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FIXED POINTS, NASH GAMES AND THEIR ORGANIZATIONS

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The concepts of (S,σ)-invariance and (S,σ,R,M)-invariance are introduced and are used to prove two existence theorems of equilibrium in the sense of Berge [2] and Nash [1, 2] using fixed point arguments. Radjef's results [8] have been extended. Conditions under which these equilibria are Nash are also shown.

Assuming that each player's strategy set is a subset of a reflexive Banach space and that the strategies can be partitioned in such a way that the argmax of each player's objective over an element of the considered partition is unique and satisfies one of the invariance properties, equilibria exist. Similar results are obtained for games with an infinite number of players.

I. Introduction

In the theory of games one is concerned with concepts of interactions between several individuals and/or several groups. From the earliest development of the theory the cooperative and noncooperative types of interactions have been considered separately. Different concepts of solution of a game exist in these two branches of game theory.

Here we will be concerned with noncooperative games and the solution concept is Berge equilibrium as related to Nash equilibrium. We wish to study the organization of Nash equilibria in noncooperative games as a foundational issue.

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In the traditional organization of a noncooperative game each player has an objective functional (which depends on all players' strategy variables, in general) which is to be maximized by selecting a strategy from his own set of strategies.

The dependency on other players' strategy variables allows for complex interactions between the functionals and it is crucial, even to the definition of Nash equilibria, that each player know the opponents' strategies. Indeed, it is this interconnection which makes it a game rather than a set of independent optimization problems.

Recently, another approach to noncooperative games has been studied by Radjef [8]. In a Radjef-type organization, each player optimizes the same objective as before, but now it is done with respect to *all* opponents' strategies. The information from opponents is shared by projections. That type of organization of a game is a special case of more general equilibria in the sense of Berge [2]. In this paper, we prove existence theorems for each of the Berge type of equilibrium, generalizing therefore the result of Radjef [8]. Moreover, conditions are shown under which Berge equilibria are also Nash equilibria points, giving yet another option when it comes to selecting Nash equilibria points.

This paper can be divided into two parts. The first part deals with games with a finite number of players. From the literature, we observe that within the traditional organization of the Nash games, two quite general existence results have been obtained. In [4], Ky Fan proved the existence of a Nash equilibrium for real-valued continuous, quasi-concave functionals on compact, convex strategy sets in real separated topological vector spaces. Later in [5], Granas applied the approach of KKM-maps to obtain the same result in another way. For reflexive Banach spaces, the present approach generalizes the earlier results even within the traditional game organization, and it places the Radjef-type organization in a general theory.

The second part treats the case of games with an infinite number of players. The results obtained here, even though similar to those of Part 1, are generalizations of the results of Part 1 and their proofs use slightly different arguments. Also, while these results generalize those found in Ma [6], they constitute, to our knowledge, the first results in the literature as far as the study of the Berge equilibria is concerned.

II. Notations and definitions

A game is defined as $\Gamma = (I, J_i, \Omega_i)$ where $I = \{1, \dots, N\}$ is the set of players and for each $i \in I$ the nonempty set Ω_i is the ith player's strategy set. We will assume that $\Omega_i \subseteq E_i$. Elements of Ω_i will be denoted by u_i, and J_i will denote the ith player's payoff functional defined on $\Omega = \prod_{i=1}^{N} \Omega_i$. The following notations will also be used:

$$-i = \{1, \ldots, i-1, i+1, \ldots, N\}, \quad u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N),$$
$$\Omega_{-i} = \Omega_1 \times \ldots \times \Omega_{i-1} \times \Omega_{i+1} \times \ldots \times \Omega_N, \quad \overline{u}_i(u) = \arg\max_{u_i \in \Omega_i} J_i(u_i, \cdot).$$

DEFINITION 1 ([2], p. 88). A feasible strategy $\widetilde{u}$ is an *equilibrium* point for Γ for the set S of players if

$$J_s(\widetilde{u}) \geq J_s(u_s, \widetilde{u}_{-s}) \quad \forall s \in S, \ \forall u \in \Omega.$$

DEFINITION 2 ([2], p. 88). A feasible strategy $\widetilde{u}$ is a *simple equilibrium* point or a *Nash equilibrium* point for Γ if

$$J_i(\widetilde{u}) \geq J_i(u_i, \widetilde{u}_{-i}) \quad \forall i \in I, \ \forall u \in \Omega.$$

REMARK 1. A Nash equilibrium point for Γ is an equilibrium for each $\{i\}$ and thus, the notion of Nash equilibrium is a special case of that of an equilibrium as defined by Berge.

DEFINITION 3 ([2], p. 88). Let $R = \{R_i : 1 \leq i \leq k\}$ be a partition of I and $S = \{S_i : 1 \leq i \leq k\}$ be subsets of I. A feasible strategy $\widetilde{u}$ is an *equilibrium point for* Γ *for the set* R *relative to the set* S if

$$J_r(\widetilde{u}) \geq J_r(u_s, \widetilde{u}_{-s}) \quad \forall r \in R_i, \ \forall s \in S_i, \ \forall u \in \Omega.$$

REMARK 2. A simple equilibrium point is an equilibrium for $R = \{\{i\} : i \in I\}$ relative to $S = \{\{i\} : i \in I\}$.

Now we introduce a theorem of Berge concerning the inverse image set of a function.

THEOREM ([2], p. 35). *Let X and Y be Hausdorff topological spaces. Let $T : X \to Y$ be a continuous multifunction with nonempty images, and let the function $f : X \times Y \to \mathbb{R}$ be continuous. Then:*

(i) *The function $\mu : X \to Y$ given by $\mu(x) = \max_{y \in T(x)} f(x, y)$ is continuous.*

(ii) *The multifunction $M : X \to Y$ given by $M(x) = \arg\max_{y \in T(x)} f(x, y)$ is u.s.c.*

The theorem stated above will be referred to throughout this paper as Berge's Maximum Theorem in the form of its corollary.

Corollary. *Let X and Y be Hausdorff topological spaces. Let $T : X \to Y$ be a continuous multifunction with nonempty images, and let the function $f : X \times Y \to \mathbb{R}$ be continuous. Then the multifunction $M : X \to Y$ given by $M(x) = \arg\max_{y \in T(x)} f(x, y)$ is continuous on X if for any $x \in X$, $M(x)$ is a singleton.*

III. Results. Part 1

There are three theorems which cover the case of a finite number of players. They are presented here in increasing order of organizational complexity.

Theorem 1. *For each $i \in I$, let Ω_i be convex, closed, bounded subsets of the reflexive Banach spaces E_i such that:*

(i) *$J_i(\cdot)$ is continuous on Ω.*
(ii) *$\overline{u}_i(u)$ is a singleton for each fixed u_{-i} in Ω_i.*

Then Γ has a Nash equilibrium.

Proof. Let $\Omega_i \subseteq (E_i, \tau_i)$ and $\tau = \prod_{i=1}^N \tau_i$ where τ_i is the weak topology of E_i. Then Ω_i is compact in (E_i, τ_i). Therefore from (i), (ii) and by Berge's Maximum Theorem, we conclude that $\overline{u}_i(\cdot)$ is continuous.

Now, consider the mapping $\widehat{u}(\cdot)$ from (Ω, τ) into itself defined by $\widehat{u}(u) = \prod_{i=1}^N \overline{u}_i(u)$. Then $\widehat{u}(\cdot)$ is continuous. We know that $\widetilde{u}$ is a Nash equilibrium of Γ if and only if $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. By Tikhonov's fixed point theorem [9], the mapping $\widehat{u}(\cdot)$ has a fixed point $\widetilde{u}$, and such a fixed point is a Nash equilibrium according to [1, p. 282]. This completes the proof.

Example 1. Consider the game of two players in which $u_i \in [-1, 1]$, $i = 1, 2$, and

$$J_1 = u_2 - 2u_1^2, \quad J_2 = u_1 - 2u_2^2.$$

Then $\overline{u}_1(u) = \{0\}$, $\overline{u}_2(u) = \{0\}$, and $\widehat{u}(u) = u$ iff $u = (0, 0)$.

Example 2. Consider the game of three players in which $u_i \in [-1, 1]$, $i = 1, 2, 3$, and

$$J_1 = -u_3^2, \quad J_2 = -u_1^2 - u_2^2, \quad J_3 = -u_1^2.$$

Then $\overline{u}_1(u) = [-1, 1]$. The traditional organization method does not work since $\overline{u}_1(u)$ is not a singleton.

Example 3. Consider the game of three players in which $u_i \in [-1, 1]$, $i = 1, 2, 3$, and

$$J_1 = -u_1^2, \quad J_2 = -u_3^2 - u_2^2, \quad J_3 = -u_1^2.$$

Then $\overline{u}_3(u) = [-1, 1]$. This game also is not amenable to traditional organization since $\overline{u}_3(u)$ is not a singleton.

DEFINITION 4. Let $S = \{S_i : 1 \le i \le N\}$ be such that $I = \bigcup_{i=1}^N S_i$ and S_i are nonempty pairwise distinct. The game Γ is said to be (S,σ)-*invariant* if there is a permutation σ on I such that:

(i) $\mathrm{Proj}_{\Omega_{\sigma(i)}}(\arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i},\cdot)) \neq \emptyset$.
(ii) $\mathrm{Proj}_{\Omega_{\sigma(i)}}(\arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i},\cdot)) \subseteq \overline{u}_{\sigma(i)}(u)$.

EXAMPLE 1 (ctd.). The game is (S,σ)-invariant iff $S_i = \{i\}$.

EXAMPLE 2 (ctd.). We have

$$\overline{u}_1(u) = [-1,1], \quad \overline{u}_2(u) = \{0\}, \quad \overline{u}_3(u) = [-1,1].$$

Let $S_1 = \{3\}$, $S_2 = \{1,2\}$ and $S_3 = \{1\}$. Define σ by $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$. Then

$$\mathrm{Proj}_{\Omega_{\sigma(1)}}(\arg\max_{u_{S_1}\in\Omega_{S_1}} J_1(u_{S_1},\cdot)) = \{0\},$$
$$\mathrm{Proj}_{\Omega_{\sigma(2)}}(\arg\max_{u_{S_2}\in\Omega_{S_2}} J_2(u_{S_2},\cdot)) = \{0\},$$
$$\mathrm{Proj}_{\Omega_{\sigma(3)}}(\arg\max_{u_{S_3}\in\Omega_{S_3}} J_3(u_{S_3},\cdot)) = \{0\},$$

and

$$\{0\} \subseteq \overline{u}_1(u), \quad \{0\} \subseteq \overline{u}_2(u), \quad \{0\} \subseteq \overline{u}_3(u).$$

Hence this game is (S,σ)-invariant.

EXAMPLE 3 (ctd.). We have

$$\overline{u}_1(u) = \{0\}, \quad \overline{u}_2(u) = \{0\}, \quad \overline{u}_3(u) = [-1,1].$$

This game cannot be (S,σ)-invariant.

THEOREM 2. *For each $i \in I$, let Ω_i be convex, closed, bounded subsets of the reflexive Banach spaces E_i such that:*

(i) *$J_i(\cdot)$ is continuous on Ω.*
(ii) *Γ is (S,σ)-invariant.*
(iii) *$\overline{u}_{S_i}(\cdot) = \arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i},\cdot)$ is a singleton for each fixed element u_{-S_i} in Ω_{-S_i}.*

Then Γ has an equilibrium point of type S which is also Nash.

PROOF. We will first show that Γ has an equilibrium $\widetilde{u}$ of type S and then complete the proof by showing that $\widetilde{u}$ is Nash.

Just as in the proof of Theorem 1, we find that $\overline{u}_{S_i}(\cdot)$ is continuous.

Let θ be any permutation on I and consider the mappings $\widehat{u}(\cdot)$ and $\widehat{u}'(\cdot)$ from (Ω,τ) into (Ω,τ) defined by

$$\widehat{u}(u) = \prod_{\theta(i)=1}^{N} \mathrm{Proj}_{\Omega_{\theta(i)}}(\overline{u}_{S_i}(u)) \quad \text{and} \quad \widehat{u}'(u) = \prod_{i=1}^{N} \overline{u}_i(u).$$

Then $\widehat{u}(\cdot)$ is continuous. Now, $\widetilde{u}$ is an equilibrium of type S as defined in [2, p. 88] for the game Γ if and only if $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. Since by Tikhonov's fixed point theorem [9], the mapping $\widehat{u}(\cdot)$ has a fixed point $\widetilde{u}$, this concludes the proof of the existence of an equilibrium point of type S.

Now, let us show that $\widetilde{u}$ is also a Nash point. Let $\theta = \sigma$. Since by (ii) any fixed point of $\widehat{u}(\cdot)$ is also a fixed point of $\widehat{u}'(\cdot)$, it follows that $\widetilde{u}$ is also a Nash equilibrium of Γ. This completes the proof of the theorem.

Remark 3. (i) As far as the existence of an equilibrium of type S is concerned, Theorem 2 is an extension of Radjef's result [8] for two reasons. While Radjef considers only the case when σ is defined by $\sigma(i) = i + 1$ for $1 \le i < N$, $\sigma(N) = 1$ and $S_i = -i$, Theorem 2 addresses the cases of any permutation and any partition of the set I.

(ii) If $\sigma(i) = i$ for $1 \le i \le N$, and $S_i = \{i\}$, then we get Theorem 1.

The following example illustrates Theorem 2.

Example 2 (ctd.). The traditional organization method does not work since $\overline{u}_1(u)$ is not unique. The following demonstrates how to apply Theorem 2 to this example.

$$\widehat{u}(u) = (\mathrm{Proj}_{\Omega_{\sigma(3)}}(\overline{u}_{S_3}(u)), \mathrm{Proj}_{\Omega_{\sigma(2)}}(\overline{u}_{S_2}(u)), \mathrm{Proj}_{\Omega_{\sigma(1)}}(\overline{u}_{S_1}(u))).$$

So $\widehat{u}(u) = u$ if and only if $u = (0, 0, 0)$.

Remark 4. In particular, by [7, Theorem 2, p. 72] the results above hold if conditions (ii) of Theorem 1 and (iii) of Theorem 2 are replaced by the following conditions respectively:

(ii) $J_i(u_i, \cdot)$ is strictly quasi-concave on Ω_i.

(iii) $J_i(u_{S_i}, \cdot)$ is strictly quasi-concave on Ω_{S_i}.

Definition 5. For $M \le N$, let $R = \{R_m : 1 \le m \le M\}$ and $S = \{S_m : 1 \le m \le M\}$ be a partition of I and a set of pairwise disjoint sets respectively, such that $I = \bigcup_{i=1}^{M} S_i$. Assume that there is a permutation σ on I such that:

(i) $\sigma(r_m) \in S_m$ for all $r_m \in R_m$ and any fixed m.

(ii) $\arg\max_{u_{S_m} \in \Omega_{S_m}} J_j(u_{S_m}, \cdot) = \arg\max_{u_{S_m} \in \Omega_{S_m}} J_k(u_{S_m}, \cdot)$ for all $k, j \in R_m$ and any fixed m.

(iii) $\mathrm{Proj}_{\Omega_{\sigma(r_m)}}(\arg\max_{u_{S_m} \in \Omega_{S_m}} J_{r_m}(u_{S_i}, \cdot)) \subseteq \overline{u}_{\sigma(r_m)}(u)$ for all $r_m \in R_m$ and any fixed m.

Then the game Γ is said to be (S, σ, R, M)-*invariant*.

Example 3 (ctd.). Set

$$S_1 = \{1\}, \quad S_2 = \{2, 3\}, \quad R_1 = \{1\}, \quad R_2 = \{2, 3\},$$

and $\sigma(i) = i$ for $i = 1, 2$. Then

$$\begin{gathered}\overline{u}_{S_1}(u) = \{0\}, \quad \overline{u}_{S_2}(u) = \{(0,0)\}, \quad \overline{u}_1(u) = \{0\}, \\ \overline{u}_2(u) = \{0\}, \quad \overline{u}_3(u) = [-1,1], \\ \mathrm{Proj}_{\Omega_1}(\operatorname*{arg\,max}_{u_{S_1}\in\Omega_{S_1}} J_1(u_{S_1},\cdot)) \subseteq \overline{u}_1(u), \\ \mathrm{Proj}_{\Omega_2}(\operatorname*{arg\,max}_{u_{S_2}\in\Omega_{S_2}} J_2(u_{S_2},\cdot)) \subseteq \overline{u}_2(u), \\ \mathrm{Proj}_{\Omega_3}(\operatorname*{arg\,max}_{u_{S_2}\in\Omega_{S_2}} J_2(u_{S_2},\cdot)) \subseteq \overline{u}_3(u),\end{gathered}$$

which shows that this game is $(S, \sigma, R, 2)$-invariant.

REMARK 5. If Γ is (S,σ)-invariant, then it is also (S, σ, R, N)-invariant.

THEOREM 3. *For each $i \in I$, let Ω_i be convex, closed, bounded subsets of the reflexive Banach spaces E_i such that:*

(i) *$J_i(\cdot)$ is continuous on Ω.*
(ii) *Γ is (S, σ, R, M)-invariant.*
(iii) *$\overline{u}_{S_m}(u) = \operatorname{arg\,max}_{u_{S_m}\in\Omega_{S_m}} J_{r_m}(u_{S_m},\cdot)$ is a singleton for all $r_m \in R_m$ and each fixed m and u_{-S_m} in Ω_{-S_m}.*

Then Γ has an equilibrium point for R relative to S which is also Nash.

PROOF. As in the proof of Theorem 2, we will first show that Γ has an equilibrium $\widetilde{u}$ for R relative to S and then complete the proof by showing that $\widetilde{u}$ is Nash.

As before, $\overline{u}_{S_m}(\cdot)$ is continuous.

Let θ be any permutation on M such that properties (i) and (ii) of Definition 3 are satisfied. Then, from (ii), the set $S' = \{S'_m : 1 \le m \le M\}$ where S'_m is the image of R_m by θ is a partition of I. Moreover, for a fixed m and any $k, j \in R_m$, we have

$$\mathrm{Proj}_{\Omega_{S'_m}} \operatorname*{arg\,max}_{u_{S_m}\in\Omega_{S_m}} J_j(u_{S_m},\cdot) = \mathrm{Proj}_{\Omega_{S'_m}} \operatorname*{arg\,max}_{u_{S_m}\in\Omega_{S_m}} J_k(u_{S_m},\cdot).$$

Let $\widehat{u}(\cdot)$ be a mapping from (Ω,τ) into itself such that the $\theta(r_m)$th component $\widehat{u}_{\theta(r_m)}(u)$ of $\widehat{u}(u)$ is defined as the projection on $\Omega_{\theta(r_m)}$ of $\overline{u}_{S_m}(u)$. Consider also the mapping $\widehat{u}'(\cdot)$ from (Ω,τ) into itself defined by $\widehat{u}'(u) = \prod_{i=1}^N \overline{u}_i(u)$. Then $\widehat{u}(\cdot)$ is continuous. Notice that $\widetilde{u}$ is an equilibrium for R relative to S for the game Γ as defined in [2, p. 88] if and only $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. Since by Tikhonov's fixed point theorem [9], the mapping $\widehat{u}(\cdot)$ has a fixed point $\widetilde{u}$, this concludes the proof of the existence of an equilibrium point for R relative to S.

Let us show that $\widetilde{u}$ is also a Nash point. Let $\theta = \sigma$. Since the game Γ is (S, σ, R, M)-invariant, any fixed point of $\widehat{u}(\cdot)$ is also a fixed point of $\widehat{u}'(\cdot)$. Then by [1] it follows that $\widetilde{u}$ is also a Nash equilibrium point of Γ. This completes the proof of the theorem.

An illustrative example for Theorem 3 will now be presented.

Example 3 (ctd.). Theorem 2 does not work because there are no N nonempty pairwise distinct subsets S_i of I such that condition (ii) of Theorem 2 is satisfied. Theorem 3 applied to this game yields the following result: $\widehat{u}(u) = u$ if and only if $u = (0, 0, 0)$.

Remark 6. (i) Theorem 3 is a generalization of Theorem 2. This follows from the previous remark. In fact, Theorem 2 can be obtained if $M = N$ and $R_i = \{i\}$ in Theorem 3.

(ii) In particular, by [7, Theorem 2, p. 72] the same result holds if assumption (iii) in Theorem 3 is replaced by the following assumption:

(iii′) $J_{r_m}(u_{S_m}, \cdot)$ is strictly quasi-concave on Ω_{S_m} for all $r_m \in R_m$, for fixed m and u_{-S_m} in Ω_{-S_m}.

IV. Results. Part 2

The following theorem is the infinite number of players version of Theorem 1 of Part 1.

Theorem 4. *Let $\{E_i\}_{i\in I}$ be a family of reflexive Banach spaces. Let $\{\Omega_i\}_{i\in I}$ be a family of closed, bounded, convex sets such that $\Omega_i \subseteq E_i$ for each $i \in I$. Let $\{J_i(\cdot)\}_{i\in I}$ be a family of real-valued continuous functions on $\Omega = \prod_{i\in I} \Omega_i$ such that $\overline{u}_i(u)$ is a singleton for each fixed element u_{-i} in Ω_{-i}. Then the game $\Gamma = (I, J_i, \Omega_i)$ has a Nash equilibrium.*

Proof. Since $\overline{u}_i(\cdot)$ is continuous, by [3, p. 51, Corollary 7.23] the composite mapping $\widehat{u}(\cdot)$ from (Ω, τ) into itself defined by $\widehat{u}(u) = \prod_{i\in I} \overline{u}_i(u)$ is continuous. By [9], $\widehat{u}(\cdot)$ has a fixed point $\widetilde{u}$ which by [1] is a Nash point for Γ. The proof is complete.

The notions of (S, σ)-invariance and (S, σ, R, M)-invariance in case I is an infinite set can naturally be generalized as follows.

Definition 6. Let $S = \{S_i\}_{i\in I}$ be such that $I = \bigcup_{i\in I} S_i$ and S_i are nonempty pairwise distinct. The game Γ is said to be (S, f)-*invariant* if there is a one-to-one function f on I such that:

(i) $\mathrm{Proj}_{\Omega_{f(i)}}(\arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i}, \cdot)) \neq \emptyset$.
(ii) $\mathrm{Proj}_{\Omega_{f(i)}}(\arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i}, \cdot)) \subseteq \overline{u}_{f(i)}(u)$.

DEFINITION 7. Let $R = \{R_m\}_{m\in M}$ be a partition of I and $S = \{S_m\}_{m\in M}$ be a set of pairwise disjoint sets such that $I = \bigcup_{m\in M} S_m$. Assume that there is a one-to-one function f on I such that:

(i) $f(r_m) \in S_m$ for all $r_m \in R_m$ and fixed m.

(ii) $\arg\max_{u_{S_m}\in\Omega_{S_m}} J_j(u_{S_m},\cdot) = \arg\max_{u_{S_m}\in\Omega_{S_m}} J_k(u_{S_m},\cdot)$ for all $k, j \in R_m$ and any fixed m.

(iii) $\mathrm{Proj}_{\Omega_{f(r_m)}}(\arg\max_{u_{S_m}\in\Omega_{S_m}} J_{r_m}(u_{S_m},\cdot)) \subseteq \overline{u}_{f(r_m)}(u)$ for all $r_m \in R_m$.

Then the game Γ is said to be (S, f, R, M)-*invariant*.

Now, let us prove the analogues of Theorems 2 and 3 in the situation of an infinite number of players.

THEOREM 5. *Let $\{E_i\}_{i\in I}$ be a family of reflexive Banach spaces. Let $\{\Omega_i\}_{i\in I}$ be a family of closed, bounded, convex sets such that $\Omega_i \subseteq E_i$ for each $i \in I$. Let $\{J_i(\cdot)\}_{i\in I}$ be a family of real-valued functions defined on $\Omega = \prod_{i\in I}\Omega_i$ such that for each $i \in I$:*

(i) *$J_i(\cdot)$ is continuous on Ω.*

(ii) *Γ is (S, f)-invariant.*

(iii) *$\overline{u}_{S_i}(\cdot) = \arg\max_{u_{S_i}\in\Omega_{S_i}} J_i(u_{S_i},\cdot)$ is a singleton.*

Then Γ has a Nash equilibrium.

PROOF. Again, $\overline{u}_{S_i}(\cdot)$ is continuous. Now let g be any one-to-one function on I and consider the mappings $\widehat{u}(\cdot)$ and $\widehat{u}'(\cdot)$ from (Ω,τ) into (Ω,τ) defined by

$$\widehat{u}(u) = \prod_{g(i)\in I} \mathrm{Proj}_{\Omega_{g(i)}}(\overline{u}_{S_i}(u)) \quad \text{and} \quad \widehat{u}'(u) = \prod_{i\in I} \overline{u}_i(u).$$

Then $\widehat{u}(\cdot)$ is continuous by [3, p. 51, Corollary 7.23]. Now, $\widetilde{u}$ is an equilibrium of type S for the game Γ as defined in [2, p. 88] if and only $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. Since by Tikhonov's fixed point theorem [9], the mapping $\widehat{u}(\cdot)$ has a fixed point $\widetilde{u}$, this concludes the proof of the existence of an equilibrium point $\widetilde{u}$ of type S.

Now, let us show that $\widetilde{u}$ is also a Nash equilibrium point. By [1], for $\widetilde{u} \in \Omega$ to be a Nash equilibrium of Γ it is sufficient that $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. Since by (ii) any fixed point of $\widehat{u}(\cdot)$ is also a fixed point of $\widehat{u}'(\cdot)$, we have just shown that $\widetilde{u}$ is also a Nash equilibrium. The proof is complete.

THEOREM 6. *Let $\{E_i\}_{i\in I}$ be a family of reflexive Banach spaces. Let $\{\Omega_i\}_{i\in I}$ be a family of closed, bounded, convex sets such that $\Omega_i \subseteq E_i$ for each $i \in I$. Let $\{J_i(\cdot)\}_{i\in I}$ be a family of real-valued functions defined on $\Omega = \prod_{i\in I}\Omega_i$ such that for each $i \in I$:*

(i) *$J_i(\cdot)$ is continuous on Ω.*

(ii) *Γ is (S, f, R, M)-invariant.*

(iii) $\overline{u}_{S_m}(u) = \arg\max_{u_{S_m} \in \Omega_{S_m}} J_{r_m}(u_{S_m}, \cdot)$ *is a singleton for all* $r_m \in R_m$ *for each fixed* m *and* u_{-S_i} *in* Ω_{-S_i}.

Then Γ *has a Nash equilibrium.*

Proof. First, $\overline{u}_{S_m}(\cdot)$ is again continuous on Ω_{S_m}. From (ii), the set $S' = \{S'_m\}_{m \in M}$ where S'_m is the image of R_m by f is a partition of I. Moreover, since Γ is (S, f, R, M)-invariant, we have

$$\mathrm{Proj}_{\Omega_{S'_m}} \operatorname*{arg\,max}_{u_{S_m} \in \Omega_{S_m}} J_j(u_{S_m}, \cdot) = \mathrm{Proj}_{\Omega_{S'_m}} \operatorname*{arg\,max}_{u_{S_m} \in \Omega_{S_m}} J_k(u_{S_m}, \cdot)$$

for each fixed m and any $j, k \in R_m$.

Let $\widehat{u}(\cdot)$ be a mapping from (Ω, τ) into itself such that the $f(r_m)$th component $\widehat{u}_{f(r_m)}(u)$ of $\widehat{u}(u)$ is defined as the projection on $\Omega_{f(r_m)}$ of $\overline{u}_{S_m}(u)$. Now consider also the mapping $\widehat{u}'(\cdot)$ from (Ω, τ) into itself defined by $u'(u) = \prod_{i \in I} \overline{u}_i(u)$. Then $\widehat{u}(\cdot)$ is continuous by [3, p. 51, Corollary 7.23]. By [1], for $\widetilde{u} \in \Omega$ to be an equilibrium of type R relative to S for the game Γ it is necessary and sufficient that $\widetilde{u}$ is a fixed point of $\widehat{u}(\cdot)$. Since by Tikhonov's theorem [9] such a fixed point exists, this implies the existence of an equilibrium point $\widetilde{u}$ of type R relative to S.

Now, since by (ii) any fixed point of $\widehat{u}(\cdot)$ is also a fixed point of $\widehat{u}'(\cdot)$ and since any fixed point of $\widehat{u}'(\cdot)$ is by definition a Nash equilibrium point we conclude that $\widetilde{u}$ is also a Nash equilibrium point. The proof is complete.

Remark 7. If in Theorems 4, 5 and 6 instead of $\overline{u}_m(u)$, $\overline{u}_{S_m}(u)$ being singletons we require that $J_m(\cdot)$ be strictly quasi-concave respectively on Ω_m and Ω_{S_m}, then these results still hold and their proofs can be obtained by using a theorem of Ma [6, p. 415, Theorem 4].

V. Conclusion

We have shown that under an invariance property that has been introduced in this paper, different types of equilibria as defined by Berge can also be Nash equilibria for N-person games. Part of the results proved in this paper generalize the Radjef-type organization. Similar results for an infinite number of players are the first obtained in the case of Berge equilibria. Even though these results have been proved under some strong assumptions, they will be of interest for some specific Nash games. Finally, our results contribute a new solution to the problem of selecting appropriate Nash equilibria.

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