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FIXED POINT INDEX FOR ITERATIONS OF MAPS, TOPOLOGICAL HORSESHOE AND CHAOS

Piotr Zgliczyński

1. Introduction

There are many examples of complicated or chaotic dynamics, but the set of examples for which chaos has been rigorously demonstrated is quite small. In most cases where chaotic dynamics has been proven, the strategy has involved analysing a simple singular map or integrable problem and then perturbing the results (see [2], [5]). This usually required some estimates on the derivatives of mappings under consideration.

Another strategy to tackle such problems is to *appropriately* homotope the given system to a model problem for which some algebraic invariants could be explicitly computed and show that these invariants remain unchanged. Nontriviality of the algebraic invariant provides a minimal description of the complexity of the dynamics of the system. In [3], [4] with the help of the discrete Conley index introduced in [6], this strategy has been applied to the Hénon map and the Lorenz equations.

In applying this strategy to a concrete problem we must answer three closely related questions: what algebraic invariants we will use, what is the model map, what are the *appropriate* homotopies.

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As the algebraic invariant we choose the fixed point index [1, Chapter VII.5] and we formulate general sufficient conditions for a homotopy to be *appropriate*. As the model maps we choose maps for which all fixed point indices of the periodic points are easily calculable. This class of maps includes for example Smale's horseshoe and one-dimensional chaotic maps. We then prove the existence of an infinite number of periodic orbits for maps which can be appropriately homotoped from model maps and their semiconjugacy to the shift automorphism on two symbols. In [4, Theorem 2.4] a very similar result was stated for a less general class of *appropriate* homotopy maps, but the proof given there has a serious gap in the part concerning "continuation".

Our results might find their application in computer assisted proofs of chaos in dynamical systems. They could be applied to any differential equation for which numerical integration gives the Poincaré map looking like Smale's horseshoe. Recently Mischaikow and Mrozek [3] performed successfully this type of calculation for the Lorenz equations (with nonclassical values of parameters $r = 54$, $s = 45$, $q = 10$). Pictures obtained by them show that the assumptions of our Theorem 3.2 are satisfied. This allows application of our Theorems 3.1 and 4.1 to strengthen their results. Similar calculations are being done by the author [9], [10] to prove the existence of chaotic dynamics in the Hénon map and Rössler equations. In both cases we consider the classical values of parameters. Work about classical Lorenz equations is in progress.

2. Continuation of fixed point indices for iterations

We denote by $\varrho(x,y)$ the distance from x to y in $\mathbb{R}^d$. For $Z \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we write $\varrho(x,Z) = \inf\{\varrho(x,y) \mid y \in Z\}$, $B(x,\varepsilon) = \{y \mid \varrho(x,y) < \varepsilon\}$ and $B(Z,\varepsilon) = \{y \mid \varrho(x,Z) < \varepsilon\}$. For mappings F defined on $[0,1] \times Z$ we write $F_\lambda(x)$ instead of $F(\lambda, x)$.

Let $f : X \to X$ be a continuous map and $Z \subset X$. We define $\operatorname{Inv}(Z,f) = \bigcap_{i=-\infty}^{\infty} f^i_{|Z}(Z)$. If Z is compact then $\operatorname{Inv}(Z,f)$ is a maximal invariant set contained in Z. We will often write simply $\operatorname{Inv}(Z)$, when f is known from the context.

We say that Z is an *isolating neighborhood* for f if Z is compact and $\operatorname{Inv}(Z,f) \subset \operatorname{int}(Z)$.

Let $Z_0, Z_1, \ldots, Z_s$ be pairwise disjoint. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{0, 1, \ldots, s\}$. Define

$$Z^\lambda_\alpha := Z_{\alpha_0} \cap F^{-1}_\lambda(Z_{\alpha_1}) \cap \ldots \cap F^{-n}_\lambda(Z_{\alpha_n}).$$

It can be easily seen that $Z^\lambda_\alpha \cap Z^\lambda_\beta = \emptyset$ for $\alpha \neq \beta$.

LEMMA 2.1. *Let $M \subset \mathbb{R}^d$ be a compact set, N a compact subset of M and $F : [0,1] \times M \to M$ a continuous map. Assume that N is an isolating neighborhood* *for F_λ for every $\lambda \in [0,1]$. Then there exists an open set D such that* $\mathrm{cl}(D) \subset \mathrm{int}(N)$ *and for every* $\lambda \in [0,1]$, $\mathrm{Inv}(N, F_\lambda) \subset D$.

PROOF. Let $x_0 \in \mathrm{bd}(N)$ and $\lambda_0 \in [0,1]$. Since N is an isolating neighborhood for F_{λ_0}, $x_0 \notin \mathrm{Inv}(N, F_{\lambda_0})$. So there exists $k > 0$ such that $F^k_{\lambda_0}(x_0) \notin N$ or $x_0 \notin F^k_{\lambda_0|N}(N)$. From compactness of N, local compactness of $\mathbb{R}^d$ and the continuity of F it follows that there exist $\delta, \varepsilon > 0$ such that

$$F^k_\lambda(x) \notin N \text{or } x \notin F^k_{\lambda_0|N}(N) \quad \text{for } x \in \mathrm{cl}(B(x_0,\varepsilon)) \text{and } |\lambda - \lambda_0| < \delta. \tag{2.1}$$

The family of sets $U_{x,\lambda} = B(x,\varepsilon) \times ((\lambda - \delta, \lambda + \delta) \cap [0,1])$ is an open covering of the compact set $\mathrm{bd}(N) \times [0,1]$. So there exists a finite U_{x_i,λ_i}, $i = 1, \dots, n$ subcovering.

Set $W := \bigcup_{i=1}^n U_{x_i,\lambda_i}$. Obviously W is open in $M \times [0,1]$, contains $\mathrm{bd}(N) \times [0,1]$, and $x \notin \mathrm{Inv}(N, F_\lambda)$ for every $(x,\lambda) \in W$.

We will construct an open, relatively compact set $V \subset M$ such that $\mathrm{bd}(N) \subset V$ and $\mathrm{cl}(V) \times [0,1] \subset W$. Let $x \in \mathrm{bd}(N)$. There exists a finite family of balls $Y_{x,i}$ with center at x and open sets Λ_i in $[0,1]$ for $i = 1, \dots, m$ such that $\mathrm{cl}(Y_{x,i}) \times \Lambda_i \subset W$ and $\{x\} \times [0,1] \subset \bigcup_{i=1}^m Y_{x,i} \times \Lambda_i$. Let $Y_x := \bigcap_{i=1}^m Y_{x,i}$. Then Y_x is obviously an open ball with center at x and $\mathrm{cl}(Y_x) \times [0,1] \subset W$.

The covering $\{Y_x : x \in \mathrm{bd}(N)\}$ of $\mathrm{bd}(N)$ has a finite subcovering Y_{x_i}, $i = 1, \dots, p$. Define $V := \bigcup_{i=1}^p Y_{x_i}$. This set is open, $\mathrm{bd}(N) \subset \mathrm{int}(V)$ and $\mathrm{cl}(V) \times [0,1] \subset W$.

Put $D := \mathrm{int}(N) \setminus \mathrm{cl}(V)$. Obviously D is open and $\mathrm{Inv}(N, F_\lambda) \subset D$ for every λ. It remains to show that $\mathrm{cl}(D) \subset \mathrm{int}(N)$. This clearly follows from the fact that $\mathrm{bd}(N) \subset V$. □

We are ready to state and prove the following theorem.

THEOREM 2.2. *Let $N = \bigcup N_s$, $s = 0, \dots, m$, where $N_s \subset \mathbb{R}^d$ are compact disjoint sets and* $\mathrm{cl}(\mathrm{int}(N_s)) = N_s$. *Let $f, g : N \to \mathbb{R}^d$ be continuous maps. Suppose that there exists the homotopy F connecting f and g and such that N is an isolating neighborhood for every* $\lambda \in [0,1]$. *Then for any finite sequence* $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \{0, \dots, m\}^{n+1}$ *the fixed point indices* $I(f^{n+1}, N_{\alpha_0} \cap f^{-1}(N_{\alpha_1}) \cap \dots \cap f^{-n}(N_{\alpha_n}))$, $I(g^{n+1}, N_{\alpha_0} \cap g^{-1}(N_{\alpha_1}) \cap \dots \cap g^{-n}(N_{\alpha_n}))$ *are defined and equal.*

PROOF. Tietze's theorem allows us to extend the homotopy F to $\widetilde{F} : [0,1] \times [-L,L]^d \to [-L,L]^d$, where L is such that $N \subset [-L,L]^d$ and $F([0,1] \times N) \subset [-L,L]^d$. Thanks to this extension one can iterate $\widetilde{F}_\lambda$ without leaving the domain of definition. In what follows we use $\widetilde{F}$ instead of the original F but we write F for this extended homotopy.

Fix n and $\alpha = (\alpha_0, \dots, \alpha_n)$. From Lemma 2.1 one can find open sets D, C such that

$$\mathrm{cl}(D) \subset C, \quad \mathrm{cl}(C) \subset \mathrm{int}(N), \quad \mathrm{Inv}(N, F_\lambda) \subset D, \quad \text{for all } \lambda \in [0,1]. \tag{2.2}$$

We define $C_i = C \cap N_i$, $D_i = D \cap N_i$. Let $\delta > 0$ be such that

$$B(D_i, \delta) \subset C_i, \quad B(C_i, \delta) \subset N_i. \tag{2.3}$$

Let $\lambda_0 \in [0, 1]$. Then there exists a set Λ open in $[0, 1]$, with $\lambda_0 \in \Lambda$, such that for every $\lambda_1, \lambda_2 \in \Lambda$,

$$\varrho(F^i_{\lambda_1}(x), F^i_{\lambda_2}(x)) \le \delta \quad \text{for } i = 1, \ldots, n,\ x \in [-L, L]^d. \tag{2.4}$$

We now show that

$$D^\lambda_\alpha \subset C^{\lambda_0}_\alpha \subset \operatorname{int}(N^\lambda_\alpha) \quad \text{for } \lambda \in \Lambda. \tag{2.5}$$

Let $x \in D^\lambda_\alpha$. Then $F^i_\lambda(x) \in D_{\alpha_i}$ for $i = 0, \ldots, n$. But from (2.3) and (2.4) it follows that $F^i_{\lambda_0}(x) \in C_{\alpha_i}$ for $i = 0, \ldots, n$. Thus $x \in C^\lambda_\alpha$. The proof of the second inclusion is analogous.

We now show that

$$F^{n+1}_\lambda(x) \ne x \quad \text{for } x \in N^\lambda_\alpha \setminus D^\lambda_\alpha. \tag{2.6}$$

Suppose there exists an $x \in N^\lambda_\alpha \setminus D^\lambda_\alpha$ such that $F^{n+1}_\lambda(x) = x$. Hence $x \in \operatorname{Inv}(N, F_\lambda)$ and for some $i = 0, \ldots, n$, $F^i_\lambda(x) \in N_{\alpha_i} \setminus D_{\alpha_i}$. This means that $\operatorname{Inv}(N, F_\lambda)$ is not contained in D, contrary to (2.2).

From (2.5) it follows that for $\lambda, \lambda_0 \in [0, 1]$ the sets $\operatorname{bd}(D^\lambda_\alpha)$, $\operatorname{bd}(C^\lambda_\alpha)$, $\operatorname{bd}(C^{\lambda_0}_\alpha)$, $\operatorname{bd}(N^\lambda_\alpha)$ are all contained in $N^\lambda_\alpha \setminus D^\lambda_\alpha$. So from (2.6) we see that the fixed point index for the maps F^{n+1}_λ relative to $D^\lambda_\alpha, C^\lambda_\alpha, N^\lambda_\alpha$ [1, Chapter VII.5] is well defined.

From the excision property of the fixed point index [1, Chapter VII, Theorem 5.4], (2.6) and (2.5) we conclude that

$$I(F^{n+1}_\lambda, D^\lambda_\alpha) = I(F^{n+1}_\lambda, C^{\lambda_0}_\alpha) = I(F^{n+1}_\lambda, N^\lambda_\alpha) \quad \text{for all } \lambda \in \Lambda. \tag{2.7}$$

Substituting $\lambda := \lambda_0$ we derive

$$I(F^{n+1}_{\lambda_0}, D^{\lambda_0}_\alpha) = I(F^{n+1}_{\lambda_0}, C^{\lambda_0}_\alpha). \tag{2.7$'$}$$

From (2.5) and (2.6) it follows that

$$\forall \lambda \in \Lambda\ \forall c \in \operatorname{bd}(C^{\lambda_0}_\alpha) \quad F^{n+1}_\lambda(x) \ne x.$$

So from the homotopy invariance of the fixed point index [1, Chapter VII, Theorem 5.8] we obtain

$$I(F^{n+1}_\lambda, C^{\lambda_0}_\alpha) = I(F^{n+1}_{\lambda_0}, C^{\lambda_0}_\alpha) \quad \text{for all } \lambda \in \Lambda. \tag{2.8}$$

From (2.7), (2.7′), (2.8) we conclude that

$$I(F^{n+1}_\lambda, D^\lambda_\alpha) = I(F^{n+1}_{\lambda_0}, D^{\lambda_0}_\alpha) \quad \text{for all } \lambda \in \Lambda. \tag{2.9}$$

From the compactness of $[0,1]$ and (2.9) we get $I(F_0^{n+1}, D_\alpha^0) = I(F_1^{n+1}, D_\alpha^1)$. And finally it follows from (2.7) that $I(F_0^{n+1}, N_\alpha^0) = I(F_1^{n+1}, N_\alpha^1)$. This finishes the proof. □

3. Topological horseshoe and chaotic behavior

In this section we discuss examples which illustrate the strength of Theorem 2.2. We concentrate on maps defined on subsets of the plane.

Let $N_0 = [-1,1] \times [-1,-0.5]$, $N_1 = [-1,1] \times [0.5, 1.0]$ and $N = N_0 \cup N_1$.

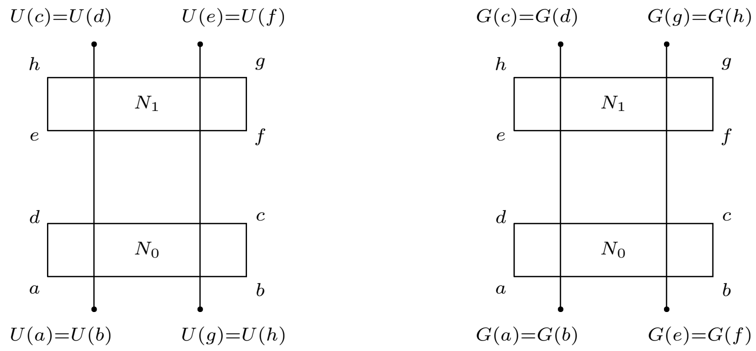

Figure 1. U-horseshoe and G-horseshoe

Definition 3.1. The mappings $U : N \to \mathbb{R}^2$, $G : N \to \mathbb{R}^2$ defined by

$$U(x,y) := \begin{cases} (-0.5, 5(x_2 + 0.75)) & \text{for } (x_1,x_2) \in N_0, \\ (0.5, -5(x_2 - 0.75)) & \text{for } (x_1,x_2) \in N_1, \end{cases}$$

$$G(x,y) := \begin{cases} (-0.5, 5(x_2 + 0.75)) & \text{for } (x_1,x_2) \in N_0, \\ (0.5, 5(x_2 - 0.75)) & \text{for } (x_1,x_2) \in N_1, \end{cases}$$

will be called respectively the U- and G-*horseshoe* (see Figure 1).

Remark 3.1. The fixed point indices $I(f^{n+1}, N_{\alpha_0} \cap f^{-1}(N_{\alpha_1}) \cap \ldots \cap f^{-n}(N_{\alpha_n}))$, where $\alpha_i \in \{0,1\}$, $f \in \{U, G\}$, are nonzero. This follows easily from the piecewise linear character of the mappings under consideration. The corresponding periodic points are hyperbolic.

Definition 3.2. Let $P \subset \mathbb{R}^2$ be a rectangle $[a,b] \times [c,d]$, $a \le b$, $c \le d$, $a,b,c,d \in \mathbb{R}$, and $\delta \ge 0$. We define

$$V(P,\delta) := [a, a+\delta] \times [c,d] \cup [b-\delta, b] \times [c,d],$$
$$H(P,\delta) := [a,b] \times [c, c+\delta] \cup [a,b] \times [d-\delta, d].$$

For any set $Z = \bigcup P_i$, $P_i = [a_i, b_i] \times [c_i, d_i]$ we define

$$V(Z,\delta) := \bigcup V(P_i,\delta), \qquad H(Z,\delta) := \bigcup H(P_i,\delta).$$

Thus $V(Z,\delta)$ is the δ-neighborhood in Z of the vertical edges of Z, and $H(Z,\delta)$ is the δ-neighborhood in Z of the horizontal edges of Z. We drop the parameter δ in the above defined symbols when $\delta = 0$. So for example $V(Z) = V(Z,0)$ is the union of the vertical edges.

Now we introduce two simple, geometrical conditions for the set N to be an isolating neighborhood for a map $f : N \to \mathbb{R}^2$. The conditions are

$$f(H(N)) \cap N = \emptyset, \tag{A}$$

$$f(N) \cap V(N) = \emptyset. \tag{B}$$

It is easy to see that if conditions (A) and (B) hold then $f(N) \cap N \cap f^{-1}(N) \subset \operatorname{int}(N)$, so N is an isolating neighborhood.

Geometrically (A) means that horizontal edges of N are mapped by f outside of N, and (B) means that vertical edges of N do not intersect the image of N.

Definition 3.3. Let $F : [0,1] \times N \to \mathbb{R}^2$ be a continuous homotopy connecting f with g. This means that

$$F(0,x_1,x_2) = f(x_1,x_2) \quad \text{and} \quad F(1,x_1,x_2) = g(x_1,x_2).$$

F will be called *appropriate* if conditions (A) and (B) hold for every map F_λ, $\lambda \in [0,1]$.

Combining Remark 3.1 and Theorem 2.2 we obtain the following theorem.

Theorem 3.1. *Let $f : N \to \mathbb{R}^2$ be a continuous map. Suppose that there exists an appropriate homotopy F connecting f with the U- or G-horseshoe. Then for any finite sequence $\alpha_0, \ldots, \alpha_n$ there exist points x, y satisfying*

$$f^i(x) \in N_{\alpha_i} \quad \text{for } i = 0,\ldots,n \quad \text{and} \quad f^{n+1}(x) = x,$$
$$f^i(y) \in N_{\alpha_i} \quad \text{for } i = 0,\ldots,n \quad \text{and} \quad f^{n+1}(y) \notin N.$$

Proof. The existence of x follows immediately from Remark 3.1 and Theorem 2.2.

It remains to show the existence of y. From the first assertion we can find x_0, x_1 such that

$$f^i(x_j) \in N_{\alpha_i} \quad \text{for } i = 0,\ldots,n,$$

and

$$f^{n+1}(x_j) \in N_j \quad \text{for } j = 0,1.$$

Consider now the line segment $p(t) = (1-t)x_0 + tx_1$. Obviously $p(0) \in N_\alpha \cap f^{-(n+1)}(N_0)$ and $p(1) \in N_\alpha \cap f^{-(n+1)}(N_1)$.

Let $t_m = \sup\{t \mid \forall 0 \le s \le t,\ p(s) \in N_\alpha \cap f^{-(n+1)}(N_0)\}$. Obviously $p(t_m) \in N_\alpha \cap f^{-(n+1)}(N_0)$ and $t_m < 1$. From (A) and (B) it follows that

$$f^i(p(t_m)) \in \operatorname{int}(N_{\alpha_i}) \quad \text{for } i = 1, \dots, n.$$

Thus

$$\exists \varepsilon > 0\ \forall 0 < \delta < \varepsilon \quad f^i(p(t_m + \delta)) \in \operatorname{int}(N_{\alpha_i}) \quad \text{for } i = 1, \dots, n. \tag{3.1}$$

But from the definition of t_m it follows that the following condition

$$f^{(n+1)}(p(t_m + \delta)) \notin N_0 \tag{3.2}$$

for some $\delta < \varepsilon$, where δ may be chosen arbitrarily small.

Since $f^{(n+1)}(p(t_m)) \in N_0$, for δ such that (3.1) and (3.2) hold, we get

$$f^{(n+1)}(p(t_m + \delta)) \notin N. \tag{3.3}$$

This finishes the proof. □

In our opinion Theorem 3.1 fully justifies the following defintion:

Definition 3.4. Maps for which the assumptions of the above theorem hold will be called *topological horseshoes*.

A natural question arises when a given mapping f defined on N for which conditions (A) and (B) hold, may be *appropriately* homotoped to Smale's horseshoe. We present a simple criterion for the existence of such a homotopy (see Figure 2).

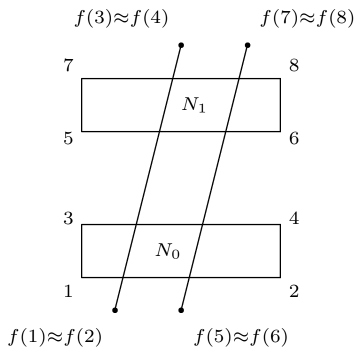

Figure 2. Sketch of the deformed G-horseshoe, obtained in [3] for the Lorenz equations

Theorem 3.2. *Let $f : N \to \mathbb{R}^2$ be such that*

$$f([-1,1] \times \{-1\}) \subset \{(x,y) \mid y < -1\},$$
$$f([-1,1] \times \{0.5\}) \subset \{(x,y) \mid y < -1\},$$
$$f([-1,1] \times \{-0.5\}) \subset \{(x,y) \mid y > 1\},$$
$$f([-1,1] \times \{1\}) \subset \{(x,y) \mid y > 1\}.$$

Suppose that there exists $\delta > 0$ such that

$$f(N) \subset \{(x,y) \mid -1+\delta < x < 1-\delta\}.$$

Then there exists an appropriate homotopy F connecting f with the G-horseshoe.

Proof. Define $F : [0,1] \times N \to \mathbb{R}^2$ by

$$F(\lambda, x) = (1-\lambda)f(x) + \lambda G(x).$$

Obviously for this F conditions (A) and (B) hold. □

Corollary 3.3. *If the assumptions of Theorem 3.2 are satisfied then the assertion of Theorem 3.1 holds.*

4. Semiconjugacy with a shift for topological horseshoes

Denote by Σ_2 the space of bi-infinite sequences of 0's and 1's with the Tikhonov topology, and by s the shift on Σ_2 given by $s((x_i)) = (x_{i-1})$.

As an application of Theorems 3.1 and 3.2 we prove the following

Theorem 4.1. *Let $f : N \to \mathbb{R}^2$ be a topological horseshoe and assume it is an injection. Then there exists a continuous surjection $\sigma : \mathrm{Inv}(N) \to \Sigma_2$ such that $\sigma \circ f = s \circ \sigma$. If $\alpha \in \Sigma_2$ is periodic, then $\sigma^{-1}(\alpha)$ contains periodic points with the same period.*

Proof. We have $\mathrm{Inv}(N) = \bigcap_{n=-\infty}^{\infty} f_{|N}^n(N)$. For any $i \in \mathbb{Z}$ and $x \in \mathrm{Inv}(N)$ we define $\sigma_i(x) = j$ if $f^i(x) \in N_j$. Obviously these maps are well defined and continuous, and yield a continuous mapping $\sigma : \mathrm{Inv}(N) \to \Sigma_2$. Obviously $\sigma \circ f = s \circ \sigma$. From Theorem 3.1 it follows that σ^{-1} of any periodic trajectory contains a periodic orbit with the same period. But periodic points are dense in $\{0,1\}^{\mathbb{Z}}$ so the entire $\{0,1\}^{\mathbb{Z}}$ lies in the image of σ. □

Remark 4.1. Exact calculations performed by Mischaikow and Mrozek in [3] for the Lorenz equations (nonclassical values of parameters $r = 54$, $s = 45$, $q = 10$) show that the assumptions of our Theorem 4.1 are satisfied (see Figure 2) for an appropriately chosen set $N = N_0 \cup N_1$ on the section $z = 53$, so the assertion of this theorem holds.

5. Concluding remarks

The horseshoes are the simplest examples of the applications of our Theorem 2.2 and conditions (A) and (B). One could easily write down many examples of such piecewise linear mappings with a larger number of components and in higher dimensional spaces. One obtains in this way models of chaotic behavior different from that of horseshoe's. For example in [10] chaotic dynamics on three symbols was proved for some Poincaré map for the Rössler system.

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P. Zgliczyński
Institute of Mathematics
Jagiellonian University
Reymonta 4, 30-059 Kraków, POLAND

E-mail address: zgliczyn@im.uj.edu.pl