Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 8, 1996, 79–93

VARIATIONAL EIGENCURVE AND BIFURCATION FOR TWO-PARAMETER NONLINEAR STURM–LIOUVILLE EQUATIONS

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1. Introduction

We consider the following two-parameter nonlinear Sturm–Liouville equation:

$$\begin{aligned} u''(x) + \mu u(x)^p &= \lambda u(x)^q, \quad x \in I = (0,1), \\ u(x) &> 0, \quad x \in I, \\ u(0) &= u(1) = 0, \end{aligned} \tag{1.1}$$

where $1 < q < p < q + 2$ and $\mu, \lambda > 0$ are eigenvalue parameters.

In order to describe and motivate the results of this paper, let us briefly recall some of the known results concerning two-parameter Sturm–Liouville problems.

There are many works concerning linear two-parameter problems. One of the main objectives is to investigate asymptotic properties of eigenvalues. In this direction, for instance, there are works of Binding and Browne [1], Faierman [2], Turyn [6] and Weinstein and Keller [7]. We also refer to Faierman [3] and the references cited therein. In particular, Binding and Browne [1] considered the following equation:

$$u''(x) + \mu r_1(x)u(x) = \lambda r_2(x)u(x), \quad x \in I. \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary 34B15.

Under suitable boundary conditions and assumptions on r_1 and r_2, they established the following asymptotic formula: as $\mu \to \infty$,

$$\frac{\lambda_n(\mu)}{\mu} \to \operatorname*{ess\,sup}_{x \in I} \frac{r_1(x)}{r_2(x)}. \tag{1.3}$$

Here $\lambda_n(\mu)$ is the nth eigenvalue of (1.2) for given $\mu \in \mathbb{R}$. The main tool used there was modified Prüfer transformation.

However, it seems that few results concerning nonlinear two-parameter problems are obtained. Recently, motivated by [1], Shibata [5] considered the nonlinear two-parameter equation of the form

$$\begin{aligned} &u''(x) + \mu u(x) = \lambda(1 + |u(x)|^{p-1})u(x), \quad x \in I, \\ &u(0) = u(1) = 0. \end{aligned} \tag{1.4}$$

By using a variational method due to Zeidler [8], the nth variational eigenvalue $\lambda = \lambda_n(\mu, \alpha)$ was defined and the asymptotic formula for $\lambda_n(\mu, \alpha)$ as $\mu \to (n\pi)^2$ was obtained:

$$\frac{\lambda_n(\mu, \alpha)}{(\mu - (n\pi)^2)^{(p+1)/2}} \to \pi^{-1/2} \frac{\Gamma((p+3)/2))}{2^p \alpha^{(p-1)/2} \Gamma((p+2)/2)}. \tag{1.5}$$

Here $\alpha > 0$ is a normalizing parameter of a general level set, which will be defined precisely later. In the proof of (1.5), the homogeneity of the left hand side of the equation (1.4) played an important role. We note here that we do not have this property any more in our problem (1.1).

In this paper, we consider a typical kind of nonlinear two-parameter problem, which is completely different from (1.4), in connection with a bifurcation problem. More precisely, we shall show the existence of an eigencurve $(\mu, \lambda(\mu), u_\mu) \in \mathbb{R}_+ \times \mathbb{R}_+ \times W_0^{1,2}(I)$ bifurcating from the trivial solution $(0, 0, 0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times W_0^{1,2}(I)$ of (1.1). Furthermore, we shall establish an asymptotic formula for $\lambda(\mu)$ as $\mu \to 0$.

We explain notations before stating our results. Let $X = W_0^{1,2}(I)$ denote the closure of $C_0^\infty(I)$ (the space of all real-valued, infinitely differentiable functions with compact support in I) in the usual Sobolev space $W^{1,2}(I)$. We equip X with the norm $\|u\|_X^2 = \int_I |u'(x)|^2\, dx$, while $\|u\|_s$ will denote the norm of $u \in L^s(I)$. We define the *general level set* $N_{\mu,\alpha}$ by

$$N_{\mu,\alpha} := \left\{ u \in X : \frac{1}{2}\|u\|_X^2 - \frac{\mu}{p+1}\|u\|_{p+1}^{p+1} = -\alpha \right\}, \tag{1.6}$$

where $\alpha > 0$ is a normalizing parameter. Hereafter, we fix $\alpha > 0$. Now we shall give the definition of variational eigenvalue $\lambda(\mu)$. We call $\lambda = \lambda(\mu)$ a *variational eigenvalue* for $\mu > 0$ if $\lambda(\mu) > 0$ and the associated eigenfunction $u_\mu \in N_{\mu,\alpha}$

satisfies the following conditions (1.7)–(1.8):

$$(\mu, \lambda(\mu), u_\mu) \in \mathbb{R}_+ \times \mathbb{R}_+ \times N_{\mu,\alpha} \text{ satisfies (1.1)}, \tag{1.7}$$

$$\frac{1}{q+1}\|u_\mu\|_{q+1}^{q+1} = \beta(\mu) := \inf_{u\in N_{\mu,\alpha}} \frac{1}{q+1}\|u\|_{q+1}^{q+1}. \tag{1.8}$$

$\lambda(\mu)$ is obtained as a Lagrange multiplier of the minimizing problem (1.8) and represented explicitly as follows:

$$\lambda(\mu) = \frac{2\alpha + \dfrac{p-1}{p+1}\mu\|u_\mu\|_{p+1}^{p+1}}{\|u_\mu\|_{q+1}^{q+1}}. \tag{1.9}$$

The latter is obtained as follows. Multiplying (1.1) by u_μ and integration by parts we obtain

$$-\|u_\mu\|_X^2 + \mu\|u_\mu\|_{p+1}^{p+1} = \lambda(\mu)\|u_\mu\|_{q+1}^{q+1}; \tag{1.10}$$

this along with the fact that $u_\mu \in N_{\mu,\alpha}$ implies (1.9).

Now we are ready to state our main results.

Theorem 1.1. *There exists a unique variational eigenvalue for $\mu > 0$, that is, if $(\mu, \lambda_1(\mu), u_{\mu,1})$ and $(\mu, \lambda_2(\mu), u_{\mu,2})$ satisfy (1.7) and (1.8) for the same $\mu > 0$, then $\lambda_1(\mu) = \lambda_2(\mu)$. Furthermore, $\lambda(\mu)$ is continuous in $\mu > 0$.*

Theorem 1.2. *As $\mu \to 0$, the following asymptotic formula holds:*

$$\lambda(\mu) = C_1\mu^{(q-1)/(p-1)} + o(\mu^{(q-1)/(p-1)}), \tag{1.11}$$

where

$$C_1 = \frac{(p-1)\|v_\infty\|_{p+1}^{p+1}}{(p+1)\|v_\infty\|_{q+1}^{q+1}}$$

and v_∞ is a unique positive solution of the minimizing problem

$$\text{Minimize } \frac{1}{q+1}\|w\|_{q+1}^{q+1} \text{ under the constraint} \tag{1.12}$$

$$w \in V_0 := \left\{ w \in X : \frac{1}{2}\|w\|_V^2 = \frac{1}{p+1}\|w\|_{p+1}^{p+1},\ w \not\equiv 0 \right\}. \tag{1.13}$$

Remark. We note that for $\mu > 0$, $N_{\mu,\alpha} \neq \emptyset$. In fact, for $t \geq 0$ and $0 \not\equiv u \in X$, we define

$$\begin{aligned} h(t) = h(t,\mu,u) &:= \frac{1}{2}\|tu\|_X^2 - \frac{1}{p+1}\mu\|tu\|_{p+1}^{p+1} \\ &= \frac{1}{2}\|u\|_X^2 t^2 - \frac{1}{p+1}\mu\|u\|_{p+1}^{p+1}t^{p+1}. \end{aligned} \tag{1.14}$$

Then it is easy to see by direct calculation that there exists a unique $t = t_u$ such that $h(t_u) = -\alpha$, that is, $t_u u \in N_{\mu,\alpha}$. On the other hand, it is clear that $N_{\mu,\alpha} = \emptyset$ for $\mu \leq 0$.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 consists of three lemmas: existence, uniqueness and continuity.

Lemma 2.1. *There exists a variational eigenvalue* $\lambda(\mu)$ *for* $\mu > 0$.

Proof. For the existence of $\lambda(\mu)$, we apply the result of Zeidler [8, Proposition 6a]. We shall check the following property: For fixed $\mu > 0$, the set

$$W_b := \left\{u \in N_{\mu,\alpha} : \frac{1}{q+1}\|u\|_{q+1}^{q+1} < b\right\} \subset X \tag{2.1}$$

is bounded for all $b > 0$. All the other conditions imposed in [8, Proposition 6a] are easily checked. By Hölder's inequality we obtain, for $u \in X$ and $0 < \gamma < q+1$,

$$\begin{aligned} \|u\|_{p+1}^{p+1} &= \int_I u^\gamma u^{p+1-\gamma}\,dx \\ &\le \left(\int_I u^{\gamma(q+1)/\gamma}\,dx\right)^{\gamma/(q+1)} \\ &\quad\times \left(\int_I u^{(p+1-\gamma)(q+1)/(q+1-\gamma)}\,dx\right)^{(q+1-\gamma)/(q+1)} \\ &= \|u\|_{q+1}^{\gamma}\|u\|_{(p+1-\gamma)(q+1)/(q+1-\gamma)}^{p+1-\gamma} \le \|u\|_{q+1}^{\gamma}\|u\|_X^{p+1-\gamma}. \end{aligned} \tag{2.2}$$

Then for $u \in W_b$ we obtain, by (2.2),

$$\|u\|_X^2 \le \frac{1}{p+1}\mu\|u\|_{p+1}^{p+1} \le \frac{1}{p+1}\mu\|u\|_{q+1}^{\gamma}\|u\|_X^{p+1-\gamma} \le \frac{1}{p+1}\mu b\|u\|_X^{p+1-\gamma}; \tag{2.3}$$

this implies that

$$\|u\|_X^{1+\gamma-p} \le \frac{1}{p+1}\mu b. \tag{2.4}$$

We choose $\gamma > 0$ satisfying $1+\gamma-p > 0$, that is, $\gamma > p-1$. Hence, γ must satisfy $p-1 < \gamma < q+1$ and it is possible to choose such $\gamma > 0$ under the condition $p < q+2$. Thus we obtain (2.1). Therefore, we can apply [8, Proposition 6a] to obtain the existence of $\lambda(\mu)$ for $\mu > 0$. $\square$

Remark 2.2. (1) It follows from the proof of Lemma 2.1 that $\beta(\mu) > 0$ for $\mu > 0$. In fact, if $\beta(\mu) = 0$ for some $\mu > 0$, then there exists $u \in N_{\mu,\alpha}$ such that $\frac{1}{q+1}\|u\|_{q+1}^{q+1} = \beta(\mu) = 0$. Then by (2.3) we obtain $\|u\|_X = 0$. However, this is impossible, since $0 \notin N_{\mu,\alpha}$.

(2) Let $0 \not\equiv u \in X$ be fixed. Furthermore, let $h(t) := h(t,\mu,u)$ be the function defined in (1.14). Then it is obvious that there exists a unique $t_{\mu,u} > 0$

such that $h(t_{\mu,u}) = -\alpha$, that is, $t_{\mu,u}u \in N_{\mu,\alpha}$. Furthermore, if $h(t) < -\alpha$, then $t_{\mu,u} < t$.

Next, we show the uniqueness of $\lambda(\mu)$.

Lemma 2.3. *Let $(\mu, \lambda_1(\mu), u_1)$ and $(\mu, \lambda_2(\mu), u_2)$ satisfy* (1.7) *and* (1.8) *for the same $\mu > 0$. Then $\lambda_1(\mu) = \lambda_2(\mu)$.*

Proof. We know from Gidas, Ni and Nirenberg [4] that the solution of (1.1) is symmetric with respect to $x = 1/2$ and $u'(x) \geq 0$ for $0 \leq x \leq 1/2$. We assume that $u_1'(0) < u_2'(0)$. Multiplying (1.1) by $u_\mu'(x)$, we obtain

$$u_\mu''(x)u_\mu'(x) + \mu u_\mu(x)^p u_\mu'(x) - \lambda(\mu)u_\mu(x)^q u_\mu'(x) = 0;$$

that is,

$$\frac{d}{dx}\left\{\frac{1}{2}u_\mu'(x)^2 + \frac{1}{p+1}\mu u_\mu(x)^{p+1} - \frac{1}{q+1}\lambda(\mu)u_\mu(x)^{q+1}\right\} = 0;$$

by putting $x = 0$ we have

$$\frac{1}{2}u_\mu'(x)^2 + \frac{1}{p+1}\mu u_\mu(x)^{p+1} - \frac{1}{q+1}\lambda(\mu)u_\mu(x)^{q+1} = \frac{1}{2}u_\mu'(0)^2. \tag{2.5}$$

Since $\|u_1\|_{q+1} = \|u_2\|_{q+1}$ by (1.8) and $0 < u_1'(0) < u_2'(0)$, there exists $x_1 \in (0, 1/2)$ such that

$$u_1(x) < u_2(x) \quad \text{for } 0 < x < x_1,$$
$$u_1(x_1) = u_2(x_1) = U \quad \text{and} \quad u_1'(x_1) \geq u_2'(x_1).$$

Then by putting $x = x_1$ in (2.5), we obtain

$$\begin{aligned}\frac{1}{2}u_1'(x_1)^2 &+ \frac{1}{p+1}\mu U^{p+1} - \frac{1}{q+1}\lambda_1(\mu)U^{q+1} \\ &= \frac{1}{2}u_1'(0)^2 < \frac{1}{2}u_2'(0)^2 = \frac{1}{2}u_2'(x_1)^2 + \frac{1}{p+1}\mu U^{p+1} - \frac{1}{q+1}\lambda_2(\mu)U^{q+1};\end{aligned}$$

this implies that

$$\frac{1}{q+1}(\lambda_1(\mu) - \lambda_2(\mu))U^{q+1} > \frac{1}{2}(u_1'(x_1)^2 - u_2'(x_1)^2) \geq 0. \tag{2.6}$$

Therefore, we obtain $\lambda_1(\mu) > \lambda_2(\mu)$.

Next, we integrate (2.5) over I to obtain

$$\frac{1}{2}\|u_\mu\|_X^2 + \frac{1}{p+1}\mu\|u_\mu\|_{p+1}^{p+1} - \frac{1}{q+1}\lambda(\mu)\|u_\mu\|_{q+1}^{q+1} = \frac{1}{2}u_\mu'(0)^2. \tag{2.7}$$

By (1.9) and (1.10) we obtain

$$\|u_\mu\|_X^2 = \frac{2}{p-1}\lambda(\mu)\|u_\mu\|_{q+1}^{q+1} - \frac{2(p+1)}{p-1}\alpha, \tag{2.8}$$

$$\mu\|u_\mu\|_{p+1}^{p+1} = \frac{p+1}{p-1}(\lambda(\mu)\|u_\mu\|_{q+1}^{q+1} - 2\alpha). \tag{2.9}$$

Now by using (2.7)–(2.9) we obtain

$$\frac{2q+3-p}{(p-1)(q+1)}\lambda(\mu)\|u_\mu\|_{q+1}^{q+1} - \frac{p+3}{p-1}\alpha = \frac{1}{2}u'_\mu(0)^2. \tag{2.10}$$

Then

$$\begin{aligned}\frac{2q+3-p}{p-1}&\lambda_1(\mu)\beta(\mu) - \frac{p+3}{p-1}\alpha \\ &= \frac{1}{2}u'_1(0)^2 < \frac{1}{2}u'_2(0)^2 = \frac{2q+3-p}{p-1}\lambda_2(\mu)\beta(\mu) - \frac{p+3}{p-1}\alpha.\end{aligned} \tag{2.11}$$

Noting that $2q+3-p > 2q+3-(q+2) = q+1 > 0$ and $\beta(\mu) > 0$, we deduce that $\lambda_1(\mu) < \lambda_2(\mu)$. This is a contradiction. Hence, $u'_1(0) < u'_2(0)$ is impossible. By this argument, we see that $u'_1(0) > u'_2(0)$ is also impossible. Hence, we obtain $u'_1(0) = u'_2(0)$. Then by (2.10),

$$\begin{aligned}\frac{2q+3-p}{p-1}&\lambda_1(\mu)\beta(\mu) - \frac{p+3}{p-1}\alpha \\ &= \frac{1}{2}u'_1(0)^2 = \frac{1}{2}u'_2(0)^2 = \frac{2q+3-p}{p-1}\lambda_2(\mu)\beta(\mu) - \frac{p+3}{p-1}\alpha;\end{aligned}$$

this along with Remark 2.2(1) implies that $\lambda_1(\mu) = \lambda_2(\mu)$. □

In order to prove continuity of $\lambda(\mu)$, we prepare the following lemma.

Lemma 2.4. *$\beta(\mu)$ is continuous in $\mu > 0$.*

Proof. We fix $\mu_0 > 0$. Let $\mu \to \mu_0$. Firstly, we show that

$$\limsup_{\mu\to\mu_0} \beta(\mu) \le \beta(\mu_0). \tag{2.12}$$

Since $u_{\mu_0} \in N_{\mu_0,\alpha}$, we have

$$\frac{1}{2}\|u_{\mu_0}\|_X^2 - \frac{1}{p+1}\mu\|u_{\mu_0}\|_{p+1}^{p+1} = -\alpha - \frac{1}{p+1}(\mu-\mu_0)\|u_{\mu_0}\|_{p+1}^{p+1}. \tag{2.13}$$

For $h(t) = h(t,\mu,u_{\mu_0})$, which is defined in (1.14), let $t_\mu > 0$ satisfy $h(t_\mu) = -\alpha$. Then $t_\mu u_{\mu_0} \in N_{\mu,\alpha}$ and we see by (1.8) that

$$\beta(\mu) \le \frac{1}{q+1}t_\mu^{q+1}\|u_{\mu_0}\|_{q+1}^{q+1} = t_\mu^{q+1}\beta(\mu_0). \tag{2.14}$$

We show that $t_\mu \to 1$ as $\mu \to \mu_0$. By definition of h, we have

$$\begin{aligned}-\alpha = h(t_\mu) &= \frac{1}{2}t_\mu^2\|u_{\mu_0}\|_X^2 - \frac{1}{p+1}\mu t_\mu^{p+1}\|u_{\mu_0}\|_{p+1}^{p+1} \\ &= t_\mu^2\left(\frac{1}{p+1}\mu_0\|u_{\mu_0}\|_{p+1}^{p+1} - \alpha\right) - \frac{1}{p+1}\mu t_\mu^{p+1}\|u_{\mu_0}\|_{p+1}^{p+1};\end{aligned} \tag{2.15}$$

that is,

$$\frac{1}{p+1}\|u_{\mu_0}\|_{p+1}^{p+1}\{(\mu_0-\mu)t_\mu^2 + \mu t_\mu^2(1-t_\mu^{p-1})\} = \alpha(t_\mu^2-1). \tag{2.16}$$

There are three cases to consider. Firstly, if there exists a subsequence of $\{t_\mu\}$, which we write $\{t_\mu\}$ again, such that $t_\mu \to \infty$ as $\mu \to \mu_0$, then it is clear that the left hand side of (2.16) tends to $-\infty$ as $\mu \to \mu_0$, while the right hand side of (2.16) tends to ∞ as $\mu \to \mu_0$. This is a contradiction.

Secondly, if there exists a subsequence of $\{t_\mu\}$, which we write $\{t_\mu\}$ again, such that $t_\mu < 1 - \delta$ for some $0 < \delta \ll 1$, then the left hand side of (2.16) is positive when μ and μ_0 are close enough, while the right hand side of (2.16) is negative. This is a contradiction. Similarly, we can show that there exists no subsequence of $\{t_\mu\}$ which satisfies $t_\mu > 1+\delta$ for some $\delta > 0$. Thus, we conclude that $t_\mu \to 1$ as $\mu \to \mu_0$. Then (2.12) follows immediately from (2.15).

Next, we show that

$$\beta(\mu_0) \le \liminf_{\mu\to\mu_0} \beta(\mu). \tag{2.17}$$

Let $t_\mu > 0$ satisfy $t_\mu u_\mu \in N_{\mu_0,\alpha}$, that is,

$$\frac{1}{2}\|t_\mu u_\mu\|_X^2 - \frac{1}{p+1}\mu_0\|t_\mu u_\mu\|_{p+1}^{p+1} = -\alpha;$$

this implies that

$$\frac{1}{p+1}\|u_\mu\|_{p+1}^{p+1}\left\{(\mu-\mu_0)t_\mu^2 + \mu_0 t_\mu^2(1-t_\mu^{p-1})\right\} = \alpha(t_\mu^2 - 1). \tag{2.18}$$

By Remark 2.2(1), there exists a constant $C > 0$ such that for $|\mu - \mu_0| \ll 1$,

$$\|u_\mu\|_{p+1} \ge \|u_\mu\|_{q+1} = (q+1)\beta(\mu) \ge C > 0.$$

Then by (2.18) and the same arguments as those used just above, we also find that $t_\mu \to 1$ as $\mu \to \mu_0$. Therefore,

$$\beta(\mu_0) \le \frac{1}{q+1}\|t_\mu u_\mu\|_{q+1}^{q+1} = t_\mu^{q+1}\beta(\mu);$$

by letting $\mu \to \mu_0$, we obtain (2.17). Now our assertion follows from (2.12) and (2.17). □

Now we are in a position to prove the continuity of $\lambda(\mu)$.

Lemma 2.5. *$\lambda(\mu)$ is continuous for $\mu > 0$.*

Proof. For convenience, we identify notations of subsequences with those of the original sequences. Let $\mu \to \mu_0 > 0$. We know from (2.3) that

$$\|u_\mu\|_X^{1+\gamma-p} \le \frac{1}{p+1}\mu\|u_\mu\|_{q+1}^\gamma = \frac{1}{p+1}\mu((q+1)\beta(\mu))^{\gamma/(q+1)}. \tag{2.19}$$

Then it follows from Lemma 2.4 and (2.19) that $\{u_\mu\} \subset X$ is bounded. Hence, by Sobolev's embedding theorem, we see that there exist a subsequence of $\{u_\mu\}$ and $u_\infty \in X$ such that

$$u_\mu \to u_\infty \text{ weakly in } X, \quad u_\mu \to u_\infty \text{ in } L^{p+1}(I), L^{q+1}(I). \tag{2.20}$$

Then

$$\|u_\infty\|_X^2 \le \liminf_{\mu\to\mu_0} \|u_\mu\|_X^2 = \lim_{\mu\to\mu_0}\left(\frac{2}{p+1}\mu\|u_\mu\|_{p+1}^{p+1} - 2\alpha\right)$$
$$= \frac{2}{p+1}\mu_0\|u_\infty\|_{p+1}^{p+1} - 2\alpha;$$

this implies that

$$-2\alpha_\infty := \|u_\infty\|_X^2 - \frac{2}{p+1}\mu_0\|u_\infty\|_{p+1}^{p+1} \le -2\alpha. \tag{2.21}$$

We show that $\alpha_\infty = \alpha$. For $t \ge 0$, let $h(t) = h(t, \mu_0, u_\infty)$, which is defined in (1.14). Then clearly, $h(t_1) \le h(t_2) < 0$ implies that $t_2 \le t_1$. Since $h(1) = -\alpha_\infty$, there exists a unique $t_\infty \le 1$ such that $h(t_\infty) = -\alpha$, that is, $t_\infty u_\infty \in N_{\mu_0,\alpha}$. Then by (1.8), (2.20) and Lemma 2.4,

$$\beta(\mu_0) \le \frac{1}{q+1}\|t_\infty u_\infty\|_{q+1}^{q+1} = t_\infty^{q+1}\lim_{\mu\to\mu_0}\frac{1}{q+1}\|u_\mu\|_{q+1}^{q+1} \tag{2.22}$$
$$= t_\infty^{q+1}\lim_{\mu\to\mu_0}\beta(\mu) = t_\infty^{q+1}\beta(\mu_0).$$

This implies that $t_\infty = 1$, that is, $\alpha_\infty = \alpha$. Therefore, $u_\infty \in N_{\mu_0,\alpha}$ and $\frac{1}{q+1}\|u_\infty\|_{q+1}^{q+1} = \beta(\mu_0)$.

It follows from (1.9), (2.19) and Remark 2.2(1) that for $|\mu - \mu_0| \ll 1$,

$$\lambda(\mu) \le \frac{2\alpha + \dfrac{p-1}{p+1}\mu\|u_\mu\|_{p+1}^{p+1}}{(q+1)\beta(\mu)} \le C. \tag{2.23}$$

Hence, we can choose a subsequence of $\{\lambda(\mu)\}$ such that $\lambda(\mu) \to \lambda_\infty$ as $\mu \to \mu_0$. We see from (1.1) that for $\varphi \in C_0^\infty(I)$,

$$-\int_I u_\mu'\varphi'\,dx + \mu\int_I u_\mu^p\varphi\,dx = \lambda(\mu)\int_I u_\mu^q\varphi\,dx; \tag{2.24}$$

by letting $\mu \to \mu_0$, we obtain

$$-\int_I u_\infty'\varphi'\,dx + \mu_0\int_I u_\infty^p\varphi\,dx = \lambda_\infty\int_I u_\infty^q\varphi\,dx. \tag{2.25}$$

Since the equation (1.1) is equivalent to its weak formulation, we find that $(\mu_0, \lambda_0, u_\infty) \in \mathbb{R}_+ \times \mathbb{R}_+ \times N_{\mu_0,\alpha}$ satisfies (1.7) and (1.8), and by the uniqueness of $\lambda(\mu_0)$, we obtain $\lambda_\infty = \lambda(\mu_0)$. Now our assertion follows from a standard compactness argument. $\square$

Theorem 1.1 follows from Lemmas 2.1, 2.3 and 2.5.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we prepare some lemmas. Hereafter, C denotes various positive constants independent of $0 < \mu \ll 1$.

Lemma 3.1. *There exists a constant $C > 0$ such that for $0 < \mu \ll 1$,*

$$C\mu^{-1/(p-1)} \le \|u_\mu\|_{q+1}. \tag{3.1}$$

Proof. We see from (2.2) that for $0 < \gamma < q + 1$,

$$\|u_\mu\|_{p+1}^{p+1} \le \|u_\mu\|_{q+1}^{\gamma}\|u_\mu\|_X^{p+1-\gamma} \le \|u_\mu\|_{q+1}^{\gamma}\left(\frac{2}{p+1}\mu\|u_\mu\|_{p+1}^{p+1}\right)^{(p+1-\gamma)/2}. \tag{3.2}$$

Let $\gamma = p - 1$. Then it follows from (3.2) that

$$\|u_\mu\|_{p+1}^{p+1} \le \frac{2}{p+1}\mu\|u_\mu\|_{q+1}^{p-1}\|u_\mu\|_{p+1}^{p+1};$$

this implies that

$$\frac{p+1}{2}\mu^{-1} \le \|u_\mu\|_{q+1}^{p-1}.$$

This yields (3.1). □

Lemma 3.2. *There exists a constant $C > 0$ such that for $0 < \mu \ll 1$,*

$$\|u_\mu\|_{q+1} \le C\mu^{-1/(p-1)}. \tag{3.3}$$

Proof. Fix $0 \not\equiv v \in X$. Put

$$h(t) = \frac{1}{2}t^2\|v\|_X^2 - \frac{1}{p+1}\mu t^{p+1}\|v\|_{p+1}^{p+1}.$$

Let $h(t_0) = -\alpha$. Furthermore, let $t = C\mu^{-1/(p-1)}$ for sufficiently large $C > 0$. Then for $0 < \mu \ll 1$,

$$h(C\mu^{-1/(p-1)}) = C^2\mu^{-2/(p-1)}\left(\frac{1}{2}\|v\|_X^2 - \frac{1}{p+1}C^{p-1}\|v\|_{p+1}^{p+1}\right) < -\alpha.$$

Therefore, by Remark 2.2(2) we obtain $t_0 < C\mu^{-1/(p-1)}$. Then by (1.8),

$$\|u_\mu\|_{q+1}^{q+1} \le \|t_0 v\|_{q+1}^{q+1} \le (C\mu^{-1/(p-1)})^{q+1}\|v\|_{q+1}^{q+1} \le C\mu^{-(q+1)/(p-1)}.$$

This implies (3.3). □

Lemma 3.3. *There exists a constant $C > 0$ such that for $0 < \mu \ll 1$,*

$$C\mu^{-1/(p-1)} \le \|u_\mu\|_{p+1} \le C^{-1}\mu^{-1/(p-1)}. \tag{3.4}$$

Proof. The first inequality follows immediately from Lemma 3.1 and Hölder's inequality. Hence, we show the second. By Lemma 3.2 and (3.2) we obtain

$$\begin{aligned} \|u_\mu\|_{p+1}^{(p+1)(1+\gamma-p)/2} &\le C\|u_\mu\|_{q+1}^{\gamma}\mu^{(p+1-\gamma)/2} \\ &\le C\mu^{-\gamma/(p-1)}\mu^{(p+1-\gamma)/2} \\ &= C\mu^{(p+1)(p-\gamma-1)/(2(p-1))}. \end{aligned} \tag{3.5}$$

Choose $\gamma > p-1$ to obtain

$$\|u_\mu\|_{p+1} \le C\mu^{-1/(p-1)}. \tag{3.6}$$ □

Now we put $v_\mu := \mu^{1/(p-1)}u_\mu$. Then by (1.9) we obtain

$$\lambda(\mu) = \frac{p-1}{p+1}\cdot\frac{\|v_\mu\|_{p+1}^{p+1}}{\|v_\mu\|_{q+1}^{q+1}}\mu^{(q-1)/(p-1)} + \frac{2\alpha}{\|v_\mu\|_{q+1}^{q+1}}\mu^{(q+1)/(p-1)}. \tag{3.7}$$

Furthermore, it follows from (1.1) that v_μ satisfies the following equation:

$$\begin{aligned} &v_\mu''(x) + v_\mu(x)^p = \lambda(\mu)\mu^{-(q-1)/(p-1)}v_\mu(x)^q, \quad x \in I,\\ &v_\mu(x) > 0, \quad x \in I,\\ &v_\mu(0) = v_\mu(1) = 0. \end{aligned} \tag{3.8}$$

Since $u_\mu \in N_{\mu,\alpha}$,

$$v_\mu \in V_{\mu,\alpha} := \left\{ v \in X : \frac{1}{2}\|v\|_X^2 - \frac{1}{p+1}\|v\|_{p+1}^{p+1} = -\alpha\mu^{2/(p-1)}. \right\} \tag{3.9}$$

Then by Lemma 3.3 we obtain

$$\frac{1}{2}\|v_\mu\|_X^2 \le \frac{1}{p+1}\|v_\mu\|_{p+1}^{p+1} \le C. \tag{3.10}$$

Hence, we can choose a subsequence of $\{v_\mu\}$, which we write $\{v_\mu\}$ again, such that

$$v_\mu \to v_\infty \text{ weakly in } X, \quad v_\mu \to v_\infty \text{ in } C(I), L^{p+1}(I), L^{q+1}(I). \tag{3.11}$$

Furthermore, it follows from (1.8) that

$$\|v_\mu\|_{q+1}^{q+1} = k(\mu) := \inf_{v\in V_{\mu,\alpha}} \|v\|_{q+1}^{q+1}. \tag{3.12}$$

We recall that V_0 is the set defined in (1.13). Let

$$k(0) := \inf_{v\in V_0} \|v\|_{q+1}^{q+1}. \tag{3.13}$$

We shall show that v_∞ is the unique positive solution of the minimizing problem (3.13). To this end, we prepare Lemmas 3.4–3.8.

Lemma 3.4. *Let*

$$k_1 := \inf_{w\in X, w\not\equiv 0}\left(\frac{p+1}{2}\cdot\frac{\|w\|_X^2\|w\|_{q+1}^{p-1}}{\|w\|_{p+1}^{p+1}}\right)^{(q+1)/(p-1)}. \tag{3.14}$$

Then $0 < k_1 \le k(0)$.

Proof. First, we show that $k_1 > 0$. By putting $\gamma = p-1$ in (3.2), for $w \in X$ we have

$$\|w\|_{p+1}^{p+1} \le \|w\|_X^2\|w\|_{q+1}^{p-1};$$

this implies that if $w \not\equiv 0$, then

$$1 \le \inf_{w \in X, w \not\equiv 0} \frac{\|w\|_X^2 \|w\|_{q+1}^{p-1}}{\|w\|_{p+1}^{p+1}}. \tag{3.15}$$

Thus our assertion follows from (3.15).

Next, we show that $k_1 \le k(0)$. Let $\{v_n\} \subset V_0$ be a minimizing sequence of the problem (3.13). Then

$$k_1 \le \left(\frac{p+1}{2} \cdot \frac{\|v_n\|_X^2 \|v_n\|_{q+1}^{p-1}}{\|v_n\|_{p+1}^{p+1}} \right)^{(q+1)/(p-1)} = \|v_n\|_{q+1}^{q+1}; \tag{3.16}$$

by letting $n \to \infty$, we obtain $k_1 \le k(0)$. □

Lemma 3.5. *$k(\mu) \to k(0)$ as $\mu \to 0$.*

Proof. Firstly, we show that

$$\limsup_{\mu \to 0} k(\mu) \le k(0). \tag{3.17}$$

Fix a small $\varepsilon > 0$. Then there exists $v_1 \in V_0$ such that

$$0 < k(0) \le \|v_1\|_{q+1}^{q+1} < k(0) + \varepsilon. \tag{3.18}$$

For $t \ge 0$, put

$$h(t) := \frac{1}{2}\|tv_1\|_X^2 - \frac{1}{p+1}\|tv_1\|_{p+1}^{p+1} = \frac{1}{p+1}\|v_1\|_{p+1}^{p+1}(t^2 - t^{p+1}). \tag{3.19}$$

It is clear that if $h(t_{0,\mu}) = -\alpha\mu^{2/(p-1)}$, then $t_{0,\mu}v_1 \in V_{\mu,\alpha}$. Furthermore, if $h(t_\mu) < -\alpha\mu^{2/(p-1)}$, then $t_{0,\mu} < t_\mu$. We put $t_\mu = 1 + C\mu^{2/(p-1)}$. Then

$$t_\mu^2 = 1 + 2C\mu^{2/(p-1)} + o(\mu^{2/(p-1)}), \tag{3.20}$$

$$t_\mu^{p+1} = 1 + (p+1)C\mu^{2/(p-1)} + o(\mu^{2/(p-1)}). \tag{3.21}$$

If we choose $C > 0$ so large that $C \ge 2(p+1)/((p-1)(k(0)+\varepsilon))$, then by (3.19)–(3.21) we obtain

$$h(t_\mu) = \frac{1}{p+1}\|v_1\|_{p+1}^{p+1}\{-(p-1)C\mu^{2/(p-1)} + o(\mu^{2/(p-1)})\} < -\alpha\mu^{2/(p-1)}. \tag{3.22}$$

Therefore, $t_{0,\mu} \le 1 + C\mu^{2/(p-1)}$. Then by (3.18) we obtain

$$\begin{aligned} \limsup_{\mu \to 0} k(\mu) &\le \limsup_{\mu \to 0} \|t_{0,\mu}v_1\|_{q+1}^{q+1} \le \limsup_{\mu \to 0} t_{0,\mu}^{q+1}(k(0) + \varepsilon) \\ &\le \limsup_{\mu \to 0} (1 + C\mu^{2/(p-1)})^{q+1}(k(0) + \varepsilon) = k(0) + \varepsilon. \end{aligned} \tag{3.23}$$

Since $\varepsilon > 0$ is arbitrary, we obtain (3.17) from (3.23).

Next, we show that

$$k(0) \le \liminf_{\mu \to 0} k(\mu). \tag{3.24}$$

For $t \geq 0$ and $0 < \mu \ll 1$ we put

$$
\begin{aligned}
(3.25)\qquad h(t, \mu) &:= \frac{1}{2}t^2\|v_\mu\|_X^2 - \frac{1}{p+1}t^{p+1}\|v_\mu\|_{p+1}^{p+1} \\
&= \left(\frac{1}{p+1}\|v_\mu\|_{p+1}^{p+1} - \alpha\mu^{2/(p-1)}\right)t^2 - \frac{1}{p+1}t^{p+1}\|v_\mu\|_{p+1}^{p+1}.
\end{aligned}
$$

Therefore, we see that $h(t_0, \mu) = 0$, where

$$
t_0 = \left(\frac{\|v_\mu\|_{p+1}^{p+1} - (p+1)\alpha\mu^{2/(p-1)}}{\|v_\mu\|_{p+1}^{p+1}}\right)^{1/(p-1)} < 1.
$$

Then $t_0 v_\mu \in V_0$ and by (3.13),

$$
k(0) \leq t_0^{q+1}\|v_\mu\|_{q+1}^{q+1} < k(\mu). \tag{3.26}
$$

By letting $\mu \to 0$ in (3.26), we obtain (3.24).

Finally, combining (3.17) and (3.24), we obtain our conclusion. □

Lemma 3.6. *Let v_∞ be the function obtained in* (3.11). *Then v_∞ is a nonnegative solution of the minimizing problem* (3.13).

Proof. We know from (3.11) that $v_\infty \geq 0$ in I. Furthermore, by (3.11) we obtain

$$
\begin{aligned}
(3.27)\qquad \frac{1}{2}\|v_\infty\|_X^2 &\leq \liminf_{\mu\to 0}\frac{1}{2}\|v_\mu\|_X^2 = \liminf_{\mu\to 0}\left(\frac{1}{p+1}\|v_\mu\|_{p+1}^{p+1} - \alpha\mu^{2/(p-1)}\right) \\
&= \frac{1}{p+1}\|v_\infty\|_{p+1}^{p+1}.
\end{aligned}
$$

We assume

$$
\frac{1}{2}\|v_\infty\|_X^2 - \frac{1}{p+1}\|v_\infty\|_{p+1}^{p+1} < 0 \tag{3.28}
$$

and derive a contradiction. For $t \geq 0$ put

$$
h(t) := \frac{1}{2}t^2\|v_\infty\|_X^2 - \frac{1}{p+1}t^{p+1}\|v_\infty\|_{p+1}^{p+1}.
$$

Then it is clear that $h(1) < 0$ and there exists a unique $t_0 < 1$ such that $h(t_0) = 0$, that is, $t_0 v_\infty \in V_0$. Then by (3.11), (3.13) and Lemma 3.5,

$$
k(0) \leq \|t_0 v_\infty\|_{q+1}^{q+1} < \|v_\infty\|_{q+1}^{q+1} = \lim_{\mu\to 0}\|v_\mu\|_{q+1}^{q+1} = \lim_{\mu\to 0}k(\mu) = k(0). \tag{3.29}
$$

This is a contradiction. Therefore, $t_0 = 1$, that is, $v_\infty \in V_0$. Furthermore, it follows from (3.29) that $\|v_\infty\|_{q+1}^{q+1} = k(0)$. □

Lemma 3.7. *Let* $v_\infty \geq 0$ *be a function defined in* (3.11). *Then* $v_\infty > 0$ *in* I.

Proof. By Lemma 3.6, we find that v_∞ satisfies the following equation:

$$\begin{aligned} & v''(x) + v(x)^p = C_v v(x)^q, \quad x \in I, \\ & v(x) \geq 0, \quad x \in I, \\ & v(0) = v(1) = 0. \end{aligned} \tag{3.30}$$

Here,

$$C_v = \frac{p-1}{p+1} \cdot \frac{\|v_\infty\|_{p+1}^{p+1}}{k(0)}$$

is a Lagrange multiplier. If there exists $x_0 \in I$ such that $v_\infty(x_0) = 0$, then clearly $v'_\infty(x_0) = 0$, since $v_\infty \geq 0$ in I. Then we deduce by the uniqueness theorem for ODE that $v_\infty \equiv 0$ in I. However, this is impossible, since we know from Lemmas 3.4 and 3.6 that $\frac{1}{q+1}\|v_\infty\|_{q+1}^{q+1} = k(0) > 0$. Hence, $v_\infty > 0$ in I. □

Lemma 3.8. *Let* $v_1, v_2 > 0$ *satisfy the minimizing problem* (3.13). *Furthermore, let* $C_j = C_{v_j}$ $(j = 1, 2)$ *be positive constants defined in* (3.30). *Then* $C_1 = C_2$.

Proof. We assume that $v'_1(0) < v'_2(0)$. Since v_1 and v_2 satisfy (3.13), there exists $x_1 \in (0, 1/2)$ such that

$$\begin{aligned} & v_1(x) \leq v_2(x) \quad \text{for any } x \in (0, x_1), \\ & v_1(x_1) = v_2(x_1) = d > 0, \quad v'_1(x_1) \geq v'_2(x_1). \end{aligned} \tag{3.31}$$

By the same argument as that used to obtain (2.5), we deduce from (3.30) that for $x \in I$ and $j = 1, 2$,

$$\frac{1}{2}v'_j(x)^2 + \frac{1}{p+1}v_j(x)^{p+1} - \frac{1}{q+1}C_j v_j(x)^{q+1} = \frac{1}{2}v'_j(0)^2. \tag{3.32}$$

Then by putting $x = x_1$ in (3.32), we infer by (3.31) that

$$\begin{aligned} 0 &\leq \frac{1}{2}(v'_1(x_1)^2 - v'_2(x_1)^2) \\ &= \frac{1}{2}v'_1(0)^2 - \frac{1}{p+1}d^{p+1} + \frac{1}{q+1}C_1 d^{q+1} \\ &\quad - \frac{1}{2}v'_2(0)^2 + \frac{1}{p+1}d^{p+1} - \frac{1}{q+1}C_2 d^{q+1}; \end{aligned} \tag{3.33}$$

this implies that

$$0 < \frac{1}{2}(v'_1(x_1)^2 - v'_2(x_1)^2) + \frac{1}{2}(v'_2(0)^2 - v'_1(0)^2) = \frac{1}{q+1}(C_1 - C_2)d^{q+1}. \tag{3.34}$$

Thus we obtain $C_1 > C_2$.

Next, by integrating (3.32) over I, we obtain, for $j = 1, 2$,

$$\frac{1}{2}\|v_j\|_X^2 + \frac{1}{p+1}\|v_j\|_{p+1}^{p+1} - \frac{1}{q+1}C_j\|v_j\|_{q+1}^{q+1} = \frac{1}{2}v'_j(0)^2. \tag{3.35}$$

Since $v_j \in V_0$, we find by (3.13) and (3.35) that for $j = 1, 2$,

$$\frac{2}{p+1}\|v_j\|_{p+1}^{p+1} - \frac{1}{q+1}C_j k(0) = \frac{1}{2}v_j'(0)^2. \tag{3.36}$$

Multiplying v_j by (3.30) and integrating by parts we obtain, for $j = 1, 2$,

$$-\|v_j\|_X^2 + \|v_j\|_{p+1}^{p+1} = C_j \|v_j\|_{q+1}^{q+1} = C_j k(0). \tag{3.37}$$

Since $v_j \in V_0$, we see by (3.37) that for $j = 1, 2$,

$$\|v_j\|_{p+1}^{p+1} = \frac{p+1}{p-1}C_j k(0). \tag{3.38}$$

Now, by using (3.36) and (3.38), for $j = 1, 2$ we obtain

$$\frac{2q+3-p}{(p-1)(q+1)}C_j k(0) = \frac{1}{2}v_j'(0)^2. \tag{3.39}$$

Since $2q + 3 - p > 2q + 3 - (q + 2) = q + 1 > 0$, it follows from (3.39) that

$$\frac{2q+3-p}{(p-1)(q+1)}C_1 k(0) = \frac{1}{2}v_1'(0)^2 < \frac{1}{2}v_2'(0)^2 = \frac{2q+3-p}{(p-1)(q+1)}C_2 k(0);$$

this implies that $C_1 < C_2$. This is a contradiction. Hence, $v_1'(0) \geq v_2'(0)$. However, by the same arguments as those used just above, we find that $v_1'(0) > v_2'(0)$ is also impossible. Hence, we obtain $v_1'(0) = v_2'(0)$. Then our assertion follows immediately from (3.39). □

PROPOSITION 3.9. *Let $v_1, v_2 > 0$ satisfy* (3.13). *Then $v_1 \equiv v_2$.*

PROOF. It follows from Lemma 3.8 that $C_1 = C_2$. Hence v_1 and v_2 satisfy (3.30) for the same $C = C_{v_1} = C_{v_2}$. By (3.39) we have $v_1'(0) = v_2'(0)$. Hence, our conclusion follows immediately from the uniqueness theorem for ODE. □

Now we are in a position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Let v_∞ be the unique positive solution of (3.13). Then by Lemma 3.8, the constant C_{v_∞} which appears in (3.30) is uniquely determined, namely, $C_1 = C_{v_\infty}$, where C_1 is the constant defined in Theorem 1.2. Then by (3.7), (3.11), Lemmas 3.6–3.8 and Proposition 3.9, we conclude that as $\mu \to 0$,

$$\lambda(\mu)\mu^{-(q-1)/(p-1)} = \frac{p-1}{p+1} \cdot \frac{\|v_\mu\|_{p+1}^{p+1}}{\|v_\mu\|_{q+1}^{q+1}} + O(\mu^{2/(p-1)}) \to \frac{p-1}{p+1} \cdot \frac{\|v_\infty\|_{p+1}^{p+1}}{\|v_\infty\|_{q+1}^{q+1}} = C_1. \ \square$$

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Manuscript received October 26, 1995

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