\varphi \in \mathcal{C}$, every connected, admissible set $K \in \mathcal{K}(\varphi)$ represents an isolated system. We say that an isolated system $K \in \mathcal{K}(\varphi)$ *contains at least one particle* if $(\varphi_{|K})^{\#} \neq 0$.

In other words, in the configuration φ, a particle is defined by

$$\sigma = \varphi_{|K},$$

where K is a connected component of $K(\varphi)$ such that $(\varphi_{|K})^{\#} \neq 0$. So each particle has its quantum numbers, which are the coefficients of $(\varphi_{|K})^{\#}$ for a fixed system of generators.

The state φ will be said a *radiation field* if $\varphi^{\#} = 0$.

On the other hand, every configuration can be written, in some sense, as the disjoint union of a set of particles and a radiation field. Indeed, we can write

$$\varphi = \Big(\bigvee \sigma_i\Big) \vee \varrho$$

where σ_i and ϱ are the restrictions of φ respectively to the admissible sets K_i and R (with disjoint interiors) such that

$$\mathbb{R}^3 = \Big(\bigcup K_i\Big) \cup R, \quad (\varphi_{|K_i})^{\#} \neq 0, \quad (\varphi_{|R})^{\#} = 0.$$

3. Functional framework. Existence of minima

Let H denote the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^n)$ with respect to the norm

$$\|\varphi\| = \|\nabla\varphi\|_{L^2} + \|\nabla\varphi\|_{L^6} \tag{3.1}$$

where

$$\|\nabla\varphi\|_{L^2} = \bigg(\sum_{j=1}^n \|\nabla\varphi^j\|_{L^2}^2\bigg)^{1/2}, \quad \|\nabla\varphi\|_{L^6} = \bigg(\sum_{j=1}^n \|\nabla\varphi^j\|_{L^6}^6\bigg)^{1/6}.$$

By this choice of H, the energy functional is coercive, namely

$$\lim_{\|\varphi\|\to\infty} E(\varphi) = \infty.$$

Moreover, by the Sobolev inequalities the space H is continuously embedded in $W^{1,6}(\mathbb{R}^3, \mathbb{R}^n)$. From this embedding we get other useful properties we will use several times.

1. There exist two constants $C_0, C_1 > 0$ such that, for every $\varphi \in H$,

$$\|\varphi\|_\infty \leq C_0\|\varphi\|, \tag{3.2}$$

   and

$$|\varphi(x) - \varphi(y)| \leq C_1|x-y|^{1/2}\|\nabla\varphi\|_{L^6}. \tag{3.3}$$

2. For every $\varphi \in H$,

$$\lim_{|x|\to\infty} \varphi(x) = 0. \tag{3.4}$$

3. If $\{\varphi_n\} \subset H$ converges weakly in H to φ, then it converges uniformly on every compact set contained in $\mathbb{R}^3$.

Moreover, as an immediate consequence of the continuous embedding of H in L^∞ (see (3.2)), we have the following property.

PROPOSITION 3.1. *There exists $\Delta^* > 0$ such that, for every $\varphi \in \Lambda$,*

$$\|\varphi\|_\infty \geq 1 \Rightarrow E(\varphi) \geq \Delta^*. \tag{3.5}$$

Now consider the configuration space Λ. It is an open subset of H; in fact, if $\varphi \in \Lambda$, then by (3.4) we have

$$0 < d = \inf_{x\in\mathbb{R}^3} \operatorname{dist}(\varphi(x), \Sigma);$$

then, by using (3.2), we deduce that there exists a small neighbourhood of φ in H contained in Λ. The boundary of Λ is given by

$$\partial\Lambda = \{\varphi \in H \mid \exists \overline{x} \in \mathbb{R}^3 \text{ such that } \varphi(\overline{x}) \in \Sigma\}.$$

We find that Λ has a rich topological structure, more precisely, it reflects the structure of $\pi_3(\Omega)$. The connected components are identified by the global homotopic invariant we have introduced in the last section:

$$\Lambda = \bigcup_{\alpha \in \pi_3(\Omega)} \Lambda_\alpha, \quad \Lambda_\alpha = \{\varphi \in \Lambda \mid \varphi^\# = \alpha\}.$$

Now we can study the second piece of the energy functional,

$$\int_{\mathbb{R}^3} V(\varphi)\, dx.$$

First we are going to study the behaviour of this integral when φ approaches the boundary of Λ.

Lemma 3.2. *Let $\{\varphi_n\} \subset \Lambda$ be bounded in the H norm and weakly converging to $\varphi \in \partial\Lambda$. Then*

$$\int_{\mathbb{R}^3} V(\varphi_n)\, dx \to \infty.$$

Proof. Since $\varphi \in \partial\Lambda$, there exists $\overline{x} \in \mathbb{R}^3$ such that $\varphi(\overline{x}) = \overline{\xi} \in \Sigma$; since V is nonnegative, it is sufficient to show that there exists a small ball centred at $\overline{x}$ such that

$$\int_{B_\varrho(\overline{x})} V(\varphi_n)\ dx \to \infty. \tag{3.6}$$

By the uniform convergence on compact sets we have

$$\lim_{n\to\infty} \varphi_n(\overline{x}) = \overline{\xi}. \tag{3.7}$$

Now we show that there exists $\varrho > 0$ such that, for every $x \in B_\varrho(\overline{x})$ and for n sufficiently large,

$$|\varphi_n(x) - \overline{\xi}| < r, \tag{3.8}$$

where r has been introduced in (1.14).

Since $\{\varphi_n\}$ is bounded in H, in particular $\{\nabla\varphi_n\}$ is bounded in L^6. Using (3.3) and the boundedness of $\{\nabla\varphi_n\}$ in L^6, we have

$$|\varphi_n(x) - \varphi_n(\overline{x})| \le \text{const}\, |x - \overline{x}|^{1/2}, \tag{3.9}$$

for every $x \in \mathbb{R}^3$. Then (3.8) easily follows from (3.7) and (3.9). We also have

$$|\varphi_n(x) - \overline{\xi}| \le \text{const}\, |x - \overline{x}|^{1/2} + o(1). \tag{3.10}$$

Now, using (3.8) and (1.14), for every $x \in B_\varrho(\overline{x})$, we have

$$V(\varphi_n(x)) \geq \frac{c}{|\varphi_n(x) - \overline{\xi}|^6};$$

then, using (3.10), we obtain

$$V(\varphi_n(x)) \geq \frac{c}{\mathrm{const}|x - \overline{x}|^3 + o(1)}.$$

Integrating on $B_\varrho(\overline{x})$, we get (3.6). □

From this lemma we immediately deduce that the sublevels of E are complete and that the following proposition holds.

PROPOSITION 3.3. *Let $\{\varphi_n\} \subset \Lambda$ be weakly converging to φ and such that $E(\varphi_n)$ is bounded. Then $\varphi \in \Lambda$.*

Moreover, it is not difficult to prove the following proposition (see [3]).

PROPOSITION 3.4. *The energy functional E is weakly lower semicontinuous and its minimum points are weak solutions of* (1.12).

The proof of our main result is based on the following proposition, in the spirit of the concentration-compactness principle for unbounded domains (see [2], [9]). We recall that

$$\Lambda^* = \bigcup_{\alpha \neq 0} \Lambda_\alpha.$$

PROPOSITION 3.5. *Let $\{\varphi_n\} \subset \Lambda^*$ be such that*

$$E(\varphi_n) \leq a. \tag{3.11}$$

There exists $l \in \mathbb{N}$ with

$$1 \leq l \leq a/\Delta^* \tag{3.12}$$

(Δ^* has been introduced in Proposition 3.1) *and there exist $\overline{\varphi}_1, \ldots, \overline{\varphi}_l \in \Lambda$, $\{x_n^1\}, \ldots, \{x_n^l\} \subset \mathbb{R}^3$, and $R_1, \ldots, R_l > 0$ such that, up to a subsequence,*

$$\varphi_n(\cdot + x_n^i) \rightharpoonup \overline{\varphi}_i, \tag{3.13}$$

$$\|\overline{\varphi}_i\|_\infty \geq 1, \tag{3.14}$$

$$|x_n^i - x_n^j| \to \infty \quad \textit{for } i \neq j, \tag{3.15}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leq \liminf_{n\to\infty} E(\varphi_n), \tag{3.16}$$

$$\forall x \in \complement\bigg(\bigcup_{i=1}^{l} B_{R_i}(x_n^i)\bigg): \quad |\varphi_n(x)| \leq 1. \tag{3.17}$$

(*Here* $\complement$ *denotes complement in* $\mathbb{R}$.) *Then we also have*

$$\varphi_n^{\#} = \sum_{i=1}^{l} \overline{\varphi}_i^{\#}; \tag{3.18}$$

$$\limsup_{n\to\infty} \bigg\| \varphi_n - \sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \bigg\|_\infty \le 1. \tag{3.19}$$

Remark 1. We notice that, from (3.14), it follows that

$$E(\overline{\varphi}_i) \ge \Delta^*. \tag{3.20}$$

Remark 2. Using the terminology introduced in Subsection 2.2, by Proposition 3.5 we can say that every sequence in Λ^* with bounded energy has (a subsequence with) the following behaviour:

- the global homotopic invariant stabilizes;
- the particles concentrate in a finite set of diverging balls.

Furthermore, it may be of interest to write φ_n in the following form:

$$\varphi_n = \psi_n + \varrho_n \tag{3.21}$$

where

$$\psi_n = \sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \in \Lambda, \tag{3.22}$$

$$\varrho_n = \varphi_n - \sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \in \Lambda. \tag{3.23}$$

Then, by (3.19), ϱ_n is a radiation field. On the other hand, ψ_n is a finite superposition of configurations with diverging supports. Unfortunately, we are not able to prove that every term in the sum (3.22) has nonzero homotopic invariant; in this sense the decomposition (3.21) is not "canonical". We suspect that it is canonical if $\{\varphi_n\}$ is a minimizing sequence.

Proof of Proposition 3.5. The proof is essentially the same as in [3]; for the sake of completeness we repeat it here.

First we introduce some notation. For every $A \subset \mathbb{R}^3$ and every $\varphi \in \Lambda$, we set

$$E_{|A}(\varphi) = \int_A \bigg(\frac{c^2}{2} |\nabla\varphi|^2 + \varepsilon \frac{c^6}{6} |\nabla\varphi|^6 + V(\varphi) \bigg)\, dx.$$

Whenever necessary, we shall tacitly consider a subsequence of $\{\varphi_n\}$.

First of all we arbitrarily choose $\gamma \in \,]0,1[$.

Let $x_n^1 \in \mathbb{R}^3$ be a maximum point for $|\varphi_n|$; by (2.1) we have $|\varphi_n(x_n^1)| > 1$. We set

$$\varphi_n^1 = \varphi_n(\cdot + x_n^1)$$

and we obtain

$$\|\varphi_n^1\|_\infty = |\varphi_n^1(0)| > 1. \tag{3.24}$$

Since $E(\varphi_n^1) = E(\varphi_n)$ and the functional E is coercive, the sequence $\{\varphi_n^1\}$ is bounded in H and we have

$$\varphi_n^1 \rightharpoonup \overline{\varphi}_1 \in H. \tag{3.25}$$

From (3.24) it follows that $\|\overline{\varphi}_1\|_\infty \geq 1$.

Since $\{\varphi_n^1\} \subset \Lambda$ and $E(\varphi_n^1)$ is bounded, by (3.25) and Proposition 3.3, we get $\overline{\varphi}_1 \in \Lambda$.

Since E is weakly lower semicontinuous, we have

$$E(\overline{\varphi}_1) \leq \liminf_{n\to\infty} E(\varphi_n^1) = \liminf_{n\to\infty} E(\varphi_n). \tag{3.26}$$

Now, using (3.4), we consider $R_1 > 0$ such that

$$\forall x \in \complement B_{R_1}(0): \quad |\overline{\varphi}_1(x)| \leq \gamma; \tag{3.27}$$

for simplicity we set $B_n^1 = B_{R_1}(x_n^1)$.

Now we distinguish two cases: either

A1) for n sufficiently large

$$\forall x \in \complement B_n^1: \quad |\varphi_n(x)| \leq 1;$$

or

B1) possibly passing to a subsequence,

$$\exists x \in \complement B_n^1 \quad \text{such that} \quad |\varphi_n(x)| > 1.$$

In the case A1) the first part of Proposition 3.5 is proved with $l = 1$; let us consider the case B1).

Let x_n^2 be a maximum point for $|\varphi_n|$ in $\mathbb{R}^3 \setminus B_n^1$; we have $|\varphi_n(x_n^2)| > 1$. We set

$$\varphi_n^2 = \varphi_n(\cdot + x_n^2)$$

and we obtain

$$\|\varphi_n^2\|_\infty = |\varphi_n^2(0)| > 1.$$

Just as for $\{\varphi_n^1\}$, we have

$$\varphi_n^2 \rightharpoonup \overline{\varphi}_2 \in \Lambda, \tag{3.28}$$

with

$$\|\overline{\varphi}_2\|_\infty \geq 1. \tag{3.29}$$

Now we have to show that

$$|x_n^2 - x_n^1| \to \infty. \tag{3.30}$$

We set $y_n = x_n^2 - x_n^1$ and, arguing by contradiction, we assume that the sequence $\{y_n\}$ is bounded in $\mathbb{R}^3$; then, up to a subsequence, $y_n \to \widetilde{y}$. Since $|y_n| = |x_n^2 - x_n^1| \geq R_1$, we have $|\widetilde{y}| \geq R_1$; then, using (3.27),

$$|\overline{\varphi}_1(\widetilde{y})| \leq \gamma < 1. \tag{3.31}$$

On the other hand, we have

$$1 \leq |\varphi_n(x_n^2)| = |\varphi_n(y_n + x_n^1)| = |\varphi_n^1(y_n)|;$$

then, by (3.31),

$$\begin{aligned} 0 < 1 - |\overline{\varphi}_1(\widetilde{y})| &\leq |\varphi_n^1(y_n)| - |\overline{\varphi}_1(\widetilde{y})| \leq |\varphi_n^1(y_n) - \overline{\varphi}_1(\widetilde{y})| \\ &\leq |\varphi_n^1(y_n) - \overline{\varphi}_1(y_n)| + |\overline{\varphi}_1(y_n) - \overline{\varphi}_1(\widetilde{y})| \\ &\leq \Big(\sup_{|y-\widetilde{y}|\leq 1} |\varphi_n^1(y) - \overline{\varphi}_1(y)|\Big) + |\overline{\varphi}_1(y_n) - \overline{\varphi}_1(\widetilde{y})|. \end{aligned}$$

Taking the limit as $n \to \infty$ we get a contradiction.

Now we show that

$$E(\overline{\varphi}_1) + E(\overline{\varphi}_2) \leq E(\varphi_n). \tag{3.32}$$

For a fixed $\eta > 0$, there exists $\varrho > 0$ such that

$$E_{|\complement B_\varrho(0)}(\overline{\varphi}_1) < \eta/2 \quad \text{and} \quad E_{|\complement B_\varrho(0)}(\overline{\varphi}_2) < \eta/2.$$

From (3.30) it follows that the spheres $B_\varrho(x_n^1)$ and $B_\varrho(x_n^2)$ are disjoint for n sufficiently large. Then we get

$$\begin{aligned} \liminf_{n\to\infty} E(\varphi_n) &\geq \liminf_{n\to\infty} (E_{|B_\varrho(x_n^1)}(\varphi_n) + E_{|B_\varrho(x_n^2)}(\varphi_n)) \\ &\geq \liminf_{n\to\infty} E_{|B_\varrho(x_n^1)}(\varphi_n) + \liminf_{n\to\infty} E_{|B_\varrho(x_n^2)}(\varphi_n) \\ &= \liminf_{n\to\infty} E_{|B_\varrho(0)}(\varphi_n^1) + \liminf_{n\to\infty} E_{|B_\varrho(0)}(\varphi_n^2) \\ &\geq E_{|B_\varrho(0)}(\overline{\varphi}_1) + E_{|B_\varrho(0)}(\overline{\varphi}_2) > E(\overline{\varphi}_1) + E(\overline{\varphi}_2) - \eta. \end{aligned}$$

From the arbitrariness of η, we get (3.32).

Finally, just as for $\overline{\varphi}_1$, from (3.4) we get $R_2 > 0$ such that

$$\forall x \in \complement B_{R_2}(0) : \quad |\overline{\varphi}_2(x)| \leq \gamma$$

and we set

$$B_n^2 = B_{R_2}(x_n^2).$$

Also in this second step we have an alternative: either

A2) for n sufficiently large,

$$\forall x \in \complement(B_n^1 \cup B_n^2) : \quad |\varphi_n(x)| \leq 1;$$

or

B2) up to a subsequence,

$$\exists x \in \complement(B_n^1 \cup B_n^2) \quad \text{such that} \quad |\varphi_n(x)| > 1.$$

If the case A2) holds true, the first part of Proposition 3.5 is proved with $l = 2$; in the case B2) we consider a maximum point of $|\varphi_n|$ in $\complement(B_n^1 \cup B_n^2)$ and we repeat the same argument used in the case B1).

This alternative process terminates in a finite number of steps. Indeed, using (3.20), (3.16) and (3.11), we get (3.12).

Now we prove (3.18). We consider n sufficiently large so that (3.17) holds and

$$B_n^i \cap B_n^j = \emptyset \quad \text{for } i \neq j. \tag{3.33}$$

Then we have, by the additive property of the local homotopic invariant,

$$\varphi_n^{\#} = (\varphi_{n|\bigcup_{i=1}^{l} B_n^i})^{\#} = \sum_{i=1}^{l} (\varphi_{n|B_n^i})^{\#} = \sum_{i=1}^{l} (\varphi^i_{n|B_{R_i}(0)})^{\#}. \tag{3.34}$$

On the other hand, for every $i \in \{1, \ldots, l\}$, since $\{\varphi_n^i\}$ converges uniformly to $\overline{\varphi}_i$ on $B_{R_i}(0)$ and

$$\forall x \in \complement B_{R_i}(0) : \quad |\overline{\varphi}_i(x)| \leq \gamma < 1,$$

we obtain, for n large enough,

$$(\varphi^i_{n|B_{R_i}(0)})^{\#} = (\overline{\varphi}^i_{|B_{R_i}(0)})^{\#} = \overline{\varphi}_i^{\#}.$$

Then, substituting in (3.34), we obtain (3.18).

Finally, in order to prove (3.19), we assume that, for every $i \in \{1, \ldots, l\}$,

$$\forall x \in B_n^i : \quad |\varphi_n(x) - \overline{\varphi}_i(x - x_n^i)| < \gamma. \tag{3.35}$$

We shall prove that, for n large enough,

$$\forall x \in \mathbb{R}^3 : \quad \left| \varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i) \right| < 1 + l\gamma. \tag{3.36}$$

Indeed, if $x \in \bigcup_{i=1}^{l} B_n^i$, then, by (3.33), there exists a unique index $j \in \{1, \ldots, l\}$ such that $x \in B_n^j$. Then

$$\begin{aligned} \left| \varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i) \right| &\leq |\varphi_n(x) - \overline{\varphi}_j(x - x_n^j)| + \sum_{i \neq j} |\overline{\varphi}_i(x - x_n^i)| \\ &< \gamma + (l-1)\gamma = l\gamma < 1 + l\gamma. \end{aligned} \tag{3.37}$$

On the other hand, if $x \notin \bigcup_{i=1}^{l} B_n^i$, then, by (3.17),

$$\left| \varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i) \right| \leq |\varphi_n(x)| + \sum_{i=1}^{l} |\overline{\varphi}_i(x - x_n^i)| \leq 1 + l\gamma.$$

Now fix $\eta > 1$; choosing γ sufficiently small we have

$$1 + l\gamma < \eta \tag{3.38}$$

(taking into account (3.12), this kind of choice can be made a priori in the proof). Substituting (3.38) in (3.36), we get

$$\forall x \in \mathbb{R}^3 : \quad \left|\varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i)\right| < \eta,$$

and, by the arbitrariness of $\eta > 1$, we obtain (3.19). □

Finally, we can give the proof of our main theorem. For every $\alpha \in \pi_3(\Omega)$, we set $E_\alpha = \inf E(\Lambda_\alpha)$.

Theorem 3.6. *The group $\pi_3(\Omega)$ is generated by*

$$A = \{\alpha \in \pi_3(\Omega) \mid E_\alpha \ \textit{is attained in}\ \Lambda_\alpha\}.$$

Proof. Denote by G the subgroup of $\pi_3(\Omega)$ generated by A, and, arguing by contradiction, assume that $B = \pi_3(\Omega) \setminus G \neq \emptyset$. Then we set

$$\Lambda_B = \bigcup_{\beta \in B} \Lambda_\beta, \quad E_B = \inf E(\Lambda_B).$$

Let $\{\varphi_n\} \subset \Lambda_B$ be such that $E(\varphi_n) \to E_B$. Since $0 \in A$, we have $\Lambda_B \subset \Lambda^*$, so we can apply Proposition 3.5. There exist $l \in \mathbb{N}$ and $\overline{\varphi}_1, \ldots, \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$E(\overline{\varphi}_i) \geq \Delta^* > 0, \tag{3.39}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leq \liminf_{n \to \infty} E(\varphi_n) = E_B, \tag{3.40}$$

$$\varphi_n^{\#} = \sum_{i=1}^{l} \overline{\varphi}_i^{\#}. \tag{3.41}$$

For simplicity we set

$$\sum_{i=1}^{l} \overline{\varphi}_i^{\#} = \sigma; \tag{3.42}$$

substituting in (3.41), we get

$$\varphi_n^{\#} = \sigma; \tag{3.43}$$

then, since $\{\varphi_n\} \subset \Lambda_B$, it follows that

$$\sigma \in B. \tag{3.44}$$

Then, since $\Lambda_\sigma \subset \Lambda_B$ and using (3.43), we get $E_B \le E_\sigma \le E(\varphi_n)$. So we conclude that $\{\varphi_n\}$ is a minimizing sequence in Λ_σ.

Now we study two cases.

If $l = 1$, then from (3.42) we get $\overline{\varphi}_1^{\#} = \sigma$, which implies $E(\overline{\varphi}_1) \ge E_\sigma$. On the other hand, by (3.40),

$$E_\sigma = \liminf_{n\to\infty} E(\varphi_n) \ge E(\overline{\varphi}_1).$$

So we get $E_\sigma = E(\overline{\varphi}_1)$. In this way we obtain $\sigma \in A$, which contradicts (3.44).

If $l \ge 2$, we get again a contradiction. First we notice that, by (3.42) and (3.44), there exists at least one index (for simplicity $i = 1$) such that $\overline{\varphi}_1^{\#} \in B$. Then, using again (3.40) and (3.39), we conclude

$$E_B \ge E(\overline{\varphi}_1) + \sum_{i=2}^{l} E(\overline{\varphi}_i) \ge E_B + (l-1)\Delta^* > E_B. \qquad \square$$

Corollary 3.7. *There exists* $\overline{\varphi} \in \Lambda^*$ *such that* $E(\overline{\varphi}) = \inf E(\Lambda^*)$.

Remark 3. We have already noticed that Theorem 3.6 is, in a wide class of cases, a multiplicity result, e.g. when $\pi_3(\Omega)$ is isomorphic to $\mathbb{Z}^k$ (see Section 1). Now we remark that the energy functional E is invariant under the action

$$\varphi(x) \mapsto \varphi(Ax + b)$$

where $\det A = \pm 1$ and $b \in \mathbb{R}^3$. Then for every nontrivial solution of (1.12) there exists a manifold of solutions.

4. Remarks on the evolution problem

By Corollary 3.7 we get at least one solution of the Cauchy problem

$$\begin{cases} \text{equations (1.11)}, \\ u(\cdot, 0) = \overline{\varphi}, \\ u_t(\cdot, 0) = 0. \end{cases}$$

In this section we first recall some stability properties of this solution which allow us to call it a soliton (for the proofs see [3]). We assume that the Cauchy problem

$$\begin{cases} \text{equations (1.11)}, \\ u(\cdot, 0) = w_0, \\ u_t(\cdot, 0) = w_1, \end{cases} \tag{4.1}$$

is well posed; we conjecture that this is the case under mild assumptions on V.

Here we confine ourselves to a simple case, namely the case where 0 is a nondegenerate minimum for the potential V. Then we can assume as "basic" the function space

$$W = H \cap L^2(\mathbb{R}^3, \mathbb{R}^n),$$

equipped with the norm

$$\|\varphi\|_W = \|\varphi\| + \|\varphi\|_{L^2}.$$

Now, under the nondegeneracy assumption, the proof we have given in H works also in W. So we can obtain $\overline{\varphi}$ as a minimum of the energy functional in

$$M = \Lambda \cap L^2(\mathbb{R}^3, \mathbb{R}^n).$$

Now the first stability property of $\overline{\varphi}$ says, roughly speaking, that if we perturb the initial data $(\overline{\varphi}, 0)$, we obtain a solution of (1.11) whose support cannot split in diverging pieces.

Theorem 4.1. *If (w_0, w_1) is a sufficiently small perturbation of $(\overline{\varphi}, 0)$, then the diameter of the support of $u(\cdot, t)$ is uniformly bounded, $u(x, t)$ being the solution of the perturbed Cauchy problem* (4.1)*.*

Now we give a second stability property of $\overline{\varphi}$, which is concerned with concentration of energy. First we need a definition.

Definition 4.2. If $K(\varphi)$ is not empty, then the *barycentre* of φ is the barycentre of $K(\varphi)$, that is,

$$\beta(\varphi) = \frac{1}{\mu(K(\varphi))} \int_{K(\varphi)} x \, dx.$$

The second stability property says that if we perturb the initial data $(\overline{\varphi}, 0)$, then we get a solution whose energy is localized in a ball, of fixed radius, centred at the barycentre of $u(\cdot, t)$.

Theorem 4.3. *For every $\eta > 0$, if (w_0, w_1) is a sufficiently small perturbation of $(\overline{\varphi}, 0)$, then there exists $R > 0$ such that, for every $t \in \mathbb{R}$,*

$$\int_{\complement B_R(\beta(\varphi_t))} \left(\frac{c^2}{2} |\nabla \varphi_t|^2 + \varepsilon \frac{c^6}{6} |\nabla \varphi_t|^6 + V(\varphi_t) \right) dx < \eta, \tag{4.2}$$

where $u(x, t)$ is the solution of the perturbed Cauchy problem (4.1) *and $\varphi_t = u(\cdot, t)$.*

Lastly, we want to remark that the topological nature of the invariant introduced in Section 2 permits us to state a conservation law for the "quantum numbers". More precisely, for any solution $u(x, t)$ of (1.11), the global homotopic invariant is constant in t. That is, if we set $\varphi_t = u(\cdot, t)$, then $\varphi_t^{\#} = \varphi_0^{\#}$ for every $t \in \mathbb{R}$. This fact immediately follows by observing that the map $t \mapsto \varphi_t$ is continuous, so it can be regarded as a homotopy in Λ.

Taking into account the local nature of the topological invariant, the above conservation law can also be stated in a suitable local framework.

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